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OPERATOR ERROR ESTIMATES FOR HOMOGENIZATION OF THE ELLIPTIC DIRICHLET PROBLEM IN A BOUNDED DOMAIN

M. A. PAKHNIN AND T. A. SUSLINA

Abstract. Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class $C^{1,1}$. In the Hilbert space $L_2(\mathcal{O};\mathbb{C}^n)$, a matrix elliptic second order differential operator $\mathcal{A}_{D,\varepsilon}$ is considered with the Dirichlet boundary condition. Here $\varepsilon > 0$ is a small parameter. The coefficients of the operator are periodic and depend on $\mathbf{x}/\varepsilon$. Approximation is found for the operator $\mathcal{A}_{D,\varepsilon}^{-1}$ in the norm of operators acting from $L_2(\mathcal{O};\mathbb{C}^n)$ to the Sobolev space $H^1(\mathcal{O};\mathbb{C}^n)$ with an error term of $O(\sqrt{\varepsilon})$. This approximation is given by the sum of the operator $(\mathcal{A}_D^0)^{-1}$ and the first order corrector, where $\mathcal{A}_D^0$ is the effective operator with constant coefficients and with the Dirichlet boundary condition.

Introduction

The paper concerns homogenization theory of periodic differential operators (DO's). A broad literature is devoted to homogenization problems in the small period limit. At the first place, we mention the books [BeLP, BaPa, ZhKO].

0.1. Operator-theoretic approach to homogenization problems. In a series of papers [BSu1, BSu2, BSu3, BSu4, BSu5] by M. Sh. Birman and T. A. Suslina, a new operator-theoretic (spectral) approach to homogenization problems was suggested and developed. By this approach, the so-called operator error estimates in homogenization problems for elliptic DO's were obtained. The objects of study were matrix elliptic DO's acting in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ and admitting a factorization of the form $\mathcal{A}_\varepsilon = b(\mathbf{D})^* g(\mathbf{x}/\varepsilon) b(\mathbf{D})$, $\varepsilon > 0$. Here $g(\mathbf{x})$ is an $(m\times m)$-matrix-valued function assumed to be bounded, uniformly positive definite, and periodic with respect to some lattice Γ. Let Ω denote the elementary cell of the lattice Γ. Assume that $m \geq n$ and $b(\mathbf{D})$ is an $(m \times n)$-matrix homogeneous first order DO such that $\operatorname{rank} b(\boldsymbol{\xi}) = n$ for $0 \neq \boldsymbol{\xi} \in \mathbb{R}^d$. The simplest example of such an operator is the scalar elliptic operator $\mathcal{A}_\varepsilon = -\operatorname{div} g(\mathbf{x}/\varepsilon)\nabla$. The operator of elasticity theory can also be written in this form. These and other examples were considered in [BSu2] in detail.

In [BSu1, BSu2, BSu3, BSu4, BSu5], the equation $\mathcal{A}_\varepsilon \mathbf{u}_\varepsilon + \mathbf{u}_\varepsilon = \mathbf{F}$, where $\mathbf{F} \in L_2(\mathbb{R}^d;\mathbb{C}^n)$, was considered. The behavior of the solution $\mathbf{u}_\varepsilon$ for small ε was studied. As $\varepsilon \to 0$, the solution $\mathbf{u}_\varepsilon$ converges in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to the solution $\mathbf{u}_0$ of the "homogenized" equation $\mathcal{A}^0\mathbf{u}_0 + \mathbf{u}_0 = \mathbf{F}$. Here $\mathcal{A}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$ is the *effective operator* with the constant effective matrix g^0. In [BSu1, BSu2], it was proved that

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathbb{R}^d)} \leq C\varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^d)}.$$

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In operator terms, this means that, as $\varepsilon \to 0$, the resolvent $(\mathcal{A}_\varepsilon + I)^{-1}$ converges in the operator norm in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to the resolvent of the effective operator, and

$$\big\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}^0 + I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le C\varepsilon. \tag{0.1}$$

In [BSu3, BSu4], more accurate approximation of the resolvent $(\mathcal{A}_\varepsilon + I)^{-1}$ in the operator norm in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ with an error term $O(\varepsilon^2)$ was obtained. (Here we do not go into details.)

In [BSu5], approximation of the resolvent $(\mathcal{A}_\varepsilon + I)^{-1}$ in the norm of operators acting from $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to the Sobolev space $H^1(\mathbb{R}^d;\mathbb{C}^n)$ was found:

$$\big\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}^0 + I)^{-1} - \varepsilon K(\varepsilon)\big\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \le C\varepsilon; \tag{0.2}$$

this corresponds to approximation of $\mathbf{u}_\varepsilon$ in the "energy" norm. Here $K(\varepsilon)$ is a corrector. It involves rapidly oscillating factors and so depends on ε.

Estimates (0.1), (0.2) are called the *operator error estimates*. They are order-sharp; the constants in estimates are controlled explicitly in terms of the problem data. The method of [BSu1, BSu2, BSu3, BSu4, BSu5] is based on the scaling transformation and the Floquet–Bloch theory. The operator $\mathcal{A} = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D})$ is expanded in the direct integral of certain operators $\mathcal{A}(\mathbf{k})$ acting in $L_2(\Omega;\mathbb{C}^n)$ and depending on the parameter $\mathbf{k}$ (the quasimomentum). The operator family $\mathcal{A}(\mathbf{k})$ has discrete spectrum and depends on $\mathbf{k}$ analytically. It can be studied by methods of the analytic perturbation theory. It turns out that only the spectral characteristics of the operator $\mathcal{A}$ near the bottom of its spectrum are important for constructing the effective operator and obtaining error estimates. This shows that homogenization can be interpreted as a spectral threshold effect.

0.2. A different approach to operator error estimates in homogenization problems was suggested by V. V. Zhikov. In [Zh1, Zh2, ZhPas, Pas], the scalar elliptic operator $-\operatorname{div} g(\mathbf{x}/\varepsilon)\nabla$ (where $g(\mathbf{x})$ is a matrix with real entries) and the system of elasticity theory were studied. Estimates of the form (0.1), (0.2) for the corresponding problems in $\mathbb{R}^d$ were obtained. The method was based on analysis of the first order approximation to the solution and on introducing an additional parameter (the shift by the vector $\boldsymbol{\omega} \in \Omega$). Besides the problems in $\mathbb{R}^d$, homogenization problems in a bounded domain $\mathcal{O} \subset \mathbb{R}^d$ with the Dirichlet or Neumann boundary condition were studied. Approximations of the solution in $H^1(\mathcal{O})$ were deduced from the corresponding result in $\mathbb{R}^d$. Due to the boundary influence, estimates in a bounded domain become worse and the error term is $O(\varepsilon^{1/2})$. The estimate $\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O})} \le C\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}$ follows directly from approximation of the solution in $H^1(\mathcal{O})$.

Similar results for the operator $-\operatorname{div} g(\mathbf{x}/\varepsilon)\nabla$ in a bounded domain with the Dirichlet or Neumann boundary condition were obtained in the papers [Gr1, Gr2] by G. Griso by the "unfolding" method.

0.3. Main results. In the present paper, we study matrix DO's $\mathcal{A}_{D,\varepsilon}$ in a bounded domain $\mathcal{O} \subset \mathbb{R}^d$ of class $C^{1,1}$. The operator $\mathcal{A}_{D,\varepsilon}$ is defined by the differential expression $b(\mathbf{D})^* g(\mathbf{x}/\varepsilon) b(\mathbf{D})$ with the Dirichlet condition on $\partial\mathcal{O}$. The effective operator $\mathcal{A}_D^0$ is given by the expression $b(\mathbf{D})^* g^0 b(\mathbf{D})$ with the Dirichlet boundary condition. We study the behavior for small ε of the solution $\mathbf{u}_\varepsilon$ of the equation $\mathcal{A}_{D,\varepsilon}\mathbf{u}_\varepsilon = \mathbf{F}$, where $\mathbf{F} \in L_2(\mathcal{O};\mathbb{C}^n)$, obtaining estimates for the H^1-norm of the difference of the solution $\mathbf{u}_\varepsilon$ and its first order approximation including a corrector. As a rough consequence of this result, we get an estimate for $\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O})}$. Here $\mathbf{u}_0$ is the solution of the equation $\mathcal{A}_D^0\mathbf{u}_0 = \mathbf{F}$.

The main results of the paper are Theorems 6.1 and 7.1. In operator terms, the following estimates are obtained:

$$\big\| \mathcal{A}_{D,\varepsilon}^{-1} - (\mathcal{A}_D^0)^{-1} - \varepsilon K_D(\varepsilon) \big\|_{L_2(\mathcal{O}) \to H^1(\mathcal{O})} \le C\varepsilon^{1/2}, \tag{0.3}$$

$$\big\| \mathcal{A}_{D,\varepsilon}^{-1} - (\mathcal{A}_D^0)^{-1} \big\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \le C\varepsilon^{1/2}. \tag{0.4}$$

Here $K_D(\varepsilon)$ is the corresponding corrector. The form of the corrector depends on the properties of the periodic solution $\Lambda(\mathbf{x})$ of the auxiliary problem (1.5). If Λ is bounded, the corrector has a standard form (see Theorem 6.1). In the general case the corrector contains an auxiliary smoothing operator (see Theorem 7.1). Besides approximation of the solution $\mathbf{u}_\varepsilon$ in $H^1(\mathcal{O};\mathbb{C}^n)$, we also approximate the "flux" $\mathbf{p}_\varepsilon := g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$ in $L_2(\mathcal{O};\mathbb{C}^m)$.

0.4. Our method is based on estimates (0.1), (0.2) for homogenization problem in $\mathbb{R}^d$ obtained in [BSu2, BSu5] and on certain tricks suggested in [Zh2, ZhPas] that make it possible to deduce estimate (0.3) from (0.1), (0.2). The main difficulties are related to estimating the "discrepancy" $\mathbf{w}_\varepsilon$, which satisfies the equation $\mathcal{A}_\varepsilon \mathbf{w}_\varepsilon = 0$ in $\mathcal{O}$ and the boundary condition $\mathbf{w}_\varepsilon = \varepsilon K_D(\varepsilon)\mathbf{F}$ on $\partial\mathcal{O}$. Note that we cannot use the facts specific for scalar elliptic equations, because we study a wide class of matrix elliptic DO's.

0.5. Error estimates in $L_2(\mathcal{O})$. It should be mentioned that estimate (0.4) is quite a rough consequence of (0.3). So, refinement of estimate (0.4) is a natural problem. In [ZhPas], for the case of the scalar elliptic operator $-\operatorname{div} g(\mathbf{x}/\varepsilon)\nabla$ (where $g(\mathbf{x})$ is a matrix with real entries), an estimate for $\big\| \mathcal{A}_{D,\varepsilon}^{-1} - (\mathcal{A}_D^0)^{-1} \big\|_{L_2 \to L_2}$ of order $\varepsilon^{\frac{d}{2d-2}}$ for $d \ge 3$ and of order $\varepsilon|\log\varepsilon|$ for $d = 2$ was obtained. The proof was based on the maximum principle, which is specific for scalar elliptic equations. In [Gr2], a sharp order estimate $\big\| \mathcal{A}_{D,\varepsilon}^{-1} - (\mathcal{A}_D^0)^{-1} \big\|_{L_2 \to L_2} \le C\varepsilon$ was established for the same scalar elliptic operator.

Using the results and technique of the present paper, one of the authors obtained a *sharp order operator error estimate*

$$\big\| \mathcal{A}_{D,\varepsilon}^{-1} - (\mathcal{A}_D^0)^{-1} \big\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \le C\varepsilon$$

for the matrix DO's under consideration. A separate paper [Su2] and a brief communication [Su1] are devoted to the proof of this result.

0.6. Organization of the paper. The paper contains seven sections. In §1, a class of operators acting in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ is introduced, the effective operator and the corrector are described, and the required results from [BSu2, BSu5] are formulated. In §2, we describe properties of the matrix-valued function Λ. In §3, we introduce the Steklov smoothing operator and prove yet another theorem for the homogenization problem in $\mathbb{R}^d$. §4 contains the statement of the problem in a bounded domain and a description of the "homogenized" problem. In §5, we prove some auxiliary statements needed for further investigation. The main results of the paper are formulated and proved in §§6 and 7. The case where $\Lambda \in L_\infty$ is studied in §6, while in §7 we treat the general case.

0.7. Notation. Let $\mathfrak{H}$ and $\mathfrak{H}_*$ be complex separable Hilbert spaces. The symbols $(\,\cdot\,,\,\cdot\,)_{\mathfrak{H}}$ and $\|\cdot\|_{\mathfrak{H}}$ stand for the inner product and the norm in $\mathfrak{H}$; the symbol $\|\cdot\|_{\mathfrak{H}\to\mathfrak{H}_*}$ denotes the norm of a continuous linear operator acting from $\mathfrak{H}$ to $\mathfrak{H}_*$.

The symbols $\langle\,\cdot\,,\,\cdot\,\rangle$ and $|\cdot|$ stand for the inner product and the norm in $\mathbb{C}^n$; $\mathbf{1} = \mathbf{1}_n$ is the identity $(n\times n)$-matrix. If a is an $(n\times n)$-matrix, the symbol $|a|$ denotes the norm of the matrix a viewed as a linear operator on $\mathbb{C}^n$. We use the notation $\mathbf{x} = (x_1,\dots,x_d) \in \mathbb{R}^d$, $iD_j = \partial_j = \partial/\partial x_j$, $j = 1,\dots,d$, $\mathbf{D} = -i\nabla = (D_1,\dots,D_d)$. The L_p-classes of $\mathbb{C}^n$-valued functions in a domain $\mathcal{O} \subset \mathbb{R}^d$ are denoted by $L_p(\mathcal{O};\mathbb{C}^n)$, $1 \le p \le \infty$. The Sobolev classes of $\mathbb{C}^n$-valued functions in a domain $\mathcal{O} \subset \mathbb{R}^d$ are denoted by $H^s(\mathcal{O};\mathbb{C}^n)$.

By $H^1_0(\mathcal{O};\mathbb{C}^n)$ we denote the closure of $C_0^\infty(\mathcal{O};\mathbb{C}^n)$ in $H^1(\mathcal{O};\mathbb{C}^n)$. If $n=1$, we write simply $L_p(\mathcal{O})$, $H^s(\mathcal{O})$, etc., but sometimes we use such abbreviated notation also for spaces of vector-valued or matrix-valued functions.

A brief communication about the results of the present paper was published in [PSu].

§1. Homogenization problem for a periodic elliptic operator in $L_2(\mathbb{R}^d;\mathbb{C}^n)$

In this section, we describe the class of matrix elliptic operators to be considered and formulate the results on the homogenization problem in $\mathbb{R}^d$ obtained in [BSu2, BSu5].

1.1. Lattices in $\mathbb{R}^d$. Let $\mathbf{a}_1,\dots,\mathbf{a}_d\in\mathbb{R}^d$ be the basis in $\mathbb{R}^d$; it gives rise to a lattice Γ:

$$\Gamma=\Big\{\mathbf{a}\in\mathbb{R}^d\ :\ \mathbf{a}=\sum_{j=1}^d \nu_j\mathbf{a}_j,\quad \nu_j\in\mathbb{Z}\Big\}.$$

Let Ω be the elementary cell of Γ:

$$\Omega:=\Big\{\mathbf{x}\in\mathbb{R}^d\ :\ \mathbf{x}=\sum_{j=1}^d \tau_j\mathbf{a}_j,\quad -\frac{1}{2}<\tau_j<\frac{1}{2}\Big\}.$$

We denote $|\Omega|=\operatorname{meas}\Omega$.

The basis $\mathbf{b}_1,\dots,\mathbf{b}_d$ in $\mathbb{R}^d$ dual to $\mathbf{a}_1,\dots,\mathbf{a}_d$ is defined by the relations $\langle\mathbf{b}_i,\mathbf{a}_j\rangle=2\pi\delta_{ij}$. This basis generates the *lattice* $\widetilde{\Gamma}$ *dual to* Γ:

$$\widetilde{\Gamma}=\Big\{\mathbf{b}\in\mathbb{R}^d\ :\ \mathbf{b}=\sum_{i=1}^d \rho_i\mathbf{b}_i,\ \rho_i\in\mathbb{Z}\Big\}.$$

We introduce the *central Brillouin zone*

$$\widetilde{\Omega}=\{\mathbf{k}\in\mathbb{R}^d\ :\ |\mathbf{k}|<|\mathbf{k}-\mathbf{b}|,\ 0\neq\mathbf{b}\in\widetilde{\Gamma}\},$$

which is a fundamental domain of $\widetilde{\Gamma}$.

Below, $\widetilde{H}^1(\Omega)$ stands for the subspace of all functions in $H^1(\Omega)$ whose Γ-periodic extension to $\mathbb{R}^d$ belongs to $H^1_{\mathrm{loc}}(\mathbb{R}^d)$. If $\varphi(\mathbf{x})$ is a Γ-periodic function in $\mathbb{R}^d$, we denote

$$\varphi^\varepsilon(\mathbf{x}):=\varphi(\varepsilon^{-1}\mathbf{x}),\quad \varepsilon>0.$$

1.2. The class of operators. In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, we consider a second order DO $\mathcal{A}_\varepsilon$ formally given by the differential expression

$$\mathcal{A}_\varepsilon=b(\mathbf{D})^*g^\varepsilon(\mathbf{x})b(\mathbf{D}),\quad \varepsilon>0. \tag{1.1}$$

Here $g(\mathbf{x})$ is a measurable $(m\times m)$-matrix-valued function (in general, with complex entries). It is assumed that $g(\mathbf{x})$ is bounded, periodic with respect to the lattice Γ, and uniformly positive definite. Next, $b(\mathbf{D})$ is a homogeneous $(m\times n)$-matrix first order DO with constant coefficients:

$$b(\mathbf{D})=\sum_{l=1}^d b_lD_l. \tag{1.2}$$

Here the b_l are constant matrices (in general, with complex entries). The *symbol* $b(\boldsymbol{\xi})=\sum_{l=1}^d b_l\xi_l$, $\boldsymbol{\xi}\in\mathbb{R}^d$, is associated with the operator $b(\mathbf{D})$. We assume that $m\geq n$ and that $\operatorname{rank} b(\boldsymbol{\xi})=n$ for all $\boldsymbol{\xi}\neq 0$. This is equivalent to the inequalities

$$\alpha_0\mathbf{1}_n\leq b(\boldsymbol{\theta})^*b(\boldsymbol{\theta})\leq\alpha_1\mathbf{1}_n,\quad \boldsymbol{\theta}\in\mathbb{S}^{d-1},\quad 0<\alpha_0\leq\alpha_1<\infty, \tag{1.3}$$

with some positive constants α_0 and α_1.

The precise definition of the operator $\mathcal{A}_\varepsilon$ is given in terms of the corresponding quadratic form

$$a_\varepsilon[\mathbf{u},\mathbf{u}] = \int_{\mathbb{R}^d} \langle g^\varepsilon(\mathbf{x}) b(\mathbf{D})\mathbf{u}, b(\mathbf{D})\mathbf{u}\rangle \, d\mathbf{x}, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n).$$

Under the above assumptions, this form is closed in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ and is nonnegative. Using the Fourier transformation and (1.3), it is easy to check that

$$c_0 \int_{\mathbb{R}^d} |\mathbf{D}\mathbf{u}|^2 \, d\mathbf{x} \le a_\varepsilon[\mathbf{u},\mathbf{u}] \le c_1 \int_{\mathbb{R}^d} |\mathbf{D}\mathbf{u}|^2 \, d\mathbf{x}, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n), \tag{1.4}$$

where $c_0 = \alpha_0 \|g^{-1}\|^{-1}_{L_\infty}$, $c_1 = \alpha_1 \|g\|_{L_\infty}$.

The simplest example of the operator (1.1) is the scalar elliptic operator

$$\mathcal{A}_\varepsilon = -\operatorname{div} g^\varepsilon(\mathbf{x})\nabla = \mathbf{D}^* g^\varepsilon(\mathbf{x})\mathbf{D}.$$

In this case we have $n = 1$, $m = d$, $b(\mathbf{D}) = \mathbf{D}$. Obviously, (1.3) is true with $\alpha_0 = \alpha_1 = 1$. The operator of elasticity theory can be also written in the form (1.1) with $n = d$, $m = d(d+1)/2$. These and other examples were considered in [BSu2] in detail.

1.3. The effective operator. In order to formulate the results, we need to introduce the effective operator $\mathcal{A}^0$.

Let an $(n \times m)$-matrix-valued function $\Lambda(\mathbf{x})$ be the (weak) Γ-periodic solution of the problem

$$b(\mathbf{D})^* g(\mathbf{x}) \left(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m \right) = 0, \qquad \int_\Omega \Lambda(\mathbf{x}) \, d\mathbf{x} = 0. \tag{1.5}$$

In other words, for the columns $\mathbf{v}_j(\mathbf{x})$, $j = 1, \dots, m$, of the matrix $\Lambda(\mathbf{x})$ the following is true: $\mathbf{v}_j \in \widetilde{H}^1(\Omega;\mathbb{C}^n)$,

$$\int_\Omega \langle g(\mathbf{x})(b(\mathbf{D})\mathbf{v}_j(\mathbf{x}) + \mathbf{e}_j), b(\mathbf{D})\boldsymbol{\eta}(\mathbf{x})\rangle \, d\mathbf{x} = 0, \quad \boldsymbol{\eta} \in \widetilde{H}^1(\Omega;\mathbb{C}^n),$$

and $\int_\Omega \mathbf{v}_j(\mathbf{x}) \, d\mathbf{x} = 0$. Here $\mathbf{e}_1, \dots, \mathbf{e}_m$ is the standard orthonormal basis in $\mathbb{C}^m$.

The so-called *effective matrix* g^0 of size $m \times m$ is defined as follows:

$$g^0 = |\Omega|^{-1} \int_\Omega g(\mathbf{x}) \left(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m \right) \, d\mathbf{x}. \tag{1.6}$$

It turns out that the matrix g^0 is positive definite. The *effective operator* $\mathcal{A}^0$ for the operator (1.1) is given by the differential expression

$$\mathcal{A}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$$

on the domain $H^2(\mathbb{R}^d;\mathbb{C}^n)$.

1.4. Properties of the effective matrix. The following properties of the effective matrix were proved in [BSu2, Chapter 3, Theorem 1.5].

Proposition 1.1. *The effective matrix g^0 satisfies the estimates*

$$\underline{g} \le g^0 \le \overline{g}. \tag{1.7}$$

Here

$$\overline{g} = |\Omega|^{-1} \int_\Omega g(\mathbf{x}) \, d\mathbf{x}, \quad \underline{g} = \left(|\Omega|^{-1} \int_\Omega g(\mathbf{x})^{-1} \, d\mathbf{x} \right)^{-1}.$$

If $m = n$, then g^0 coincides with $\underline{g}$.

In homogenization theory, for specific DO's estimates (1.7) are well known as the Voight–Reuss bracketing. Now we distinguish the cases where one of the inequalities in (1.7) becomes an identity. The following statements were checked in [BSu2, Chapter 3, Propositions 1.6 and 1.7].

Proposition 1.2. *The identity $g^0 = \overline{g}$ is equivalent to the relations*

$$b(\mathbf{D})^* \mathbf{g}_k(\mathbf{x}) = 0, \quad k = 1, \dots, m, \tag{1.8}$$

where the $\mathbf{g}_k(\mathbf{x})$, $k = 1, \dots, m$, are the columns of the matrix $g(\mathbf{x})$.

Proposition 1.3. *The identity $g^0 = \underline{g}$ is equivalent to the representations*

$$\mathbf{l}_k(\mathbf{x}) = \mathbf{l}_k^0 + b(\mathbf{D})\mathbf{w}_k, \quad \mathbf{l}_k^0 \in \mathbb{C}^m, \quad \mathbf{w}_k \in \widetilde{H}^1(\Omega; \mathbb{C}^n), \quad k = 1, \dots, m, \tag{1.9}$$

where the $\mathbf{l}_k(\mathbf{x})$, $k = 1, \dots, m$, are the columns of the matrix $g(\mathbf{x})^{-1}$.

Obviously, (1.7) implies the following estimates for the norms of the matrices g^0 and $(g^0)^{-1}$:

$$|g^0| \le \|g\|_{L_\infty}, \quad |(g^0)^{-1}| \le \|g^{-1}\|_{L_\infty}. \tag{1.10}$$

1.5. The smoothing operator. We need an auxiliary smoothing operator Π_ε acting in $L_2(\mathbb{R}^d; \mathbb{C}^m)$ and defined by

$$(\Pi_\varepsilon \mathbf{u})(\mathbf{x}) = (2\pi)^{-d/2} \int_{\widetilde{\Omega}/\varepsilon} e^{i\langle \mathbf{x}, \boldsymbol{\xi} \rangle} \widehat{\mathbf{u}}(\boldsymbol{\xi})\, d\boldsymbol{\xi}, \tag{1.11}$$

where $\widehat{\mathbf{u}}(\boldsymbol{\xi})$ is the Fourier image of $\mathbf{u}(\mathbf{x})$. In other words, Π_ε is the pseudodifferential operator whose symbol $\chi_{\widetilde{\Omega}/\varepsilon}(\boldsymbol{\xi})$ is the indicator of the set $\widetilde{\Omega}/\varepsilon$. Obviously, Π_ε is the orthogonal projection in each space $H^s(\mathbb{R}^d; \mathbb{C}^m)$, $s \ge 0$. Moreover, $D^\alpha \Pi_\varepsilon \mathbf{u} = \Pi_\varepsilon D^\alpha \mathbf{u}$ for $\mathbf{u} \in H^s(\mathbb{R}^d; \mathbb{C}^m)$ and any multiindex α such that $|\alpha| \le s$. Below, we shall need the following statement.

Proposition 1.4. *For any $\mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^m)$ we have*

$$\|\Pi_\varepsilon \mathbf{u} - \mathbf{u}\|_{L_2(\mathbb{R}^d; \mathbb{C}^m)} \le \varepsilon r_0^{-1} \|\mathbf{D}\mathbf{u}\|_{L_2(\mathbb{R}^d)},$$

where r_0 is the radius of the ball inscribed in $\operatorname{clos} \widetilde{\Omega}$.

Proof. If $\boldsymbol{\xi} \in \mathbb{R}^d \setminus (\widetilde{\Omega}/\varepsilon)$, then $|\boldsymbol{\xi}| \ge r_0 \varepsilon^{-1}$. Hence,

$$\begin{aligned} \|\Pi_\varepsilon \mathbf{u} - \mathbf{u}\|^2_{L_2(\mathbb{R}^d; \mathbb{C}^m)} &= \int\limits_{\mathbb{R}^d \setminus (\widetilde{\Omega}/\varepsilon)} |\widehat{\mathbf{u}}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \\ &\le \varepsilon^2 r_0^{-2} \int_{\mathbb{R}^d} |\boldsymbol{\xi}|^2 |\widehat{\mathbf{u}}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} = \varepsilon^2 r_0^{-2} \int_{\mathbb{R}^d} |\mathbf{D}\mathbf{u}(\mathbf{x})|^2 \, d\mathbf{x}. \qquad \square \end{aligned}$$

The following property was proved in [BSu5, Subsection 10.2].

Proposition 1.5. *Let $f(\mathbf{x})$ be a Γ-periodic function in $\mathbb{R}^d$ such that $f \in L_2(\Omega)$. Let $[f^\varepsilon]$ denote the operator of multiplication by the function $f(\varepsilon^{-1}\mathbf{x})$. Then the operator $[f^\varepsilon]\Pi_\varepsilon$ is continuous in $L_2(\mathbb{R}^d; \mathbb{C}^m)$, and*

$$\|[f^\varepsilon]\Pi_\varepsilon\|_{L_2(\mathbb{R}^d; \mathbb{C}^m) \to L_2(\mathbb{R}^d; \mathbb{C}^m)} \le |\Omega|^{-1/2} \|f\|_{L_2(\Omega)}.$$

1.6. Results on homogenization problems in $\mathbb{R}^d$. Consider the following elliptic equation in $\mathbb{R}^d$:

$$\mathcal{A}_\varepsilon \mathbf{u}_\varepsilon + \mathbf{u}_\varepsilon = \mathbf{F}, \tag{1.12}$$

where $\mathbf{F} \in L_2(\mathbb{R}^d;\mathbb{C}^n)$. It is known that, as $\varepsilon \to 0$, the solution $\mathbf{u}_\varepsilon$ converges in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to the solution of the "homogenized" equation

$$\mathcal{A}^0 \mathbf{u}_0 + \mathbf{u}_0 = \mathbf{F}. \tag{1.13}$$

The following result was obtained in [BSu2, Chapter 4, Theorem 2.1].

Theorem 1.6. *Let $\mathbf{u}_\varepsilon$ be the solution of equation* (1.12)*, and let $\mathbf{u}_0$ be the solution of equation* (1.13)*. Then*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)} \le C_1\varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)}, \quad 0<\varepsilon\le 1,$$

or, in operator terms,

$$\big\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}^0 + I)^{-1}\big\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)\to L_2(\mathbb{R}^d;\mathbb{C}^n)} \le C_1\varepsilon, \quad 0<\varepsilon\le 1.$$

The constant C_1 depends only on the norms $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the constants α_0, α_1 occurring in (1.3)*, and the parameters of the lattice Γ.*

In order to find approximation of the solution $\mathbf{u}_\varepsilon$ in $H^1(\mathbb{R}^d;\mathbb{C}^n)$, it is necessary to take the first order corrector into account. We put

$$K(\varepsilon) = [\Lambda^\varepsilon]\Pi_\varepsilon b(\mathbf{D})(\mathcal{A}^0 + I)^{-1}. \tag{1.14}$$

Here $[\Lambda^\varepsilon]$ is the operator of multiplication by the matrix-valued function $\Lambda(\varepsilon^{-1}\mathbf{x})$, and Π_ε is the smoothing operator defined by (1.11). The operator (1.14) is continuous from $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to $H^1(\mathbb{R}^d;\mathbb{C}^n)$. This fact can easily be checked by using Proposition 1.5 and the fact that $\Lambda \in \widetilde{H}^1(\Omega)$. Here, $\varepsilon\|K(\varepsilon)\|_{L_2\to H^1} = O(1)$.

The "first order approximation" of the solution $\mathbf{u}_\varepsilon$ is given by

$$\mathbf{v}_\varepsilon = \mathbf{u}_0 + \varepsilon\Lambda^\varepsilon\Pi_\varepsilon b(\mathbf{D})\mathbf{u}_0 = (\mathcal{A}^0 + I)^{-1}\mathbf{F} + \varepsilon K(\varepsilon)\mathbf{F}. \tag{1.15}$$

The following result was obtained in [BSu5, Theorem 10.6].

Theorem 1.7. *Let $\mathbf{u}_\varepsilon$ be the solution of equation* (1.12)*, and let $\mathbf{u}_0$ be the solution of equation* (1.13)*. Let $\mathbf{v}_\varepsilon$ be the function defined by* (1.15)*. Then*

$$\|\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon\|_{H^1(\mathbb{R}^d;\mathbb{C}^n)} \le C_2\varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)}, \quad 0<\varepsilon\le 1, \tag{1.16}$$

or, in operator terms,

$$\big\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}^0 + I)^{-1} - \varepsilon K(\varepsilon)\big\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)\to H^1(\mathbb{R}^d;\mathbb{C}^n)} \le C_2\varepsilon, \quad 0<\varepsilon\le 1.$$

The constant C_2 depends only on m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.

Now we distinguish the case where the corrector is equal to zero. The next statement follows from Theorem 1.7, Proposition 1.2, and equation (1.5).

Proposition 1.8. *If $g^0 = \overline{g}$, i.e., relations* (1.8) *are satisfied, then $\Lambda = 0$ and $K(\varepsilon) = 0$. In this case we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{H^1(\mathbb{R}^d;\mathbb{C}^n)} \le C_2\varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)}, \quad 0<\varepsilon\le 1.$$

It turns out that, under certain assumptions on the solution of problem (1.5), the smoothing operator Π_ε in the corrector (1.14) can be removed (replaced by the identity operator). We impose the following condition.

Condition 1.9. *Suppose that the Γ-periodic solution $\Lambda(\mathbf{x})$ of problem* (1.5) *is bounded*: $\Lambda \in L_\infty$.

Put

$$K^0(\varepsilon) = [\Lambda^\varepsilon] b(\mathbf{D})(\mathcal{A}^0 + I)^{-1}.$$

In [BSu5], it was shown that under Condition 1.9 the operator $K^0(\varepsilon)$ is a continuous mapping of $L_2(\mathbb{R}^d; \mathbb{C}^n)$ into $H^1(\mathbb{R}^d; \mathbb{C}^n)$. (This fact can also be deduced easily from Corollary 2.4 proved below.)

Instead of (1.15), we consider another approximation of the solution $\mathbf{u}_\varepsilon$:

$$\check{\mathbf{v}}_\varepsilon = \mathbf{u}_0 + \varepsilon \Lambda^\varepsilon b(\mathbf{D}) \mathbf{u}_0 = (\mathcal{A}^0 + I)^{-1} \mathbf{F} + \varepsilon K^0(\varepsilon) \mathbf{F}. \tag{1.17}$$

The following result was obtained in [BSu5, Theorem 10.8].

Theorem 1.10. *Suppose that Condition* 1.9 *is satisfied. Let* $\mathbf{u}_\varepsilon$ *be the solution of equation* (1.12)*, and let* $\mathbf{u}_0$ *be the solution of equation* (1.13)*. Let* $\check{\mathbf{v}}_\varepsilon$ *be the function defined by* (1.17)*. Then*

$$\|\mathbf{u}_\varepsilon - \check{\mathbf{v}}_\varepsilon\|_{H^1(\mathbb{R}^d;\mathbb{C}^n)} \leq C_3 \varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)}, \quad 0 < \varepsilon \leq 1,$$

or, in operator terms,

$$\left\| (\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}^0 + I)^{-1} - \varepsilon K^0(\varepsilon) \right\|_{L_2(\mathbb{R}^d;\mathbb{C}^n) \to H^1(\mathbb{R}^d;\mathbb{C}^n)} \leq C_3 \varepsilon, \quad 0 < \varepsilon \leq 1.$$

The constant C_3 *depends only on* m, d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the parameters of the lattice* Γ, *and the norm* $\|\Lambda\|_{L_\infty}$.

In some cases Condition 1.9 is valid automatically. The following statement was checked in [BSu5, Lemma 8.7].

Proposition 1.11. *Condition* 1.9 *is valid a fortiori if at least one of the following assumptions is fulfilled*:

1° *the dimension does not exceed two, i.e.,* $d \leq 2$;

2° *the operator under study acts in* $L_2(\mathbb{R}^d)$, $d \geq 1$, *and has the form* $\mathcal{A}_\varepsilon = \mathbf{D}^* g^\varepsilon(\mathbf{x}) \mathbf{D}$, *where* $g(\mathbf{x})$ *is a matrix with real entries*;

3° *the dimension is arbitrary and* $g^0 = \underline{g}$, *i.e., relations* (1.9) *are satisfied.*

Note that Condition 1.9 can also be ensured by the assumption that the matrix $g(\mathbf{x})$ is sufficiently smooth.

§2. Properties of the matrix-valued function Λ

The following statement is proved by analogy with the proof of Lemma 8.3 in [BSu5].

Lemma 2.1. *Let* $\Lambda(\mathbf{x})$ *be the* Γ*-periodic solution of problem* (1.5)*. Then for any function* $u \in C_0^\infty(\mathbb{R}^d)$ *we have*

$$\int_{\mathbb{R}^d} |\mathbf{D}\Lambda(\mathbf{x})|^2 |u|^2 \, d\mathbf{x} \leq \beta_1 \|u\|^2_{L_2(\mathbb{R}^d)} + \beta_2 \int_{\mathbb{R}^d} |\Lambda(\mathbf{x})|^2 |\mathbf{D}u|^2 \, d\mathbf{x}. \tag{2.1}$$

The constants β_1 *and* β_2 *are defined below in* (2.12) *and depend only on* m, d, α_0, α_1, $\|g\|_{L_\infty}$, *and* $\|g^{-1}\|_{L_\infty}$.

Proof. Let $\mathbf{v}_j(\mathbf{x})$, $j = 1, \dots, m$, be the columns of the matrix $\Lambda(\mathbf{x})$. By (1.5), for any function $\boldsymbol{\eta} \in H^1(\mathbb{R}^d; \mathbb{C}^n)$ such that $\boldsymbol{\eta}(\mathbf{x}) = 0$ for $|\mathbf{x}| > r$ (with some $r > 0$) we have

$$\int_{\mathbb{R}^d} \langle g(\mathbf{x}) \left(b(\mathbf{D}) \mathbf{v}_j(\mathbf{x}) + \mathbf{e}_j \right), b(\mathbf{D}) \boldsymbol{\eta}(\mathbf{x}) \rangle \, d\mathbf{x} = 0. \tag{2.2}$$

Let $u \in C_0^\infty(\mathbb{R}^d)$. We put $\boldsymbol{\eta}(\mathbf{x}) = \mathbf{v}_j(\mathbf{x}) |u(\mathbf{x})|^2$. By (1.2),

$$b(\mathbf{D}) \boldsymbol{\eta}(\mathbf{x}) = \left(b(\mathbf{D}) \mathbf{v}_j(\mathbf{x}) \right) |u(\mathbf{x})|^2 + \sum_{l=1}^d b_l \mathbf{v}_j(\mathbf{x}) D_l |u(\mathbf{x})|^2. \tag{2.3}$$

Substituting (2.3) in (2.2), we obtain

$$\int_{\mathbb{R}^d} \langle g(\mathbf{x}) \left(b(\mathbf{D})\mathbf{v}_j(\mathbf{x}) + \mathbf{e}_j\right), b(\mathbf{D})\mathbf{v}_j(\mathbf{x})\rangle \, |u|^2 \, d\mathbf{x} + \int_{\mathbb{R}^d} \sum_{l=1}^{d} \langle g(\mathbf{x}) \left(b(\mathbf{D})\mathbf{v}_j(\mathbf{x}) + \mathbf{e}_j\right), b_l \mathbf{v}_j(\mathbf{x})\rangle \left(D_l u \, \overline{u} + u \, D_l \overline{u}\right) \, d\mathbf{x} = 0.$$

Hence,

$$\begin{aligned} J := \int_{\mathbb{R}^d} \left| g^{1/2} b(\mathbf{D})\mathbf{v}_j \right|^2 |u|^2 \, d\mathbf{x} = &- \int_{\mathbb{R}^d} \left\langle g^{1/2}\mathbf{e}_j, g^{1/2} b(\mathbf{D})\mathbf{v}_j \right\rangle |u|^2 \, d\mathbf{x} \\ &- \int_{\mathbb{R}^d} \sum_{l=1}^{d} \langle g^{1/2} b(\mathbf{D})\mathbf{v}_j, g^{1/2} b_l \mathbf{v}_j \rangle \left(D_l u \, \overline{u} + u \, D_l \overline{u}\right) \, d\mathbf{x} \\ &- \int_{\mathbb{R}^d} \sum_{l=1}^{d} \langle g\mathbf{e}_j, b_l \mathbf{v}_j \rangle \left(D_l u \, \overline{u} + u \, D_l \overline{u}\right) \, d\mathbf{x}. \end{aligned} \tag{2.4}$$

We denote the summands on the right by J_1, J_2, J_3. The first term J_1 can be estimated as follows:

$$|J_1| \le \int_{\mathbb{R}^d} \left(\left| g^{1/2}\mathbf{e}_j \right|^2 + \frac{1}{4} \left| g^{1/2} b(\mathbf{D})\mathbf{v}_j \right|^2 \right) |u|^2 \, d\mathbf{x} \le \|g\|_{L_\infty} \|u\|^2_{L_2(\mathbb{R}^d)} + \frac{1}{4} J. \tag{2.5}$$

Next, from (1.3) it follows that

$$|b_l| \le \alpha_1^{1/2}, \quad l = 1, \dots, d. \tag{2.6}$$

Taking (2.6) into account, we estimate the second term J_2:

$$\begin{aligned} |J_2| &\le 2 \int_{\mathbb{R}^d} \left| g^{1/2} \ b(\mathbf{D})\mathbf{v}_j \right| |u| \left(\sum_{l=1}^{d} \left| g^{1/2} b_l \mathbf{v}_j \right| |D_l u| \right) d\mathbf{x} \\ &\le \frac{1}{4} J + 4d\alpha_1 \|g\|_{L_\infty} \int_{\mathbb{R}^d} |\mathbf{v}_j|^2 |\mathbf{D}u|^2 \, d\mathbf{x}. \end{aligned} \tag{2.7}$$

Finally, the term J_3 satisfies the estimate

$$\begin{aligned} |J_3| &\le 2 \int_{\mathbb{R}^d} |g\mathbf{e}_j| \ |u| \left(\sum_{l=1}^{d} |b_l \mathbf{v}_j| \, |D_l u| \right) d\mathbf{x} \\ &\le \|g\|_{L_\infty} \|u\|^2_{L_2(\mathbb{R}^d)} + d\alpha_1 \|g\|_{L_\infty} \int_{\mathbb{R}^d} |\mathbf{v}_j|^2 |\mathbf{D}u|^2 \, d\mathbf{x}. \end{aligned} \tag{2.8}$$

Combining (2.4), (2.5), (2.7), and (2.8), we obtain

$$\frac{1}{2} J \le 2\|g\|_{L_\infty} \|u\|^2_{L_2(\mathbb{R}^d)} + 5d\alpha_1 \|g\|_{L_\infty} \int_{\mathbb{R}^d} |\mathbf{v}_j|^2 |\mathbf{D}u|^2 \, d\mathbf{x}. \tag{2.9}$$

Now, we show how the required estimate can be deduced from (2.9). By the Fourier transformation, from the left inequality in (1.3) it follows that

$$\int_{\mathbb{R}^d} |\mathbf{D}(\mathbf{v}_j u)|^2 \ d\mathbf{x} \le \alpha_0^{-1} \int_{\mathbb{R}^d} |b(\mathbf{D})(\mathbf{v}_j u)|^2 \ d\mathbf{x}.$$

By (1.2),

$$b(\mathbf{D})(\mathbf{v}_j u) = (b(\mathbf{D})\mathbf{v}_j)u + \sum_{l=1}^{d} b_l \mathbf{v}_j D_l u.$$

Then, using (2.6) and the expression for J (see (2.4)), we get

$$\begin{aligned}\int_{\mathbb{R}^d} |\mathbf{D}(\mathbf{v}_j u)|^2\, d\mathbf{x} &\le 2\alpha_0^{-1} \int_{\mathbb{R}^d} |b(\mathbf{D})\mathbf{v}_j|^2\, |u|^2\, d\mathbf{x} + 2\alpha_0^{-1}\alpha_1 d \int_{\mathbb{R}^d} |\mathbf{v}_j|^2 |\mathbf{D}u|^2\, d\mathbf{x} \\ &\le 2\alpha_0^{-1}\|g^{-1}\|_{L_\infty} J + 2\alpha_0^{-1}\alpha_1 d \int_{\mathbb{R}^d} |\mathbf{v}_j|^2 |\mathbf{D}u|^2\, d\mathbf{x}.\end{aligned} \tag{2.10}$$

Obviously,

$$\int_{\mathbb{R}^d} |\mathbf{D}\mathbf{v}_j|^2\, |u|^2\, d\mathbf{x} \le 2 \int_{\mathbb{R}^d} |\mathbf{D}(\mathbf{v}_j u)|^2\, d\mathbf{x} + 2 \int_{\mathbb{R}^d} |\mathbf{v}_j|^2 |\mathbf{D}u|^2\, d\mathbf{x}. \tag{2.11}$$

Relations (2.9)–(2.11) imply that

$$\begin{aligned}\int_{\mathbb{R}^d} |\mathbf{D}\mathbf{v}_j|^2\, |u|^2\, d\mathbf{x} &\le 16\alpha_0^{-1}\|g^{-1}\|_{L_\infty}\|g\|_{L_\infty}\|u\|^2_{L_2(\mathbb{R}^d)} \\ &\quad + 2\left(1 + 2d\alpha_0^{-1}\alpha_1 + 20d\alpha_0^{-1}\alpha_1\|g^{-1}\|_{L_\infty}\|g\|_{L_\infty}\right) \int_{\mathbb{R}^d} |\mathbf{v}_j|^2 |\mathbf{D}u|^2\, d\mathbf{x}.\end{aligned}$$

Summing over j, we arrive at estimate (2.1) with

$$\begin{aligned}\beta_1 &= 16m\alpha_0^{-1}\|g^{-1}\|_{L_\infty}\|g\|_{L_\infty}, \\ \beta_2 &= 2\left(1 + 2d\alpha_0^{-1}\alpha_1 + 20d\alpha_0^{-1}\alpha_1\|g^{-1}\|_{L_\infty}\|g\|_{L_\infty}\right).\end{aligned} \tag{2.12}$$

□

Corollary 2.2. *Under Condition* 1.9*, for any $u \in H^1(\mathbb{R}^d)$ we have*

$$\int_{\mathbb{R}^d} |\mathbf{D}\Lambda(\mathbf{x})|^2 |u|^2\, d\mathbf{x} \le \beta_1\|u\|^2_{L_2(\mathbb{R}^d)} + \beta_2\|\Lambda\|^2_{L_\infty} \int_{\mathbb{R}^d} |\mathbf{D}u|^2\, d\mathbf{x}. \tag{2.13}$$

Proof. Indeed, the second integral on the right-hand side of (2.1) can be estimated by $\|\Lambda\|^2_{L_\infty}\int_{\mathbb{R}^d}|\mathbf{D}u|^2\,d\mathbf{x}$. Then (2.13) is valid for any $u \in C_0^\infty(\mathbb{R}^d)$. By continuity, inequality (2.13) extends from the dense set $C_0^\infty(\mathbb{R}^d)$ to the entire space $H^1(\mathbb{R}^d)$. □

The next statement follows from Lemma 2.1 by a scaling transformation.

Lemma 2.3. *Under the assumptions of Lemma* 2.1*, we have*

$$\int_{\mathbb{R}^d} |(\mathbf{D}\Lambda)^\varepsilon(\mathbf{x})|^2\, |u(\mathbf{x})|^2\, d\mathbf{x} \le \beta_1\|u\|^2_{L_2(\mathbb{R}^d)} + \beta_2\varepsilon^2 \int_{\mathbb{R}^d} |\Lambda^\varepsilon(\mathbf{x})|^2\, |\mathbf{D}u|^2\, d\mathbf{x}.$$

Proof. By the changes $\mathbf{y} = \varepsilon^{-1}\mathbf{x}$ and $u(\mathbf{x}) = v(\mathbf{y})$, from (2.1) it follows that

$$\begin{aligned}\int_{\mathbb{R}^d} \left|(\mathbf{D}\Lambda)(\varepsilon^{-1}\mathbf{x})\right|^2 |u(\mathbf{x})|^2\, d\mathbf{x} &= \int_{\mathbb{R}^d} |(\mathbf{D}\Lambda)(\mathbf{y})|^2\, |v(\mathbf{y})|^2\, \varepsilon^d\, d\mathbf{y} \\ &\le \beta_1 \int_{\mathbb{R}^d} |v(\mathbf{y})|^2 \varepsilon^d d\mathbf{y} + \beta_2 \int_{\mathbb{R}^d} |\Lambda(\mathbf{y})|^2\, |\mathbf{D}_y v(\mathbf{y})|^2 \varepsilon^d\, d\mathbf{y} \\ &= \beta_1 \int_{\mathbb{R}^d} |u(\mathbf{x})|^2\, d\mathbf{x} + \beta_2\varepsilon^2 \int_{\mathbb{R}^d} \left|\Lambda(\varepsilon^{-1}\mathbf{x})\right|^2 |\mathbf{D}_{\mathbf{x}} u(\mathbf{x})|^2\, d\mathbf{x}.\end{aligned}$$

□

Corollary 2.4. *Under Condition* 1.9*, for any $u \in H^1(\mathbb{R}^d)$ we have*

$$\int_{\mathbb{R}^d} |(\mathbf{D}\Lambda)^\varepsilon(\mathbf{x})|^2 |u|^2\, d\mathbf{x} \le \beta_1\|u\|^2_{L_2(\mathbb{R}^d)} + \beta_2\|\Lambda\|^2_{L_\infty}\varepsilon^2 \int_{\mathbb{R}^d} |\mathbf{D}u|^2\, d\mathbf{x}.$$

In conclusion of this section, we give two estimates for the matrix-valued function Λ, obtained in [BSu4, (6.28) and Subsection 7.3]:

$$\|\Lambda\|_{L_2(\Omega)} \le |\Omega|^{1/2} m^{1/2} (2r_0)^{-1} \alpha_0^{-1/2} \|g\|^{1/2}_{L_\infty} \|g^{-1}\|^{1/2}_{L_\infty}, \tag{2.14}$$

$$\|\mathbf{D}\Lambda\|_{L_2(\Omega)} \le |\Omega|^{1/2} m^{1/2} \alpha_0^{-1/2} \|g\|^{1/2}_{L_\infty} \|g^{-1}\|^{1/2}_{L_\infty}. \tag{2.15}$$

§3. Smoothing in Steklov's sense. Yet another result for homogenization problem in $\mathbb{R}^d$

In [Zh2, ZhPas], smoothing in Steklov's sense was used instead of the smoothing operator (1.11). It turns out that smoothing in Steklov's sense is more convenient for the study of homogenization problem in a bounded domain. In this section, we show that for the problem in $\mathbb{R}^d$ both versions fit, i.e., Theorem 1.7 remains true if in the corrector (1.14) we replace the operator Π_ε by the Steklov smoothing operator.

3.1. Smoothing in Steklov's sense. In $L_2(\mathbb{R}^d;\mathbb{C}^m)$, we consider the operator S_ε defined by

$$(S_\varepsilon \mathbf{u})(\mathbf{x}) = |\Omega|^{-1}\int_\Omega \mathbf{u}(\mathbf{x}-\varepsilon\mathbf{z})\,d\mathbf{z}; \tag{3.1}$$

it is called the *Steklov smoothing operator*. Note that $\|S_\varepsilon\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le 1$. Obviously, $D^\alpha S_\varepsilon \mathbf{u} = S_\varepsilon D^\alpha \mathbf{u}$ for $\mathbf{u}\in H^s(\mathbb{R}^d;\mathbb{C}^m)$ and any multiindex α such that $|\alpha|\le s$.

We need some properties of the operator (3.1), cf. [ZhPas, Lemmas 1.1 and 1.2].

Proposition 3.1. *For any* $\mathbf{u}\in H^1(\mathbb{R}^d;\mathbb{C}^m)$ *we have*

$$\|S_\varepsilon \mathbf{u}-\mathbf{u}\|_{L_2(\mathbb{R}^d;\mathbb{C}^m)} \le \varepsilon r_1 \|\mathbf{D}\mathbf{u}\|_{L_2(\mathbb{R}^d)}, \tag{3.2}$$

where $2r_1 = \operatorname{diam}\Omega$.

Proof. By the Cauchy inequality,

$$\begin{aligned}\|S_\varepsilon \mathbf{u}-\mathbf{u}\|^2_{L_2(\mathbb{R}^d;\mathbb{C}^m)} &= \int_{\mathbb{R}^d} d\mathbf{x}\left| |\Omega|^{-1}\int_\Omega (\mathbf{u}(\mathbf{x}-\varepsilon\mathbf{z})-\mathbf{u}(\mathbf{x}))\,d\mathbf{z}\right|^2 \\ &\le |\Omega|^{-1}\int_{\mathbb{R}^d} d\mathbf{x}\int_\Omega |\mathbf{u}(\mathbf{x}-\varepsilon\mathbf{z})-\mathbf{u}(\mathbf{x})|^2\,d\mathbf{z}.\end{aligned} \tag{3.3}$$

Using the Fourier transformation, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} |\mathbf{u}(\mathbf{x}-\varepsilon\mathbf{z})-\mathbf{u}(\mathbf{x})|^2\,d\mathbf{x} &= \int_{\mathbb{R}^d} |\exp(-i\varepsilon\langle \mathbf{z},\boldsymbol{\xi}\rangle)-1|^2\,|\hat{\mathbf{u}}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi} \\ &\le \varepsilon^2|\mathbf{z}|^2\int_{\mathbb{R}^d} |\boldsymbol{\xi}|^2|\hat{\mathbf{u}}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi} = \varepsilon^2|\mathbf{z}|^2\int_{\mathbb{R}^d}|\mathbf{D}\mathbf{u}(\mathbf{x})|^2\,d\mathbf{x}.\end{aligned}$$

Integrating this inequality over $\mathbf{z}\in\Omega$, we conclude that

$$\int_\Omega d\mathbf{z}\int_{\mathbb{R}^d} |\mathbf{u}(\mathbf{x}-\varepsilon\mathbf{z})-\mathbf{u}(\mathbf{x})|^2\,d\mathbf{x} \le \varepsilon^2 r_1^2|\Omega|\int_{\mathbb{R}^d}|\mathbf{D}\mathbf{u}(\mathbf{x})|^2\,d\mathbf{x}.$$

Combined with (3.3), this implies (3.2). ☐

Proposition 3.2. *Let $f(\mathbf{x})$ be a Γ-periodic function in $\mathbb{R}^d$ such that $f\in L_2(\Omega)$. Then the operator $[f^\varepsilon]S_\varepsilon$ is continuous in $L_2(\mathbb{R}^d;\mathbb{C}^m)$, and*

$$\|[f^\varepsilon]S_\varepsilon\|_{L_2(\mathbb{R}^d;\mathbb{C}^m)\to L_2(\mathbb{R}^d;\mathbb{C}^m)} \le |\Omega|^{-1/2}\|f\|_{L_2(\Omega)}.$$

Proof. By the Cauchy inequality and a change of variables, from (3.1) it follows that

$$\begin{aligned}\int_{\mathbb{R}^d} |f^\varepsilon(\mathbf{x})(S_\varepsilon\mathbf{u})(\mathbf{x})|^2\,d\mathbf{x} &\le |\Omega|^{-1}\int_{\mathbb{R}^d} d\mathbf{x}\,|f(\varepsilon^{-1}\mathbf{x})|^2\int_\Omega |\mathbf{u}(\mathbf{x}-\varepsilon\mathbf{z})|^2\,d\mathbf{z} \\ &= |\Omega|^{-1}\int_{\mathbb{R}^d} d\mathbf{y}\int_\Omega |f(\varepsilon^{-1}\mathbf{y}+\mathbf{z})|^2|\mathbf{u}(\mathbf{y})|^2\,d\mathbf{z} = |\Omega|^{-1}\|f\|^2_{L_2(\Omega)}\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}.\end{aligned}$$ ☐

3.2. We put

$$\widetilde{K}(\varepsilon) = [\Lambda^\varepsilon] S_\varepsilon b(\mathbf{D})(\mathcal{A}^0 + I)^{-1}. \tag{3.4}$$

The operator (3.4) is continuous from $L_2(\mathbb{R}^d; \mathbb{C}^n)$ to $H^1(\mathbb{R}^d; \mathbb{C}^n)$. Indeed, the operator $b(\mathbf{D})(\mathcal{A}^0 + I)^{-1}$ is a continuous mapping of $L_2(\mathbb{R}^d; \mathbb{C}^n)$ into $H^1(\mathbb{R}^d; \mathbb{C}^m)$. By using Proposition 3.2 and relation $\Lambda \in \widetilde{H}^1(\Omega)$, it is easy to check that the operator $[\Lambda^\varepsilon] S_\varepsilon$ is continuous from $H^1(\mathbb{R}^d; \mathbb{C}^m)$ to $H^1(\mathbb{R}^d; \mathbb{C}^n)$.

Let $\mathbf{u}_\varepsilon$ be the solution of equation (1.12). Instead of (1.15) we consider another first order approximation of $\mathbf{u}_\varepsilon$:

$$\tilde{\mathbf{v}}_\varepsilon = \mathbf{u}_0 + \varepsilon \Lambda^\varepsilon S_\varepsilon b(\mathbf{D}) \mathbf{u}_0 = (\mathcal{A}^0 + I)^{-1} \mathbf{F} + \varepsilon \widetilde{K}(\varepsilon) \mathbf{F}. \tag{3.5}$$

Along with Theorem 1.7, the following statement is true.

Theorem 3.3. *Let $\mathbf{u}_\varepsilon$ be the solution of equation* (1.12), *and let $\mathbf{u}_0$ be the solution of equation* (1.13). *Let $\tilde{\mathbf{v}}_\varepsilon$ be the function defined by* (3.5). *Then*

$$\|\mathbf{u}_\varepsilon - \tilde{\mathbf{v}}_\varepsilon\|_{H^1(\mathbb{R}^d;\mathbb{C}^n)} \leq \widetilde{C}_2 \varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)}, \quad 0 < \varepsilon \leq 1, \tag{3.6}$$

or, in operator terms,

$$\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}^0 + I)^{-1} - \varepsilon \widetilde{K}(\varepsilon)\|_{L_2(\mathbb{R}^d;\mathbb{C}^n) \to H^1(\mathbb{R}^d;\mathbb{C}^n)} \leq \widetilde{C}_2 \varepsilon, \quad 0 < \varepsilon \leq 1.$$

The constant $\widetilde{C}_2$ depends only on m, d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.

Theorem 3.3 will be deduced from Theorem 1.7.

Lemma 3.4. *For any $\mathbf{u} \in H^2(\mathbb{R}^d; \mathbb{C}^n)$ we have*

$$\begin{aligned} \int_{\mathbb{R}^d} |(\mathbf{D}\Lambda)^\varepsilon|^2 |(\Pi_\varepsilon - S_\varepsilon) b(\mathbf{D}) \mathbf{u}|^2 \, d\mathbf{x} &\leq \beta_1 \int_{\mathbb{R}^d} |(\Pi_\varepsilon - S_\varepsilon) b(\mathbf{D}) \mathbf{u}|^2 \, d\mathbf{x} \\ &\quad + \beta_2 \varepsilon^2 \sum_{j=1}^d \int_{\mathbb{R}^d} |\Lambda^\varepsilon|^2 |(\Pi_\varepsilon - S_\varepsilon) b(\mathbf{D}) \partial_j \mathbf{u}|^2 \, d\mathbf{x}. \end{aligned} \tag{3.7}$$

Proof. By Propositions 1.5, 3.2 and relation $\Lambda \in \widetilde{H}^1(\Omega)$, all terms in (3.7) are continuous functionals of $\mathbf{u}$ in the norm of $H^2(\mathbb{R}^d; \mathbb{C}^n)$. Since $C_0^\infty(\mathbb{R}^d; \mathbb{C}^n)$ is dense in $H^2(\mathbb{R}^d; \mathbb{C}^n)$, it suffices to check (3.7) for $\mathbf{u} \in C_0^\infty(\mathbb{R}^d; \mathbb{C}^n)$.

We fix a function $\zeta \in C^\infty(\mathbb{R}_+)$ such that $0 \leq \zeta(t) \leq 1$, $\zeta(t) = 1$ for $0 \leq t \leq 1$, and $\zeta(t) = 0$ for $t \geq 2$. We put $\zeta_R(\mathbf{x}) = \zeta(R^{-1}|\mathbf{x}|)$, $\mathbf{x} \in \mathbb{R}^d$, $R > 0$. Let $\mathbf{u} \in C_0^\infty(\mathbb{R}^d; \mathbb{C}^n)$. Then $\zeta_R (\Pi_\varepsilon - S_\varepsilon) b(\mathbf{D}) \mathbf{u} \in C_0^\infty(\mathbb{R}^d; \mathbb{C}^m)$ and Lemma 2.3 yields

$$\begin{aligned} \int_{\mathbb{R}^d} |(\mathbf{D}\Lambda)^\varepsilon|^2 |\zeta_R (\Pi_\varepsilon - S_\varepsilon) b(\mathbf{D}) \mathbf{u}|^2 \, d\mathbf{x} &\leq \beta_1 \int_{\mathbb{R}^d} |\zeta_R (\Pi_\varepsilon - S_\varepsilon) b(\mathbf{D}) \mathbf{u}|^2 \, d\mathbf{x} \\ + \beta_2 \varepsilon^2 \sum_{j=1}^d \int_{\mathbb{R}^d} \left|\Lambda^\varepsilon\right|^2 &\left|(\partial_j \zeta_R)(\Pi_\varepsilon - S_\varepsilon) b(\mathbf{D}) \mathbf{u} + \zeta_R (\Pi_\varepsilon - S_\varepsilon) b(\mathbf{D}) \partial_j \mathbf{u}\right|^2 d\mathbf{x}. \end{aligned}$$

Observe that $\max |\partial_j \zeta_R| \leq c R^{-1}$. Now (3.7) follows from the inequality above by a limit passage as $R \to \infty$, by the Lebesgue theorem. □

From Proposition 1.5 and estimate (2.14) it follows that

$$\begin{aligned} \|[\Lambda^\varepsilon] \Pi_\varepsilon\|_{L_2(\mathbb{R}^d;\mathbb{C}^m) \to L_2(\mathbb{R}^d;\mathbb{C}^n)} &\leq |\Omega|^{-1/2} \|\Lambda\|_{L_2(\Omega)} \\ &\leq m^{1/2} (2r_0)^{-1} \alpha_0^{-1/2} \|g\|_{L_\infty}^{1/2} \|g^{-1}\|_{L_\infty}^{1/2} =: M. \end{aligned} \tag{3.8}$$

Similarly, Proposition 3.2 implies

$$\|[\Lambda^\varepsilon] S_\varepsilon\|_{L_2(\mathbb{R}^d;\mathbb{C}^m) \to L_2(\mathbb{R}^d;\mathbb{C}^n)} \leq M. \tag{3.9}$$

Lemma 3.5. *We have*

$$\|\varepsilon\Lambda^\varepsilon(\Pi_\varepsilon - S_\varepsilon)b(\mathbf{D})\mathbf{u}_0\|_{H^1(\mathbb{R}^d;\mathbb{C}^n)} \le \check{C}\varepsilon\|\mathbf{u}_0\|_{H^2(\mathbb{R}^d;\mathbb{C}^n)}. \tag{3.10}$$

The constant $\check{C}$ *is defined below in* (3.16) *and depends only on* m, d, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, α_0, α_1, *and the parameters of the lattice* Γ.

Proof. From (1.3), (3.8), and (3.9), it follows that

$$\|\varepsilon\Lambda^\varepsilon(\Pi_\varepsilon - S_\varepsilon)b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)} \le 2M\alpha_1^{1/2}\varepsilon\|\mathbf{u}_0\|_{H^1(\mathbb{R}^d;\mathbb{C}^n)}. \tag{3.11}$$

Consider the derivatives

$$\frac{\partial}{\partial x_j}(\varepsilon\Lambda^\varepsilon(\Pi_\varepsilon - S_\varepsilon)b(\mathbf{D})\mathbf{u}_0) = \Big(\frac{\partial\Lambda}{\partial x_j}\Big)^\varepsilon(\Pi_\varepsilon - S_\varepsilon)b(\mathbf{D})\mathbf{u}_0 + \varepsilon\Lambda^\varepsilon(\Pi_\varepsilon - S_\varepsilon)b(\mathbf{D})\partial_j\mathbf{u}_0, \quad j = 1,\dots,d.$$

We have

$$\begin{aligned}\sum_{j=1}^d \|\partial_j(\varepsilon\Lambda^\varepsilon(\Pi_\varepsilon - S_\varepsilon)b(\mathbf{D})\mathbf{u}_0)\|^2_{L_2(\mathbb{R}^d)} &\le 2\int_{\mathbb{R}^d} |(\mathbf{D}\Lambda)^\varepsilon|^2|(\Pi_\varepsilon - S_\varepsilon)b(\mathbf{D})\mathbf{u}_0|^2\,d\mathbf{x} \\ &+ 2\varepsilon^2\sum_{j=1}^d\int_{\mathbb{R}^d} |\Lambda^\varepsilon(\Pi_\varepsilon - S_\varepsilon)b(\mathbf{D})\partial_j\mathbf{u}_0|^2\,d\mathbf{x}.\end{aligned} \tag{3.12}$$

The second summand on the right-hand side of (3.12) is estimated by using (1.3), (3.8), and (3.9):

$$2\varepsilon^2\sum_{j=1}^d\int_{\mathbb{R}^d} |\Lambda^\varepsilon(\Pi_\varepsilon - S_\varepsilon)b(\mathbf{D})\partial_j\mathbf{u}_0|^2\,d\mathbf{x} \le 8\varepsilon^2M^2\alpha_1\|\mathbf{u}_0\|^2_{H^2(\mathbb{R}^d;\mathbb{C}^n)}. \tag{3.13}$$

The first summand on the right-hand side of (3.12) is estimated with the help of Lemma 3.4:

$$\begin{aligned}2\int_{\mathbb{R}^d} |(\mathbf{D}\Lambda)^\varepsilon|^2|(\Pi_\varepsilon - S_\varepsilon)b(\mathbf{D})\mathbf{u}_0|^2\,d\mathbf{x} &\le 2\beta_1\int_{\mathbb{R}^d} |(\Pi_\varepsilon - S_\varepsilon)b(\mathbf{D})\mathbf{u}_0|^2\,d\mathbf{x} \\ &+ 2\beta_2\varepsilon^2\sum_{j=1}^d\int_{\mathbb{R}^d} |\Lambda^\varepsilon|^2|(\Pi_\varepsilon - S_\varepsilon)b(\mathbf{D})\partial_j\mathbf{u}_0|^2\,d\mathbf{x}.\end{aligned} \tag{3.14}$$

Next, using Propositions 1.4, 3.1 and relation (1.3), we get

$$\|(\Pi_\varepsilon - S_\varepsilon)b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathbb{R}^d)} \le \varepsilon(r_0^{-1} + r_1)\alpha_1^{1/2}\|\mathbf{u}_0\|_{H^2(\mathbb{R}^d;\mathbb{C}^n)}. \tag{3.15}$$

The second summand on the right-hand side of (3.14) is estimated with the help of (3.13). Finally, combining (3.12)–(3.15) yields

$$\sum_{j=1}^d \big\|\partial_j(\varepsilon\Lambda^\varepsilon(\Pi_\varepsilon - S_\varepsilon)b(\mathbf{D})\mathbf{u}_0)\big\|^2_{L_2(\mathbb{R}^d)} \le \varepsilon^2\left(8M^2(1+\beta_2) + 2\beta_1(r_0^{-1} + r_1)^2\right)\alpha_1\|\mathbf{u}_0\|^2_{H^2(\mathbb{R}^d;\mathbb{C}^n)}.$$

Together with (3.11), this implies (3.10) with the constant

$$\check{C} = \alpha_1^{1/2}\left(M^2(8\beta_2 + 12) + 2\beta_1(r_0^{-1} + r_1)^2\right)^{1/2}. \tag{3.16}$$

□

Proof of Theorem 3.3. By (1.3) and (1.10), we obtain the following lower estimate for the symbol of the effective operator:

$$b(\boldsymbol{\xi})^* g^0 b(\boldsymbol{\xi}) \geq c_0 |\boldsymbol{\xi}|^2 \mathbf{1}_n, \quad \boldsymbol{\xi} \in \mathbb{R}^d, \ \ c_0 = \alpha_0 \|g^{-1}\|^{-1}_{L_\infty}. \tag{3.17}$$

Using the Fourier transformation and (3.17), we estimate the norm of the function $\mathbf{u}_0 = (\mathcal{A}^0 + I)^{-1}\mathbf{F}$ in $H^2(\mathbb{R}^d;\mathbb{C}^n)$:

$$\begin{aligned} \|\mathbf{u}_0\|^2_{H^2(\mathbb{R}^d;\mathbb{C}^n)} &= \int_{\mathbb{R}^d} (1+|\boldsymbol{\xi}|^2)^2 \left|(b(\boldsymbol{\xi})^* g^0 b(\boldsymbol{\xi}) + \mathbf{1}_n)^{-1} \widehat{\mathbf{F}}(\boldsymbol{\xi})\right|^2 d\boldsymbol{\xi} \\ &\leq \int_{\mathbb{R}^d} (1+|\boldsymbol{\xi}|^2)^2 (c_0|\boldsymbol{\xi}|^2 + 1)^{-2} |\widehat{\mathbf{F}}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \leq (1 + c_0^{-1})^2 \|\mathbf{F}\|^2_{L_2(\mathbb{R}^d;\mathbb{C}^n)}. \end{aligned}$$

Combining this with (1.15), (3.5), and (3.10), we obtain

$$\|\mathbf{v}_\varepsilon - \tilde{\mathbf{v}}_\varepsilon\|_{H^1(\mathbb{R}^d;\mathbb{C}^n)} \leq \check{C}\varepsilon \|\mathbf{u}_0\|_{H^2(\mathbb{R}^d;\mathbb{C}^n)} \leq (1 + c_0^{-1}) \check{C} \varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)}. \tag{3.18}$$

Relations (1.16) and (3.18) imply (3.6) with $\widetilde{C}_2 = C_2 + (1 + c_0^{-1})\check{C}$. □

§4. Dirichlet problem homogenization in a bounded domain: preliminaries

4.1. Statement of the problem. Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class $C^{1,1}$. In $L_2(\mathcal{O};\mathbb{C}^n)$, we consider the operator $\mathcal{A}_{D,\varepsilon}$ formally given by the differential expression $b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})$ with the Dirichlet condition on $\partial\mathcal{O}$. More precisely, $\mathcal{A}_{D,\varepsilon}$ is the self-adjoint operator in $L_2(\mathcal{O};\mathbb{C}^n)$ generated by the quadratic form

$$a_{D,\varepsilon}[\mathbf{u},\mathbf{u}] = \int_{\mathcal{O}} \langle g^\varepsilon(\mathbf{x}) b(\mathbf{D})\mathbf{u}, b(\mathbf{D})\mathbf{u}\rangle \, d\mathbf{x}, \quad \mathbf{u} \in H^1_0(\mathcal{O};\mathbb{C}^n).$$

This form is closed and positive definite. Indeed, extend $\mathbf{u}$ by zero to $\mathbb{R}^d \setminus \mathcal{O}$. Then $\mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n)$. Applying (1.4), we obtain

$$c_0 \int_{\mathcal{O}} |\mathbf{D}\mathbf{u}|^2 \, d\mathbf{x} \leq a_{D,\varepsilon}[\mathbf{u},\mathbf{u}] \leq c_1 \int_{\mathcal{O}} |\mathbf{D}\mathbf{u}|^2 \, d\mathbf{x}, \quad \mathbf{u} \in H^1_0(\mathcal{O};\mathbb{C}^n). \tag{4.1}$$

It remains to note that the functional $\|\mathbf{D}\mathbf{u}\|_{L_2(\mathcal{O})}$ determines a norm in $H^1_0(\mathcal{O};\mathbb{C}^n)$ equivalent to the standard one.

Our goal is to find approximation for small ε for the operator $\mathcal{A}^{-1}_{D,\varepsilon}$ in the norm of operators acting from $L_2(\mathcal{O};\mathbb{C}^n)$ to $H^1(\mathcal{O};\mathbb{C}^n)$. In terms of solutions, we are interested in the behavior of the weak solution $\mathbf{u}_\varepsilon \in H^1_0(\mathcal{O};\mathbb{C}^n)$ of the Dirichlet problem

$$b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) \mathbf{u}_\varepsilon(\mathbf{x}) = \mathbf{F}(\mathbf{x}), \quad \mathbf{x} \in \mathcal{O}; \quad \mathbf{u}_\varepsilon|_{\partial\mathcal{O}} = 0, \tag{4.2}$$

where $\mathbf{F} \in L_2(\mathcal{O};\mathbb{C}^n)$. Then $\mathbf{u}_\varepsilon = \mathcal{A}^{-1}_{D,\varepsilon}\mathbf{F}$.

4.2. The energy inequality. Now, we consider problem (4.2) with the right-hand side of class $H^{-1}(\mathcal{O};\mathbb{C}^n)$ and prove the energy inequality. Recall that $H^{-1}(\mathcal{O};\mathbb{C}^n)$ is defined as the space dual to $H^1_0(\mathcal{O};\mathbb{C}^n)$ with respect to the $L_2(\mathcal{O};\mathbb{C}^n)$-coupling. If $\mathbf{f} \in H^{-1}(\mathcal{O};\mathbb{C}^n)$ and $\boldsymbol{\eta} \in H^1_0(\mathcal{O};\mathbb{C}^n)$, we write $\int_{\mathcal{O}} \langle \mathbf{f}, \boldsymbol{\eta}\rangle \, d\mathbf{x}$ for the value of the functional $\mathbf{f}$ on the element $\boldsymbol{\eta}$. We have

$$\left| \int_{\mathcal{O}} \langle \mathbf{f}, \boldsymbol{\eta}\rangle \, d\mathbf{x} \right| \leq \|\mathbf{f}\|_{H^{-1}(\mathcal{O};\mathbb{C}^n)} \|\boldsymbol{\eta}\|_{H^1(\mathcal{O};\mathbb{C}^n)}. \tag{4.3}$$

Lemma 4.1. *Let $\mathbf{f} \in H^{-1}(\mathcal{O};\mathbb{C}^n)$, and let $\mathbf{z}_\varepsilon \in H^1_0(\mathcal{O};\mathbb{C}^n)$ be the weak solution of the Dirichlet problem*

$$b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) \mathbf{z}_\varepsilon(\mathbf{x}) = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathcal{O}; \quad \mathbf{z}_\varepsilon|_{\partial\mathcal{O}} = 0.$$

In other words, $\mathbf{z}_\varepsilon$ *satisfies the identity*

$$\int_{\mathcal{O}} \langle g^\varepsilon(\mathbf{x}) b(\mathbf{D})\mathbf{z}_\varepsilon, b(\mathbf{D})\boldsymbol{\eta}\rangle \, d\mathbf{x} = \int_{\mathcal{O}} \langle \mathbf{f}, \boldsymbol{\eta}\rangle \, d\mathbf{x}, \qquad \boldsymbol{\eta} \in H^1_0(\mathcal{O};\mathbb{C}^n). \tag{4.4}$$

Then the following estimate called the "energy inequality" is true:

$$\|\mathbf{z}_\varepsilon\|_{H^1(\mathcal{O};\mathbb{C}^n)} \le \widehat{C}\|\mathbf{f}\|_{H^{-1}(\mathcal{O};\mathbb{C}^n)}. \tag{4.5}$$

Here $\widehat{C} = (1 + (\operatorname{diam}\mathcal{O})^2)\alpha_0^{-1}\|g^{-1}\|_{L_\infty}$.

Proof. By the lower estimate in (4.1), we have

$$\|\mathbf{D}\mathbf{z}_\varepsilon\|^2_{L_2(\mathcal{O})} \le c_0^{-1} \left(g^\varepsilon b(\mathbf{D})\mathbf{z}_\varepsilon, b(\mathbf{D})\mathbf{z}_\varepsilon\right)_{L_2(\mathcal{O})}. \tag{4.6}$$

Next, (4.3) and (4.4) with $\boldsymbol{\eta} = \mathbf{z}_\varepsilon$ imply

$$\left(g^\varepsilon b(\mathbf{D})\mathbf{z}_\varepsilon, b(\mathbf{D})\mathbf{z}_\varepsilon\right)_{L_2(\mathcal{O})} = \int_{\mathcal{O}} \langle \mathbf{f}, \mathbf{z}_\varepsilon\rangle \, d\mathbf{x} \le \|\mathbf{f}\|_{H^{-1}(\mathcal{O})}\|\mathbf{z}_\varepsilon\|_{H^1(\mathcal{O})}. \tag{4.7}$$

By the Friedrichs inequality,

$$\|\mathbf{z}_\varepsilon\|_{L_2(\mathcal{O})} \le (\operatorname{diam}\mathcal{O})\|\mathbf{D}\mathbf{z}_\varepsilon\|_{L_2(\mathcal{O})}. \tag{4.8}$$

Finally, combining relations (4.6)–(4.8), we obtain

$$\|\mathbf{z}_\varepsilon\|^2_{H^1(\mathcal{O})} \le (1 + (\operatorname{diam}\mathcal{O})^2)\|\mathbf{D}\mathbf{z}_\varepsilon\|^2_{L_2(\mathcal{O})} \le (1 + (\operatorname{diam}\mathcal{O})^2)c_0^{-1}\|\mathbf{f}\|_{H^{-1}(\mathcal{O})}\|\mathbf{z}_\varepsilon\|_{H^1(\mathcal{O})}.$$

This implies (4.5). □

The following statement is a direct consequence of Lemma 4.1.

Corollary 4.2. *The operator* $\mathcal{A}_{D,\varepsilon}^{-1}$ *is continuous from* $L_2(\mathcal{O};\mathbb{C}^n)$ *to* $H^1_0(\mathcal{O};\mathbb{C}^n)$, *and*

$$\|\mathcal{A}_{D,\varepsilon}^{-1}\|_{L_2(\mathcal{O};\mathbb{C}^n)\to H^1(\mathcal{O};\mathbb{C}^n)} \le \widehat{C}.$$

In what follows, we shall need the next claim, which is deduced from Lemma 4.1.

Lemma 4.3. *Let* $\boldsymbol{\psi} \in H^1(\mathcal{O};\mathbb{C}^n)$, *and let* $\mathbf{r}_\varepsilon \in H^1(\mathcal{O};\mathbb{C}^n)$ *be the weak solution of the problem*

$$b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})\mathbf{r}_\varepsilon(\mathbf{x}) = 0, \quad \mathbf{x} \in \mathcal{O}; \;\; \mathbf{r}_\varepsilon|_{\partial\mathcal{O}} = \boldsymbol{\psi}|_{\partial\mathcal{O}}. \tag{4.9}$$

Then

$$\|\mathbf{r}_\varepsilon\|_{H^1(\mathcal{O};\mathbb{C}^n)} \le \gamma_0\|\boldsymbol{\psi}\|_{H^1(\mathcal{O};\mathbb{C}^n)}, \quad \gamma_0 = 1 + \widehat{C} d^{1/2}\alpha_1\|g\|_{L_\infty}. \tag{4.10}$$

Proof. By (4.9), the function $\mathbf{r}_\varepsilon - \boldsymbol{\psi}$ is the solution of the Dirichlet problem

$$\mathcal{A}_\varepsilon(\mathbf{r}_\varepsilon - \boldsymbol{\psi}) = -\mathcal{A}_\varepsilon\boldsymbol{\psi} \;\text{ in } \mathcal{O}; \quad (\mathbf{r}_\varepsilon - \boldsymbol{\psi})|_{\partial\mathcal{O}} = 0. \tag{4.11}$$

Here the right-hand side in the equation belongs to $H^{-1}(\mathcal{O};\mathbb{C}^n)$, and

$$\begin{aligned} \|\mathcal{A}_\varepsilon\boldsymbol{\psi}\|_{H^{-1}(\mathcal{O})} &= \sup_{0\ne\boldsymbol{\varphi}\in H^1_0(\mathcal{O};\mathbb{C}^n)} \frac{\left|\left(g^\varepsilon b(\mathbf{D})\boldsymbol{\psi}, b(\mathbf{D})\boldsymbol{\varphi}\right)_{L_2(\mathcal{O})}\right|}{\|\boldsymbol{\varphi}\|_{H^1(\mathcal{O})}} \\ &\le \alpha_1^{1/2}\|g\|_{L_\infty}\|b(\mathbf{D})\boldsymbol{\psi}\|_{L_2(\mathcal{O})}. \end{aligned} \tag{4.12}$$

We have used the inequality $\|b(\mathbf{D})\boldsymbol{\varphi}\|_{L_2(\mathcal{O})} \le \alpha_1^{1/2}\|\mathbf{D}\boldsymbol{\varphi}\|_{L_2(\mathcal{O})}$, which can be checked as follows. Extend $\boldsymbol{\varphi} \in H^1_0(\mathcal{O};\mathbb{C}^n)$ by zero to $\mathbb{R}^d \setminus \mathcal{O}$, keeping the same notation $\boldsymbol{\varphi}$. Then $\boldsymbol{\varphi} \in H^1(\mathbb{R}^d;\mathbb{C}^n)$. Using the Fourier transformation and the upper estimate in (1.3), we obtain

$$\begin{aligned} \|b(\mathbf{D})\boldsymbol{\varphi}\|^2_{L_2(\mathcal{O})} &= \|b(\mathbf{D})\boldsymbol{\varphi}\|^2_{L_2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |b(\boldsymbol{\xi})\widehat{\boldsymbol{\varphi}}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \\ &\le \alpha_1 \int_{\mathbb{R}^d} |\boldsymbol{\xi}|^2|\widehat{\boldsymbol{\varphi}}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} = \alpha_1\|\mathbf{D}\boldsymbol{\varphi}\|^2_{L_2(\mathbb{R}^d)} = \alpha_1\|\mathbf{D}\boldsymbol{\varphi}\|^2_{L_2(\mathcal{O})}. \end{aligned} \tag{4.13}$$

Next, by (1.2) and (2.6),

$$\|b(\mathbf{D})\boldsymbol{\psi}\|_{L_2(\mathcal{O})} \le \alpha_1^{1/2} \sum_{l=1}^{d} \|D_l \boldsymbol{\psi}\|_{L_2(\mathcal{O})} \le \alpha_1^{1/2} d^{1/2} \|\boldsymbol{\psi}\|_{H^1(\mathcal{O})}. \tag{4.14}$$

From (4.12) and (4.14) it follows that

$$\|\mathcal{A}_\varepsilon \boldsymbol{\psi}\|_{H^{-1}(\mathcal{O})} \le \alpha_1 d^{1/2} \|g\|_{L_\infty} \|\boldsymbol{\psi}\|_{H^1(\mathcal{O})}. \tag{4.15}$$

Applying Lemma 4.1 to problem (4.11), we obtain

$$\|\mathbf{r}_\varepsilon - \boldsymbol{\psi}\|_{H^1(\mathcal{O})} \le \widehat{C} \|\mathcal{A}_\varepsilon \boldsymbol{\psi}\|_{H^{-1}(\mathcal{O})}. \tag{4.16}$$

Now, (4.15) and (4.16) imply (4.10). □

Remark 4.4. The claims of Lemma 4.1 and Corollary 4.2 remain true for any bounded domain $\mathcal{O} \subset \mathbb{R}^d$ (without the assumption that $\partial\mathcal{O} \in C^{1,1}$). The same is true for Lemma 4.3 if the problem (4.9) is understood as the identity

$$\int_{\mathcal{O}} \langle g^\varepsilon(\mathbf{x}) b(\mathbf{D}) \mathbf{r}_\varepsilon, b(\mathbf{D})\boldsymbol{\eta} \rangle \, d\mathbf{x} = 0, \quad \boldsymbol{\eta} \in H^1_0(\mathcal{O};\mathbb{C}^n),$$

and the relation $\mathbf{r}_\varepsilon - \boldsymbol{\psi} \in H^1_0(\mathcal{O};\mathbb{C}^n)$.

4.3. The "homogenized" problem. In $L_2(\mathcal{O};\mathbb{C}^n)$, we consider the selfadjoint operator $\mathcal{A}_D^0$ generated by the quadratic form

$$\int_{\mathcal{O}} \left\langle g^0 b(\mathbf{D})\mathbf{u}, b(\mathbf{D})\mathbf{u} \right\rangle \, d\mathbf{x}, \quad \mathbf{u} \in H^1_0(\mathcal{O};\mathbb{C}^n).$$

Here g^0 is the effective matrix defined by (1.6). Applying Corollary 4.2 with g^ε replaced by g^0 and taking (1.10) into account, we see that the operator $(\mathcal{A}_D^0)^{-1}$ is continuous from $L_2(\mathcal{O};\mathbb{C}^n)$ to $H^1_0(\mathcal{O};\mathbb{C}^n)$, and

$$\|(\mathcal{A}_D^0)^{-1}\|_{L_2(\mathcal{O};\mathbb{C}^n) \to H^1(\mathcal{O};\mathbb{C}^n)} \le \widehat{C}, \tag{4.17}$$

where $\widehat{C}$ is the constant defined in Lemma 4.1. Note that this fact is valid in any bounded domain $\mathcal{O} \subset \mathbb{R}^d$ (without the assumption $\partial\mathcal{O} \in C^{1,1}$).

Let $\mathbf{u}_0 \in H^1_0(\mathcal{O};\mathbb{C}^n)$ be the weak solution of the Dirichlet problem

$$b(\mathbf{D})^* g^0 b(\mathbf{D}) \mathbf{u}_0(\mathbf{x}) = \mathbf{F}(\mathbf{x}), \quad \mathbf{x} \in \mathcal{O}; \;\; \mathbf{u}_0|_{\partial\mathcal{O}} = 0, \tag{4.18}$$

where $\mathbf{F} \in L_2(\mathcal{O};\mathbb{C}^n)$. Then $\mathbf{u}_0 = (\mathcal{A}_D^0)^{-1}\mathbf{F}$.

Since $\partial\mathcal{O} \in C^{1,1}$, for the solution $\mathbf{u}_0$ of (4.18) we have $\mathbf{u}_0 \in H^1_0(\mathcal{O};\mathbb{C}^n) \cap H^2(\mathcal{O};\mathbb{C}^n)$, and

$$\|\mathbf{u}_0\|_{H^2(\mathcal{O};\mathbb{C}^n)} \le \widehat{c} \|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)}. \tag{4.19}$$

Here the constant $\widehat{c}$ depends only on α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the domain $\mathcal{O}$. To justify these properties, it suffices to note that the operator $b(\mathbf{D})^* g^0 b(\mathbf{D})$ is a *strongly elliptic* matrix DO and to apply theorems about regularity of solutions of strongly elliptic systems (see, e.g., [McL, Chapter 4]).

It follows that the operator $\mathcal{A}_D^0$ is given by the differential expression $b(\mathbf{D})^* g^0 b(\mathbf{D})$ on the domain $H^1_0(\mathcal{O};\mathbb{C}^n) \cap H^2(\mathcal{O};\mathbb{C}^n)$, and that the inverse operator satisfies the estimate

$$\|(\mathcal{A}_D^0)^{-1}\|_{L_2(\mathcal{O};\mathbb{C}^n) \to H^2(\mathcal{O};\mathbb{C}^n)} \le \widehat{c}. \tag{4.20}$$

Below we shall see that the solution $\mathbf{u}_\varepsilon$ of problem (4.2) converges in $L_2(\mathcal{O};\mathbb{C}^n)$ to the solution $\mathbf{u}_0$ of the "homogenized" problem (4.18), as $\varepsilon \to 0$. Our *main goal* is to approximate $\mathbf{u}_\varepsilon$ in the norm of $H^1(\mathcal{O};\mathbb{C}^n)$; for this, it is necessary to take the first order corrector into account.

§5. Auxiliary statements

In this section, we prove several auxiliary statements needed for further considerations.

Lemma 5.1. *Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class C^1. Denote*

$$B_\varepsilon = \{\mathbf{x} \in \mathcal{O} \,:\, \operatorname{dist}\{\mathbf{x}, \partial\mathcal{O}\} < \varepsilon\}.$$

Then there exists a number $\varepsilon_0 \in (0,1]$ depending on the domain $\mathcal{O}$ such that for any $u \in H^1(\mathcal{O})$ we have

$$\int_{B_\varepsilon} |u|^2 d\mathbf{x} \le \beta\varepsilon \|u\|_{H^1(\mathcal{O})} \|u\|_{L_2(\mathcal{O})}, \quad 0 < \varepsilon \le \varepsilon_0. \tag{5.1}$$

The constant $\beta = \beta(\mathcal{O})$ depends only on the domain $\mathcal{O}$.

Proof. We start with a model problem in the semiball

$$\mathcal{D}_0 = \left\{\mathbf{x} \in \mathbb{R}^d \,:\, |\mathbf{x}| < 1,\ x_d > 0\right\}.$$

For $\mathbf{x} \in \mathbb{R}^d$ we write $\mathbf{x} = (\mathbf{x}', x_d)$, where $\mathbf{x}' = (x_1, \dots, x_{d-1})$. We introduce the following notation:

$$\mathcal{D}_t = \{\mathbf{x} \in \mathbb{R}^d \,:\, |\mathbf{x}| < 1,\ x_d > t\}, \quad \Sigma_t = \{\mathbf{x} \in \partial\mathcal{D}_t \,:\, x_d = t\}, \quad 0 \le t \le \varepsilon;$$
$$\Upsilon_\varepsilon = \{\mathbf{x} \in \mathbb{R}^d \,:\, |\mathbf{x}| < 1,\ 0 < x_d < \varepsilon\}, \quad \Sigma = \{\mathbf{x} \in \partial\mathcal{D}_0 \,:\, |\mathbf{x}| = 1\}.$$

Assume that $u \in H^1(\mathcal{D}_0)$ and $u = 0$ on Σ. Let $0 \le t \le \varepsilon$. The Green formula in the domain $\mathcal{D}_t$ yields

$$\int_{\mathcal{D}_t} \frac{\partial u}{\partial x_d}\, \bar{u}\, d\mathbf{x}' dx_d = -\int_{\Sigma_t} |u|^2\, d\mathbf{x}' - \int_{\mathcal{D}_t} u\, \frac{\partial \bar{u}}{\partial x_d}\, d\mathbf{x}' dx_d.$$

Hence,

$$\int_{\Sigma_t} |u(\mathbf{x}', t)|^2\, d\mathbf{x}' \le \int_{\mathcal{D}_t} 2\left|\frac{\partial u}{\partial x_d}\right| |u|\, d\mathbf{x} \le 2\biggl(\int_{\mathcal{D}_0} \left|\frac{\partial u}{\partial x_d}\right|^2 d\mathbf{x}\biggr)^{1/2} \biggl(\int_{\mathcal{D}_0} |u|^2\, d\mathbf{x}\biggr)^{1/2}.$$

Integrating over $t \in (0, \varepsilon)$, we obtain

$$\int_{\Upsilon_\varepsilon} |u|^2\, d\mathbf{x} \le 2\varepsilon\biggl(\int_{\mathcal{D}_0} \left|\frac{\partial u}{\partial x_d}\right|^2 d\mathbf{x}\biggr)^{1/2} \biggl(\int_{\mathcal{D}_0} |u|^2\, d\mathbf{x}\biggr)^{1/2}.$$

Now, estimate (5.1) in the case of a bounded domain $\mathcal{O}$ of class C^1 is deduced in a standard way with the help of local charts, diffeomorphisms rectifying the boundary, and partitions of unity. Note that the space H^1 is invariant under diffeomorphisms of class C^1. The number ε_0 must be such that the set B_{ε_0} admits covering by a finite number of open sets admitting diffeomorphisms rectifying the boundary. Thus, the number ε_0 depends only on the domain $\mathcal{O}$. ☐

The next statement is a direct consequence of Lemma 5.1.

Lemma 5.2. *Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class C^1. Denote*

$$(\partial\mathcal{O})_\varepsilon = \left\{\mathbf{x} \in \mathbb{R}^d \,:\, \operatorname{dist}\{\mathbf{x}, \partial\mathcal{O}\} < \varepsilon\right\}.$$

Let $\varepsilon_1 \in (0,1]$ be such that the set $(\partial\mathcal{O})_{\varepsilon_1}$ can be covered by a finite number of open sets admitting diffeomorphisms of class C^1 rectifying the boundary $\partial\mathcal{O}$. Then for any $u \in H^1(\mathbb{R}^d)$ we have

$$\int_{(\partial\mathcal{O})_\varepsilon} |u|^2\, d\mathbf{x} \le \beta^0\varepsilon \|u\|_{H^1(\mathbb{R}^d)} \|u\|_{L_2(\mathbb{R}^d)}, \quad 0 < \varepsilon \le \varepsilon_1. \tag{5.2}$$

The constant $\beta^0 = \beta^0(\mathcal{O})$ depends only on the domain $\mathcal{O}$.

Proof. We apply Lemma 5.1 to the domain $\mathcal{O}$ and to the domain $\mathcal{B}\setminus\bar{\mathcal{O}}$, where $\mathcal{B}$ is some open ball containing $\bar{\mathcal{O}}\cup\overline{(\partial\mathcal{O})_{\varepsilon_1}}$. Then (5.2) is true with $\beta^0=\max\{\beta(\mathcal{O}),\beta(\mathcal{B}\setminus\bar{\mathcal{O}})\}$. □

The following statement is similar to Lemma 2.6 in [ZhPas].

Lemma 5.3. *Let S_ε be the operator* (3.1)*. Suppose that the domain $\mathcal{O}$ and the number ε_1 satisfy the assumptions of Lemma* 5.2*. Assume that $f(\mathbf{x})$ is a Γ-periodic function on $\mathbb{R}^d$ such that $f\in L_2(\Omega)$. Then for any $\mathbf{u}\in H^1(\mathbb{R}^d;\mathbb{C}^m)$ we have*

$$\int_{(\partial\mathcal{O})_\varepsilon}|f^\varepsilon(\mathbf{x})|^2|(S_\varepsilon\mathbf{u})(\mathbf{x})|^2\,d\mathbf{x} \le \beta_*\varepsilon|\Omega|^{-1}\|f\|^2_{L_2(\Omega)}\|\mathbf{u}\|_{H^1(\mathbb{R}^d;\mathbb{C}^m)}\|\mathbf{u}\|_{L_2(\mathbb{R}^d;\mathbb{C}^m)},\quad 0<\varepsilon\le\varepsilon_2, \tag{5.3}$$

where $\varepsilon_2=\varepsilon_1(1+r_1)^{-1}$, $\beta_*=\beta^0(1+r_1)$, $2r_1=\operatorname{diam}\Omega$.

Proof. Using (3.1), the Cauchy inequality, and a change of variables, we obtain

$$\begin{aligned}\int_{(\partial\mathcal{O})_\varepsilon}|f^\varepsilon(\mathbf{x})|^2|(S_\varepsilon\mathbf{u})(\mathbf{x})|^2\,d\mathbf{x} &\le |\Omega|^{-1}\int_{(\partial\mathcal{O})_\varepsilon}d\mathbf{x}\,|f(\varepsilon^{-1}\mathbf{x})|^2\int_\Omega|\mathbf{u}(\mathbf{x}-\varepsilon\mathbf{z})|^2\,d\mathbf{z}\\ &\le |\Omega|^{-1}\int_{(\partial\mathcal{O})_{\widetilde{\varepsilon}}}d\mathbf{y}\int_\Omega d\mathbf{z}\,|f(\varepsilon^{-1}\mathbf{y}+\mathbf{z})|^2|\mathbf{u}(\mathbf{y})|^2\\ &\le |\Omega|^{-1}\|f\|^2_{L_2(\Omega)}\int_{(\partial\mathcal{O})_{\widetilde{\varepsilon}}}|\mathbf{u}(\mathbf{y})|^2\,d\mathbf{y}.\end{aligned}$$

Here $\widetilde{\varepsilon}=\varepsilon(1+r_1)$. Applying Lemma 5.2, we arrive at (5.3). □

§6. Results in the case of bounded Λ

6.1. We start with the case where Condition 1.9 is satisfied. Denote

$$K_D^0(\varepsilon)=[\Lambda^\varepsilon]b(\mathbf{D})(\mathcal{A}_D^0)^{-1}. \tag{6.1}$$

Relation (4.20) shows that $b(\mathbf{D})(\mathcal{A}_D^0)^{-1}$ is a continuous mapping of $L_2(\mathcal{O};\mathbb{C}^n)$ into $H^1(\mathcal{O};\mathbb{C}^m)$. Under Condition 1.9, the operator $[\Lambda^\varepsilon]$ of multiplication by the matrix-valued function $\Lambda^\varepsilon(\mathbf{x})$ is continuous from $H^1(\mathcal{O};\mathbb{C}^m)$ to $H^1(\mathcal{O};\mathbb{C}^n)$. This follows easily from Corollary 2.4. Consequently, the operator (6.1) is continuous from $L_2(\mathcal{O};\mathbb{C}^n)$ to $H^1(\mathcal{O};\mathbb{C}^n)$.

Let $\mathbf{u}_\varepsilon$ be the solution of problem (4.2), and let $\mathbf{u}_0$ be the solution of problem (4.18). The "first order approximation" of $\mathbf{u}_\varepsilon$ is given by

$$\check{\mathbf{v}}_\varepsilon=\mathbf{u}_0+\varepsilon\Lambda^\varepsilon b(\mathbf{D})\mathbf{u}_0=(\mathcal{A}_D^0)^{-1}\mathbf{F}+\varepsilon K_D^0(\varepsilon)\mathbf{F}. \tag{6.2}$$

The next theorem is *our main result* in the case where $\Lambda\in L_\infty$.

Theorem 6.1. *Suppose that $\mathcal{O}\subset\mathbb{R}^d$ is a bounded domain of class $C^{1,1}$. Suppose that $g(\mathbf{x})$ and $b(\mathbf{D})$ satisfy the assumptions of Subsection* 1.2*. Let $\mathbf{u}_\varepsilon$ be the solution of problem* (4.2)*, and let $\mathbf{u}_0$ be the solution of problem* (4.18) *with $\mathbf{F}\in L_2(\mathcal{O};\mathbb{C}^n)$. Suppose that $\Lambda(\mathbf{x})$ is the Γ-periodic solution of problem* (1.5) *and that Condition* 1.9 *is satisfied. Let $\check{\mathbf{v}}_\varepsilon$ be the function defined by* (6.2)*. Then there exists a number $\varepsilon_1\in(0,1]$ depending on the domain $\mathcal{O}$ such that we have*

$$\|\mathbf{u}_\varepsilon-\check{\mathbf{v}}_\varepsilon\|_{H^1(\mathcal{O};\mathbb{C}^n)}\le C_0\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)},\quad 0<\varepsilon\le\varepsilon_1, \tag{6.3}$$

or, in operator terms,

$$\|\mathcal{A}_{D,\varepsilon}^{-1}-(\mathcal{A}_D^0)^{-1}-\varepsilon K_D^0(\varepsilon)\|_{L_2(\mathcal{O};\mathbb{C}^n)\to H^1(\mathcal{O};\mathbb{C}^n)}\le C_0\varepsilon^{1/2},\quad 0<\varepsilon\le\varepsilon_1.$$

The flux $\mathbf{p}_\varepsilon:=g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$ admits the approximation

$$\|\mathbf{p}_\varepsilon-\widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathcal{O};\mathbb{C}^m)}\le C_0'\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)},\quad 0<\varepsilon\le\varepsilon_1, \tag{6.4}$$

where $\widetilde{g}(\mathbf{x}) := g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m)$. *The constants* C_0, C_0' *depend only on* m, d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the parameters of the lattice* Γ, *the norm* $\|\Lambda\|_{L_\infty}$, *and the domain* $\mathcal{O}$.

Recall that some sufficient conditions ensuring Condition 1.9 were given above in Proposition 1.11. In particular, the statement of Theorem 6.1 is true for all operators of the form (1.1) if $d \le 2$, and also for the scalar elliptic operator $\mathcal{A}_\varepsilon = -\operatorname{div} g^\varepsilon(\mathbf{x})\nabla$ in arbitrary dimension, where $g(\mathbf{x})$ is a matrix with real entries.

The next statement is a rough consequence of Theorem 6.1.

Corollary 6.2. *Under the assumptions of Theorem* 6.1, *we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O};\mathbb{C}^n)} \le \widetilde{C}_0 \varepsilon^{1/2} \|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)}, \quad 0 < \varepsilon \le \varepsilon_1, \tag{6.5}$$

or, in operator terms,

$$\left\| \mathcal{A}_{D,\varepsilon}^{-1} - (\mathcal{A}_D^0)^{-1} \right\|_{L_2(\mathcal{O};\mathbb{C}^n) \to L_2(\mathcal{O};\mathbb{C}^n)} \le \widetilde{C}_0 \varepsilon^{1/2}, \quad 0 < \varepsilon \le \varepsilon_1.$$

Here $\widetilde{C}_0 = C_0 + \widehat{C}\alpha_1^{1/2}\|\Lambda\|_{L_\infty}$, *where* $\widehat{C}$ *is the constant defined in Lemma* 4.1.

Proof. From (6.2) and (6.3) it follows that

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O})} \le C_0 \varepsilon^{1/2} \|\mathbf{F}\|_{L_2(\mathcal{O})} + \varepsilon \|\Lambda^\varepsilon b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathcal{O})}, \quad 0 < \varepsilon \le \varepsilon_1. \tag{6.6}$$

Under Condition 1.9, we have

$$\|\Lambda^\varepsilon b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathcal{O})} \le \|\Lambda\|_{L_\infty} \|b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathcal{O})}. \tag{6.7}$$

As in (4.13),

$$\|b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathcal{O})} \le \alpha_1^{1/2} \|\mathbf{D}\mathbf{u}_0\|_{L_2(\mathcal{O})}. \tag{6.8}$$

Combining (6.7) and (6.8) and using (4.17), we obtain

$$\|\Lambda^\varepsilon b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathcal{O})} \le \alpha_1^{1/2} \|\Lambda\|_{L_\infty} \|\mathbf{D}\mathbf{u}_0\|_{L_2(\mathcal{O})} \le \widehat{C}\alpha_1^{1/2} \|\Lambda\|_{L_\infty} \|\mathbf{F}\|_{L_2(\mathcal{O})}.$$

Together with (6.6) this implies (6.5). □

Now we distinguish special cases. The next statement follows from Theorem 6.1 and Propositions 1.2 and 1.3.

Proposition 6.3. 1°. *If* $g^0 = \overline{g}$, *i.e., relations* (1.8) *are satisfied, then* $\Lambda = 0$ *and* $K_D^0(\varepsilon) = 0$. *In this case we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{H^1(\mathcal{O};\mathbb{C}^n)} \le C_0 \varepsilon^{1/2} \|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)}, \quad 0 < \varepsilon \le \varepsilon_1.$$

2°. *If* $g^0 = \underline{g}$, *i.e., relations* (1.9) *are satisfied, then* $\widetilde{g} = g^0$. *In this case we have*

$$\|\mathbf{p}_\varepsilon - g^0 b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathcal{O};\mathbb{C}^m)} \le C_0' \varepsilon^{1/2} \|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)}, \quad 0 < \varepsilon \le \varepsilon_1.$$

6.2. The proof of Theorem 6.1 is based on results for homogenization problem in $\mathbb{R}^d$ (Theorems 1.6 and 1.10) and on the tricks suggested in [Zh2, ZhPas], which make it possible to carry such results over to the case of a bounded domain.

We fix a continuous linear extension operator

$$P_\mathcal{O} : H^2(\mathcal{O};\mathbb{C}^n) \to H^2(\mathbb{R}^d;\mathbb{C}^n), \tag{6.9}$$

and put $\widetilde{\mathbf{u}}_0 = P_\mathcal{O}\mathbf{u}_0$. Then

$$\|\widetilde{\mathbf{u}}_0\|_{H^2(\mathbb{R}^d;\mathbb{C}^n)} \le C_\mathcal{O} \|\mathbf{u}_0\|_{H^2(\mathcal{O};\mathbb{C}^n)}, \tag{6.10}$$

where $C_\mathcal{O}$ is the norm of the operator (6.9). Denote

$$\mathbf{v}_\varepsilon^{(1)}(\mathbf{x}) = \widetilde{\mathbf{u}}_0(\mathbf{x}) + \varepsilon\Lambda^\varepsilon(\mathbf{x}) b(\mathbf{D})\widetilde{\mathbf{u}}_0(\mathbf{x}). \tag{6.11}$$

Then $\check{\mathbf{v}}_\varepsilon = \mathbf{v}_\varepsilon^{(1)}|_\mathcal{O}$.

The following statement is proved with the help of Theorems 1.6 and 1.10.

Lemma 6.4. *Let $\mathbf{u}_0$ be the solution of problem* (4.18), *and let $\check{\mathbf{v}}_\varepsilon$ be the function defined by* (6.2). *Then for $0 < \varepsilon \le 1$ we have*

$$\big\|\mathcal{A}_\varepsilon \check{\mathbf{v}}_\varepsilon - \mathcal{A}^0 \mathbf{u}_0\big\|_{H^{-1}(\mathcal{O};\mathbb{C}^n)} \le C_4 \varepsilon \|\mathbf{u}_0\|_{H^2(\mathcal{O};\mathbb{C}^n)}. \tag{6.12}$$

The constant C_4 depends only on m, d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, the norm $\|\Lambda\|_{L_\infty}$, and the domain $\mathcal{O}$.

Proof. In the case of a bounded domain, the required estimate is deduced from a similar inequality in $\mathbb{R}^d$. Let $\mathbf{v}_\varepsilon^{(1)}$ be defined by (6.11). We check that

$$\big\|\mathcal{A}_\varepsilon \mathbf{v}_\varepsilon^{(1)} - \mathcal{A}^0 \widetilde{\mathbf{u}}_0\big\|_{H^{-1}(\mathbb{R}^d;\mathbb{C}^n)} \le \widetilde{C}_4 \varepsilon \|\widetilde{\mathbf{u}}_0\|_{H^2(\mathbb{R}^d;\mathbb{C}^n)}, \quad 0 < \varepsilon \le 1. \tag{6.13}$$

Clearly,

$$\widetilde{\mathbf{F}} := \mathcal{A}^0 \widetilde{\mathbf{u}}_0 + \widetilde{\mathbf{u}}_0 \in L_2(\mathbb{R}^d;\mathbb{C}^n). \tag{6.14}$$

Using the Fourier transformation and (1.3), (1.10), we obtain

$$\begin{aligned} \|\widetilde{\mathbf{F}}\|^2_{L_2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \big|(b(\boldsymbol{\xi})^* g^0 b(\boldsymbol{\xi}) + \mathbf{1})\widehat{\mathbf{u}}_0(\boldsymbol{\xi})\big|^2 \, d\boldsymbol{\xi} \\ &\le \int_{\mathbb{R}^d} (\alpha_1 |g^0| |\boldsymbol{\xi}|^2 + 1)^2 |\widehat{\mathbf{u}}_0(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \le \left(\max\{\alpha_1 \|g\|_{L_\infty}, 1\}\right)^2 \|\widetilde{\mathbf{u}}_0\|^2_{H^2(\mathbb{R}^d)}. \end{aligned} \tag{6.15}$$

Here $\widehat{\mathbf{u}}_0(\boldsymbol{\xi})$ is the Fourier image of the function $\widetilde{\mathbf{u}}_0(\mathbf{x})$.

Let $\mathbf{s}_\varepsilon \in H^1(\mathbb{R}^d;\mathbb{C}^n)$ be the weak solution of the equation

$$\mathcal{A}_\varepsilon \mathbf{s}_\varepsilon + \mathbf{s}_\varepsilon = \widetilde{\mathbf{F}}. \tag{6.16}$$

Theorems 1.6 and 1.10 imply

$$\|\mathbf{s}_\varepsilon - \widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)} \le C_1 \varepsilon \|\widetilde{\mathbf{F}}\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)}, \quad 0 < \varepsilon \le 1, \tag{6.17}$$

$$\|\mathbf{s}_\varepsilon - \mathbf{v}_\varepsilon^{(1)}\|_{H^1(\mathbb{R}^d;\mathbb{C}^n)} \le C_3 \varepsilon \|\widetilde{\mathbf{F}}\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)}, \quad 0 < \varepsilon \le 1. \tag{6.18}$$

By (6.14) and (6.16),

$$\mathcal{A}_\varepsilon \mathbf{v}_\varepsilon^{(1)} - \mathcal{A}^0 \widetilde{\mathbf{u}}_0 = \mathcal{A}_\varepsilon(\mathbf{v}_\varepsilon^{(1)} - \mathbf{s}_\varepsilon) + \mathcal{A}_\varepsilon \mathbf{s}_\varepsilon - \mathcal{A}^0 \widetilde{\mathbf{u}}_0 = \mathcal{A}_\varepsilon(\mathbf{v}_\varepsilon^{(1)} - \mathbf{s}_\varepsilon) - (\mathbf{s}_\varepsilon - \widetilde{\mathbf{u}}_0).$$

Hence,

$$\big\|\mathcal{A}_\varepsilon \mathbf{v}_\varepsilon^{(1)} - \mathcal{A}^0 \widetilde{\mathbf{u}}_0\big\|_{H^{-1}(\mathbb{R}^d)} \le \big\|\mathcal{A}_\varepsilon(\mathbf{v}_\varepsilon^{(1)} - \mathbf{s}_\varepsilon)\big\|_{H^{-1}(\mathbb{R}^d)} + \big\|\mathbf{s}_\varepsilon - \widetilde{\mathbf{u}}_0\big\|_{H^{-1}(\mathbb{R}^d)}. \tag{6.19}$$

Next, taking (1.3) into account, we obtain

$$\begin{aligned} \big\|\mathcal{A}_\varepsilon(\mathbf{v}_\varepsilon^{(1)} - \mathbf{s}_\varepsilon)\big\|_{H^{-1}(\mathbb{R}^d)} &= \sup_{0 \ne \boldsymbol{\eta} \in H^1(\mathbb{R}^d;\mathbb{C}^n)} \frac{\big|\big(g^\varepsilon b(\mathbf{D})(\mathbf{v}_\varepsilon^{(1)} - \mathbf{s}_\varepsilon), b(\mathbf{D})\boldsymbol{\eta}\big)_{L_2(\mathbb{R}^d)}\big|}{\|\boldsymbol{\eta}\|_{H^1(\mathbb{R}^d)}} \\ &\le \alpha_1 \|g\|_{L_\infty} \|\mathbf{v}_\varepsilon^{(1)} - \mathbf{s}_\varepsilon\|_{H^1(\mathbb{R}^d)}. \end{aligned}$$

Combining this with (6.17)–(6.19), we see that

$$\big\|\mathcal{A}_\varepsilon \mathbf{v}_\varepsilon^{(1)} - \mathcal{A}^0 \widetilde{\mathbf{u}}_0\big\|_{H^{-1}(\mathbb{R}^d)} \le (C_1 + C_3 \alpha_1 \|g\|_{L_\infty}) \varepsilon \|\widetilde{\mathbf{F}}\|_{L_2(\mathbb{R}^d)}, \quad 0 < \varepsilon \le 1. \tag{6.20}$$

Now, (6.15) and (6.20) imply (6.13) with the constant

$$\widetilde{C}_4 = (C_1 + C_3 \alpha_1 \|g\|_{L_\infty}) \max\{\alpha_1 \|g\|_{L_\infty}, 1\}.$$

Returning to the case of a bounded domain, observe that if $\mathbf{f} \in H^{-1}(\mathcal{O};\mathbb{C}^n)$ and $\widetilde{\mathbf{f}} \in H^{-1}(\mathbb{R}^d;\mathbb{C}^n)$ are such that $\widetilde{\mathbf{f}}|_{\mathcal{O}} = \mathbf{f}$, then

$$\|\mathbf{f}\|_{H^{-1}(\mathcal{O})} = \sup_{0\neq\boldsymbol{\varphi}\in C_0^\infty(\mathcal{O})} \frac{\left| \int_{\mathcal{O}} \langle \mathbf{f}, \boldsymbol{\varphi}\rangle \, d\mathbf{x}\right|}{\|\boldsymbol{\varphi}\|_{H^1(\mathcal{O})}} = \sup_{0\neq\boldsymbol{\varphi}\in C_0^\infty(\mathcal{O})} \frac{\left| \int_{\mathbb{R}^d} \langle \widetilde{\mathbf{f}}, \boldsymbol{\varphi}\rangle \, d\mathbf{x}\right|}{\|\boldsymbol{\varphi}\|_{H^1(\mathbb{R}^d)}}$$
$$\leq \sup_{0\neq\boldsymbol{\varphi}\in C_0^\infty(\mathbb{R}^d)} \frac{\left| \int_{\mathbb{R}^d} \langle \widetilde{\mathbf{f}}, \boldsymbol{\varphi}\rangle \, d\mathbf{x}\right|}{\|\boldsymbol{\varphi}\|_{H^1(\mathbb{R}^d)}} = \|\widetilde{\mathbf{f}}\|_{H^{-1}(\mathbb{R}^d)}.$$

Hence,

$$\left\|\mathcal{A}_\varepsilon \check{\mathbf{v}}_\varepsilon - \mathcal{A}^0 \mathbf{u}_0\right\|_{H^{-1}(\mathcal{O})} \leq \left\|\mathcal{A}_\varepsilon \mathbf{v}_\varepsilon^{(1)} - \mathcal{A}^0 \widetilde{\mathbf{u}}_0\right\|_{H^{-1}(\mathbb{R}^d)}.$$

Together with (6.13) and (6.10), this yields

$$\left\|\mathcal{A}_\varepsilon \check{\mathbf{v}}_\varepsilon - \mathcal{A}^0 \mathbf{u}_0\right\|_{H^{-1}(\mathcal{O})} \leq \widetilde{C}_4 \varepsilon \|\widetilde{\mathbf{u}}_0\|_{H^2(\mathbb{R}^d)} \leq \widetilde{C}_4 C_{\mathcal{O}} \varepsilon \|\mathbf{u}_0\|_{H^2(\mathcal{O})}.$$

Thus, inequality (6.12) holds true with $C_4 = \widetilde{C}_4 C_{\mathcal{O}}$. □

6.3. The first order approximation $\check{\mathbf{v}}_\varepsilon$ of the solution $\mathbf{u}_\varepsilon$, defined by (6.2), does not satisfy the Dirichlet condition on $\partial\mathcal{O}$. We consider the "discrepancy" $\check{\mathbf{w}}_\varepsilon$ which is the generalized solution of the problem

$$\mathcal{A}_\varepsilon \check{\mathbf{w}}_\varepsilon = 0 \ \text{ in } \mathcal{O}, \quad \check{\mathbf{w}}_\varepsilon|_{\partial\mathcal{O}} = \check{\mathbf{v}}_\varepsilon|_{\partial\mathcal{O}} = \varepsilon \Lambda^\varepsilon b(\mathbf{D}) \mathbf{u}_0|_{\partial\mathcal{O}}. \tag{6.21}$$

Here the equation is understood in the weak sense: the function $\check{\mathbf{w}}_\varepsilon \in H^1(\mathcal{O};\mathbb{C}^n)$ satisfies the identity

$$\int_{\mathcal{O}} \langle g^\varepsilon(\mathbf{x}) b(\mathbf{D}) \check{\mathbf{w}}_\varepsilon, b(\mathbf{D})\boldsymbol{\eta}\rangle \, d\mathbf{x} = 0, \quad \boldsymbol{\eta} \in H_0^1(\mathcal{O};\mathbb{C}^n).$$

The boundary condition in (6.21) is understood in the sense of the trace theorem: under the assumptions of Theorem 6.1 one has $\Lambda^\varepsilon b(\mathbf{D})\mathbf{u}_0 \in H^1(\mathcal{O};\mathbb{C}^n)$, whence $\Lambda^\varepsilon b(\mathbf{D})\mathbf{u}_0|_{\partial\mathcal{O}} \in H^{1/2}(\partial\mathcal{O};\mathbb{C}^n)$.

By (4.2) and (4.18), $\mathcal{A}_\varepsilon(\mathbf{u}_\varepsilon - \check{\mathbf{v}}_\varepsilon) = \mathcal{A}^0\mathbf{u}_0 - \mathcal{A}_\varepsilon \check{\mathbf{v}}_\varepsilon$. Consequently, by (6.21), the function $\mathbf{u}_\varepsilon - \check{\mathbf{v}}_\varepsilon + \check{\mathbf{w}}_\varepsilon$ is the solution of the following Dirichlet problem

$$\mathcal{A}_\varepsilon(\mathbf{u}_\varepsilon - \check{\mathbf{v}}_\varepsilon + \check{\mathbf{w}}_\varepsilon) = \mathcal{A}^0\mathbf{u}_0 - \mathcal{A}_\varepsilon \check{\mathbf{v}}_\varepsilon \ \text{ in } \mathcal{O}, \quad (\mathbf{u}_\varepsilon - \check{\mathbf{v}}_\varepsilon + \check{\mathbf{w}}_\varepsilon)|_{\partial\mathcal{O}} = 0.$$

The right-hand side in the equation belongs to $H^{-1}(\mathcal{O};\mathbb{C}^n)$. Then, applying Lemmas 4.1 and 6.4, for $0 < \varepsilon \leq 1$ we obtain

$$\|\mathbf{u}_\varepsilon - \check{\mathbf{v}}_\varepsilon + \check{\mathbf{w}}_\varepsilon\|_{H^1(\mathcal{O};\mathbb{C}^n)} \leq \widehat{C} \left\|\mathcal{A}^0\mathbf{u}_0 - \mathcal{A}_\varepsilon \check{\mathbf{v}}_\varepsilon\right\|_{H^{-1}(\mathcal{O};\mathbb{C}^n)} \leq \widehat{C} C_4 \varepsilon \|\mathbf{u}_0\|_{H^2(\mathcal{O};\mathbb{C}^n)}.$$

Together with (4.19) this implies that

$$\|\mathbf{u}_\varepsilon - \check{\mathbf{v}}_\varepsilon + \check{\mathbf{w}}_\varepsilon\|_{H^1(\mathcal{O};\mathbb{C}^n)} \leq \widehat{C} C_4 \widehat{c} \varepsilon \|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)}, \quad 0 < \varepsilon \leq 1. \tag{6.22}$$

Therefore, the proof of estimate (6.3) in Theorem 6.1 is reduced to estimation of $\check{\mathbf{w}}_\varepsilon$ in $H^1(\mathcal{O};\mathbb{C}^n)$.

Assume that $0 < \varepsilon \leq \varepsilon_1$, where the number $\varepsilon_1 \in (0,1]$ is defined in Lemma 5.2. Fix a smooth cut-off function $\theta_\varepsilon(\mathbf{x})$ in $\mathbb{R}^d$ supported in the ε–vicinity of the boundary $\partial\mathcal{O}$ and such that

$$\begin{gathered} \theta_\varepsilon \in C_0^\infty(\mathbb{R}^d), \quad \operatorname{supp}\theta_\varepsilon \subset (\partial\mathcal{O})_\varepsilon, \quad 0 \leq \theta_\varepsilon(\mathbf{x}) \leq 1, \\ \theta_\varepsilon(\mathbf{x})|_{\partial\mathcal{O}} = 1, \quad \varepsilon\,|\nabla\theta_\varepsilon(\mathbf{x})| \leq \kappa = \text{const}. \end{gathered} \tag{6.23}$$

Consider the following function in $\mathbb{R}^d$:

$$\check{\boldsymbol{\phi}}_\varepsilon(\mathbf{x}) = \varepsilon\theta_\varepsilon(\mathbf{x})\Lambda^\varepsilon(\mathbf{x}) b(\mathbf{D})\widetilde{\mathbf{u}}_0(\mathbf{x}). \tag{6.24}$$

Then $\check{\boldsymbol{\phi}}_\varepsilon \in H^1(\mathbb{R}^d;\mathbb{C}^n)$ and $\check{\boldsymbol{\phi}}_\varepsilon|_{\partial\mathcal{O}} = \varepsilon\Lambda^\varepsilon b(\mathbf{D})\mathbf{u}_0|_{\partial\mathcal{O}}$. The problem (6.21) can be rewritten as: $\mathcal{A}_\varepsilon \check{\mathbf{w}}_\varepsilon = 0$ in $\mathcal{O}$, $\check{\mathbf{w}}_\varepsilon|_{\partial\mathcal{O}} = \check{\boldsymbol{\phi}}_\varepsilon|_{\partial\mathcal{O}}$. Applying Lemma 4.3, we obtain

$$\|\check{\mathbf{w}}_\varepsilon\|_{H^1(\mathcal{O};\mathbb{C}^n)} \leq \gamma_0 \|\check{\boldsymbol{\phi}}_\varepsilon\|_{H^1(\mathcal{O};\mathbb{C}^n)}. \tag{6.25}$$

Thus, the proof of the required estimate for the norm of $\check{\mathbf{w}}_\varepsilon$ in $H^1(\mathcal{O};\mathbb{C}^n)$ is reduced to the next statement.

Lemma 6.5. *Under the assumptions of Theorem* 6.1*, suppose that* $0 < \varepsilon \leq \varepsilon_1$*, where* $\varepsilon_1 \in (0,1]$ *is the number defined in Lemma* 5.2*. Let* $\check{\boldsymbol{\phi}}_\varepsilon$ *be the function defined in accordance with* (6.23)*,* (6.24)*. Then*

$$\|\check{\boldsymbol{\phi}}_\varepsilon\|_{H^1(\mathcal{O};\mathbb{C}^n)} \leq C_5 \varepsilon^{1/2} \|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)}, \quad 0 < \varepsilon \leq \varepsilon_1. \tag{6.26}$$

The constant C_5 *depends only on* m*,* d*,* α_0*,* α_1*,* $\|g\|_{L_\infty}$*,* $\|g^{-1}\|_{L_\infty}$*, the norm* $\|\Lambda\|_{L_\infty}$*, and the domain* $\mathcal{O}$*.*

Proof. The norm of $\check{\boldsymbol{\phi}}_\varepsilon$ in $L_2(\mathcal{O};\mathbb{C}^n)$ is estimated with the help of Condition 1.9 and relations (4.17), (6.8), and (6.23):

$$\begin{aligned} \|\check{\boldsymbol{\phi}}_\varepsilon\|_{L_2(\mathcal{O})} &\leq \varepsilon \|\Lambda^\varepsilon b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathcal{O})} \leq \varepsilon \|\Lambda\|_{L_\infty} \|b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathcal{O})} \\ &\leq \varepsilon \alpha_1^{1/2} \|\Lambda\|_{L_\infty} \|\mathbf{u}_0\|_{H^1(\mathcal{O})} \leq \varepsilon \widehat{C} \alpha_1^{1/2} \|\Lambda\|_{L_\infty} \|\mathbf{F}\|_{L_2(\mathcal{O})}. \end{aligned} \tag{6.27}$$

Consider the derivatives

$$\frac{\partial \check{\boldsymbol{\phi}}_\varepsilon}{\partial x_j} = \varepsilon \frac{\partial \theta_\varepsilon}{\partial x_j} \Lambda^\varepsilon b(\mathbf{D})\tilde{\mathbf{u}}_0 + \theta_\varepsilon \Big(\frac{\partial \Lambda}{\partial x_j}\Big)^\varepsilon b(\mathbf{D})\tilde{\mathbf{u}}_0 + \varepsilon \theta_\varepsilon \Lambda^\varepsilon (b(\mathbf{D})\partial_j \tilde{\mathbf{u}}_0), \quad j = 1, \dots, d.$$

Then

$$\begin{aligned} \|\mathbf{D}\check{\boldsymbol{\phi}}_\varepsilon\|^2_{L_2(\mathcal{O})} &\leq 3\varepsilon^2 \int_{\mathcal{O}} |\nabla\theta_\varepsilon|^2 |\Lambda^\varepsilon b(\mathbf{D})\mathbf{u}_0|^2 \, d\mathbf{x} \\ &+ 3 \int_{\mathcal{O}} |(\mathbf{D}\Lambda)^\varepsilon|^2 |\theta_\varepsilon b(\mathbf{D})\tilde{\mathbf{u}}_0|^2 \, d\mathbf{x} + 3\varepsilon^2 \sum_{j=1}^d \int_{\mathcal{O}} |\theta_\varepsilon|^2 |\Lambda^\varepsilon b(\mathbf{D}) D_j \mathbf{u}_0|^2 \, d\mathbf{x}. \end{aligned} \tag{6.28}$$

We denote the terms on the right-hand side of (6.28) by $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$, respectively.

It is easy to estimate $\mathcal{I}_3$. By (6.23), Condition 1.9, and (1.2), (2.6), we have

$$\|\theta_\varepsilon \Lambda^\varepsilon b(\mathbf{D}) D_j \mathbf{u}_0\|^2_{L_2(\mathcal{O})} \leq \|\Lambda\|^2_{L_\infty} \alpha_1 d \sum_{l=1}^d \|D_l D_j \mathbf{u}_0\|^2_{L_2(\mathcal{O})}.$$

Together with (4.19) this yields

$$\mathcal{I}_3 \leq 3\varepsilon^2 \|\Lambda\|^2_{L_\infty} \alpha_1 d \|\mathbf{u}_0\|^2_{H^2(\mathcal{O})} \leq \gamma_3 \varepsilon^2 \|\mathbf{F}\|^2_{L_2(\mathcal{O})}, \tag{6.29}$$

where $\gamma_3 = 3\widehat{c}^2 \alpha_1 d \|\Lambda\|^2_{L_\infty}$.

In order to estimate the first term on the right-hand side of (6.28), we apply (6.23), Condition 1.9, and Lemma 5.1, obtaining

$$\mathcal{I}_1 \leq 3\kappa^2 \|\Lambda\|^2_{L_\infty} \int_{B_\varepsilon} |b(\mathbf{D})\mathbf{u}_0|^2 \, d\mathbf{x} \leq 3\kappa^2 \|\Lambda\|^2_{L_\infty} \beta\varepsilon \|b(\mathbf{D})\mathbf{u}_0\|_{H^1(\mathcal{O})} \|b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathcal{O})}.$$

Using (4.17), (4.19), (6.8), and the estimate

$$\|b(\mathbf{D})\mathbf{u}_0\|_{H^1(\mathcal{O})} = \Big\|\sum_{l=1}^d b_l D_l \mathbf{u}_0\Big\|_{H^1(\mathcal{O})} \leq \alpha_1^{1/2} d^{1/2} \|\mathbf{u}_0\|_{H^2(\mathcal{O})},$$

we arrive at the inequality

$$\mathcal{I}_1 \leq 3\varepsilon\kappa^2 \|\Lambda\|^2_{L_\infty} \beta \alpha_1 d^{1/2} \|\mathbf{u}_0\|_{H^1(\mathcal{O})} \|\mathbf{u}_0\|_{H^2(\mathcal{O})} \leq \gamma_1 \varepsilon \|\mathbf{F}\|^2_{L_2(\mathcal{O})}, \tag{6.30}$$

where $\gamma_1 = 3\widehat{c}\widehat{C}\kappa^2\|\Lambda\|^2_{L_\infty}\beta\alpha_1 d^{1/2}$.

It remains to consider the second term on the right-hand side of (6.28). By Corollary 2.4,

$$\begin{aligned}\mathcal{I}_2 \le 3\int_{\mathbb{R}^d} |(\mathbf{D}\Lambda)^\varepsilon|^2|\theta_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0|^2\,d\mathbf{x} &\le 3\beta_1\,\|\theta_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|^2_{L_2(\mathbb{R}^d)}\\ &+ 3\beta_2\|\Lambda\|^2_{L_\infty}\varepsilon^2\int_{\mathbb{R}^d}\sum_{l=1}^d\Big|\frac{\partial}{\partial x_l}(\theta_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0)\Big|^2\,d\mathbf{x}.\end{aligned}\tag{6.31}$$

Since

$$\frac{\partial}{\partial x_l}(\theta_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0) = \frac{\partial\theta_\varepsilon}{\partial x_l}b(\mathbf{D})\widetilde{\mathbf{u}}_0 + \theta_\varepsilon\frac{\partial}{\partial x_l}(b(\mathbf{D})\widetilde{\mathbf{u}}_0),$$

we can use (6.23) and (6.31) to get

$$\mathcal{I}_2 \le 3\left(\beta_1 + 2\beta_2\|\Lambda\|^2_{L_\infty}\kappa^2\right)\int_{(\partial\mathcal{O})_\varepsilon}|b(\mathbf{D})\widetilde{\mathbf{u}}_0|^2\,d\mathbf{x} + 6\beta_2\|\Lambda\|^2_{L_\infty}\varepsilon^2\int_{\mathbb{R}^d}\sum_{l=1}^d|b(\mathbf{D})D_l\widetilde{\mathbf{u}}_0|^2\,d\mathbf{x}.$$

Combining this with Lemma 5.2 and condition (1.3), we obtain

$$\begin{aligned}\mathcal{I}_2 \le\ & 3\left(\beta_1 + 2\beta_2\|\Lambda\|^2_{L_\infty}\kappa^2\right)\beta^0\varepsilon\|b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{H^1(\mathbb{R}^d)}\|b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)}\\ &+ 6\beta_2\|\Lambda\|^2_{L_\infty}\varepsilon^2\alpha_1\|\widetilde{\mathbf{u}}_0\|^2_{H^2(\mathbb{R}^d)}\\ \le\ & 3\varepsilon\left(\beta_1 + 2\beta_2\|\Lambda\|^2_{L_\infty}\kappa^2\right)\beta^0\alpha_1\|\widetilde{\mathbf{u}}_0\|_{H^2(\mathbb{R}^d)}\|\widetilde{\mathbf{u}}_0\|_{H^1(\mathbb{R}^d)}\\ &+ 6\varepsilon^2\beta_2\|\Lambda\|^2_{L_\infty}\alpha_1\|\widetilde{\mathbf{u}}_0\|^2_{H^2(\mathbb{R}^d)}.\end{aligned}\tag{6.32}$$

Taking (4.19) and (6.10) into account, from (6.32) we deduce that

$$\mathcal{I}_2 \le \gamma_2\varepsilon\|\mathbf{F}\|^2_{L_2(\mathcal{O})},\tag{6.33}$$

where $\gamma_2 = 3(\widehat{c}C_{\mathcal{O}})^2\left(\left(\beta_1 + 2\beta_2\|\Lambda\|^2_{L_\infty}\kappa^2\right)\beta^0\alpha_1 + 2\beta_2\alpha_1\|\Lambda\|^2_{L_\infty}\right)$.

Now, relations (6.28)–(6.30) and (6.33) imply

$$\left\|\mathbf{D}\check{\boldsymbol{\phi}}_\varepsilon\right\|^2_{L_2(\mathcal{O})} \le \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 \le \varepsilon(\gamma_1+\gamma_2+\gamma_3)\|\mathbf{F}\|^2_{L_2(\mathcal{O})},\quad 0<\varepsilon\le\varepsilon_1.\tag{6.34}$$

Finally, (6.27) and (6.34) yield estimate (6.26) with

$$C_5 = \left(\widehat{C}^2\alpha_1\|\Lambda\|^2_{L_\infty} + \gamma_1+\gamma_2+\gamma_3\right)^{1/2}.$$

□

Now it is easy to complete the proof of Theorem 6.1.

Proof of Theorem 6.1. From (6.22), (6.25), and (6.26) it follows that

$$\|\mathbf{u}_\varepsilon - \check{\mathbf{v}}_\varepsilon\|_{H^1(\mathcal{O};\mathbb{C}^n)} \le \widehat{C}C_4\widehat{c}\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)} + \gamma_0C_5\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)},\quad 0<\varepsilon\le\varepsilon_1,$$

which implies (6.3) with $C_0 = \widehat{C}C_4\widehat{c} + \gamma_0C_5$.

It remains to check (6.4). By (6.3), (1.2), and (2.6), we have

$$\begin{aligned}\left\|\mathbf{p}_\varepsilon - g^\varepsilon b(\mathbf{D})\mathbf{u}_0 - g^\varepsilon b(\mathbf{D})(\varepsilon\Lambda^\varepsilon b(\mathbf{D})\mathbf{u}_0)\right\|_{L_2(\mathcal{O})} &\le \|g\|_{L_\infty}\alpha_1^{1/2}d^{1/2}C_0\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})},\\ &\qquad 0<\varepsilon\le\varepsilon_1.\end{aligned}\tag{6.35}$$

Using (1.2) and the definition of the matrix $\widetilde{g}$, we see that

$$g^\varepsilon b(\mathbf{D})\mathbf{u}_0 + g^\varepsilon b(\mathbf{D})(\varepsilon\Lambda^\varepsilon b(\mathbf{D})\mathbf{u}_0) = \widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0 + \varepsilon g^\varepsilon\sum_{l=1}^d b_l\Lambda^\varepsilon b(\mathbf{D})D_l\mathbf{u}_0.\tag{6.36}$$

Applying Condition 1.9 and relations (1.2), (2.6), and (4.19), we obtain

$$\left\|\varepsilon g^\varepsilon\sum_{l=1}^d b_l\Lambda^\varepsilon b(\mathbf{D})D_l\mathbf{u}_0\right\|_{L_2(\mathcal{O})} \le \varepsilon\|g\|_{L_\infty}\|\Lambda\|_{L_\infty}\alpha_1 d\widehat{c}\|\mathbf{F}\|_{L_2(\mathcal{O})}.\tag{6.37}$$

Now, relations (6.35)–(6.37) imply (6.4) with the constant

$$C_0' = \|g\|_{L_\infty}\alpha_1^{1/2}d^{1/2}C_0 + \|g\|_{L_\infty}\|\Lambda\|_{L_\infty}\alpha_1 d\widehat{c}. \qquad \square$$

§7. Results in the general case

7.1. Now we lift the assumption that $\Lambda(\mathbf{x})$ is bounded. Then we need to include a smoothing operator in the corrector.

Let $P_{\mathcal{O}}$ be the extension operator (6.9), and let S_ε be the Steklov smoothing operator defined by (3.1). We denote by $R_{\mathcal{O}}$ the operator of restriction of functions in $\mathbb{R}^d$ to the domain $\mathcal{O}$. We put

$$K_D(\varepsilon) = R_{\mathcal{O}}[\Lambda^\varepsilon]S_\varepsilon b(\mathbf{D})P_{\mathcal{O}}(\mathcal{A}_D^0)^{-1}. \tag{7.1}$$

The operator $b(\mathbf{D})P_{\mathcal{O}}(\mathcal{A}_D^0)^{-1}$ is a continuous mapping of $L_2(\mathcal{O};\mathbb{C}^n)$ into $H^1(\mathbb{R}^d;\mathbb{C}^m)$. As was mentioned in Subsection 3.2, the operator $[\Lambda^\varepsilon]S_\varepsilon$ is continuous from $H^1(\mathbb{R}^d;\mathbb{C}^m)$ to $H^1(\mathbb{R}^d;\mathbb{C}^n)$. Hence, the operator (7.1) is continuous from $L_2(\mathcal{O};\mathbb{C}^n)$ to $H^1(\mathcal{O};\mathbb{C}^n)$.

Let $\mathbf{u}_\varepsilon$ be the solution of problem (4.2), and let $\mathbf{u}_0$ be the solution of problem (4.18). As above, we denote $\widetilde{\mathbf{u}}_0 = P_{\mathcal{O}}\mathbf{u}_0$. We put

$$\mathbf{v}_\varepsilon^{(2)}(\mathbf{x}) = \widetilde{\mathbf{u}}_0(\mathbf{x}) + \varepsilon\Lambda^\varepsilon(\mathbf{x})(S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0)(\mathbf{x}),$$

and $\mathbf{v}_\varepsilon := \mathbf{v}_\varepsilon^{(2)}|_{\mathcal{O}}$. Then

$$\mathbf{v}_\varepsilon = (\mathcal{A}_D^0)^{-1}\mathbf{F} + \varepsilon K_D(\varepsilon)\mathbf{F}. \tag{7.2}$$

The next theorem is *our main result* in the general case.

Theorem 7.1. *Suppose that $\mathcal{O} \subset \mathbb{R}^d$ is a bounded domain of class $C^{1,1}$, and that $g(\mathbf{x})$ and $b(\mathbf{D})$ satisfy the assumptions of Subsection* 1.2. *Let $\mathbf{u}_\varepsilon$ be the solution of problem* (4.2), *and let $\mathbf{u}_0$ be the solution of problem* (4.18) *with $\mathbf{F} \in L_2(\mathcal{O};\mathbb{C}^n)$. Let $\mathbf{v}_\varepsilon$ be the function defined by* (7.1), (7.2). *Then there exists a number $\varepsilon_2 \in (0,1]$ depending on the domain $\mathcal{O}$ and the lattice Γ and such that*

$$\|\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon\|_{H^1(\mathcal{O};\mathbb{C}^n)} \le C\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)}, \quad 0 < \varepsilon \le \varepsilon_2, \tag{7.3}$$

or, in operator terms,

$$\left\|\mathcal{A}_{D,\varepsilon}^{-1} - (\mathcal{A}_D^0)^{-1} - \varepsilon K_D(\varepsilon)\right\|_{L_2(\mathcal{O};\mathbb{C}^n)\to H^1(\mathcal{O};\mathbb{C}^n)} \le C\varepsilon^{1/2}.$$

The flux $\mathbf{p}_\varepsilon := g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$ admits the approximation

$$\left\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\right\|_{L_2(\mathcal{O};\mathbb{C}^m)} \le C'\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)}, \quad 0 < \varepsilon \le \varepsilon_2, \tag{7.4}$$

where $\widetilde{g}(\mathbf{x}) := g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m)$. The constants C, C' depend only on m, d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, and the domain $\mathcal{O}$.

The next statement is a rough consequence of Theorem 7.1.

Corollary 7.2. *Under the assumptions of Theorem* 7.1, *for $0 < \varepsilon \le \varepsilon_2$ we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O};\mathbb{C}^n)} \le \widetilde{C}\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)}, \tag{7.5}$$

or, in operator terms,

$$\left\|\mathcal{A}_{D,\varepsilon}^{-1} - (\mathcal{A}_D^0)^{-1}\right\|_{L_2(\mathcal{O};\mathbb{C}^n)\to L_2(\mathcal{O};\mathbb{C}^n)} \le \widetilde{C}\varepsilon^{1/2}.$$

The constant $\widetilde{C}$ is given by

$$\widetilde{C} = C + C_{\mathcal{O}}\widehat{c}m^{1/2}(2r_0)^{-1}\alpha_0^{-1/2}\alpha_1^{1/2}\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty}^{1/2},$$

where $\widehat{c}$ is the constant occurring in (4.19), *$C_{\mathcal{O}}$ is the norm of the extension operator $P_{\mathcal{O}}$, and r_0 is the radius of the ball inscribed in* $\operatorname{clos}\widetilde{\Omega}$.

Proof. From (7.2) and (7.3) it follows that

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O})} \le C\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})} + \varepsilon\|\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathcal{O})}. \tag{7.6}$$

By (3.9) and (1.3),

$$\begin{aligned} \|\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathcal{O})} &\le \|\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} \\ &\le M\|b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} \le M\alpha_1^{1/2}\|\widetilde{\mathbf{u}}_0\|_{H^1(\mathbb{R}^d)}. \end{aligned} \tag{7.7}$$

Taking (4.19) and (6.10) into account, we obtain

$$\|\widetilde{\mathbf{u}}_0\|_{H^1(\mathbb{R}^d)} \le \|\widetilde{\mathbf{u}}_0\|_{H^2(\mathbb{R}^d)} \le C_\mathcal{O}\|\mathbf{u}_0\|_{H^2(\mathcal{O})} \le C_\mathcal{O}\widehat{c}\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{7.8}$$

Now, relations (7.6)–(7.8) imply

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O})} \le C\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})} + \varepsilon M\alpha_1^{1/2}C_\mathcal{O}\widehat{c}\|\mathbf{F}\|_{L_2(\mathcal{O})}.$$

Recalling the expression for M (see (3.8)), we arrive at (7.5). □

7.2. We start the proof of Theorem 7.1. The next statement is similar to Lemma 6.4.

Lemma 7.3. *Let* $\mathbf{u}_0$ *be the solution of problem* (4.18)*, and let* $\mathbf{v}_\varepsilon$ *be the function defined by* (7.1), (7.2)*. Then for* $0 < \varepsilon \le 1$ *we have*

$$\big\|\mathcal{A}_\varepsilon\mathbf{v}_\varepsilon - \mathcal{A}^0\mathbf{u}_0\big\|_{H^{-1}(\mathcal{O};\mathbb{C}^n)} \le C_6\varepsilon\|\mathbf{u}_0\|_{H^2(\mathcal{O};\mathbb{C}^n)}.$$

Here the constant C_6 *is given by*

$$C_6 = C_\mathcal{O}(C_1 + \widetilde{C}_2\alpha_1\|g\|_{L_\infty})\max\{\alpha_1\|g\|_{L_\infty}, 1\}$$

and depends only on m*,* d*,* α_0*,* α_1*,* $\|g\|_{L_\infty}$*,* $\|g^{-1}\|_{L_\infty}$*, the parameters of the lattice* Γ*, and the domain* $\mathcal{O}$*.*

Proof. Lemma 7.3 can be proved much as Lemma 6.4. The only difference is that Theorem 3.3 should be applied instead of Theorem 1.10. □

Next, by analogy with the proof of Theorem 6.1, we consider the "discrepancy" $\mathbf{w}_\varepsilon \in H^1(\mathcal{O};\mathbb{C}^n)$, which is the weak solution of the problem

$$\mathcal{A}_\varepsilon\mathbf{w}_\varepsilon = 0 \ \text{ in } \mathcal{O}, \quad \mathbf{w}_\varepsilon|_{\partial\mathcal{O}} = \mathbf{v}_\varepsilon|_{\partial\mathcal{O}} = \varepsilon\Lambda^\varepsilon(S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0)|_{\partial\mathcal{O}}. \tag{7.9}$$

The equation in (7.9) is understood in the weak sense, and the boundary condition in the sense of the trace theorem. Observe that $\Lambda^\varepsilon(S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0) \in H^1(\mathcal{O};\mathbb{C}^n)$.

By (4.2), (4.18), and (7.9), the function $\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon + \mathbf{w}_\varepsilon$ is the solution of the following problem:

$$\mathcal{A}_\varepsilon(\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon + \mathbf{w}_\varepsilon) = \mathcal{A}^0\mathbf{u}_0 - \mathcal{A}_\varepsilon\mathbf{v}_\varepsilon \ \text{ in } \ \mathcal{O}, \quad (\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon + \mathbf{w}_\varepsilon)|_{\partial\mathcal{O}} = 0.$$

Applying Lemmas 4.1 and 7.3, for $0 < \varepsilon \le 1$ we obtain

$$\|\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon + \mathbf{w}_\varepsilon\|_{H^1(\mathcal{O};\mathbb{C}^n)} \le \widehat{C}\|\mathcal{A}^0\mathbf{u}_0 - \mathcal{A}_\varepsilon\mathbf{v}_\varepsilon\|_{H^{-1}(\mathcal{O};\mathbb{C}^n)} \le \widehat{C}C_6\varepsilon\|\mathbf{u}_0\|_{H^2(\mathcal{O};\mathbb{C}^n)}.$$

Together with (4.19), this implies

$$\|\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon + \mathbf{w}_\varepsilon\|_{H^1(\mathcal{O};\mathbb{C}^n)} \le \widehat{C}C_6\widehat{c}\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)}, \quad 0 < \varepsilon \le 1. \tag{7.10}$$

7.3. By (7.10), the proof of estimate (7.3) is reduced to estimating the H^1-norm of $\mathbf{w}_\varepsilon$. As in Subsection 6.3, we fix a cut-off function $\theta_\varepsilon(\mathbf{x})$ satisfying conditions (6.23). We assume that $0 < \varepsilon \le \varepsilon_2$, where $\varepsilon_2 \in (0, 1]$ is the number defined in Lemma 5.3. Consider the following function on $\mathbb{R}^d$:

$$\boldsymbol{\phi}_\varepsilon(\mathbf{x}) = \varepsilon\theta_\varepsilon(\mathbf{x})\Lambda^\varepsilon(\mathbf{x})(S_\varepsilon b(\mathbf{D})\tilde{\mathbf{u}}_0)(\mathbf{x}). \tag{7.11}$$

As in (6.25), Lemma 4.3 yields

$$\|\mathbf{w}_\varepsilon\|_{H^1(\mathcal{O};\mathbb{C}^n)} \le \gamma_0\|\boldsymbol{\phi}_\varepsilon\|_{H^1(\mathcal{O};\mathbb{C}^n)}. \tag{7.12}$$

Thus, our problem reduces to the proof of the following statement.

Lemma 7.4. *Under the assumptions of Theorem* 7.1, *let* $0 < \varepsilon \le \varepsilon_2$, *where* $\varepsilon_2 \in (0, 1]$ *is the number defined in Lemma* 5.3. *Let* $\boldsymbol{\phi}_\varepsilon$ *be the function defined in accordance with* (6.23), (7.11). *Then*

$$\|\boldsymbol{\phi}_\varepsilon\|_{H^1(\mathcal{O};\mathbb{C}^n)} \le C_7\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)}, \quad 0 < \varepsilon \le \varepsilon_2. \tag{7.13}$$

The constant C_7 *depends only on* m, d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the parameters of the lattice* Γ, *and the domain* $\mathcal{O}$.

Proof. We start with estimating the norm of the function (7.11) in $L_2(\mathcal{O};\mathbb{C}^n)$. Using (1.3), (3.9), (6.23), and (7.8), we get

$$\begin{aligned}\|\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})} &\le \varepsilon\|\Lambda^\varepsilon(S_\varepsilon b(\mathbf{D})\tilde{\mathbf{u}}_0)\|_{L_2(\mathbb{R}^d)} \\ &\le \varepsilon M\alpha_1^{1/2}\|\tilde{\mathbf{u}}_0\|_{H^1(\mathbb{R}^d)} \le \varepsilon M\alpha_1^{1/2}C_{\mathcal{O}}\hat{c}\|\mathbf{F}\|_{L_2(\mathcal{O})}.\end{aligned} \tag{7.14}$$

Consider the derivatives

$$\frac{\partial\boldsymbol{\phi}_\varepsilon}{\partial x_j} = \varepsilon\frac{\partial\theta_\varepsilon}{\partial x_j}\Lambda^\varepsilon(S_\varepsilon b(\mathbf{D})\tilde{\mathbf{u}}_0) + \theta_\varepsilon\Big(\frac{\partial\Lambda}{\partial x_j}\Big)^\varepsilon(S_\varepsilon b(\mathbf{D})\tilde{\mathbf{u}}_0) + \varepsilon\theta_\varepsilon\Lambda^\varepsilon(S_\varepsilon b(\mathbf{D})\partial_j\tilde{\mathbf{u}}_0), \quad j = 1, \dots, d.$$

Then

$$\begin{aligned}\|\mathbf{D}\boldsymbol{\phi}_\varepsilon\|^2_{L_2(\mathcal{O})} \le 3\varepsilon^2\int_{\mathcal{O}}|\nabla\theta_\varepsilon|^2|\Lambda^\varepsilon(S_\varepsilon b(\mathbf{D})\tilde{\mathbf{u}}_0)|^2 d\mathbf{x} + 3\int_{\mathcal{O}}|(\mathbf{D}\Lambda)^\varepsilon|^2|\theta_\varepsilon(S_\varepsilon b(\mathbf{D})\tilde{\mathbf{u}}_0)|^2 d\mathbf{x} \\ + 3\varepsilon^2\sum_{j=1}^d\int_{\mathcal{O}}|\theta_\varepsilon|^2|\Lambda^\varepsilon(S_\varepsilon b(\mathbf{D})D_j\tilde{\mathbf{u}}_0)|^2\, d\mathbf{x}.\end{aligned} \tag{7.15}$$

The summands on the right-hand side of (7.15) are denoted by $\mathcal{J}_1$, $\mathcal{J}_2$, and $\mathcal{J}_3$, respectively.

It is easy to estimate $\mathcal{J}_3$. From (1.3), (3.9), and (6.23) it follows that

$$\mathcal{J}_3 \le 3\varepsilon^2\sum_{j=1}^d\|\Lambda^\varepsilon(S_\varepsilon b(\mathbf{D})D_j\tilde{\mathbf{u}}_0)\|^2_{L_2(\mathbb{R}^d)} \le 3\varepsilon^2M^2\alpha_1\|\tilde{\mathbf{u}}_0\|^2_{H^2(\mathbb{R}^d)}.$$

Combining this with (4.19) and (6.10), we obtain

$$\mathcal{J}_3 \le \hat{\gamma}_3\varepsilon^2\|\mathbf{F}\|^2_{L_2(\mathcal{O})}, \tag{7.16}$$

where $\hat{\gamma}_3 = 3M^2\alpha_1(C_{\mathcal{O}}\hat{c})^2$.

The first term on the right-hand side of (7.15) is estimated with the help of (6.23) and Lemma 5.3. For $0 < \varepsilon \le \varepsilon_2$ we have

$$\begin{aligned}\mathcal{J}_1 &\le 3\kappa^2\int_{(\partial\mathcal{O})_\varepsilon}|\Lambda^\varepsilon\,(S_\varepsilon b(\mathbf{D})\tilde{\mathbf{u}}_0)\,|^2\, d\mathbf{x} \\ &\le 3\kappa^2\beta_*\varepsilon|\Omega|^{-1}\|\Lambda\|^2_{L_2(\Omega)}\|b(\mathbf{D})\tilde{\mathbf{u}}_0\|_{H^1(\mathbb{R}^d)}\|b(\mathbf{D})\tilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)}.\end{aligned}$$

Combining this with (1.3), (4.19), (6.10), and estimate (2.14), we arrive at the inequality

$$\mathcal{J}_1 \leq \widehat{\gamma}_1 \varepsilon \|\mathbf{F}\|^2_{L_2(\mathcal{O})}, \tag{7.17}$$

where $\widehat{\gamma}_1 = 3\kappa^2 \beta_* (C_{\mathcal{O}}\widehat{c})^2 m (2r_0)^{-2} \alpha_0^{-1} \alpha_1 \|g\|_{L_\infty} \|g^{-1}\|_{L_\infty}$.

It remains to consider the second term on the right-hand side of (7.15). By (6.23),

$$\mathcal{J}_2 \leq 3 \int_{(\partial\mathcal{O})_\varepsilon} |(\mathbf{D}\Lambda)^\varepsilon|^2 |S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0|^2 \, d\mathbf{x}.$$

By Lemma 5.3, for $0 < \varepsilon \leq \varepsilon_2$ we have

$$\mathcal{J}_2 \leq 3\beta_* \varepsilon |\Omega|^{-1} \|\mathbf{D}\Lambda\|^2_{L_2(\Omega)} \|b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{H^1(\mathbb{R}^d)} \|b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)}.$$

Combined with (1.3), (2.15), (4.19), and (6.10), this implies that

$$\mathcal{J}_2 \leq \widehat{\gamma}_2 \varepsilon \|\mathbf{F}\|^2_{L_2(\mathcal{O})}, \quad 0 < \varepsilon \leq \varepsilon_2, \tag{7.18}$$

where $\widehat{\gamma}_2 = 3\beta_* (C_{\mathcal{O}}\widehat{c})^2 m \alpha_0^{-1} \alpha_1 \|g\|_{L_\infty} \|g^{-1}\|_{L_\infty}$.

Finally, relations (7.15)–(7.18) yield

$$\|\mathbf{D}\boldsymbol{\phi}_\varepsilon\|^2_{L_2(\mathcal{O})} \leq \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 \leq (\widehat{\gamma}_1 + \widehat{\gamma}_2 + \widehat{\gamma}_3)\varepsilon \|\mathbf{F}\|^2_{L_2(\mathcal{O})}, \quad 0 < \varepsilon \leq \varepsilon_2.$$

Now, recalling (7.14), we obtain (7.13) with

$$C_7 = (M^2 \alpha_1 (C_{\mathcal{O}}\widehat{c})^2 + \widehat{\gamma}_1 + \widehat{\gamma}_2 + \widehat{\gamma}_3)^{1/2}. \qquad \square$$

Proof of Theorem 7.1. From (7.10), (7.12), and (7.13) it follows that

$$\|\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon\|_{H^1(\mathcal{O};\mathbb{C}^n)} \leq \widehat{C} C_6 \widehat{c} \varepsilon \|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)} + \gamma_0 C_7 \varepsilon^{1/2} \|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)}, \quad 0 < \varepsilon \leq \varepsilon_2.$$

This implies (7.3) with $C = \widehat{C} C_6 \widehat{c} + \gamma_0 C_7$.

It remains to check (7.4). Taking (1.2) and (2.6) into account, from (7.3) we deduce the inequality

$$\begin{aligned} &\|\mathbf{p}_\varepsilon - g^\varepsilon b(\mathbf{D})\mathbf{u}_0 - g^\varepsilon b(\mathbf{D})(\varepsilon \Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0)\|_{L_2(\mathcal{O})} \\ &\quad \leq \|g\|_{L_\infty} \alpha_1^{1/2} d^{1/2} C \varepsilon^{1/2} \|\mathbf{F}\|_{L_2(\mathcal{O})}, \quad 0 < \varepsilon \leq \varepsilon_2. \end{aligned} \tag{7.19}$$

Using Proposition 3.1 and relations (1.3), (4.19), and (6.10), we conclude that

$$\begin{aligned} \big\| g^\varepsilon b(\mathbf{D})\mathbf{u}_0 - g^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0 \big\|_{L_2(\mathcal{O})} &\leq \|g\|_{L_\infty} \big\| b(\mathbf{D})\widetilde{\mathbf{u}}_0 - S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0 \big\|_{L_2(\mathbb{R}^d)} \\ &\leq \varepsilon r_1 \|g\|_{L_\infty} \alpha_1^{1/2} \|\widetilde{\mathbf{u}}_0\|_{H^2(\mathbb{R}^d)} \leq \varepsilon r_1 \|g\|_{L_\infty} \alpha_1^{1/2} C_{\mathcal{O}} \widehat{c} \|\mathbf{F}\|_{L_2(\mathcal{O})}. \end{aligned} \tag{7.20}$$

Relation (1.2) and the definition of the matrix $\widetilde{g}$ show that

$$g^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0 + g^\varepsilon b(\mathbf{D})(\varepsilon \Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0) = \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0 + \varepsilon g^\varepsilon \sum_{l=1}^d b_l \Lambda^\varepsilon S_\varepsilon b(\mathbf{D}) D_l \widetilde{\mathbf{u}}_0. \tag{7.21}$$

Taking (1.3), (2.6), (3.9), (4.19), and (6.10) into account, we obtain

$$\begin{aligned} \left\| \varepsilon g^\varepsilon \sum_{l=1}^d b_l \Lambda^\varepsilon S_\varepsilon b(\mathbf{D}) D_l \widetilde{\mathbf{u}}_0 \right\|_{L_2(\mathcal{O})} &\leq \varepsilon \|g\|_{L_\infty} M \alpha_1 d^{1/2} \|\widetilde{\mathbf{u}}_0\|_{H^2(\mathbb{R}^d)} \\ &\leq \varepsilon \|g\|_{L_\infty} M \alpha_1 d^{1/2} C_{\mathcal{O}} \widehat{c} \|\mathbf{F}\|_{L_2(\mathcal{O})}. \end{aligned} \tag{7.22}$$

Now, relations (7.19)–(7.22) imply (7.4) with the constant

$$C' = \|g\|_{L_\infty} \alpha_1^{1/2} d^{1/2} C + \|g\|_{L_\infty} C_{\mathcal{O}} \widehat{c} (r_1 \alpha_1^{1/2} + M \alpha_1 d^{1/2}). \qquad \square$$

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Department of Physics, St. Petersburg State University, Ul′yanovskaya 3, Petrodvorets, St. Petersburg 198504, Russia

E-mail address: mpakhnin@yandex.ru

Department of Physics, St. Petersburg State University, Ul′yanovskaya 3, Petrodvorets, St. Petersburg 198504, Russia

E-mail address: suslina@list.ru

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