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SUPERSYMMETRIC STRUCTURES FOR SECOND ORDER DIFFERENTIAL OPERATORS

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In memory of Vladimir Buslaev

Abstract. Necessary and sufficient conditions are obtained for a real semiclassical partial differential operator of order two to possess a supersymmetric structure. For the operator coming from a chain of oscillators coupled to two heat baths, it is shown that no smooth supersymmetric structure can exist for a suitable interaction potential, provided that the temperatures of the baths are different.

§1. Introduction and statement of results

In a large number of problems coming from statistical or quantum mechanics, involving real partial differential operators of order two with the spectrum contained in the right half-plane, one is often interested in the splitting between the two smallest real parts of the eigenvalues, at least when it is possible to show that the first eigenvalue is simple and isolated. This kind of issue may concern the Schrödinger operator or the Witten Laplacian in quantum mechanics, and the Kramers–Fokker–Planck operator, or more generally, some models of chains of oscillators coupled to heat baths, in statistical mechanics.

In the semiclassical context, the tools available for studying the splitting are very powerful, and allow for a detailed analysis of models where a natural small parameter is in play, e.g., the Planck constant or the low temperature parameter. In some cases the splitting may be exponentially small with respect to the parameter, and a so-called tunneling effect may appear, related to a very low rate of convergence for the associated evolution problem. It is often related to some degenerate geometry in the model, such as the presence of multiple confining wells.

In fact, for some of the preceding semiclassical differential operators, a particularly convenient structure is sometimes available, which simplifies considerably the analysis of the splitting and of the tunneling effect. Indeed, when the operators have a Hodge Laplacian type structure of the form $P = d^*d$, where d is a (perhaps modified) de Rham operator, the eigenspaces have some natural stability properties with respect to the operator d, and the study of the eigenvalues can be reduced to the study of the singular values of d on finite-dimensional subspaces. This type of operators is sometimes called *supersymmetric* (see in particular [21] and the full definition given below), and this method was successfully employed for determining the splitting corresponding to some models mentioned above.

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The natural question raised in this paper is to determine under which conditions a given (not necessarily selfadjoint) operator is of supersymmetric type. The answer given here implies that for some models describing systems out of equilibrium, there is no such property, and we give a concrete example in the last section of the paper. The significance of this kind of result is that the study of the splitting for these models may be far more complicated, in particular in the nonselfadjoint case, and to the best of our knowledge, no such analysis has been performed in the nonselfadjoint case so far.

By way of introduction and also for motivation, we mention now some examples of supersymmetric differential operators and related results about the eigenvalue splitting. The first example is the Witten–Hodge Laplacian,

$$W = -h^2\Delta + (\partial_x V(x))^2 - h\Delta V(x), \tag{1.1}$$

on $\mathbb{R}^n_x$. Here $V \in C^\infty(\mathbb{R}^n;\mathbb{R})$ is a Morse function with $\partial^\alpha V \in L^\infty(\mathbb{R}^n)$ for $|\alpha| \geq 2$, and such that $|\partial_x V(x)| \geq 1/C$ when $|x| \geq C > 0$. The second example is the Kramers–Fokker–Planck operator given by

$$K = -h^2\Delta_y + y^2 - hn + y\cdot h\partial_x - \partial_x V(x)\cdot h\partial_y, \tag{1.2}$$

on $\mathbb{R}^{2n}_{x,y}$, where the potential V has the same properties as above. This is a kinetic (nonselfadjoint) model for an oscillator coupled to a heat bath.

For these two models, the splitting can be evaluated very precisely in the semiclassical limit $h \to 0$. Considering W and K as unbounded operators on L^2, it was shown in [9, 8] for the Witten Laplacian, and in [10, 11] for the Kramers–Fokker–Planck operator, that, under suitable assumptions, it is exponentially small. We notice here that in both cases, the operators have non-negative symmetric parts and have $\mu_1 = 0$ as a simple eigenvalue, under the additional assumption that $V(x) \to \infty$ as $x \to \infty$. For simplicity, here we recall the result in the case where the potential V has two local minima $m_\pm$ and a saddle point s_0 of signature $(n-1,1)$, although a far more general result was given in [12], in the Kramers–Fokker–Planck case. Let $S_\pm = V(s_0) - V(m_\pm)$ and $S = \min(S_+, S_-)$. Then there exists $c > 0$ and $h_0 > 0$ such that for all $0 < h \leq h_0$ there are precisely two eigenvalues of K in the open disk $D(0, ch)$, one of them being $\mu_1 = 0$, and the second is real positive, given by

$$\mu_2 = hl(h)e^{-S/h}, \tag{1.3}$$

where $l(h) = l_0 + hl_1 + h^2 l_2 + h^3 l_3 \ldots$, with $l_0 > 0$.

We shall now discuss the notion of a supersymmetric structure more precisely. Let X be $\mathbb{R}^n$ or a smooth compact manifold of dimension n, equipped with a smooth strictly positive volume density $\omega(dx)$. We can choose local coordinates $x_1, \ldots, x_n$ near any given point on X such that $\omega(dx) = dx_1 \cdot \ldots \cdot dx_n$.

Let $d : C^\infty(X;\wedge^k T^*X) \to C^\infty(X;\wedge^{k+1}T^*X)$ be the de Rham complex, and let $\delta : C^\infty(X;\wedge^{k+1}TX) \to C^\infty(X;\wedge^k TX)$ be the adjoint of d with respect to the natural pointwise duality between $\wedge^k T^*X$ and $\wedge^k TX$, integrated against ω. In the special local coordinates above, we have

$$d = \sum_{j=1}^n dx_j^\wedge \circ \frac{\partial}{\partial x_j}, \quad \delta = -\sum_{j=1}^n \frac{\partial}{\partial x_j} \circ dx_j^\rfloor. \tag{1.4}$$

Let $A(x) : T_x^*X \to T_xX$ be a linear map depending smoothly on $x \in X$. The k-fold exterior product $\wedge^k A$ maps $\wedge^k T_x^*X \to \wedge^k T_xX$, and by abuse of notation, we shall sometimes write simply A instead of $\wedge^k A$. By convention, $\wedge^0 A$ is the identity map on $\mathbb{R}$. Associated with A, we have the bilinear product on the space of compactly supported smooth k-forms,

$$(u|v)_A = (Au|v) = \int (A(x)u(x)|v(x))\omega(dx), \quad u, v \in C_0^\infty(X; \wedge^k T_x^* X). \tag{1.5}$$

Since we shall restrict the attention to real-valued sections and operators, there is no need for complex conjugations in (1.5). When the map $A(x)$ is bijective for each x, there is a natural way of defining the formal adjoint of a linear operator B taking k-forms to ℓ-forms on X. This adjoint, denoted by $B^{A,*}$, maps ℓ-forms to k-forms and is given by

$$(Bu|v)_A = (u|B^{A,*}v)_A,$$

and more explicitly, by the following expression:

$$B^{A,*} = (\wedge^k A^{\mathrm{t}})^{-1} B^* \wedge^\ell A^{\mathrm{t}}.$$

Here B^* is the usual adjoint taking ℓ-vectors to k-vectors. Notice that for $k = 0$, we can still define the adjoint $B^{A,*}$, even if A is not everywhere invertible. In the case where $B = d$, the de Rham differentiation, we get

$$d^{A,*} = (A^{\mathrm{t}})^{-1} \delta A^{\mathrm{t}},$$

and for the adjoint of the restriction of d to 0-forms, we get

$$d^{A,*} = \delta A^{\mathrm{t}}.$$

We can write, in the special local coordinates above,

$$d^{A,*} = -\sum_{j=1}^n \frac{\partial}{\partial x_j} \circ dx_j^{\rfloor} \circ A^{\mathrm{t}}.$$

Let $\varphi \in C^\infty(X; \mathbb{R})$, and let us introduce the Witten complex, given by the weighted semiclassical de Rham differentiation

$$d_{\varphi,h} = e^{-\varphi/h} \circ hd \circ e^{\varphi/h} = hd + (d\varphi)^\wedge \,:\, C_0^\infty(X; \wedge^k T^* X) \to C_0^\infty(X; \wedge^{k+1} T^* X). \tag{1.6}$$

Considering the adjoint of $d_{\varphi,h}$ with respect to the bilinear product (1.5), we can introduce the operator

$$d_{\varphi,h}^{A,*} = e^{\varphi/h} \circ hd^{A,*} \circ e^{-\varphi/h}, \tag{1.7}$$

which on 1-forms becomes

$$d_{\varphi,h}^{A,*} = (h\delta + d\varphi^{\rfloor}) \circ A^{\mathrm{t}}.$$

Now we are able to give a precise definition of the notion of a supersymmetric differential operator. In doing so, it will be convenient to distinguish between the nonsemiclassical case, corresponding to the situation where the parameter $h > 0$ is kept fixed, say, $h = 1$, and the semiclassical case, where we shall let $h \in (0, h_0]$, $h_0 > 0$.

Definition 1.1. (i) (The nonsemiclassical case.) Let $P = P(x, D_x)$ be a second order scalar real differential operator on X, with C^∞-coefficients. We say that P has a supersymmetric structure on X if there exists a linear map $A(x) : T_x^* X \to T_x X$ smooth in $x \in X$, and functions $\varphi, \psi \in C^\infty(X; \mathbb{R})$ such that

$$P = d_\psi^{A,*} d_\varphi.$$

Here $d_\varphi = d_{\varphi,1}$, and similarly for d_ψ.

(ii) (The semiclassical case.) Let $P = P(x, hD_x; h)$ be a second order scalar real semiclassical differential operator on X. We say that P has a supersymmetric structure on X in the semiclassical sense if there exists a linear h-dependent map $A(x; h) : T_x^* X \to$

T_xX smooth in $x \in X$, and functions $\varphi = \varphi(x;h)$ and $\psi = \psi(x;h)$ smooth in x, such that $\varphi = \varphi_0(x) + \mathcal{O}(h)$, $\psi = \psi_0(x) + \mathcal{O}(h)$ in the C^∞-sense, and for which we have

$$P = d_{\psi,h}^{A,*} d_{\varphi,h}, \tag{1.8}$$

for all $h \in (0, h_0]$, $h_0 > 0$.

Remark. Notice that no control on the behavior of $A(x;h)$ as $h \to 0$ is required in the semiclassical case in Definition 1.1. It is an interesting problem to determine when we can find $A(x;h)$ with some uniform control as $h \to 0$. As we shall see from the proof of Theorem 1.2, the symmetric part of A can be determined immediate, while the antisymmetric part arises as a solution of a δ-problem with exponential weights.

Assuming that the linear map $A(x;h)$ is invertible for each $x \in X$ and all $h \in (0, h_0]$, for an arbitrary degree $k \geq 0$ we put

$$\square_{A,\psi,\varphi}^{(k)} = d_{\psi,h}^{A,*} d_{\varphi,h} + d_{\varphi,h} d_{\psi,h}^{A,*}. \tag{1.9}$$

The supersymmetric semiclassical operator P in (1.8) agrees with the restriction of $\square_{A,\psi,\varphi}$ to 0-forms. The analysis of the eigenvalue splitting of P strongly depends on the following formal intertwining relations:

$$\square_{A,\psi,\varphi}^{(k+1)} d_{\varphi,h} = d_{\varphi,h} \square_{A,\psi,\varphi}^{(k)}, \quad d_{\psi,h}^{A,*} \square_{A,\psi,\varphi}^{(k+1)} = \square_{A,\psi,\varphi}^{(k)} d_{\psi,h}^{A,*}. \tag{1.10}$$

As a clarifying example, let us now come back to the case of the Witten Laplacian W in (1.1). A straightforward computation shows that the operator W enjoys a semiclassical supersymmetric structure in the sense of Definition 1.1,

$$W = (hd + (dV)^\wedge)^{\mathrm{I},*}(hd + (dV)^\wedge) = d_{V,h}^{\mathrm{I},*} d_{V,h},$$

with $\varphi = \psi = V$, and $A = \mathrm{I}$. In the case of the Kramers–Fokker–Planck operator K, the semiclassical supersymmetric structure was observed in [1, 2], and in [20]. It will be recalled in the next section, following [10].

The first result of this paper is a necessary and sufficient condition for a second order real differential operator to be supersymmetric, both in the non-semiclassical and in the semiclassical sense, either locally or globally on X.

Theorem 1.2. (i) (The nonsemiclassical case.) *Let P be a second order scalar real differential operator on X that can be written in local coordinates as follows:*

$$P = -\sum_{j,k=1}^{n} \partial_{x_j} \circ B_{j,k}(x) \circ \partial_{x_k} + \sum_{j=1}^{n} v_j(x) \circ \partial_{x_j} + v_0,$$

where $(B_{j,k})$ is symmetric, and the $(B_{j,k})$, v_j, v_0 are real-valued and smooth. In order for P to have a supersymmetric structure on X, it is necessary that there should exist $\varphi, \psi \in C^\infty(X;\mathbb{R})$ such that

$$P(e^{-\varphi}) = 0, \quad P^*(e^{-\psi}) = 0. \tag{1.11}$$

Here P^* is the formal adjoint of P with respect to the L^2-scalar product determined by the density of integration $\omega(dx)$. If (1.11) *holds and the δ-complex is exact in degree* 1 *for smooth sections, then P has a supersymmetric structure in the sense of Definition* 1.1.

(ii) (The semiclassical case.) *Let P be a second order scalar real semiclassical differential operator on X that can be written in local coordinates as follows:*

$$P = -\sum_{j,k=1}^{n} h\partial_{x_j} \circ B_{j,k}(x;h) \circ h\partial_{x_k} + \sum_{j=1}^{n} v_j(x;h) \circ h\partial_{x_j} + v_0(x;h), \tag{1.12}$$

where $(B_{j,k})$ is symmetric, and the $(B_{j,k})$, v_j, v_0 are real-valued, h-dependent, and smooth in $x \in X$. In order for P to have a supersymmetric structure on X in the semiclassical *sense, it is necessary that there should exist functions $\varphi(x;h), \psi(x;h)$ smooth in $x \in X$, with $\varphi = \varphi_0(x) + \mathcal{O}(h)$, $\psi = \psi_0(x) + \mathcal{O}(h)$ in the C^∞-sense, such that*

$$P(e^{-\varphi/h}) = 0, \quad P^*(e^{-\psi/h}) = 0, \tag{1.13}$$

for all $h \in (0, h_0]$, $h_0 > 0$. If (1.13) *holds and the δ-complex is exact in degree* 1 *for smooth sections, then P has a semiclassical supersymmetric structure in the sense of Definition* 1.1.

Remark. Assume that X is a compact Riemannian manifold and that $P = -\Delta + v$, where v is a smooth real vector field. Then (1.11) holds, with $\varphi = \psi = 0$, precisely when the vector field v is divergence free.

As a fundamental example of a second order semiclassical operator for which the supersymmetric structure may break down, now we shall discuss the case of an operator associated with a model of a chain of anharmonic oscillators coupled to two heat baths. Referring to the next section for the construction of this model, here we consider the following real semiclassical differential operator of order two:

$$\widetilde{P}_W = \frac{\gamma}{2} \sum_{j=1}^{2} \alpha_j (-h\partial_{z_j}) \Big(h\partial_{z_j} + \frac{2}{\alpha_j}(z_j - x_j) \Big) + y \cdot h\partial_x - (\partial_x W(x) + x - z) \cdot h\partial_y. \tag{1.14}$$

Here $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $z = (z_1, z_2)$ belong to $\mathbb{R}^{2n}$. Furthermore, $\gamma > 0$, $\alpha_j > 0$, $j = 1, 2$. The effective potential W is of the form $W(x) = W_1(x_1) + W_2(x_2) + \delta W(x_1, x_2)$, where $W_j \in C^\infty(\mathbb{R}^n; \mathbb{R})$, $j = 1, 2$, and $\delta W \in C^\infty(\mathbb{R}^{2n}; \mathbb{R})$. The parameters α_j are proportional to the temperatures in the baths, $j = 1, 2$. As we shall see, the structure of the operator $\widetilde{P}_W$ is completely different depending on whether $\alpha_1 = \alpha_2$ or not. The following is the second main result of this paper — see also Theorem 3.1 below for a more precise statement.

Theorem 1.3. *Consider the operator $\widetilde{P}_W$ defined in* (1.14). *We have the following two cases.*

i) *If $\alpha_1 = \alpha_2$* (*the equilibrium case*), *or $\delta W \equiv 0$* (*the decoupled case*), *then $\widetilde{P}_W$ has a semiclassical supersymmetric structure on $\mathbb{R}^{6n}$, in the sense of Definition* 1.1.

ii) *Take $\gamma = 1$ and assume that $\alpha_1 \neq \alpha_2$. Then there exists a Morse function $W_1(x_1)$ with two local minima and a saddle point, a positive definite quadratic form $W_2(x_2)$, and $\delta W(x_1, x_2) \in C_0^\infty(\mathbb{R}^{2n})$, arbitrarily small in the uniform norm, such that the operator $\widetilde{P}_W$ has no supersymmetric structure on $\mathbb{R}^{6n}$ in the semiclassical sense* (*see Definition* 1.1).

The study of the tunneling effect for the Witten Laplacian through its supersymmetric properties was first performed in [9], following the supersymmetry observation pointed out in [21]. As already mentioned, this structure is of great help in the study of the eigenvalue splitting, since in particular it allows one to avoid the study near the so-called "nonresonant wells", such as the saddle point in the example mentioned above. The results were subsequently generalized in [3, 4], in [8] with a full asymptotic expansion, in [14, 15] in cases with boundary, and in [16, 17] in the case of forms of higher degree. Notice that the computation of the exponentially small eigenvalues is performed here by using the singular values of the Witten differential.

For the Kramers–Fokker–Planck operator, the supersymmetric structure was first observed in [1, 2] and in [20], and was used for a complete study of the tunneling effect in [10] and [12]. This structure helps substantially in this nonselfadjoint setting, and using also an additional symmetry of a PT-type, we were able to compute the singular values, in order to get results in the form given in (1.3). We also remark that if the supersymmetric method is not available, a natural approach to the tunneling analysis would proceed by means of exponentially weighted estimates of Agmon type, where the notion of the Agmon distance should be replaced by a degenerate Finsler distance, see [18].

In fact, in [10, 11, 12] it was shown, under additional assumptions, that the tunneling of any supersymmetric operator of order two could be studied by using the method given there, and the case of chains of oscillators coupled to two heat baths was given there as an example. This model was first introduced in [6] and was shown to be supersymmetric in the case of the equal temperatures in [11]. For general second order nonnegative operators, the question of intrinsic geometric structures and their relations to the spectral gap in asymptotic regimes was also studied in [7] in the elliptic case, and this paper was one of the motivations for our study of supersymmetric hidden structures here.

The paper is organized as follows. In §2, first we give a necessary and sometimes sufficient condition for a general real partial differential operator of order two to be supersymmetric, both in the nonsemiclassical and the semiclassical cases, and establish Theorem 1.2. Then we review the three supersymmetric examples and, in particular, discuss the equilibrium and the uncoupled cases for the chains (case i) of Theorem 1.3. §3 is devoted to the study of the case of different temperatures for the chains and to the proof of part ii) of Theorem 1.3.

§2. Generalities on supersymmetric structures and some examples

At the beginning of this section we shall establish Theorem 1.2. We begin with the nonsemiclassical case where $h = 1$. Let X be $\mathbb{R}^n$ or a smooth compact manifold of dimension n, equipped with a smooth strictly positive volume density $\omega(dx)$. Assume that local coordinates $x_1, \dots, x_n$ have been chosen so that $\omega(dx) = dx_1 \cdot \ldots \cdot dx_n$. If

$$v = \sum_{j=1}^{n} v_j(x)\partial_{x_j} \tag{2.1}$$

is a smooth vector field on X, then the divergence of v is well defined by the choice of ω, and in the special coordinates as above, we have

$$\operatorname{div} v = \sum_{j=1}^{n} \partial_{x_j}(v_j). \tag{2.2}$$

Let d be the de Rham exterior differentiation, and let δ be its adjoint, defined in the Introduction. Then using the local expression (1.4), we obtain

$$\delta v = -\operatorname{div}(v).$$

Now, let $A(x) : T_x^*X \to T_xX$ depend smoothly on $x \in X$, and consider the bilinear product defined in (1.5) and the induced action of A on k-forms. We now let P be a second order real differential operator on X, which we can write in the special local coordinates above as follows,

$$P = -\sum_{j,k=1}^{n} \partial_{x_j} \circ B_{j,k}(x) \circ \partial_{x_k} + \sum_{j=1}^{n} v_j(x) \circ \partial_{x_j} + v_0. \tag{2.3}$$

Here $(B_{j,k})$ is symmetric and the $B_{j,k}$, v_j, and v_0 are real-valued and smooth. Viewing P as acting on 0-forms, first we ask whether there exists a smooth map $A(x)$ as above such that

$$P = d^{A,*}d = \delta A^{\mathrm{t}} d, \tag{2.4}$$

either locally or globally on X.

Proposition 2.1. *In order to have* (2.4), *it is necessary that*

$$P(1)=0 \quad \textit{and} \quad P^*(1)=0. \tag{2.5}$$

Here P^* is the adjoint of our scalar operator P with respect to the L^2-scalar product determined by the density of integration $\omega(dx)$.

Proof. If (2.4) holds, then clearly $P(1)=0$, because $d(1)=0$. Since $P^*=\delta A d$, we also have $P^*(1)=0$. □

Proposition 2.2. *Property* (2.5) *is equivalent to the following*:

$$v_0=0 \quad \textit{and} \quad \operatorname{div} v=0. \tag{2.6}$$

Proof. In the special local coordinates, we see that

$$P^* = -\sum_{j,k=1}^{n} \partial_{x_j}\circ B_{j,k}(x)\circ\partial_{x_k} - \sum_{j=1}^{n} v_j\circ\partial_{x_j} - \operatorname{div} v + v_0.$$

Thus the property $P(1)=0$ is equivalent to $v_0=0$ and the property $P^*(1)=0$ is equivalent to $v_0-\operatorname{div}(v)=0$. □

Now we look for a smooth map $A=A(x)$ such that (2.4) holds. In the special coordinates above, let $A=(A_{j,k}(x))$ be the matrix of A. Then

$$\delta A^{\mathrm{t}} d = -\sum_{j,k=1}^{n} \partial_{x_j}\circ A_{k,j}(x)\circ\partial_{x_k}.$$

Write $A=\widetilde{B}+C$, where $\widetilde{B}$ is symmetric, $\widetilde{B}^{\mathrm{t}}=\widetilde{B}$, and C is antisymmetric, $C^{\mathrm{t}}=-C$. Then

$$\delta A^{\mathrm{t}} d = -\sum_{j,k=1}^{n} \partial_{x_j}\circ \widetilde{B}_{j,k}(x)\circ\partial_{x_k} + \sum_{k=1}^{n}\Big(\sum_{j=1}^{n}\partial_{x_j}(C_{j,k})\Big)\partial_{x_k}. \tag{2.7}$$

In order to have (2.4), we see that $\widetilde{B}$ must be equal to B, and we assume that from now on, so that

$$A=B+C, \quad B^{\mathrm{t}}=B, \quad C^{\mathrm{t}}=-C, \tag{2.8}$$

where B is the matrix appearing in (2.3), and the antisymmetric matrix C remains to be determined. In order to have (2.4), it is necessary and sufficient that

$$v_k=\sum_{j=1}^{n}\partial_{x_j}(C_{j,k}). \tag{2.9}$$

Consider the 2-vector form

$$\Gamma=\sum_{j,k=1}^{n} C_{j,k}\partial_{x_j}\wedge\partial_{x_k}.$$

By a straightforward calculation, see (1.4), we obtain

$$\delta\Gamma = -\sum_{\nu=1}^{n}\frac{\partial}{\partial x_\nu}dx_\nu^{\lrcorner}\Big(\sum_{j,k=1}^{n} C_{j,k}\partial_{x_j}\wedge\partial_{x_k}\Big) = -2\sum_{j,k=1}^{n}\frac{\partial}{\partial x_j}(C_{j,k})\partial_{x_k},$$

so (2.9) amounts to

$$v=-\frac{1}{2}\delta\Gamma. \tag{2.10}$$

As we have seen, the necessary condition (2.5) is equivalent to (2.6), which contains the assumption that $\operatorname{div} v = 0$, i.e., that $\delta v = 0$. Since the δ-complex is locally exact in degree 1, we get the following result.

Proposition 2.3. *If* (2.5) *holds, then locally, we can find a smooth matrix A such that* (2.4) *holds. More precisely, we can find a smooth matrix A so that* (2.4) *holds, in any open subset $\widetilde{X} \subset X$ where the δ-complex is exact in degree* 1 *for smooth sections.*

Remark. If $A = B + C$ is a solution of (2.4), then $A = B + \widetilde{C}$ is another solution if and only if $\widetilde{C} - C$ (identified with a 2-vector form) is δ-closed.

Now we shall replace the assumption (2.5) by the more general assumption that there exist smooth strictly positive functions $e^{-\varphi}$ and $e^{-\psi}$ in the kernels of P and P^*, respectively,

$$P(e^{-\varphi}) = 0, \quad P^*(e^{-\psi}) = 0. \tag{2.11}$$

Put $\widetilde{P} = e^{-\psi} \circ P \circ e^{-\varphi}$, so that $\widetilde{P}^* = e^{-\varphi} \circ P^* \circ e^{-\psi}$. *Then $\widetilde{P}$ satisfies* (2.5). Hence, if $\widetilde{X} \subset X$ is an open connected subset where the δ-complex is exact in degree 1, we have a smooth matrix $\widetilde{A}$ on $\widetilde{X}$ such that

$$\widetilde{P} = \delta \widetilde{A}^{\mathrm{t}} d \quad \text{in} \quad \widetilde{X}. \tag{2.12}$$

Putting $d_\varphi = e^{-\varphi} \circ d \circ e^{\varphi}$, $d_\psi = e^{-\psi} \circ d \circ e^{\psi}$, we get with $A = e^{\varphi+\psi}\widetilde{A}$:

$$P = e^{\psi}\widetilde{P}e^{\varphi} = e^{\psi}\delta e^{-\psi}A^{\mathrm{t}}e^{-\varphi}de^{\varphi} = d_\psi^* A^{\mathrm{t}} d_\varphi = (d_\psi)^{A,*} d_\varphi.$$

Summarizing the discussion so far, we see that now we have established the first part of Theorem 1.2, addressing the nonsemiclassical case. Restoring now the semiclassical parameter $h \in (0, h_0]$ and recalling the notion of a supersymmetric structure in the semiclassical sense given in Definition 1.1, we see that the arguments given above can be applied to the conjugated operator $e^{-\psi/h} \circ P(x, hD_x; h) \circ e^{-\varphi/h}$, where $\varphi = \varphi_0(x) + \mathcal{O}(h)$, $\psi = \psi_0(x) + \mathcal{O}(h)$, for each fixed value of $h \in (0, h_0]$. The second statement of Theorem 1.2 follows and the proof of Theorem 1.2 is therefore complete.

Corollary 2.4. *Let $P = P(x, hD_x; h)$ be a second order real semiclassical differential operator on X that can be written in local coordinates as follows*:

$$P = -\sum_{j,k=1}^{n} h\partial_{x_j} \circ B_{j,k}(x) \circ h\partial_{x_k} + \sum_{j=1}^{n} v_j(x) \circ h\partial_{x_j} + v_0(x) + hw_0(x).$$

Here $(B_{j,k})$ is symmetric, and the $(B_{j,k})$, v_j, v_0, w_0 are real-valued, smooth, and independent of h. Then a necessary condition for the operator P to be supersymmetric in the semiclassical sense is that there should exist smooth h-independent functions φ_0 and ψ_0 satisfying the following eikonal equations on X:

$$\sum_{j,k=1}^{n} B_{j,k}(x)\partial_{x_j}\varphi_0\partial_{x_k}\varphi_0 + \sum_{j=1}^{n} v_j(x)\partial_{x_j}\varphi_0(x) - v_0(x) = 0,$$

and

$$-\sum_{j,k=1}^{n} B_{j,k}(x)\partial_{x_j}\psi_0\partial_{x_k}\psi_0 + \sum_{j=1}^{n} v_j(x)\partial_{x_j}\psi_0(x) + v_0(x) = 0.$$

Remark. Note that if an operator P possesses a supersymmetric structure in the semiclassical sense, then so does $e^{f/h} \circ P \circ e^{-f/h}$, for a smooth h–independent function f. Indeed, if $P = d^{A,*}_{\psi,h} d_{\varphi,h}$, then

$$e^{f/h} \circ P \circ e^{-f/h} = d^{A,*}_{\psi+f,h} d_{\varphi-f,h}.$$

We shall end this section by coming back to the three examples mentioned in the introduction, namely, the Witten Laplacian, the Kramers–Fokker–Planck operator, and the model of chains of oscillators coupled to heat baths. All three of them come from stochastic differential equations, and we refer to §5 in [11] for a complete discussion concerning their derivation. Here we shall merely quote some results given there. The associated time-dependent equation,

$$(h\partial_t + P)\, f(t,x) = 0, \quad t \geq 0, \tag{2.13}$$

describes the evolution of the particle density, and due to the stochastic origins of this model, the equation $P^*(1) = 0$ is always satisfied, so that for $\psi = 0$, the equation $P^*(e^{-\psi/h}) = 0$ is given for free. By Theorem 1.2, the only remaining thing to check in order to obtain the existence of a semiclassical supersymmetric structure for P is then the existence of a smooth function $\varphi(x;h) = \varphi_0(x) + \mathcal{O}(h)$ such that $P(e^{-2\varphi/h}) = 0$, for all $h > 0$ sufficiently small.

The Witten Laplacian. We begin with the semiclassical Witten case. It corresponds to an evolution equation with a gradient field $-\gamma\nabla V(x)$ and a diffusion force coming from a heat bath at a temperature $T = h/2$. The stochastic differential equation corresponding to this model is of the form

$$dx = -\gamma\partial_x V\, dt + \sqrt{\gamma h}\, dw.$$

Here $x \in \mathbb{R}^n$ is the spatial variable, the parameter $\gamma > 0$ is a friction coefficient, and w is an n-dimensional Wiener process of mean 0 and variance 1. Then equation (2.13) for the particle density in this case is

$$h\partial_t f - \frac{\gamma}{2} h\partial_x \cdot (h\partial_x + 2\partial_x V) f = 0. \tag{2.14}$$

We see that $P^*(1) = 0$, and also that $P(e^{-2\varphi/h}) = 0$ for all $h > 0$. Here we have written $\varphi = V$. An application of Theorem 1.2 shows that P has a supersymmetric structure on $\mathbb{R}^n$ in the semiclassical sense. The function $\mathcal{M}(x) = e^{-2\varphi(x)/h}$ is sometimes called the (non-normalized) Maxwellian of the system and corresponds to a stationary density, provided that it is integrable on $\mathbb{R}^n$.

Writing $f = \mathcal{M}^{1/2} u$, from (2.14) we obtain

$$h\partial_t u + \frac{\gamma}{2}(-h\partial_x + \partial_x V) \cdot (h\partial_x + \partial_x V) u = 0, \tag{2.15}$$

and here we recognize the Witten operator $W = (-h\partial_x + \partial_x V) \cdot (h\partial_x + \partial_x V)$ of the Introduction. The supersymmetric structure of W is given by

$$A = \frac{\gamma}{2} \mathrm{I}_n, \quad \varphi(x) = \psi(x) = V(x).$$

The Kramers–Fokker–Planck operator. The stochastic differential equation associated with this model is

$$\begin{cases} dx = y dt, \\ dy = -\gamma y dt - \partial_x V(x) dt + \sqrt{\gamma h} dw. \end{cases} \tag{2.16}$$

The parameter $\gamma > 0$ is a friction coefficient, and the particle of position $x \in \mathbb{R}^n$ and velocity $y \in \mathbb{R}^n$ is submitted to an external force field derived from a potential V, with w

being an n-dimensional Wiener process of mean 0 and variance 1. Then the corresponding equation for the particle density (2.13) is

$$h\partial_t f - \frac{\gamma}{2} h\partial_y \cdot (h\partial_y + 2y) f + y \cdot h\partial_x f - \partial_x V \cdot h\partial_y f = 0. \tag{2.17}$$

We have $P^*(1) = 0$, and posing $\varphi(x, y) = y^2/2 + V(x)$, we also see that $P(e^{-2\varphi/h}) = 0$. An application of Theorem 1.2 shows that P is semiclassically supersymmetric on $\mathbb{R}^{2n}$. The (nonnormalized) Maxwellian is given by $\mathcal{M}(x, y) = e^{-2\varphi(x,y)/h}$.

If we write $f = \mathcal{M}^{1/2} u$, then (2.17) becomes

$$h\partial_t u + \frac{\gamma}{2}(-h\partial_y + y) \cdot (h\partial_y + y) u + y \cdot h\partial_x u - \partial_x V \cdot h\partial_y u = 0. \tag{2.18}$$

Taking $\gamma = 2$, we arrive at the semiclassical Kramers–Fokker–Planck operator K mentioned in the Introduction,

$$K = y \cdot h\partial_x - \partial_x V(x) \cdot h\partial_y - h^2 \Delta_y + y^2 - hn.$$

The supersymmetric structure of K is given by

$$A = \frac{1}{2} \begin{pmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & \gamma \end{pmatrix} \text{ and } \varphi(x, y) = y^2/2 + V(x), \quad \psi(x, y) = y^2/2 + V(x).$$

Chains of oscillators. This is a model for a system of particles described by their respective position and velocity $(x_j, y_j) \in \mathbb{R}^{2n}$ corresponding to two oscillators. We suppose that for each oscillator $j \in \{1, 2\}$, the particles are submitted to an external force derived from an effective potential $W_j(x_j)$, and that there is a coupling between the two oscillators derived from an effective potential $\delta W(x_1, x_2)$. We denote by W the sum

$$W(x) = W_1(x_1) + W_2(x_2) + \delta W(x_1, x_2),$$

where $x = (x_1, x_2)$, and we also write $y = (y_1, y_2)$. By z_j, $j \in \{1, 2\}$, we denote the variables describing the state of the particles in each of the heat baths, and we set $z = (z_1, z_2)$. Suppose that the particles in each bath are submitted to a coupling with the nearest oscillator, a friction force, and a thermal diffusion at a temperature $T_j = \alpha_j h/2$, $j = 1, 2$. We denote by w_j, $j \in \{1, 2\}$, two n-dimensional Brownian motions of mean 0 and variance 1, and set $w = (w_1, w_2)$. The stochastic differential equation for this model is the following:

$$\begin{cases} dx_1 = y_1\, dt, \\ dy_1 = -\partial_{x_1} W(x)\, dt + (z_1 - x_1)\, dt, \\ dz_1 = -\gamma z_1 dt + \gamma x_1\, dt - \sqrt{\gamma \alpha_1 h}\, dw_1, \\ dz_2 = -\gamma z_1 dt + \gamma x_2\, dt - \sqrt{\gamma \alpha_2 h}\, dw_2, \\ dy_2 = -\partial_{x_2} W(x)\, dt + (z_2 - x_2)\, dt, \\ dx_2 = y_2\, dt. \end{cases} \tag{2.19}$$

The parameter $\gamma > 0$ is the friction coefficient in the baths. Then the corresponding semiclassical equation (2.13) for the particle density is

$$\begin{aligned} h\partial_t f + \widetilde{P}_W f := h\partial_t f &+ \frac{\gamma}{2} \alpha_1 (-h\partial_{z_1}) \big(h\partial_{z_1} + 2(z_1 - x_1)/\alpha_1 \big) f \\ &+ \frac{\gamma}{2} \alpha_2 (-h\partial_{z_2}) \big(h\partial_{z_2} + 2(z_2 - x_2)/\alpha_2 \big) f \\ &+ \big(y \cdot h\partial_x - (\partial_x W(x) + x - z) \cdot h\partial_y \big) f = 0. \end{aligned} \tag{2.20}$$

We have $\widetilde{P}_W^*(1) = 0$. In order to exhibit a semiclassical supersymmetric structure for $\widetilde{P}_W$, we need to find a smooth function $\varphi(x; h) = \varphi_0(x) + \mathcal{O}(h)$ such that $\widetilde{P}_W(e^{-2\varphi/h}) = 0$ for all $h > 0$ sufficiently small. With a general configuration corresponding to different temperatures and nontrivial coupling $\delta W \neq 0$, the existence of a Maxwellian of the form $e^{-2\varphi/h}$ with φ as above is not clear, and in fact we shall show in the next section that in some cases, there is no such a Maxwellian, thus establishing part ii) of Theorem 1.3. In the remainder of this section, we shall be concerned with two cases for the model (2.20) where a supersymmetric structure can be found.

Set $W_0(x) = W_1(x_1) + W_2(x_2)$. The smooth function

$$\varphi_0(x,y,z) = \sum_{j=1}^{2} \frac{1}{\alpha_j}\Big(\frac{y_j^2}{2} + W_j(x_j) + \frac{(x_j - z_j)^2}{2}\Big). \tag{2.21}$$

will be of central importance in the following discussion.

Proof of Theorem 1.3, i) *in the case of equal temperatures.* We start with the case where $\alpha_1 = \alpha_2 =: \alpha$. In this case it is immediate to check that if we define

$$\varphi = \varphi_0 + \frac{1}{\alpha}\delta W = \sum_{j=1}^{2} \frac{1}{\alpha_j}\Big(\frac{y_j^2}{2} + W_j(x_j) + \frac{(x_j - z_j)^2}{2}\Big) + \frac{1}{\alpha}\delta W(x),$$

then $\widetilde{P}_W(e^{-2\varphi/h}) = 0$. An application of Theorem 1.2 then shows that the operator $\widetilde{P}_W$ is supersymmetric on $\mathbb{R}^{6n}$, in the semiclassical sense. Then the associated Maxwellian is defined up to a constant by

$$\mathcal{M}(x,y,z) = e^{-2\varphi/h}.$$

We also give the conjugated version of the time-dependent problem. If we write

$$f = \mathcal{M}^{1/2}u,$$

equation (2.20) becomes

$$\begin{aligned} h\partial_t u &+ \frac{\gamma}{2}\alpha\Big(-h\partial_{z_1} + \frac{1}{\alpha}(z_1 - x_1)\Big)\Big(h\partial_{z_1} + \frac{1}{\alpha}(z_1 - x_1)\Big)u \\ &+ \frac{\gamma}{2}\alpha\Big(-h\partial_{z_2} + \frac{1}{\alpha}(z_2 - x_2)\Big)\Big(h\partial_{z_2} + \frac{1}{\alpha}(z_2 - x_2)\Big)u \\ &+ \big(y\cdot h\partial_x - (\partial_x W(x) + x - z)\cdot h\partial_y\big)u = 0. \end{aligned} \tag{2.22}$$

Then the conjugated operator

$$\begin{aligned} P_W = \frac{\gamma}{2}\sum_{j=1}^{2}\alpha\Big(-h\partial_{z_j} + \frac{1}{\alpha}(z_j - x_j)\Big)\Big(h\partial_{z_j} + \frac{1}{\alpha}(z_j - x_j)\Big) \\ + y\cdot h\partial_x - (\partial_x W(x) + x - z)\cdot h\partial_y \end{aligned} \tag{2.23}$$

enjoys a supersymmetric structure of the form $P_W = d^{A,*}_{\varphi,h} d_{\varphi,h}$, with

$$\varphi = \varphi_0 + \frac{1}{\alpha}\delta W = \frac{1}{\alpha}\big(W(x) + y^2/2 + (z-x)^2/2\big) \tag{2.24}$$

and a nondegenerate (constant) matrix A given by

$$A = \frac{\alpha}{2}\begin{pmatrix} 0 & \mathrm{I}_n & 0 \\ -\mathrm{I}_n & 0 & 0 \\ 0 & 0 & \gamma\mathrm{I}_n \end{pmatrix}. \tag{2.25}$$

□

Proof of Theorem 1.3, i) *in the case where* $\delta W \equiv 0$. Now we consider the case where $\delta W \equiv 0$, so that $W = W_0$, while α_1 and α_2 may be unequal. In this case it is immediate to check that, with φ_0 defined in (2.21), we have $\widetilde{P}_{W_0}(e^{-2\varphi_0/h}) = 0$. Following Theorem 1.2, we conclude therefore that the operator $\widetilde{P}_{W_0}$ is supersymmetric, in the semiclassical sense. Then the Maxwellian associated with the problem is defined up to a constant by

$$\mathcal{M}(x,y,z) = e^{-2\varphi_0/h}.$$

If we write as before,

$$f = \mathcal{M}^{1/2}u,$$

then equation (2.20) becomes

$$\begin{aligned} h\partial_t u &+ \frac{\gamma}{2}\alpha_1\Big(- h\partial_{z_1} + \frac{1}{\alpha_1}(z_1 - x_1)\Big)\Big(h\partial_{z_1} + \frac{1}{\alpha_1}(z_1 - x_1)\Big)u \\ &+ \frac{\gamma}{2}\alpha_2\Big(- h\partial_{z_2} + \frac{1}{\alpha_2}(z_2 - x_2)\Big)\Big(h\partial_{z_2} + \frac{1}{\alpha_2}(z_2 - x_2)\Big)u \\ &+ \big(y\cdot h\partial_x - (\partial_x W_0 + x - z)\cdot h\partial_y\big)u = 0. \end{aligned} \tag{2.26}$$

We have a supersymmetric structure for the conjugated operator occurring in (2.26), with

$$\varphi = \psi = \varphi_0 = \sum_{j=1}^{2} \frac{1}{\alpha_j}\big(W_j(x) + y_j^2/2 + (z_j - x_j)^2/2\big), \tag{2.27}$$

and a nondegenerate (constant) matrix A given by

$$A = \frac{1}{2}\begin{pmatrix} 0 & 0 & \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & 0 & 0 \\ -\alpha_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma\alpha_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma\alpha_2 \end{pmatrix}. \qquad \square$$

§3. Nonexistence of supersymmetric structures for chains of oscillators

Our purpose in this section is to discuss the question of nonexistence of smooth supersymmetric structures for the model of a chain of anharmonic oscillators, in the case where the temperatures of the baths are unequal and the effective interaction potential δW is nonvanishing. This will establish part ii) of Theorem 1.3.

Following (2.20), we are interested in the equation

$$\big(h\partial_t + \widetilde{P}_W\big)f(t,x,y,z) = 0, \quad t \geq 0, \tag{3.1}$$

where

$$\widetilde{P}_W = \frac{\gamma}{2}\sum_{j=1}^{2}\alpha_j(-h\partial_{z_j})\Big(h\partial_{z_j} + \frac{2}{\alpha_j}(z_j - x_j)\Big) + y\cdot h\partial_x - (\partial_x W(x) + x - z)\cdot h\partial_y. \tag{3.2}$$

Here the equation $\widetilde{P}_W^*(1) = 0$ is satisfied automatically. In order to establish the breakdown of the semiclassical supersymmetry for $\widetilde{P}_W$, we only need to show that there does not exist a function $\varphi \in C^\infty(\mathbb{R}^{6n};\mathbb{R})$ with $\varphi(x;h) = \varphi_0(x) + \mathcal{O}(h)$ in the C^∞-sense, such that $\widetilde{P}_W(e^{-2\varphi/h}) = 0$ for all $h > 0$ sufficiently small, for a suitable choice of the effective potentials W_1, W_2, and δW.

In what follows, when $W = W_0$, we shall find it convenient to work with the conjugated operator

$$P_0 = P_{W_0} := e^{\varphi_0/h}\circ \widetilde{P}_{W_0}\circ e^{-\varphi_0/h}, \tag{3.3}$$

where we recall that the function φ_0 is given by

$$\varphi_0 = \sum_{j=1}^{2}\frac{1}{\alpha_j}\Big(\frac{y_j^2}{2} + W_j(x_j) + \frac{(x_j - z_j)^2}{2}\Big).$$

Following (2.26), we obtain

$$P_0 = \frac{\gamma}{2}\sum_{j=1}^{2}\alpha_j\Big(-h\partial_{z_j} + \frac{1}{\alpha_j}(z_j - x_j)\Big)\Big(h\partial_{z_j} + \frac{1}{\alpha_j}(z_j - x_j)\Big) + y\cdot h\partial_x - (\partial_x W_0(x) + x - z)\cdot h\partial_y, \tag{3.4}$$

with the function $e^{-\varphi_0/h}$ being in the kernel of P_0 and its adjoint.

The leading part $p_0 = p_0(x, y, z, \xi, \eta, \zeta)$ of the semiclassical symbol ofthe operator $P_0 = P_0(x, y, z, hD_x, hD_y, hD_z; h)$ is given by

$$p_0 = \frac{\gamma}{2}\sum_{j=1}^{2}\alpha_j\Big(\zeta_j^2 + \frac{1}{\alpha_j^2}(x_j - z_j)^2\Big) + iy\cdot\xi - i(\partial_x W_0(x) + x - z)\cdot\eta. \tag{3.5}$$

We may notice here that $\operatorname{Re} p_0 \geq 0$. Associated with the operator P_0 in (3.4) is the real symbol

$$q_0(x, y, z, \xi, \eta, \zeta) = -p_0(x, y, z, i\xi, i\eta, i\zeta),$$

with

$$P_0(x, y, z, hD_x, hD_y, hD_z) = -q_0(x, y, z, -h\partial_x, -h\partial_y, -h\partial_z)$$

to leading order. We have

$$q_0 = \frac{\gamma}{2}\sum_{j=1}^{2}\alpha_j\Big(\zeta_j^2 - \frac{1}{\alpha_j^2}(x_j - z_j)^2\Big) + y\cdot\xi - (\partial_x W_0(x) + x - z)\cdot\eta. \tag{3.6}$$

The phase function $\varphi_0 \in C^\infty(\mathbb{R}^{6n})$ satisfies the eikonal equation

$$q_0(x, y, z, \partial_x\varphi_0, \partial_y\varphi_0, \partial_z\varphi_0) = 0, \tag{3.7}$$

reflecting the fact that $P_0(x, y, z, -ih\partial_x, -ih\partial_y, -ih\partial_z; h)\left(e^{-\varphi_0/h}\right) = 0$.

Let $x_0 \in \mathbb{R}^{2n}$ be a nondegenerate critical point of $W_0(x)$, and let $\rho \in T^*\mathbb{R}^{6n}$ be the corresponding point in the phase space, given by $x = x_0$, $y = 0$, $z = x_0$, $\xi = \eta = \zeta = 0$. Furthermore, let $F_{p_0} = F_{p_0,\rho}$ be the fundamental matrix of the quadratic approximation of p_0 at the doubly characteristic point ρ. Then, as explained in [10, 11], the spectrum of F_{p_0} avoids the real axis and is of the form $\pm\lambda_k$, $1 \leq k \leq 6n$, $\operatorname{Im}\lambda_k > 0$.

Remark. The preceding result comes from the fact that near ρ, the average of $\operatorname{Re} p_0$ along the Hamilton flow of $\operatorname{Im} p_0$ has a positive definite quadratic part in its Taylor expansion at ρ, see [10, 13]. Following [13] and [19], this can also be interpreted by saying that the so-called singular space S, defined by

$$S = \Big(\bigcap_{j=0}^{\infty}\operatorname{Ker}\big[\operatorname{Re} F_{p_0}(\operatorname{Im} F_{p_0})^j\big]\Big) \cap T^*\mathbb{R}^{6n},$$

is trivial in this case.

Next, let $\rho \in T^*\mathbb{R}^{6n}$ and F_{p_0} be as above, and let F_{q_0} be the fundamental matrix of the quadratic approximation of q_0 at the point ρ. Following, e.g. [10], we see that the eigenvalues of F_{q_0} and $i^{-1}F_{p_0}$ are the same and are of the form $\pm\frac{1}{i}\lambda_k$, $1 \leq k \leq 6n$, where

$$\operatorname{Re}\Big(\frac{1}{i}\lambda_k\Big) > 0, \quad 1 \leq k \leq 6n.$$

As explained in [10], an application of the stable manifold theorem allows us to conclude that in a suitable neighborhood of the point $(x_0, 0, x_0) = \pi(\rho)$, the eikonal equation

$$q_0(x, y, z, \partial_x\varphi, \partial_y\varphi, \partial_z\varphi) = 0, \tag{3.8}$$

has a unique smooth solution $\varphi(x,y,z)$ such that $\varphi(\pi(\rho)) = \varphi'(\pi(\rho)) = 0$ and $\varphi''(\pi(\rho))$ is positive definite. Here $\pi((x,y,z,\xi,\eta,\zeta)) = (x,y,z)$. Indeed, the function $\varphi(x,y,z)$ is obtained as the generating function for the unstable manifold through ρ for the H_{q_0}-flow. Applying this to the case where x_0 is a nondegenerate local minimum of $W_0(x)$, we see that, necessarily, we have $\varphi = \varphi_0$ in a neighborhood of $\pi(\rho)$.

Throughout this section, we shall assume that $W_0(x) = W_1(x_1) + W_2(x_2)$, where W_2 is a positive definite quadratic form on $\mathbb{R}^n$. As for W_1, we assume that W_1 is a Morse function on $\mathbb{R}^n$ such that $\partial_{x_1}^\alpha W_1 = \mathcal{O}(1)$ for all $\alpha \in \mathbf{N}^n$ with $|\alpha| \geq 2$, and such that $|\nabla W_1(x_1)| \geq 1/C$ for $|x_1| \geq C > 0$. Furthermore, assume that W_1 is a double well potential, so that it has precisely three critical points: two local minima $m_\pm$ and a saddle point s_0 of signature $(n-1,1)$. Then the critical points of W_0 are the local minima $M_\pm = (m_\pm, 0)$ and the saddle point $S_0 = (s_0, 0)$, of signature $(2n-1,1)$. Furthermore, $W_0(x) \to +\infty$ as $|x| \to \infty$. From (2.21) we see that the restriction of φ_0 to the subspace

$$L = \{(x,y,z) \in \mathbb{R}^{6n};\ z = x,\ y = 0\}$$

can be identified with W_0, and that the Hessian of φ_0 in the directions orthogonal to L is positive definite. Consequently, φ_0 is also a Morse function on $\mathbb{R}^{6n}$, tending to $+\infty$ as $(x,y,z) \to \infty$, with precisely three critical points given by the local minima $\widetilde{M}_\pm = (M_\pm, 0, M_\pm)$ and the saddle point $\widetilde{S}_0 = (S_0, 0, S_0)$, of signature $(6n-1,1)$.

Let $\delta W(x) \in C^\infty(\mathbb{R}^{2n})$. Thinking about $\delta W(x)$ as a perturbation, we introduce the perturbed effective potential $W = W_0 + \delta W$. Then the corresponding operator $\widetilde{P}_W$ in (3.2) is of the form

$$\widetilde{P}_W = \widetilde{P}_{W_0} - \partial_x\left(\delta W(x)\right) \cdot h\partial_y, \tag{3.9}$$

and after conjugation, we obtain the operator

$$\begin{aligned} P_W := e^{\varphi_0/h} \circ \widetilde{P}_W \circ e^{-\varphi_0/h} &= P_0 - e^{\varphi_0/h} \circ \partial_x\left(\delta W(x)\right) \cdot h\partial_y \circ e^{-\varphi_0/h} \\ &= P_0 - \partial_x\left(\delta W(x)\right) \cdot (h\partial_y - \partial_y\varphi_0). \end{aligned} \tag{3.10}$$

Associated with the operator (3.10), we have the perturbed real Hamiltonian

$$q(x,y,z,\xi,\eta,\zeta) = q_0(x,y,z,\xi,\eta,\zeta) - \partial_x\left(\delta W(x)\right) \cdot \left(\eta + \partial_y\varphi_0\right). \tag{3.11}$$

We are interested in the question whether the perturbed conjugated operator P_W still possesses a smooth supersymmetric structure on $\mathbb{R}^{6n}$, in the semiclassical sense. By Corollary 2.5, a necessary condition for that is the existence of a smooth solution φ of the eikonal equation

$$q(x,y,z,\partial_x\varphi,\partial_y\varphi,\partial_z\varphi) = 0. \tag{3.12}$$

The following is the main result of this section.

Theorem 3.1. *With a suitable choice of W_1 and W_2 as above, there exists $\delta W \in C^\infty(\mathbb{R}^{2n})$ with M_+ and $S_0 \notin \operatorname{supp}(\delta W)$ such that the eikonal equation* (3.12) *does not have any solution $\varphi \in C^\infty(\Omega)$ with $\varphi(\widetilde{M}_+) = \varphi'(\widetilde{M}_+) = 0$, $\varphi''(\widetilde{M}_+) > 0$, for any open set $\Omega \subset \mathbb{R}^{6n}$ such that $\Omega_+ \Subset \Omega$. Here Ω_+ is the connected component of the set $\varphi_0^{-1}((-\infty, \varphi_0(\widetilde{S}_0)))$ that contains $\widetilde{M}_+$.*

Corollary 3.2. *The perturbed operator $\widetilde{P}_W$ in* (3.9) *does not possess any smooth supersymmetric structure in the open set Ω, in the semiclassical sense.*

Remark. The idea behind the result of Theorem 3.1 is that the solvability of the eikonal equation (3.12) does not seem to be an issue near the local minimum $\widetilde{M}_+$, but if we start by solving the problem near this point and try to extend the solution, we may run into some trouble when approaching the saddle point $\widetilde{S}_0$.

When establishing Theorem 3.1, it will be convenient to write $\varphi = \varphi_0 + \psi$ and to consider the corresponding eikonal equation for ψ. A direct computation using (2.21), (3.6), and (3.11) shows that $\psi \in C^\infty(\Omega)$ should satisfy the following equation:

$$\nu\psi + \frac{\gamma}{2}\sum_{j=1}^{2}\alpha_j(\partial_{z_j}\psi)^2 - \partial_x\delta W \cdot \partial_y\psi = 2\partial_x\delta W \cdot \partial_y\varphi_0. \tag{3.13}$$

Here the real vector field ν is given by

$$\nu = \gamma(z-x)\cdot\partial_z + y\cdot\partial_x - (\partial_x W_0(x) + x - z)\cdot\partial_y, \tag{3.14}$$

and we notice that ν can be identified with the Hamilton vector field H_{q_0} along Λ_{φ_0}, via the projection $\pi|_{\Lambda_{\varphi_0}}$. Here the Lagrangian manifold $\Lambda_{\varphi_0} \subset T^*\mathbb{R}^{6n}$ is given by $(\xi, \eta, \zeta) = (\partial_x\varphi_0, \partial_y\varphi_0, \partial_z\varphi_0)$.

We shall prepare for the analysis of the eikonal equation (3.13) by studying the behavior of the integral curves of ν, in particular, close to the stationary points of ν. Here we may also notice that the flow of ν is complete.

Without using the assumption that W_0 is a direct sum, we see that the vector field ν vanishes at a point (x, y, z) precisely when

$$y = 0, \quad z = x, \quad \partial_x W_0(x) = 0. \tag{3.15}$$

Recall that for simplicity we assumed $\gamma = 1$, and let x_0 be a critical point of W_0. Then at the corresponding point $(x_0, y_0, z_0) = (x_0, 0, x_0)$, the linearization of ν has the block matrix

$$N = \begin{pmatrix} 0 & 1 & 0 \\ -W''(x_0) - 1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix}. \tag{3.16}$$

Let λ be an eigenvalue of N, and let $(x\ y\ z)^t$ be a corresponding eigenvector. Then we get the system

$$\begin{cases} -\lambda x + y = 0, \\ (W''(x_0)+1)x + \lambda y - z = 0, \\ -x + (1-\lambda)z = 0. \end{cases}$$

Observing that $\lambda = 1$ is not an eigenvalue, we obtain

$$y = \lambda x, \ z = (1-\lambda)^{-1}x,$$

where x satisfies

$$W''(x_0)x = \Big(\frac{\lambda}{1-\lambda} - \lambda^2\Big)x.$$

Thus, each eigenvalue w of $W''(x_0)$ gives rise to three eigenvalues λ of N, given by the equation

$$F(\lambda) := \frac{\lambda}{1-\lambda} - \lambda^2 = w. \tag{3.17}$$

We have

$$F'(\lambda) = \frac{G(\lambda)}{(1-\lambda)^2}, \quad \lambda \neq 1, \tag{3.18}$$

where $G(\lambda) = 1 - 2\lambda(1-\lambda)^2$. It is easily seen that $G(\lambda) > 0$ for $\lambda < 1$, and therefore, $F(\lambda)$ is strictly increasing from $-\infty$ to $+\infty$ on the interval $(-\infty, 1)$, with $F(0) = 0$. Next, $G(\lambda)$ is strictly decreasing from 1 to $-\infty$ on the interval $(1, \infty)$, and therefore, on this interval, the function F has a unique critical point $m > 1$, with $F(m) < 0$. (Numerical computations show that $m = 1{,}5652$.)

Equation (3.17) can be written as a polynomial equation $\lambda^3 - \lambda^2 + (1+w)\lambda - w = 0$, so that the three roots λ_j satisfy

$$\sum_{j=1}^{3} \lambda_j = 1. \tag{3.19}$$

Therefore, we get the following information about the solutions of (3.17).

- If $w \leq F(m)$, then the three roots are real, and two are greater than 1. The remaining root is negative.
- If $w > F(m)$, then we have precisely one real root λ_1, and it belongs to the interval $(-\infty, 1)$. The other two roots are of the form $a \pm ib$, with $a, b \in \mathbb{R}$, and from relation (3.19) we see that $2a + \lambda_1 = 1$, implying that $a > 0$.

The main conclusion concerning the three roots λ_j is as follows.

- If $w > 0$, then all three roots have positive real parts.
- If $w = 0$, then one root vanishes and the other two have positive real parts.
- If $w < 0$, then one root is negative and the other two have positive real parts.
- For each eigenvalue w, the x-component of the eigenvector $(x, y, z)^t$ of N is the corresponding eigenvector of $W''(x_0)$.

As the next step in the analysis of the ν-flow, we shall show that φ_0 is strictly increasing along the integral curves of ν in the region where $W_0'(x) \neq 0$. When doing so, it will be convenient to write

$$\nu = \nu_1(w_1, \partial_{w_1}) + \nu_2(w_2, \partial_{w_2}), \quad w_j = (x_j, y_j, z_j).$$

Using (2.21) and (3.14) and still assuming that $\gamma = 1$, we get

$$\nu(\varphi_0) = \sum_{j=1}^{2} \frac{1}{\alpha_j}(z_j - x_j)^2, \tag{3.20}$$

$$\nu^2(\varphi_0) = \sum_{j=1}^{2} \frac{2}{\alpha_j}(z_j - x_j)\nu_j(z_j - x_j), \tag{3.21}$$

$$\nu^3(\varphi_0) = \sum_{j=1}^{2} \frac{2}{\alpha_j}\left((\nu_j(z_j - x_j))^2 + (z_j - x_j)\cdot \nu_j^2(z_j - x_j)\right), \tag{3.22}$$

$$\nu^4(\varphi_0) = \sum_{j=1}^{2} \Big(\frac{6}{\alpha_j}(\nu_j^2(z_j - x_j))\nu_j(z_j - x_j) + \frac{2}{\alpha_j}(z_j - x_j)\nu_j^3(z_j - x_j)\Big), \tag{3.23}$$

$$\begin{aligned}\nu^5(\varphi_0) = \sum_{j=1}^{2} \Big(&\frac{6}{\alpha_j}(\nu_j^2(z_j - x_j))^2 + \frac{8}{\alpha_j}(\nu_j^3(z_j - x_j))\nu_j(z_j - x_j) \\ &+ \frac{2}{\alpha_j}(z_j - x_j)\nu_j^4(z_j - x_j)\Big).\end{aligned} \tag{3.24}$$

Here

$$\nu(z - x) = \gamma(z - x) - y,$$
$$\nu^2(z - x) = (\gamma^2 - 1)(z - x) - \gamma y + W_0'(x).$$

Still working in the region where $W_0'(x) \neq 0$, we see that

1) $\nu(\varphi_0) \geq 0$, with equality precisely when $z = x$;
2) if $\nu(\varphi_0) = 0$ and $y \neq 0$, then $\nu^3(\varphi_0) > 0$;
3) if $\nu(\varphi_0) = 0$ and $y = 0$, then $\nu^2(z - x) \neq 0$ and $\nu^5(\varphi_0) > 0$.

This leads to the statement that for every compact set where $W_0'(x) \neq 0$, there exists a constant $C > 0$ such that

$$\nu(\varphi_0) \geq \frac{1}{C} \quad \text{or} \quad \nu^3(\varphi_0) \geq \frac{1}{C} \quad \text{or} \quad \nu^5(\varphi_0) \geq \frac{1}{C}. \tag{3.25}$$

Proposition 3.3. *For every compact set $K \subset \mathbb{R}^{6n}_{x,y,z}$, where $W_0'(x) \neq 0$ there exists a constant $C > 0$ such that*

$$\varphi_0 \circ \exp t\nu(x) - \varphi_0(x) \geq t^5/C, \quad x \in K, \quad 0 \leq t \leq 1/C. \tag{3.26}$$

Proof. Consider the function $f(s) = \nu(\varphi_0) \circ \exp s\nu(x)$ along an integral curve $(-\frac{1}{C}, \frac{1}{C}) \ni t \mapsto \exp t\nu(x)$ of ν. If $f(0) \geq$ const > 0, we get $\int_0^t f(s)\, ds \geq t/C$ for $0 \leq t \ll 1$ and (3.26) follows (for this value of x).

If $0 \leq f(0) \ll 1$ and $f''(0) \geq$ const > 0, then $f(s)$ is a strictly convex and nonnegative function, whence there exists a unique point s_0 close to 0 where $f(t)$ attains its minimum, and we have $f(s) \geq (s-s_0)^2/C$. Integrating this inequality, we see that $\int_0^t f(s)\, ds \geq t^3/C$, and (3.26) follows.

It remains to treat the case where $0 \leq f(0) \ll 1$, $f''(0) \leq \epsilon \ll 1$, and $f^{(4)}(0) \geq$ const > 0. The function $g(s) = \frac{1}{2}(f(s) + f(-s))$ is even and has the Taylor expansion

$$g(s) = f(0) + \frac{1}{2} f''(0) s^2 + \frac{1}{4!} f^{(4)}(0) s^4 + \mathcal{O}(s^6),$$

and is therefore a smooth and strictly convex function of s^2, nonnegative for $s^2 \geq 0$. We denote this function by $k(s^2)$ and restrict our attention to the interval $[0, t^2]$. If $k'(0) \geq 0$, strict convexity implies that $k(\tau) \geq \tau^2/C$ on $[0, t^2]$. If $k'(t^2) \leq 0$, we have $k(\tau) \geq (\tau - t^2)^2/C$. In the remaining case when $k'(0) \leq 0$ and $k'(t^2) \geq 0$, there exists $\tau_0 \in [0, t^2]$ such that $k(\tau) \geq (\tau - \tau_0)^2/C$ on $[0, t^2]$ and this is finally the conclusion in all three cases.

Thus, $g(s) \geq \frac{1}{C}(s^2 - \tau_0)^2$ for $-t \leq s \leq t$ and an easy calculation shows that

$$\int_{-t}^{t} f(s)\, ds = \int_{-t}^{t} g(s)\, ds \geq \frac{1}{C} \int_{-t}^{t} (s^2 - \tau_0)^2\, ds \geq \frac{t^5}{\widetilde{C}}, \quad 0 \leq t \ll 1.$$

In other words, $(\varphi_0 \circ \exp t\nu - \varphi_0 \circ \exp(-t\nu))(x)$ has a lower bound as in (3.26), and we are allowed to vary the point x in a small neighborhood, so we get the same conclusion for $(\varphi_0 \circ \exp 2t\nu - \varphi_0)(x)$, and after replacing $2t$ by t, for $(\varphi_0 \circ \exp t\nu - \varphi_0)(x)$. $\square$

From the statement of Theorem 3.1, we recall the set $\Omega_+ \subset \mathbb{R}^{6n}$, defined as the connected component of the set

$$\varphi_0^{-1}((-\infty, \varphi_0(\widetilde{S}_0)))$$

that contains $\widetilde{M}_+$. Property 1) prior to (3.25) implies that

$$\exp(t\nu)(\Omega_+) \subset \Omega_+, \quad t \leq 0,$$

and Proposition 3.3 shows that $\exp(t\nu)(K)$ converges to $\{\widetilde{M}_+\}$ as $t \to -\infty$, for every $K \Subset \Omega_+$.

The stable manifold theorem tells us that in a suitable neighborhood of $\widetilde{S}_0$, there is a unique curve Γ (manifold of dimension 1) that is stable under the ν-flow in the sense that if $x \in \Gamma$, then $\exp(t\nu)(x)$ converges to $\widetilde{S}_0$ exponentially fast as $t \to +\infty$. The set $\Gamma_+ := \Gamma \cap \Omega_+$ is also invariant under the forward ν-flow and is the image of a (connected) curve. For $w_0 \in \Gamma_+$, put $\gamma(t) = \exp(t\nu)(w_0)$, $t \in \mathbf{R}$. Then γ is a smooth curve in Ω_+, and we have

$$\gamma(t) \to \begin{cases} \widetilde{M}_+, & t \to -\infty, \\ \widetilde{S}_0, & t \to +\infty. \end{cases} \tag{3.27}$$

The trajectory $\gamma(t)$ will play a crucial role in the proof of Theorem 3.1.

Now we resume the analysis of the eikonal equation (3.13). The perturbation $\delta W(x)$ will be chosen so that $M_+ \notin \mathrm{supp}(\delta W)$, and since we are interested in smooth solutions φ of (3.12), for which $\varphi(\widetilde{M_+}) = \varphi'(\widetilde{M_+}) = 0$ and $\varphi''(\widetilde{M_+}) > 0$, as we saw above, we have $\varphi = \varphi_0$ in a neighborhood of $\widetilde{M_+} = (M_+, 0, M_+)$. Therefore, we study the solvability of problem (3.13), assuming that $\psi = 0$ in a neighborhood of $(M_+, 0, M_+)$.

Proposition 3.4. *Let $\delta W \in C^\infty(\mathbb{R}^{2n})$ be such that $M_+ \notin \mathrm{supp}(\delta W)$, and assume that $\delta W(x_1, x_2)$ is a homogeneous polynomial of degree $m \geq 3$ in x_2. For $\Omega_+ \Subset \Omega$, if $\psi \in C^\infty(\Omega)$ satisfies (3.13) with $\psi = 0$ near $(M_+, 0, M_+)$, then in Ω_+ we have*

$$\psi(x, y, z) = \mathcal{O}((x_2, y_2, z_2)^2).$$

Proof. We shall view (3.13) as a Hamilton–Jacobi equation of the form

$$p(x, y, z, \psi'_{x,y,z}) = 0,$$

where

$$\begin{aligned} p(x, y, z, \xi, \eta, \zeta) &= y \cdot \xi + \gamma(z - x) \cdot \zeta - \big(\partial_x W_0(x) + x - z\big) \cdot \eta \\ &\quad + \frac{\gamma}{2} \sum_{j=1}^{2} \alpha_j \zeta_j^2 - \partial_x \delta W \cdot \eta - 2\partial_x \delta W \cdot \partial_y \varphi_0. \end{aligned} \tag{3.28}$$

In what follows, it will be convenient to write $w = (w_1, w_2)$, $w_j = (x_j, y_j, z_j)$, and $\omega = (\omega_1, \omega_2)$, $\omega_j = (\xi_j, \eta_j, \zeta_j)$, for $j = 1, 2$. We know that the Lagrangian manifold $\Lambda_\psi = \{(w, \psi'(w))\}$ is the H_p-flowout of the set

$$\Lambda_\psi \cap \mathrm{neigh}\big((M_+, 0, M_+; 0, 0, 0), T^*\mathbb{R}^{6n}\big).$$

To be precise, let $\rho(0) = (w(0); \omega(0)) \in \mathrm{neigh}\big((M_+, 0, M_+; 0, 0, 0), T^*\mathbb{R}^{6n}\big)$ be such that $\omega(0) = 0$, so that $\rho(0) \in \Lambda_\psi \subset p^{-1}(0)$, and consider the corresponding Hamiltonian trajectory

$$(w(t); \omega(t)) = \rho(t) = \exp(tH_p)(\rho_0) \in \Lambda_\psi.$$

We shall be interested in the trajectories $\rho(t)$ for which $w_2(0) = 0$. From (2.21) and (3.28) it follows that

$$\partial_{w_2} p(w, \omega) = \mathcal{O}((w_2, \omega_2)), \quad \partial_{\omega_2} p(w, \omega) = \mathcal{O}((w_2, \omega_2));$$

hence, the Hamilton equations

$$\dot{w_2}(t) = \partial_{\omega_2} p(w(t), \omega(t)), \quad \dot{\omega_2}(t) = -\partial_{w_2} p(w(t), \omega(t))$$

imply that

$$(\dot{w_2}(t), \dot{\omega_2}(t)) = \mathcal{O}((w_2(t), \omega_2(t))).$$

Along a trajectory for which $w_2(0) = \omega_2(0) = 0$, we have therefore $w_2(t) \equiv 0$, $\omega_2(t) \equiv 0$. A straightforward computation shows next that along the set where $w_2 = 0$, $\omega_2 = 0$, we have

$$\partial_{w_1} p = \mathcal{O}(\omega_1),$$

while

$$\partial_{\omega_1} p \cdot \partial_{w_1} = \nu_1(w_1, \partial_{w_1}) + \mathcal{O}(\omega_1) \cdot \partial_{w_1},$$

where $\nu_1(w_1, \partial_{w_1}) = \gamma(z_1 - x_1) \cdot \partial_{z_1} + y_1 \cdot \partial_{x_1} - (\partial_{x_1} W_1(x_1) + x_1 - z_1) \cdot \partial_{y_1}$. It follows that an H_p-trajectory $\rho(t) = (w_1(t), w_2(t); \omega_1(t), \omega_2(t))$ for which $w_2(0) = 0$, $\omega(0) = 0$, satisfies

$$\rho(t) = (\exp(t\nu_1)(w_1(0)), 0; 0, 0).$$

From the definition of Λ_ψ we know that

$$d\psi = \omega \, dw, \tag{3.29}$$

so that $\psi'(w_1(t), w_2(t)) = (\omega_1(t), \omega_2(t)) = (0,0)$. Thus, $\psi'(w_1, 0) = 0$ for all w_1 such that $(w_1, 0) \in \Omega_+$. Now by a classical formula given, for instance, in Chapter 1 of [5], from (3.29) we also have

$$\psi(w(t)) = \psi(w(0)) + \int_0^t \omega(s) \cdot \partial_\omega p(w(s), \omega(s))\, ds, \tag{3.30}$$

where we know that $\psi(w(0)) = 0$, so that $\psi(w_1, 0) = 0$ again for all w_1 such that $(w_1, 0) \in \Omega_+$. Using this and $\psi'(w_1, 0) = 0$, we see that $\psi(w) = \mathcal{O}(w_2^2)$ and the proof is complete. $\square$

In what follows, we shall assume that $\delta W \in C^\infty(\mathbb{R}^n)$ has the properties described in Proposition 3.4. Using also the fact that

$$\partial_{y_j} \varphi_0 = \frac{1}{\alpha_j} y_j, \quad j = 1, 2,$$

we see that the right-hand side of (3.13) is homogeneous of degree m in $w_2 = (x_2, y_2, z_2)$. If problem (3.13) has a smooth solution $\psi \in C^\infty(\Omega)$, then we have a Taylor expansion at $w_2 = 0$, writing also $w_1 = (x_1, y_1, z_1)$,

$$\psi(w_1, w_2) \simeq \sum_{k=0}^{\infty} \psi_k(w_1, w_2). \tag{3.31}$$

Here $\psi_k(w_1, w_2)$ is homogeneous of degree k in w_2, with C^∞-coefficients in w_1, and by Proposition 3.4 we know that ψ_0 and ψ_1 vanish in Ω_+. From (3.14) it follows that $\nu\psi_k$ is also homogeneous of degree k in w_2. Then we see that the term homogeneous of degree $\mu \geq 0$ on the left-hand side of (3.13) is given by

$$\begin{aligned} \nu\psi_\mu + \frac{\gamma}{2}\alpha_1 \sum_{k=0}^{\mu} (\partial_{z_1}\psi_k)(\partial_{z_1}\psi_{\mu-k}) + \frac{\gamma}{2}\alpha_2 \sum_{k=0}^{\mu} (\partial_{z_2}\psi_{k+1})(\partial_{z_2}\psi_{\mu-k+1}) \\ - \partial_{x_1}\delta W \cdot \partial_{y_1}\psi_{\mu-m} - \partial_{x_2}\delta W \cdot \partial_{y_2}\psi_{\mu+2-m}. \end{aligned} \tag{3.32}$$

Here it is understood that $\psi_j \equiv 0$, for $j < 0$.

Now it is easy to conclude that ψ_μ all vanish, for $\mu < m$, on the open set $\Omega_+ \subset \mathbb{R}^{6n}$. Indeed, taking $\mu = 2$ in (3.32), we see that the sum

$$\sum_{k=0}^{2} (\partial_{z_1}\psi_k)(\partial_{z_1}\psi_{2-k})$$

vanishes in Ω_+, while the only nonvanishing term in the sum

$$\sum_{k=0}^{2} (\partial_{z_2}\psi_{k+1})(\partial_{z_2}\psi_{2-k+1})$$

is given by $(\partial_{z_2}\psi_2)^2$. We get the equation

$$\nu\psi_2 + \frac{\gamma}{2}\alpha_2(\partial_{z_2}\psi_2)^2 = 0, \tag{3.33}$$

where we also know that ψ_2 vanishes near $(M_+, 0, M_+)$. The preceding equation can be viewed as a first order ordinary differential equation for ψ_2 along the integral curves of ν in $\Omega_+ \cap \{(w_1, w_2)\ ;\ w_2 = 0\}$, and we can conclude that $\psi_2 = 0$ in Ω_+.

Now we shall argue inductively to see that $\psi_\mu = 0$ in Ω_+, for $\mu < m$. Indeed, assume that $\psi_0 = \psi_1 = \psi_2 = \cdots = \psi_{\mu-1} = 0$, for some $2 < \mu < m$. Then

$$\sum_{k=0}^{\mu} (\partial_{z_1}\psi_k)(\partial_{z_1}\psi_{\mu-k}) = 0.$$

As for the sum
$$\sum_{k=0}^{\mu}(\partial_{z_2}\psi_{k+1})(\partial_{z_2}\psi_{\mu-k+1}),$$
we see that the only term here that is not clearly vanishing corresponds to the case where $k+1=\mu$. In this case, the corresponding term is equal to $(\partial_{z_2}\psi_\mu)\partial_{z_2}\psi_2$, which vanishes after all. Consequently, the sum above vanishes, and from (3.32) we get the equation
$$\nu\psi_\mu = 0, \quad \mu < m.$$
It follows that $\psi_\mu = 0$ in Ω_+, for $\mu < m$. We conclude that a smooth solution ψ of (3.13) such that $\psi = 0$ near $(M_+, 0, M_+)$ has the following form in Ω_+:
$$\psi(w_1, w_2) = \psi_m(w_1, w_2) + \mathcal{O}(w_2^{m+1}).$$
Combining (3.13) and (3.32), we see that ψ_m should satisfy the following nonhomogeneous transport equation in Ω_+:
$$\nu\psi_m = 2\partial_x \delta W \cdot \partial_y \varphi_0. \tag{3.34}$$

Therefore, the proof of Theorem 3.1 will be concluded, once we establish the following result.

Proposition 3.5. *There exists a positive definite quadratic form $W_2(x_2)$, a Morse function $W_1(x_1)$ with two local minima and a saddle point, and a perturbation $\delta W \in C^\infty(\mathbb{R}^{2n})$ with $M_+ \notin \operatorname{supp}(\delta W)$, $S_0 \notin \operatorname{supp}(\delta W)$, such that $\delta W(x_1, x_2)$ is a homogeneous polynomial of degree $m \geq 3$ in x_2, for which the transport equation* (3.34) *does not have a smooth solution in $\Omega_+ \cup \{\widetilde{S}_0\}$.*

Proof. With the notation $w_j = (x_j, y_j, z_j)$, $j = 1, 2$, we write $\nu = \sum_{j=1}^2 \nu_j(w_j, \partial_{w_j})$. The preceding discussion shows that there exists an integral curve γ_1 of ν_1 such that
$$\gamma_1(t) \to \begin{cases} (m_+, 0, m_+), & t \to -\infty, \\ (s_0, 0, s_0), & t \to +\infty. \end{cases}$$

Let N_2 be the coefficient matrix of the linear vector field ν_2, which we shall view as the linearization at $w_2 = 0$, and let $\lambda_1, \dots, \lambda_{3n}$ be the corresponding eigenvalues, so that $\operatorname{Re}\lambda_j > 0$, $1 \leq j \leq 3n$. Let us assume, as we may, that N_2 has no Jordan blocks, and after a linear change of variables, we may therefore assume that $w_2 = (\omega_1, \dots, \omega_{3n})$ and
$$\nu_2 = \sum_{j=1}^{3n} \lambda_j \omega_j \partial_{\omega_j}.$$
Then, writing $\lambda = (\lambda_1, \dots, \lambda_{3n}) \in \mathbf{C}^{3n}$, we have
$$\nu_2(\omega^\alpha) = (\lambda \cdot \alpha)\, \omega^\alpha.$$

Consider equation (3.34),
$$(\nu_1 + \nu_2)(\psi_m) = \frac{2}{\alpha_1} y_1 \cdot \partial_{x_1}\delta W + \frac{2}{\alpha_2} y_2 \cdot \partial_{x_2}\delta W, \tag{3.35}$$
and put
$$\psi_m = \frac{2}{\alpha_1}\delta W(x) + u.$$
Then we get
$$(\nu_1 + \nu_2)(u) = \Big(\frac{2}{\alpha_2} - \frac{2}{\alpha_1}\Big) y_2 \cdot \partial_{x_2}\delta W, \tag{3.36}$$
where we recall that $\alpha_2 \neq \alpha_1$. Using the fact that $\delta W(x_1, x_2)$ is a homogeneous polynomial of degree $m \geq 3$ in x_2 for every x_1, we may write
$$\Big(\frac{2}{\alpha_2} - \frac{2}{\alpha_1}\Big) y_2 \cdot \partial_{x_2}\delta W = \sum_{|\alpha| = m} g_\alpha(x_1)\omega^\alpha,$$

and if (3.36) has a smooth solution u, we can assume without loss of generality that

$$u = \sum_{|\alpha|=m} u_\alpha(w_1)\omega^\alpha.$$

Then equation (3.36) reduces to the following decoupled system of equations:

$$(\nu_1(w_1, \partial_{w_1}) + \lambda \cdot \alpha)\, u_\alpha(w_1) = g_\alpha(x_1), \quad |\alpha| = m. \tag{3.37}$$

We shall choose $\delta W(x_1, x_2) = \pm\psi(x_1)v(x_2)$, where $0 \leq \psi \in C_0^\infty(\mathbb{R}^n)$, $m_+, s_0 \notin \operatorname{supp}\psi$, while $\psi > 0$ somewhere on the image of γ_1, and where v is a homogeneous polynomial of degree m. Then for some $\alpha = \alpha_0$ of length m, we have $0 \leq g_\alpha \in C_0^\infty(\mathbb{R}^n)$, $m_+, s_0 \notin \operatorname{supp} g_\alpha$, and g_α is positive somewhere on the image of γ_1.

Equation (3.37) along γ_1 with $\alpha = \alpha_0$ has a compactly supported right-hand side with constant sign, not identically equal to 0, so that the solution $\widetilde{u} = u_{\alpha_0}$ is nonzero either on $\gamma_1 \cap \operatorname{neigh}(m_+, 0, m_+)$, or on $\gamma_1 \cap \operatorname{neigh}(s_0, 0, s_0)$.

In the first case we use the natural parametrization $\gamma_1(t) = \exp t\nu_1(w_0)$, $-\infty < t < +\infty$, where w_0 is some fixed point on γ_1. Then there is a constant $C > 0$ such that $\operatorname{dist}(\gamma_1(t), (m_+, 0, m_+)) \leq Ce^{-|t|/C}$ for $t \leq 0$. For t large and negative, we have

$$\Big(\frac{d}{dt} + \lambda \cdot \alpha\Big) u_\alpha(\gamma_1(t)) = 0,$$

whence $u_\alpha(\gamma(t)) = Ce^{-(\lambda\cdot\alpha)t}$, $C \neq 0$. Thus, u_α is unbounded near $(m_+, 0, m_+)$, and hence cannot be smooth near that point.

In the second case, we recall that γ_1 is a part of the one-dimensional stable manifold through $(s_0, 0, s_0)$ for the ν_1-flow. We can find new smooth local coordinates $x = (x_1, x'')$ centered at $(s_0, 0, s_0)$ and such that this stable manifold is given by $x'' = 0$, whence

$$\nu_1 = a_1(x)\partial_{x_1} + \sum_{j=2}^{3n} a_j(x)\partial_{x_j},$$

where $a_j(x_1, 0) = 0$ for $j \geq 2$. Furthermore, $a_1(x_1, 0) = -\mu_1(x_1 + f(x_1))$, where $\mu_1 > 0$, $f(x_1) = \mathcal{O}(x_1^2)$, and we may assume that γ_1 coincides with the positive x_1-axis near $(s_0, 0, s_0)$. Along γ_1 and near $(s_0, 0, s_0)$ we know that u_α is a nonvanishing solution of the equation

$$(-\mu_1(x_1 + f(x_1))\partial_{x_1} + \lambda \cdot \alpha)u_\alpha = 0, \quad 0 < x_1 \ll 1,$$

so that

$$u_\alpha(x_1, 0) = C \exp\left(\frac{\lambda\cdot\alpha}{\mu_1}\int_{x_1^0}^{x_1} \frac{1}{s + f(s)}\, ds\right),$$

where $C \neq 0$ and $x_1^0 > 0$ is small and fixed. Here

$$\frac{1}{s + f(s)} = \frac{1}{s}\,\frac{1}{1 + \frac{f(s)}{s}} = \frac{1}{s} - \frac{f(s)}{s^2} + \frac{f(s)^2}{s^3} - \ldots,$$

so

$$\int_{x_1^0}^{x_1} \frac{1}{s + f(s)}\, ds = \ln x_1 + g(x_1),$$

where g is smooth near $x_1 = 0$. Thus, $u_\alpha(x_1, 0) = C\, x_1^{\frac{\lambda\cdot\alpha}{\mu_1}} e^{g(x_1)}$, $C \neq 0$, and if $\frac{\lambda\cdot\alpha}{\mu_1} \notin \mathbb{N}$ (which can be arranged by choosing the parameters suitably, cf. (3.17)), we conclude that u_α cannot be smooth near $(s_0, 0, s_0)$. The proof of Proposition 3.5 is complete. $\square$

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