Алгебра и анализ
Том 25 (2013), № 3

St. Petersburg Math. J.
Vol. 25 (2014), No. 3, Pages 421–446
S 1061-0022(2014)01297-6
Article electronically published on May 16, 2014

ESTIMATES FOR FUNCTIONALS WITH A KNOWN FINITE SET OF MOMENTS IN TERMS OF HIGH ORDER MODULI OF CONTINUITY IN SPACES OF FUNCTIONS DEFINED ON A SEGMENT

O. L. VINOGRADOV AND V. V. ZHUK

Dedicated to Boris Mikhaĭlovich Makarov

Abstract. A new technique is developed for estimating functionals in terms of the quantities mentioned in the title. The constants in estimates are indicated explicitly. As examples, Jackson type inequalities for approximations by polynomials and splines can be mentioned, along with estimates of error terms for interpolation formulas and for formulas of numerical differentiation and integration. One of results can be stated as follows. Let E be a segment, $|E|$ its length, E_{n-1} the best uniform approximation by polynomials of degree at most $n-1$, and ω_{2m} the uniform modulus of continuity of order $2m$. Let $\mathcal{K}_r = \frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu(r+1)}}{(2\nu+1)^{r+1}}$ be the Favard constants, $\mathcal{W}_{2m}$ the Whitney constants, and $\nu_m = \frac{8}{\binom{2m}{m}}\sum_{l=0}^{\lfloor (m-1)/2\rfloor}\frac{\binom{2m}{m-2l-1}}{(2l+1)^2}$. Let $m \ge 2$, $n \ge 2m$, $\gamma > 0$, $f \in C(E)$. Then

$$E_{n-1}(f) \le \Bigg(\frac{1}{\binom{2m}{m}}\Bigg(1 + \frac{\nu_m}{\gamma^2}\frac{\mathcal{K}_2}{4} + \sum_{k=2}^{m-1}\frac{\mathcal{K}_{2k}}{2^{2k}}\frac{(2m-2k)!\,(2m)^{2k}}{(2m)!}\frac{\nu_m^k}{\gamma^{2k}}\Bigg) + \frac{\mathcal{K}_{2m}}{2^{2m}}\frac{(2m)^{2m}}{(2m)!}\frac{\nu_m^m}{4^m\gamma^{2m}}\Bigg)(2^{2m}-1)\mathcal{W}_{2m}\omega_{2m}\Big(f,\frac{\gamma|E|}{n}\Big).$$

§1. Introduction

In the papers [1, 2, 3], the authors developed a technique of estimating functionals in terms of high order moduli of continuity. In this paper, this technique is applied to estimates in spaces of functions defined on a segment. We establish inequalities of the type

$$\Phi(f) \le K \cdot \omega_{2m}\big(f^{(s)}, h\big)_{L_p[a,b]},$$

where the constants are written explicitly. The constants are expressed in terms of the moments of the functional Φ, i.e., its lowest upper bounds on the Sobolev classes.

If it is not stated otherwise, E is a nondegenerate segment, J is an interval, $|J|$ is its length. In what follows, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$ are the sets of reals, nonnegative reals, integers, nonnegative integers, and positive integers, respectively; $[a:b] = [a,b] \cap \mathbb{Z}$.

Function spaces are denoted as follows: $C(E)$ is the space of functions continuous on E, with the uniform norm, $C^{(r)}(E)$ is the set of r times continuously differentiable functions on E, $C^{(0)}(E) = C(E)$. If $p \in [1,+\infty)$, then $L_p(J)$ is the space of measurable

---

2010 *Mathematics Subject Classification.* Primary 41A15, 41A17, 41A35.
Key words and phrases. Best approximation, modulus of continuity, sharp constants, numerical differentiation and integration.

functions f integrable on J with pth power, with the norm $\|f\|_{L_p(J)} = \left(\int_J |f|^p\right)^{1/p}$; $L_\infty(J)$ is the space of measurable functions essentially bounded on J, with the vrai sup-norm; if $p \in [1,+\infty]$, then $L_{p,\mathrm{loc}}(J)$ is the set of functions that belong to $L_p(E)$ for each segment $E \subset J$, $W_p^{(r)}(J)$ and $W_{p,\mathrm{loc}}^{(r)}(J)$ are the sets of functions that belong to $L_p(J)$ and $L_{p,\mathrm{loc}}(J)$ and are the r-fold integrals of functions lying in $L_p(J)$ and $L_{p,\mathrm{loc}}(J)$, respectively; $W_p^{(0)}(J) = L_p(J)$, $W_{p,\mathrm{loc}}^{(0)}(J) = L_{p,\mathrm{loc}}(J)$. Unless implied otherwise by the context, function spaces may be real or complex; to specify the field of scalars, we write, e.g., $\mathbb{R}C(E)$.

Next, I is the identity operator, Δ_t^r and δ_t^r are the operators of forward and central difference of order $r \in \mathbb{Z}_+$ with step t, i.e.,

$$\Delta_t^r(f,x) = \sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k} f(x+kt), \quad \delta_t^r(f,x) = \Delta_t^r\Big(f, x - \frac{rt}{2}\Big).$$

The moduli of continuity in the spaces $L_p(J)$ are defined by

$$\omega_r(f,h)_{L_p(J)} = \sup_{0\le t\le h}\big\|\Delta_t^r(f)\big\|_{L_p(J\cap(J-rt))}.$$

The moduli of continuity in the spaces $C(E)$ are defined in a similar way. The symbols $E_n(f)_{C(E)}$ and $E_n(f)_{L_p(E)}$ denote the best approximations of a function f by algebraic polynomials of degree at most n.

Let

$$\nu_m = \frac{8}{\binom{2m}{m}} \sum_{l=0}^{\lfloor (m-1)/2\rfloor} \frac{\binom{2m}{m-2l-1}}{(2l+1)^2},$$

where $\lfloor x\rfloor$ is the integral part of x. Functions are assumed to be extended to the points of a removable break by continuity; in other cases the symbol $\frac{0}{0}$ is understood as 0. If $\mathfrak{M}$ is a vector space, then $\mathcal{F}(\mathfrak{M})$ denotes the set of semiadditive functionals $\Phi : \mathfrak{M} \to \mathbb{R}_+$, i.e., those satisfying $\Phi(f+g) \le \Phi(f) + \Phi(g)$ for each $f, g \in \mathfrak{M}$.

§2. General assertions

Suppose that $(\mathfrak{M}, P)$ is a space with a seminorm, $\mathfrak{L}, \mathfrak{N}$ are subspaces of $\mathfrak{M}$, $\Phi \in \mathcal{F}(\mathfrak{N})$, and $A : \mathfrak{L} \to \mathfrak{M}$ is a linear operator. We put

$$N_P(\Phi, A) = \sup_{f\in\mathfrak{L}\cap\mathfrak{N}} \frac{\Phi(f)}{P(Af)}. \tag{2.1}$$

We formulate separately a special case of (2.1) when A is the operator D^k of k-fold differentiation. If $p \in [1,+\infty]$, $(\mathfrak{M}, P)$ is a space with a seminorm, $\mathfrak{M} \subset L_{p,\mathrm{loc}}(J)$, $k \in \mathbb{Z}_+$, then we put

$$\mathfrak{M}^{(k)} = \big\{g \in \mathfrak{M} \cap W_{p,\mathrm{loc}}^{(k)}(J) \,:\, g^{(k)} \in \mathfrak{M}\big\}.$$

Moreover, assume that $\mathfrak{N}$ is a subspace of $\mathfrak{M}$, $\Phi \in \mathcal{F}(\mathfrak{N})$. The quantities

$$m_k(\Phi)_P = N_P(\Phi, D^k) = \sup_{g\in\mathfrak{M}^{(k)}\cap\mathfrak{N}} \frac{\Phi(g)}{P(g^{(k)})}$$

are called the *moments* of Φ.

In particular, if $s \in \mathbb{Z}_+$ and $k \ge s$, $\Phi \in \mathcal{F}\big(W_p^{(s)}(E)\big)$, then

$$m_k(\Phi)_p = \sup_{f\in W_p^{(k)}(E)} \frac{\Phi(f)}{\|f^{(k)}\|_{L_p(E)}},$$

and if $\Phi \in \mathcal{F}\big(C^{(s)}(E)\big)$, then

$$m_k(\Phi)_C = \sup_{f\in C^{(k)}(E)} \frac{\Phi(f)}{\|f^{(k)}\|_{C(E)}}.$$

Now we show that, in some cases, when estimating functionals we may restrict ourselves to classes of continuously differentiable functions. For this, we need the technique of Steklov finctions.

For $E = [a, b]$, $f \in L_1(E)$, $x \in E$, $\tau \in (0, |E|]$ we put (see [4, 5] and also [6, p. 18], [7, p. 102], and [8, p. 63]),

$$f_\tau(x) = \frac{1}{\tau}\int_0^\tau f\Big(x - \tau\frac{x-a}{b-a} + t\Big)\,dt.$$

The functions f_τ and their generalizations for primitives and differences of order $r \geq 2$ are called the *modified Steklov functions* or the *Steklov–Sendov functions.*

We list several obvious properites of the modified Steklov functions, without aiming at the maximal generality. In a more general situation, these properties were proved in [6, p. 18–20].

Lemma A. 1. *If $s \in \mathbb{Z}_+$, $p \in [1, +\infty]$, $f \in W_p^{(s)}(E)$, then $f_\tau \in W_p^{(s+1)}(E)$.*

2. *If $k \in \mathbb{N}$, $s \in [\,0 : k-1]$, $f \in W_\infty^{(k)}(E)$, then*

$$\big\|f_\tau^{(s)} - f^{(s)}\big\|_{C(E)} \underset{\tau\to 0}{\longrightarrow} 0.$$

3. *If $s \in \mathbb{Z}_+$, $f \in W_1^{(s)}(E)$, then*

$$\big\|f_\tau^{(s)} - f^{(s)}\big\|_{L_1(E)} \underset{\tau\to 0}{\longrightarrow} 0.$$

4. *If $r, s \in \mathbb{Z}_+$, $h > 0$, $f \in W_\infty^{(s)}(E)$, then*

$$\omega_r\big(f_\tau^{(s)}, h\big)_{C(E)} \leq \Big(1 - \frac{\tau}{|E|}\Big)^s \omega_r\big(f^{(s)}, h\big)_{L_\infty(E)}.$$

Lemma 2.1. *Suppose $r, s \in \mathbb{Z}_+$, $k \in \mathbb{N}$, $k > s$, $h > 0$, $\Phi \in \mathcal{F}\big(C^{(s)}(E)\big)$, $m_s(\Phi)_C < +\infty$. Then*

$$\sup_{f\in W_\infty^{(k)}(E)} \frac{\Phi(f)}{\omega_r(f^{(k)}, h)_{L_\infty(E)}} = \sup_{f\in C^{(k)}(E)} \frac{\Phi(f)}{\omega_r(f^{(k)}, h)_{C(E)}}. \tag{2.2}$$

In particular, for $r = 0$ we have

$$\sup_{f\in W_\infty^{(k)}(E)} \frac{\Phi(f)}{\|f^{(k)}\|_{L_\infty(E)}} = \sup_{f\in C^{(k)}(E)} \frac{\Phi(f)}{\|f^{(k)}\|_{C(E)}}.$$

Proof. Clearly, the right-hand side of (2.2) is not greater than the quantity on the left. We prove the reverse inequality, denoting the right-hand side by M. If $M = +\infty$, the inequality is trivial. Let $M < +\infty$. Then for every $f \in C^{(k)}(E)$ we have

$$\Phi(f) \leq M\omega_r(f^{(k)}, h)_{C(E)}.$$

Let $f \in W_\infty^{(k)}(E)$. We take a sequence τ_n of positive numbers tending to zero, and denote by f_n the Steklov–Sendov functions of f with the step τ_n. Then $f_n \in C^{(k)}(E)$ and, by statement 4 of Lemma A,

$$\Phi(f_n) \leq M\omega_r\big(f_n^{(k)}, h\big)_{C(E)} \leq M\,\omega_r\big(f^{(k)}, h\big)_{L_\infty(E)}. \tag{2.3}$$

Statement 2 of Lemma A and the finiteness of $m_s(\Phi)_C$ imply that

$$\big|\Phi(f_n) - \Phi(f)\big| \leq m_s(\Phi)_C\big\|f_n^{(s)} - f^{(s)}\big\|_{C(E)} \underset{n\to\infty}{\longrightarrow} 0.$$

Passing to the limit in (2.3), we get the required inequality. □

To obtain an analog of Lemma 2.1 for $k = s$, we assume that the functional in question has an additional property. We say that a functional $\Phi \in \mathcal{F}\big(W_\infty^{(s)}(E)\big)$ has the *generalized Fatou property* if $\Phi(f) \le K$ whenever $f \in W_\infty^{(s)}(E)$, $f_n \in C^{(s)}(E)$, $f_n^{(s)} \to f^{(s)}$ almost everywhere on E, the sequence $\big\{\|f_n^{(s)}\|_{L_\infty(E)}\big\}$ is bounded, and $\Phi(f_n) \le K$ for every $n \in \mathbb{N}$.

Lemma 2.2. *Assume that $r, s \in \mathbb{Z}_+$, $h > 0$, $\Phi \in \mathcal{F}\big(W_\infty^{(s)}(E)\big)$, Φ has the generalized Fatou property. Then*

$$\sup_{f \in W_\infty^{(s)}(E)} \frac{\Phi(f)}{\omega_r(f^{(s)}, h)_{L_\infty(E)}} = \sup_{f \in C^{(s)}(E)} \frac{\Phi(f)}{\omega_r(f^{(s)}, h)_{C(E)}}.$$

In particular, for $r = 0$ we have

$$\sup_{f \in W_\infty^{(s)}(E)} \frac{\Phi(f)}{\|f^{(s)}\|_{L_\infty(E)}} = \sup_{f \in C^{(s)}(E)} \frac{\Phi(f)}{\|f^{(s)}\|_{C(E)}}.$$

Proof. As in the proof of Lemma 2.1, we have

$$\Phi(f_n) \le M\omega_r\big(f_n^{(s)}, h\big)_{C(E)} \le M\,\omega_r\big(f^{(s)}, h\big)_{L_\infty(E)}.$$

Moreover, by statement 3 of Lemma A, the sequence f_n can be chosen so that $f_n^{(s)} \to f^{(s)}$ almost everywhere on E. It remains to apply the the generalized Fatou property. □

Remark 2.1. We present examples of functionals with the generalized Fatou property. In these examples, a sequence $\{f_n\}$ and a number K are taken as in the definition of the generalized Fatou property.

1. Let X be a finite-dimensional subspace of $L_\infty(E)$, and let $E_X(\,\cdot\,)_{L_\infty(E)}$ be the functional of best approximation by the subspace X. It is known (see [9, pp. 121–122, Proposition 3.4.5]) that

$$E_X(f)_{L_\infty(E)} = \sup_{\substack{g \perp X,\\ \|g\|_{L_1(E)} \le 1}} \left| \int_E fg \right|.$$

Then for every function $g \in L_1(E)$ such that $\|g\|_{L_1(E)} \le 1$ and $g \perp X$, the inequality $\big|\int_E f_n g\big| \le K$ holds true. The Lebesgue dominated convergence theorem yields $\big|\int_E fg\big| \le K$. Consequently, $E_X(f)_{L_\infty(E)} \le K$.

2. Consider an operator $V : L_\infty(E) \to L_\infty(E)$ of the form

$$V(f, x) = f(x) + \int_E f(t)F(x, t)\,dt,$$

$\Phi(f) = \|V(f^{(s)})\|_{L_\infty(E)}$ for $f \in W_\infty^{(s)}(E)$. Then $V(f_n^{(s)}) \to V(f^{(s)})$ almost everywhere, whence $\Phi(f) \le K$.

Remark 2.2. The density of algebraic polynomials in $C(E)$ implies that the upper bounds in Lemmas 2.1 and 2.2 may be taken over the set of polynomials.

The following fact was proved in [3, Theorem 2.2].

Theorem B. *Assume that $(\mathfrak{M}, P)$ is a space with a seminorm, $\mathfrak{N}$ is a subspace of $\mathfrak{M}$, $\Psi \in \mathcal{F}(\mathfrak{N})$, $m \in \mathbb{N}$, $U : \mathfrak{N} \to \mathfrak{N}$, $A_k : U^k(\mathfrak{N}) \to \mathfrak{M}$ are linear operators, and $N_P(\Psi, A_k) < +\infty$ for every $k \in [0 : m]$. Then for every $f \in \mathfrak{N}$ we have*

$$\Psi(f) \le \sum_{k=0}^{m-1} N_P(\Psi, A_k) P\big(A_k U^k (I - U) f\big) + N_P(\Psi, A_m) P\big(A_m U^m f\big).$$

Theorem B will be applied to the spaces $W_{p,\mathrm{loc}}(\mathbb{R})$ equipped with the seminorms $P(g) = \big\| g|_E \big\|_{L_p(E)}$, $p \in [1, +\infty]$. The problems of estimates of functionals on the spaces of functions defined on a segment can be reduced to theorem B with the help of extension.

For the role of A_k we take the operators D^{s+2k}, where D is the differentiation operator, and for the role of U linear combinations of Steklov averages:

$$U_{h,m} = \frac{2}{\binom{2m}{m}} \sum_{j=1}^{m} (-1)^{j-1} \binom{2m}{m-j} S_{jh}^2.$$

The Steklov averages of the second order are defined by

$$S_h^2(f, x) = \frac{1}{h} \int_{-h}^{h} f(x-t) \Big(1 - \frac{|t|}{h}\Big)\, dt.$$

The following estimates of the quantities related to the operators $U_{h,m}$ for a class of shift-invariant spaces more general than $L_p(\mathbb{R})$, are mainly contained in [1]. The only difference is that in [1] it was assumed that a function g itself belongs to the space. Formulas below show that it suffices that the differences of g belong to the space.

The identity

$$g - U_{h,m} g = \frac{2(-1)^m}{h\binom{2m}{m}} \int_0^h \Big(1 - \frac{t}{h}\Big) \delta_t^{2m} g\, dt$$

implies that if $\delta_t^{2m} g \in L_p(\mathbb{R})$ for all $t \in [0, h]$, then $g - U_{h,m} g \in L_p(\mathbb{R})$ and

$$\|g - U_{h,m} g\|_{L_p(\mathbb{R})} \le \frac{2}{h\binom{2m}{m}} \int_0^h \big\|\delta_t^{2m} f\big\|_{L_p(\mathbb{R})} \Big(1 - \frac{t}{h}\Big)\, dt \le \frac{1}{\binom{2m}{m}} \omega_{2m}(g, h)_{L_p(\mathbb{R})}. \tag{2.4}$$

The operator $D^2 U_{h,m}$ is a summatory operator of the form

$$D^2 U_{h,m}(g, x) = \sum_{s \in \mathbb{Z}} a_s g(x + sh)$$

with finitely many nonzero coefficients a_s, and its norm is estimated as

$$\|D^2 U_{h,m}\|_{L_p(\mathbb{R}) \to L_p(\mathbb{R})} \le \frac{\nu_m}{h^2}.$$

So,

$$\|D^{2k} U_{h,m}^k\|_{L_p(\mathbb{R}) \to L_p(\mathbb{R})} \le \frac{\nu_m^k}{h^{2k}}.$$

On the other hand, the operator $D^2 U_{h,m}$ admits separation of the difference factor:

$$D^2 U_{h,m} = \frac{1}{h^2} \delta_h^2 W_{h,m},$$

where $W_{h,m}$ is a summatory operator of the same form, and

$$\|W_{h,m}\|_{L_p(\mathbb{R}) \to L_p(\mathbb{R})} \le \frac{\nu_m}{4}.$$

Consequently, if $\delta_h^{2k} g \in L_p(\mathbb{R})$, then $D^{2k} U_{h,m}^k g \in L_p(\mathbb{R})$ and

$$\|D^{2k} U_{h,m}^k g\|_{L_p(\mathbb{R})} \le \frac{\nu_m^k}{4^k h^{2k}} \|\delta_h^{2k} g\|_{L_p(\mathbb{R})}. \tag{2.5}$$

Remark 2.3. *A fortiori*, estimates (2.4) and (2.5) are valid if the norms on their left-hand sides are taken in the space $L_p(E)$.

Remark 2.4. In [10], the authors established the inequality

$$\frac{8}{\pi^2} \sqrt{\frac{\pi}{2}} \frac{\sqrt{2m}}{2m+1} \le 1 - \frac{\nu_m}{\pi^2} \le \frac{8}{\pi^2} \sqrt{\frac{\pi}{2}} \frac{\sqrt{2m+1}}{2m}.$$

Definition. Let $r \in \mathbb{N}$, $p \in [1, +\infty]$. The quantities

$$\mathcal{W}_{r,p} = \sup_{f \in L_p(E)} \frac{E_{r-1}(f)_{L_p(E)}}{\omega_r\big(f, \frac{|E|}{r}\big)_{L_p(E)}},$$

$$\mathcal{W}_{r,C} = \sup_{f \in C(E)} \frac{E_{r-1}(f)_{C(E)}}{\omega_r\big(f, \frac{|E|}{r}\big)_{C(E)}}$$

are called the *Whitney constants.*

Obviously, the Whitney constants do not depend on a segment E.

Remark 2.5. Nowadays, the study of the Whitney constants forms an independent part of approximation theory, with a lot of papers by many authors. See, e.g., [6, 8, 11] for the details, where some known results are proved and bibliographic notes are given. The case where $r = 1$ stands apart and will not be discussed in this paper. For $r \geq 2$, explicit estimates for the Whitney constants were usually formulated without specializing whether the real or the complex space is considered. It is known that $\mathbb{R}\mathcal{W}_{2,C} = \frac{1}{2}$, $\frac{8}{15} \leq \mathbb{R}\mathcal{W}_{3,C} \leq \frac{7}{10}$ [12]; $\mathcal{W}_{r,C} \leq 1$ for $r \leq 8$ [13, 14, 11] (this result of [14] was cited in [11]); $\mathcal{W}_{r,C} \leq 2$ for $r \leq 82000$, $\mathcal{W}_{r,C} \leq 2 + e^{-2}$ for all r [15]; $\mathcal{W}_{r,1} \leq 6,4$, $\mathcal{W}_{r,p} \leq 9$ [16]. As far as we can judge, all these estimates except for the first two, were established by linear approximation methods and are valid both in the real and complex space.

Remark 2.6. By Lemma 2.2, $\mathcal{W}_{r,C} = \mathcal{W}_{r,\infty}$. Whitney proved his theorem for the space $\ell_\infty(E)$ of bounded functions with sup-norm. Later, inequalities like

$$E_{r-1}(f)_{\ell_\infty(E)} \leq \mathcal{W}_{r,C}\, \omega_r\Big(f, \frac{|E|}{r}\Big)_{\ell_\infty(E)}$$

were mainly stated for continuous functions, and their validity on subspaces of $\ell_\infty(E)$, wider than $C(E)$ was not always mentioned. Moreover, some proofs were correct for arbitrary bounded functions, while some other (based on integral interpolation) required integrability. We do not know whether the Whitney constants for the spaces $C(E)$ and $\ell_\infty(E)$ are equal in the general case.

Definition. Let $r \in \mathbb{N}$, $h > 0$, $p \in [1, +\infty]$. The values

$$\lambda_{r,E}(h)_p = \sup_{f \in L_p(E)} \inf_{\substack{g \in L_{p,\mathrm{loc}}(\mathbb{R}) \\ g|_E = f}} \frac{\omega_r(g, h)_{L_p(\mathbb{R})}}{\omega_r(f, h)_{L_p(E)}},$$

$$\lambda_{r,E}(h)_C = \sup_{f \in C(E)} \inf_{\substack{g \in L_{\infty,\mathrm{loc}}(\mathbb{R}) \\ g|_E = f}} \frac{\omega_r(g, h)_{L_\infty(\mathbb{R})}}{\omega_r(f, h)_{C(E)}}$$

are called the *extension constants.*

Lemma 2.3. *Under the conditions of the definition, we have*

$$\begin{aligned} \lambda_{r,E}(h)_p &\leq \big(2(2^r - 1)^p \mathcal{W}_{r,p}^p + 1\big)^{1/p}, \quad p \in [1, +\infty), \\ \lambda_{r,E}(h)_\infty &\leq \max\left\{(2^r - 1)\mathcal{W}_{r,\infty}, 1\right\}, \end{aligned} \tag{2.6}$$

where the $\mathcal{W}_{r,p}$ are the Whitney constants.

Proof. Suppose $E = [a, b]$, $p \in [1, +\infty)$, $0 < h \leq \frac{|E|}{r}$. We denote by $\pi_\pm$ the best approximation polynomials of degree $r - 1$ for a function f on the segments $[a, a + rh]$

and $[b-rh,b]$, and put

$$g(x)=g_h(x)=\begin{cases}\pi_-(x), & x<a,\\ f(x), & x\in E,\\ \pi_+(x), & x>b.\end{cases}$$

Since the rth difference of a polymonial of degree at most $r-1$ equals zero, for all $t\in(0,h]$ we have

$$\|\Delta_t^r g\|^p_{L_p(\mathbb{R})}=\|\Delta_t^r g\|^p_{L_p[a-rt,a]}+\|\Delta_t^r f\|^p_{L_p(E)}+\|\Delta_t^r g\|^p_{L_p[b,b+rt]}.$$

The second term does not exceed $\omega_r^p(f,h)_{L_p(E)}$. We estimate the first term:

$$\begin{aligned}\|\Delta_t^r g\|_{L_p[a-rt,a]}&=\|\Delta_t^r(g-\pi_-)\|_{L_p[a-rt,a]}\\&=\left\|\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}(g-\pi_-)(\,\cdot\,+kt)\right\|_{L_p[a-rt,a]}\\&\le\sum_{k=1}^{r}\binom{r}{k}\|(g-\pi_-)(\,\cdot\,+kt)\|_{L_p[a-rt,a]}\le(2^r-1)E_{r-1}(f)_{L_p[a,a+rh]}.\end{aligned}$$

Similarly, we prove that

$$\|\Delta_t^r g\|_{L_p[b-rt,b]}\le(2^r-1)E_{r-1}(f)_{L_p[b-rh,b]}.$$

By the definition of the Whitney constants, the quantities $E_{r-1}(f)_{L_p[a,a+rh]}$ and $E_{r-1}(f)_{L_p[b-rh,b]}$ do not exceed $\mathcal{W}_{r,p}\omega_r(f,h)_{L_p(E)}$. Collecting the estimates, we see that

$$\|\Delta_t^r g\|_{L_p(\mathbb{R})}\le\left(2(2^r-1)^p\mathcal{W}_{r,p}^p+1\right)^{1/p}\omega_r(f,h)_{L_p(E)}\tag{2.7}$$

for all $t\in(0,h]$.

For $h>\frac{|E|}{r}$, put $g=g_h=g_{|E|/r}$ and denote by π the polynomial of the best approximation of f on E. For $t\in\left(0,\frac{|E|}{r}\right]$, inequality (2.7) is proved as before. For $t\in\left(\frac{|E|}{r},h\right]$, we have

$$\|\Delta_t^r g\|^p_{L_p(\mathbb{R})}=\|\Delta_t^r(g-\pi)\|^p_{L_p[a-rt,b-rt]}+\|\Delta_t^r(g-\pi)\|^p_{L_p(E)}\le 2(2^r-1)^pE^p_{r-1}(f)_{L_p(E)}.$$

Recalling the definition of the Whitney constants, we conclude that inequality (2.7) is valid for all $t\in(0,h]$.

The case of $p=+\infty$ is studied similarly, but the norms of the differences on the segments should be estimated separately. □

Remark 2.7. Clearly, $\lambda_{r,E}(h)_C\le\lambda_{r,E}(h)_\infty$. We emphasize that in the definition of $\lambda_{r,E}(h)_C$ an extension is not required to be continuous. The proof of (2.6) remains valid for the space $\ell_\infty(E)$. We do not know whether the extension constants of the spaces $C(E)$, $L_\infty(E)$, and $\ell_\infty(E)$ are equal.

Remark 2.8. For the first time, the method of extension of a function used in the proof of Lemma 2.3 was applied in [17]; the case of $r=2$ was later included in [18]. In this case $\mathbb{R}\lambda_{2,E}(h)_C=\frac{3}{2}$. The upper bound was obtained by Zhuk in [17, 18], and the lower bound by Vinogradov in [19]. In [20], for $p\in[1,+\infty)$, the inequality $\lambda_{r,E}(h)_p\le 2(2^r-1)W_{r,p}+1$ was proved. For $p=\infty$, estimate (2.6) for $\lambda_{r,E}(h)_C$ was obtained in [17, Section 1, §1].

Remark 2.9. Clearly, the dependence of the extension constants on E and h consists only in their dependence on the quotient $\frac{h}{|E|}$. Therefore, they will be denoted by $\lambda_r\left(\frac{h}{|E|}\right)_p$ and $\lambda_r\left(\frac{h}{|E|}\right)_C$. We also mention that the estimates of the extension constants in Lemma 2.3 do not depend on h. For $r\ge 3$ we do not know whether the constants themselves depend on h.

Now we formulate the main theorem of the paper.

Theorem 2.1. *Suppose $s \in \mathbb{Z}_+$, $m \in \mathbb{N}$, $h > 0$.*

1. *If $p \in [1, +\infty]$, $\Phi \in \mathcal{F}\big(W_p^{(s)}(E)\big)$, and $m_{s+2k}(\Phi)_p < +\infty$ for all $k \in [0, m]$, then for every $f \in W_p^{(s)}(E)$ we have*

$$\Phi(f) \le \left(\frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} m_{s+2k}(\Phi)_p \frac{\nu_m^k}{h^{2k}} + m_{s+2m}(\Phi)_p \frac{\nu_m^m}{4^m h^{2m}}\right) \lambda_{2m,E}(h)_p\, \omega_{2m}\big(f^{(s)}, h\big)_{L_p(E)}.$$

2. *If $\Phi \in \mathcal{F}\big(C^{(s)}(E)\big)$ and $m_{s+2k}(\Phi)_C < +\infty$ for all $k \in [0, m]$, then for every $f \in C^{(s)}(E)$ we have*

$$\Phi(f) \le \left(\frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} m_{s+2k}(\Phi)_C \frac{\nu_m^k}{h^{2k}} + m_{s+2m}(\Phi)_C \frac{\nu_m^m}{4^m h^{2m}}\right) \lambda_{2m,E}(h)_C\, \omega_{2m}\big(f^{(s)}, h\big)_{C(E)}.$$

Proof. 1. In Theorem B we put $\mathfrak{M} = L_{p,\mathrm{loc}}(\mathbb{R})$, $P(g) = \big\|g|_E\big\|_{L_p(E)}$ $(g \in \mathfrak{M})$, $\mathfrak{N} = W_{p,\mathrm{loc}}^{(s)}(\mathbb{R})$, $\Psi(g) = \Phi(g|_E)$ $(g \in \mathfrak{N})$, $A_k = D^{s+2k}$, $U = U_{h,m}$. Let $f \in W_p^{(s)}(E)$, and let $g \in \mathfrak{M}$ be an extension of f with $\omega_{2m}(g^{(s)}, h)_{L_p(\mathbb{R})} < +\infty$. Applying theorem B and Remark 2.3, we obtain

$$\Phi(f) = \Psi(g) \le \left(\frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} m_{2k}(\Psi)_p \frac{\nu_m^k}{h^{2k}} + m_{2m}(\Psi)_p \frac{\nu_m^m}{4^m h^{2m}}\right) \omega_{2m}(g^{(s)}, h)_{L_p(\mathbb{R})}.$$

By the definition, the moments of the functionals Φ and Ψ coincide. Passing to the infimum over all extensions of f, we get the required claim.

2. Now in Theorem B we put $\mathfrak{M} = L_{\infty,\mathrm{loc}}(\mathbb{R})$, $P(g) = \big\|g|_E\big\|_{L_\infty(E)}$ $(g \in \mathfrak{M})$,

$$\mathfrak{N} = \left\{g \in W_{\infty,\mathrm{loc}}^{(s)}(\mathbb{R}) : g|_E \in C^{(s)}(E)\right\},$$

$\Psi(g) = \Phi(g|_E)$ $(g \in \mathfrak{N})$, $A_k = D^{s+2k}$, $U = U_{h,m}$. Then $m_j(\Psi)_\infty = m_j(\Phi)_C$ for $j = s$ by the definition of the moments, and for $j > s$ by Lemma 2.1 we have

$$\begin{aligned} m_j(\Psi)_\infty &= \sup_{g \in W_{\infty,\mathrm{loc}}^{(j)}(\mathbb{R})} \frac{\Psi(g)}{P(g^{(j)})} = \sup_{g \in W_{\infty,\mathrm{loc}}^{(j)}(\mathbb{R})} \frac{\Phi(g|_E)}{\|g^{(j)}\|_{L_\infty(E)}} \\ &= \sup_{f \in W_\infty^{(j)}(E)} \frac{\Phi(f)}{\|f^{(j)}\|_{L_\infty(E)}} = m_j(\Phi)_C. \end{aligned}$$

The remaining part of the proof is carried out in a similar way. □

We list the known estimates of functionals by moduli of continuity of order greater than 2 in which the constants were written explicitly. Recall that in the present paper we only consider spaces of functions defined on a segment.

In [21, p. 23, Theorem 2], for $p \in [1, +\infty]$, and $h \in \big(0, \frac{|E|}{r}\big]$, the following estimate was obtained:

$$\Phi(f) \le \left(2^{1/p} r\, m_s(\Phi)_p + \left(r^r \sum_{k=1}^{r} \frac{\binom{r}{k}}{k^r}\right) \frac{m_{r+s}(\Phi)_p}{h^r}\right) \omega_r\big(f^{(s)}, h\big)_{L_p(E)}. \tag{2.8}$$

A similar estimate where the factor $2^{1/p}$ should be replaced by 1 was obtained for the space $C(E)$. Later this result was included in the book [7] , see [7, p. 107, Theorem 2].

In the proof of (2.8), linear combinations of the Steklov–Sendov functions were used. For the further estimates, the techniques of extensions and the Whitney theorem were employed. Citing these results, we leave the extension constants on the right-hand sides.

In the original works these constants were estimated by the Whitney constants in the spirit of Lemma 2.3; for this topic see Remark 2.8.

The next inequality was obtained in [17, p. 9, Theorem 1] for $h \in \left(0, \frac{|E|}{2m}\right]$:

(2.9)

$$\Phi(f) \le \frac{1}{\binom{2m}{m}}\left(m_s(\Phi)_C + \left(2m^{2m}\sum_{k=1}^{m}\frac{\binom{2m}{m-k}}{k^{2m}}\right)\frac{m_{2m+s}(\Phi)_C}{h^{2m}}\right)\lambda_{2m,E}(h)_C\omega_{2m}\left(f^{(s)},h\right)_{C(E)}.$$

This result was not published later elsewhere. Its generalization for the spaces $L_p(E)$ does not meet any difficulties, but at that time no explicit estimates were known for the extension constants for these spaces.

In [20, Theorem 3], the inequality

(2.10)

$$\Phi(f) \le \frac{1}{(2m+1)!}\left(\sum_{k=0}^{m}\binom{2m+1}{2k+1}|M_{2m}^{(2k+1)}(0)|\frac{m_{2k}(\Phi)_p}{h^{2k}}\right)\lambda_{2m,E}(h)_p\,\omega_{2m}(f,h)_{L_p(E)},$$

was obtained for $p \in [1,+\infty)$, where $M_{2m}(y) = y\prod_{j=1}^{m}(y^2-j^2)$. A similar estimate was obtained for the space $C(E)$.

For $m=1$, i.e., for the second modulus of continuity, the constants in Theorem 2.1 and in inequality (2.9) coincide, and for $s=0$ they coincide with the constant in (2.10). Applying Lemma 2.3 and estimates for the Whitney constants, it is easy to show that for even $r=2m$ the constant in (2.8) is always greater than that in (2.9). Indeed, the coefficients of each moment in (2.8) are greater. For the remaining estimates it is generally difficult to answer which constant is smaller, because the answers may be different in dependence on the parameters; in particular, the answers may differ for different functionals.

For $s=0$, denote the constants before $\omega_{2m}(f,h)$ in Theorem 2.1 and in inequalities (2.9) and (2.10) by $A_{2m}(h)$, $B_{2m}(h)$, and $C_{2m}(h)$, respectively. It can be shown that, for $m \ge 2$ fixed and h sufficiently small, the inequality $C_{2m}(h) < A_{2m}(h) < B_{2m}(h)$ holds true, while for h sufficiently large the reverse inequality is valid. At the same time, there exist situations where $A_{2m}(h)$ is the smallest of the three quantities. The calculations are similar to those performed in [22, §4].

To simplify the constants in the inequalities of the form (2.10) up to numerical values, one should know estimates for all the intermediate moments of the functional, while for the type (2.8) and (2.9) — only for the two extreme moments. For this reason, it turns out to be possible to specify inequalities of the type (2.8) and (2.9) for many functionals. On the other hand, in some important cases, taking into account information about a greater set of moments leads to better estimates.

At the end of the section we mention yet another simple method of estimates by moduli of continuity, see [8, Theorem 2.7, p. 60]. Let $m_r(\Phi)_p < +\infty$. Then Φ vanishes on the polynomials of degree at most $r-1$. Therefore, denoting by $P_{r-1}(f)_{L_p(E)}$ the polynomial of best approximation, we have

$$\Phi(f) = \Phi\left(f - P_{r-1}(f)_{L_p(E)}\right) \le m_0(\Phi)_p E_{r-1}(f)_{L_p(E)}.$$

Then we may apply the Jackson-type inequality (see Subsection 3.1), where $E_{r-1}(f)_{L_p(E)}$ is estimated by $\omega_r(f,h)_{L_p(E)}$. For $h = \frac{|E|}{r}$ it coincides with the Whitney inequality.

This method utilizes only poor information on a functional and, as a rule, leads to greately overstated constants. However, if the nature of a functional allows us to divide a segment E into parts and apply the above considerations to these parts, then the estimates obtained may be quite satisfactory. For example, the error terms of composite quadrature formulas have such a structure. They were estimated in this way in [8].

§3. Jackson type inequalities for approximation by algebraic polynomials and splines

In this section, we denote by $\mathcal{K}_r$ the Favard constants:

$$\mathcal{K}_r = \frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu(r+1)}}{(2\nu+1)^{r+1}}.$$

3.1. Jackson type inequalities for approximation by algebraic polynomials. In this subsection, Theorem 2.1 is applied to the best approximation functional $\Phi = E_{n-1}(\,\cdot\,)_{L_\infty(E)}$, $n \in \mathbb{N}$. For uniformity, it is convenient to denote the norm of a continuous function f by $\|f\|_{L_\infty(E)}$ if we do not want to emphasize that we consider the space $C(E)$.

The following Sinwel inequality is known [23] (see also [24, Chapter 3, §2, pp. 108–115]). Suppose $n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $n \ge r$, $f \in W_\infty^{(r)}(E)$. Then

$$E_{n-1}(f)_{L_\infty(E)} \le \mathcal{K}_r \frac{(n-r)!}{n!}\Big(\frac{|E|}{2}\Big)^r \big\|f^{(r)}\big\|_{L_\infty(E)}. \tag{3.1}$$

For $r = 2$, $n \ge 1$ and for $r = n$ the following stronger inequalities are known:

$$E_{n-1}(f)_{L_\infty(E)} \le \frac{\mathcal{K}_2}{n^2}\Big(\frac{|E|}{2}\Big)^2 \big\|f''\big\|_{L_\infty(E)}, \tag{3.2}$$

$$E_{n-1}(f)_{L_\infty(E)} \le \frac{1}{2^{n-1}n!}\Big(\frac{|E|}{2}\Big)^n \big\|f^{(n)}\big\|_{L_\infty(E)}. \tag{3.3}$$

Inequality (3.2) is contained in [24, p. 107], and (3.3) in [25, p. 163]. In those sources, the inequalities are formulated for functions of class $C^{(r)}$, but Lemma 2.1 shows that they are valid for functions of class $W_\infty^{(r)}$. Inequality (3.3) is sharp for every n, and (3.1) is sharp for fixed r and all n in the aggregate, i.e., the constant $\mathcal{K}_r$ in it cannot be replaced by a smaller constant independent of n.

Theorem 3.1. *Let $n, m \in \mathbb{N}$, $\gamma > 0$.*

1. *If $s \in \mathbb{Z}_+$ and $n \ge s + 2m$, then for every $f \in W_\infty^{(s)}(E)$ we have*

$$\begin{aligned} E_{n-1}(f)_{L_\infty(E)} \le \Big(\frac{|E|}{n}\Big)^s \Big(\frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \frac{\mathcal{K}_{s+2k}}{2^{s+2k}} \frac{(n-s-2k)!\, n^{s+2k}}{n!} \frac{\nu_m^k}{\gamma^{2k}} \\ + \frac{\mathcal{K}_{s+2m}}{2^{s+2m}} \frac{(n-s-2m)!\, n^{s+2m}}{n!} \frac{\nu_m^m}{4^m\gamma^{2m}} \Big) \lambda_{2m}\Big(\frac{\gamma}{n}\Big)_C \omega_{2m}\Big(f^{(s)}, \frac{\gamma|E|}{n}\Big)_{L_\infty(E)}. \end{aligned} \tag{3.4}$$

2. *If $n \ge 2 + 2m$, then for every $f \in W_\infty^{(2)}(E)$ we have*

$$\begin{aligned} E_{n-1}(f)_{L_\infty(E)} \le \Big(\frac{|E|}{n}\Big)^2 \Big(\frac{1}{\binom{2m}{m}} \Big(\frac{\mathcal{K}_2}{4} + \sum_{k=1}^{m-1} \frac{\mathcal{K}_{2+2k}}{2^{2+2k}} \frac{(n-2-2k)!\, n^{2+2k}}{n!} \frac{\nu_m^k}{\gamma^{2k}} \Big) \\ + \frac{\mathcal{K}_{2+2m}}{2^{2+2m}} \frac{(n-2-2m)!\, n^{2+2m}}{n!} \frac{\nu_m^m}{4^m\gamma^{2m}} \Big) \lambda_{2m}\Big(\frac{\gamma}{n}\Big)_C \omega_{2m}\Big(f'', \frac{\gamma|E|}{n}\Big)_{L_\infty(E)}. \end{aligned}$$

3. *If $m \ge 2$ and $n \ge 2m$, then for every $f \in L_\infty(E)$ we have*

$$\begin{aligned} E_{n-1}(f)_{L_\infty(E)} \le \Big(\frac{1}{\binom{2m}{m}} \Big(1 + \frac{\nu_m}{\gamma^2}\frac{\mathcal{K}_2}{4} + \sum_{k=2}^{m-1} \frac{\mathcal{K}_{2k}}{2^{2k}} \frac{(n-2k)!\, n^{2k}}{n!} \frac{\nu_m^k}{\gamma^{2k}} \Big) \\ + \frac{\mathcal{K}_{2m}}{2^{2m}} \frac{(n-2m)!\, n^{2m}}{n!} \frac{\nu_m^m}{4^m\gamma^{2m}} \Big) \lambda_{2m}\Big(\frac{\gamma}{n}\Big)_C \omega_{2m}\Big(f, \frac{\gamma|E|}{n}\Big)_{L_\infty(E)}. \end{aligned}$$

Proof. By Lemmas 2.1 and 2.2 and Remark 2.1, it suffices to prove the above inequalities for $f \in C^{(s)}(E)$. We prove the first assertion. Applying Theorem 2.1 to the functional $E_{n-1}(\,\cdot\,)_{C(E)}$ and using estimates (3.1), we get

$$E_{n-1}(f)_{C(E)} \le \biggl(\frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \mathcal{K}_{s+2k} \frac{(n-s-2k)!}{n!} \Bigl(\frac{|E|}{2}\Bigr)^{s+2k} \frac{\nu_m^k}{h^{2k}} + \mathcal{K}_{s+2m} \frac{(n-s-2m)!}{n!} \Bigl(\frac{|E|}{2}\Bigr)^{s+2m} \frac{\nu_m^m}{4^m h^{2m}}\biggr) \lambda_{2m,E}(h)_C\, \omega_{2m}\bigl(f^{(s)}, h\bigr)_{C(E)}.$$

It remains to take $h = \frac{\gamma|E|}{n}$ and to perform simplifications.

To prove the second and third assertions, one should use estimate (3.2). □

Remark 3.1. A direct way to prove Theorem 3.1 is to apply Theorem 2.1 to the functional $E_{n-1}(\,\cdot\,)_{L_\infty(E)}$ and a function $f \in W_\infty^{(s)}(E)$. This leads to an estimate that involves the extension constant $\lambda_{2m,E}(h)_\infty$. Since we do not know whether it coincides with $\lambda_{2m,E}(h)_C$, first we proved the inequality for functions in $C^{(s)}(E)$, and then used Lemmas 2.1 and 2.2 and Remark 2.1.

This remark also concerns Theorem 3.2 about approximation by splines.

For each $q \in \mathbb{N}$ the sequence $\bigl\{\frac{(n-q)!\, n^q}{n!}\bigr\}_{n=q}^\infty$ is monotone decreasing. Applying, moreover, estimate (2.6), from Theorem 3.1 we shall deduce inequalities of the type

$$E_{n-1}(f)_{L_\infty(E)} \le \Bigl(\frac{|E|}{n}\Bigr)^s A_{m,s}(\gamma)\, \omega_{2m}\Bigl(f^{(s)}, \frac{\gamma|E|}{n}\Bigr)_{L_\infty(E)},$$

where the quantity $A_{m,s}(\gamma)$ depends only on the written parameters.

Corollary 3.1. *Suppose $n, m \in \mathbb{N}$, $\gamma > 0$.*

1. *If $s \in \mathbb{Z}_+$ and $n \ge s + 2m$, then for every $f \in W_\infty^{(s)}(E)$ we have*

$$E_{n-1}(f)_{L_\infty(E)} \le \Bigl(\frac{|E|}{n}\Bigr)^s \biggl(\frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \frac{\mathcal{K}_{s+2k}}{2^{s+2k}} \frac{(2m-2k)!\,(s+2m)^{s+2k}}{(s+2m)!} \frac{\nu_m^k}{\gamma^{2k}} + \frac{\mathcal{K}_{s+2m}}{2^{s+2m}} \frac{(s+2m)^{s+2m}}{(s+2m)!} \frac{\nu_m^m}{4^m\gamma^{2m}}\biggr) (2^{2m}-1)\mathcal{W}_{2m,\infty}\omega_{2m}\Bigl(f^{(s)}, \frac{\gamma|E|}{n}\Bigr)_{L_\infty(E)}.$$

2. *If $n \ge 2 + 2m$, then for every $f \in W_\infty^{(2)}(E)$ we have*

$$E_{n-1}(f)_{L_\infty(E)} \le \Bigl(\frac{|E|}{n}\Bigr)^2 \biggl(\frac{1}{\binom{2m}{m}} \biggl(\frac{\mathcal{K}_2}{4} + \sum_{k=1}^{m-1} \frac{\mathcal{K}_{2+2k}}{2^{2+2k}} \frac{(2m-2k)!\,(2+2m)^{2+2k}}{(2+2m)!} \frac{\nu_m^k}{\gamma^{2k}}\biggr) + \frac{\mathcal{K}_{2+2m}}{2^{2+2m}} \frac{(2+2m)^{2+2m}}{(2+2m)!} \frac{\nu_m^m}{4^m\gamma^{2m}}\biggr) (2^{2m}-1)\mathcal{W}_{2m,\infty}\omega_{2m}\Bigl(f'', \frac{\gamma|E|}{n}\Bigr)_{L_\infty(E)}.$$

3. *If $m \ge 2$ and $n \ge 2m$, then for every $f \in L_\infty(E)$ we have*

$$E_{n-1}(f)_{L_\infty(E)} \le \biggl(\frac{1}{\binom{2m}{m}} \biggl(1 + \frac{\nu_m}{\gamma^2}\frac{\mathcal{K}_2}{4} + \sum_{k=2}^{m-1} \frac{\mathcal{K}_{2k}}{2^{2k}} \frac{(2m-2k)!\,(2m)^{2k}}{(2m)!} \frac{\nu_m^k}{\gamma^{2k}}\biggr) + \frac{\mathcal{K}_{2m}}{2^{2m}} \frac{(2m)^{2m}}{(2m)!} \frac{\nu_m^m}{4^m\gamma^{2m}}\biggr) (2^{2m}-1)\mathcal{W}_{2m,\infty}\omega_{2m}\Bigl(f, \frac{\gamma|E|}{n}\Bigr)_{L_\infty(E)}.$$

Remark 3.2. For $n = s + 2m$, the factor $\mathcal{K}_{s+2m}$ in inequality (3.4) and its special cases can be replaced by $\frac{1}{2^{n-1}}$. This refinement is ensured by inequality (3.3).

Remark 3.3. We write separately the improvement of Corollary 3.1 in the special case where $s=0$, $m=1$, $n-1\in\mathbb{N}$, which is not contained in the general formulas:

$$E_{n-1}(f)_{L_\infty(E)} \le \Big(\frac{3}{4}+\frac{3\pi^2}{64\gamma^2}\Big)\omega_2\Big(f, \frac{\gamma|E|}{n}\Big)_{L_\infty(E)}, \quad f\in\mathbb{R}L_\infty(E).$$

To proof this, we should take into account that $\mathcal{K}_2=\frac{\pi^2}{8}$, $\nu_1=4$, $\mathbb{R}\lambda_2(h)_C=\frac{3}{2}$.

For $n=2$, inequality (3.3) takes the form

$$E_1(f)_{L_\infty(E)} \le \frac{1}{4}\|f''\|_{L_\infty(E)}.$$

So, the constant $\frac{\pi^2}{8}$ can be lowered to 1:

$$E_1(f)_{L_\infty(E)} \le \Big(\frac{3}{4}+\frac{3}{8\gamma^2}\Big)\omega_2\Big(f, \frac{\gamma|E|}{n}\Big)_{L_\infty(E)}, \quad f\in\mathbb{R}L_\infty(E).$$

Remark 3.4. Since $\lim_{n\to\infty}\dfrac{(n-q)!\,n^q}{n!}=1$, for large n the constants in Corollary 3.1 may turn out to be greatly overstated in comparison with Theorem 3.1.

Remark 3.5. By applying the known inequalities

$$E_{n-1}(f)_{L_p(E)} \le \frac{(3\pi)^r}{r!\binom{n}{r}}\Big(\frac{|E|}{2}\Big)^r\|f^{(r)}\|_{L_p(E)}, \quad p\in[1,+\infty) \tag{3.5}$$

(see [21, p. 30, Corollary 3′]), a certain analog of Theorem 3.1 can be obtained for the spaces $L_p(E)$. Since the constants in (3.5) are probably far from the best, we do not formulate this analog explicitly.

In [20, Theorem 4], Jackson type inequalities were deduced from (2.10). The comparison of constants was discussed at the end of §2.

3.2. Jackson type inequalities for approximation by splines. We use the following notation: $S_{N,\mu}$ is the space of splines of order μ and defect 1 on a segment $E=[a,b]$ with the knots $a+\frac{k(b-a)}{N}$ ($k\in[\,0:N]$), $E_{N,\mu}$ is the best approximation by the space $S_{N,\mu}$,

$$\varphi_r(t)=\frac{4}{\pi^{r+1}}\sum_{\nu=0}^{\infty}\frac{\sin\big((2\nu+1)\pi t-r\pi/2\big)}{(2\nu+1)^{r+1}}, \quad r\in\mathbb{Z}_+,$$

is the Euler perfect spline, $\varphi_{r,E}(x)=\varphi_r\big(\frac{x}{|E|}\big)$. Clearly,

$$\|\varphi_r\|_{L_\infty[\,0,1]}=\frac{\mathcal{K}_r}{\pi^r}, \quad \|\varphi_{r,E}\|_{L_p(E)}=|E|^{1/p}\|\varphi_r\|_{L_p[\,0,1]}. \tag{3.6}$$

The following estimates are known for best approximations of functions defined on a segment by splines of minimal defect [9, Theorem 4.1.17, p. 150]. In [9], these estimates are formulated for the segment $[\,0,1]$, the case of an arbitrary segment in easily obtained by a linear change of the variable. If $r,\mu\in\mathbb{Z}_+$, $\mu\ge r-1$, then

$$E_{N,\mu}(f)_{L_\infty(E)} \le \frac{\mathcal{K}_r|E|^r}{(\pi N)^r}\|f^{(r)}\|_{L_\infty(E)}, \quad f\in W^{(r)}_\infty(E). \tag{3.7}$$

If $r,\mu\in\mathbb{Z}_+$, $\mu\ge r-1$, and $p\in[1,+\infty]$, then

$$E_{N,\mu}(f)_{L_1(E)} \le \frac{|E|^r}{N^r}\|\varphi_{r,E}\|_{L_{p'}(E)}\|f^{(r)}\|_{L_p(E)}, \quad f\in W^{(r)}_p(E). \tag{3.8}$$

We mention that, by (3.6), for $p=1$, the constant in (3.8) coincides with that in (3.7).

Theorem 3.2. *Suppose $s \in \mathbb{Z}_+$, $N, \mu, m \in \mathbb{N}$, $\mu \ge s + 2m$, and $\gamma > 0$.*

1. *For every $f \in W_\infty^{(s)}(E)$ we have*

$$E_{N,\mu}(f)_{L_\infty(E)} \le \left(\frac{|E|}{N}\right)^s \left(\frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \frac{\mathcal{K}_{s+2k}}{\pi^{s+2k}} \frac{\nu_m^k}{\gamma^{2k}} + \frac{\mathcal{K}_{s+2m}}{\pi^{s+2m}} \frac{\nu_m^m}{4^m \gamma^{2m}}\right) \lambda_{2m}\left(\frac{\gamma}{N}\right)_C \omega_{2m}\left(f^{(s)}, \frac{\gamma|E|}{N}\right)_{L_\infty(E)}.$$

2. *If $p \in [1, +\infty]$, then for every $f \in W_p^{(s)}(E)$ we have*

$$E_{N,\mu}(f)_{L_1(E)} \le \left(\frac{|E|}{N}\right)^s \left(\frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \|\varphi_{s+2k,E}\|_{L_{p'}(E)} \frac{\nu_m^k}{\gamma^{2k}} + \|\varphi_{s+2m,E}\|_{L_{p'}(E)} \frac{\nu_m^m}{4^m \gamma^{2m}}\right) \lambda_{2m}\left(\frac{\gamma}{N}\right)_p \omega_{2m}\left(f^{(s)}, \frac{\gamma|E|}{N}\right)_{L_p(E)}.$$

For $p = +\infty$ the quantity $\lambda_{2m}(\,\cdot\,)_\infty$ can be replaced by $\lambda_{2m}(\,\cdot\,)_C$.

Proof. By Lemmas 2.1 and 2.2 and Remark 2.1, it suffices to prove the claims stated for $f \in W_\infty^{(s)}(E)$ only for $f \in C^{(s)}(E)$. Applying Theorem 2.1 to the functionals $E_{N,\mu}(\,\cdot\,)_{C(E)}$ and $E_{N,\mu}(\,\cdot\,)_{L_1(E)}$, and using estimates (3.7) and (3.8), respectively, we get

$$E_{N,\mu}(f)_{C(E)} \le \left(\frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \frac{\mathcal{K}_{s+2k}|E|^{s+2k}}{(\pi N)^{s+2k}} \frac{\nu_m^k}{h^{2k}} + \frac{\mathcal{K}_{s+2m}|E|^{s+2m}}{(\pi N)^{s+2m}} \frac{\nu_m^m}{4^m h^{2m}}\right) \lambda_{2m,E}(h)_C\, \omega_{2m}\big(f^{(s)}, h\big)_{C(E)},$$

$$E_{N,\mu}(f)_{L_1(E)} \le \left(\frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \frac{|E|^{s+2k}}{N^{s+2k}} \|\varphi_{s+2k,E}\|_{L_{p'}(E)} \frac{\nu_m^k}{h^{2k}} + \frac{|E|^{s+2m}}{N^{s+2m}} \|\varphi_{s+2m,E}\|_{L_{p'}(E)} \frac{\nu_m^m}{4^m h^{2m}}\right) \lambda_{2m,E}(h)_p\, \omega_{2m}\big(f^{(s)}, h\big)_{L_p(E)}.$$

It remains to take $h = \frac{\gamma|E|}{N}$ and to perform simplifications. □

§4. Estimates for values related to interpolation, numerical differentiation and integration

In this section, the remainder terms of different formulas will be written in the integral form. First, we state a general assertion that provides such a possibility.

Lemma 4.1. *Let $r \in \mathbb{N}$; suppose that a vector space X of functions defined on a segment E and a linear functional Q defined on X satisfy the following conditions.*

1) *$W_1^{(r)}(E) \subset X$.*

2) *For almost every $t \in E$ the function $x \mapsto (x-t)_+^{r-1}$ belongs to X, and the function $t \mapsto Q\big((\,\cdot\, - t)^{r-1}\big)$ belongs to $L_\infty(E)$.*

3) *For each $g \in L_1(E)$, we have*

$$Q\left(\int_E g(t)(\,\cdot\, - t)_+^{r-1}\,dt\right) = \int_E g(t) Q\big((\,\cdot\, - t)_+^{r-1}\big)\,dt.$$

4) *If f is a polynomial of degree at most $r-1$, then $Q(f) = 0$.*

Then

$$Q(f) = \frac{1}{(r-1)!} \int_E f^{(r)}(t) Q\big((\,\cdot\, - t)_+^{r-1}\big)\,dt$$

for every $f \in W_1^{(r)}(E)$.

Proof. The proof follows immediately if we write the Taylor formula with the integral remainder term

$$f(x) = T_{r-1,a}(f,x) + \frac{1}{(r-1)!}\int_E f^{(r)}(t)(x-t)_+^{r-1}\,dt$$

and use the conditions of the lemma. □

The functionals to be dealt with in the sequel are linear combinations of the values of a function itself and its derivatives at a point, and also integrals. Obviously, such functionals satisfy the conditions of Lemma 4.1; this fact will not be mentioned specially.

4.1. Estimates of remainder terms for interpolation formulas and for their derivatives. Assume that $n \in \mathbb{Z}_+$, $x_0, \dots, x_n \in E$, L is the operator that takes a function f to its interpolational polynomial with the nodes x_k. Some of the x_k may coincide; in this case the nodes are treated as multiple. For brevity, we call L an interpolational operator of degree n. Denote by ρ the maximal multiplicity of the nodes; then $\rho \le n+1$. By Lemma 4.1, for every $r \in [\rho : n+1]$, $x \in E$, and $f \in W_1^{(r)}(E)$, we have

$$f(x) - L(f,x) = \int_E f^{(r)}(t)G_r(x,t)\,dt, \tag{4.1}$$

where

$$G_r(x,t) = \frac{(x-t)_+^{r-1}}{(r-1)!} - L\Big(\frac{(\,\cdot - t)_+^{r-1}}{(r-1)!}, x\Big).$$

The kernel is well defined for every t except, possibly, the nodes of interpolation. For uniformity of notation, for $r=0$ we agree to treat the function $t \mapsto \frac{(x-t)^{r-1}}{(r-1)!}$ as a δ-measure, and to treat the norm $\|G_0(x,\,\cdot\,)\|_{L_1(E)}$ as the variation of the signed measure $G_0(x,\,\cdot\,)$, i.e., the norm of the functional $f \mapsto f(x) - L(f,x)$ defined on $C(E)$. Clearly, if x does not coincide with any of the interpolation nodes, then $\|G_0(x,\,\cdot\,)\|_{L_1(E)} = 1 + \|L(\,\cdot\,,x)\|_{C^*(E)}$.

If $\alpha \in \mathbb{N}$, $\max\{\alpha+1,\rho\} \le r \le n+1$, and $f \in W_1^{(r)}(E)$, then, differentiating formula (4.1) α times, we get

$$f^{(\alpha)}(x) - L^{(\alpha)}(f,x) = \int_E f^{(r)}(t)G_r^{(\alpha,0)}(x,t)\,dt. \tag{4.2}$$

Here $G_r^{(\alpha,0)}$ means the α-fold derivative of the function G_r with respect to the first argument. For $r=\alpha$, we treat the function $G_r^{(\alpha,0)}(x,\,\cdot\,)$ in (4.2) as a signed measure, and $\|G_r^{(\alpha,0)}(x,\,\cdot\,)\|_{L_1(E)}$ as its variation.

Formulas (4.1) and (4.2) imply that if $\max\{\alpha+1,\rho\} \le r \le n+1$, $p \in [1,+\infty]$, and $f \in W_p^{(r)}(E)$, then

$$\big|f^{(\alpha)}(x) - L^{(\alpha)}(f,x)\big| \le \big\|G_r^{(\alpha,0)}(x,\,\cdot\,)\big\|_{L_{p'}(E)}\big\|f^{(r)}\big\|_{L_p(E)}. \tag{4.3}$$

Moreover, if $\max\{\alpha,\rho-1\} \le r \le n+1$, and $f \in C^{(r)}(E)$, then

$$\big|f^{(\alpha)}(x) - L^{(\alpha)}(f,x)\big| \le \big\|G_r^{(\alpha,0)}(x,\,\cdot\,)\big\|_{L_1(E)}\big\|f^{(r)}\big\|_{C(E)}. \tag{4.4}$$

Inequalities (4.3) and (4.4) are sharp.

Applying Theorem 2.1 to the functional Φ defined by $\Phi(f) = \big|f^{(\alpha)}(x) - L^{(\alpha)}(f,x)\big|$, we arrive at the folowing result.

Theorem 4.1. *Assume that $s, n, \alpha \in \mathbb{Z}_+$, $\rho, m \in \mathbb{N}$, L is an interpolational operator of degree n, $\max\{\alpha+1, \rho\} \le s \le n+1-2m$, $x \in E$, $h > 0$, and $p \in [1, +\infty]$. Then for every $f \in W_p^{(s)}(E)$ we have*

$$\begin{aligned}\left|f^{(\alpha)}(x) - L^{(\alpha)}(f,x)\right| \le \Bigg(\frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \left\|G_{s+2k}^{(\alpha,0)}(x,\,\cdot\,)\right\|_{L_{p'}(E)} \frac{\nu_m^k}{h^{2k}} \\ + \left\|G_{s+2m}^{(\alpha,0)}(x,\,\cdot\,)\right\|_{L_{p'}(E)} \frac{\nu_m^m}{4^m h^{2m}}\Bigg) \lambda_{2m,E}(h)_p\, \omega_{2m}\big(f^{(s)}, h\big)_{L_p(E)}.\end{aligned}$$

If $p = +\infty$ and $s = \max\{\alpha, \rho-1\}$, then the claim is valid for every $f \in C^{(s)}(E)$. For $p = +\infty$ the value $\lambda_{2m,E}(h)_\infty$ can be replaced by $\lambda_{2m,E}(h)_C$.

The possibility to replace $\lambda_{2m,E}(h)_\infty$ by $\lambda_{2m,E}(h)_C$ is provided by Lemma 2.1.

We list the estimates of moments that we know. All the nodes in these estimates are assumed to be simple. It is possible to generalize the majority of relations to the case of multiple nodes by passage to the limit, but we shall not perform this. We denote by $\Omega(x) = \prod_{i=0}^n (x - x_i)$ the nodal polynomial.

The following identity is commonly known:

$$\|G_{n+1}(x,\,\cdot\,)\|_{L_1(E)} = \frac{|\Omega(x)|}{(n+1)!}.$$

Turetskiĭ studied the norms of the functional $L(x,\cdot)$ defined on $C(E)$ and of the operator L as an operator from $C(E)$ to $C(E)$. We formulate his results for the segment $E = [-1,1]$. Suppose $l \in [0:n-1]$, $\alpha \in [0,1]$, $x_k = -1 + \frac{2k}{n}$ for $k \in [0:n]$, $x = -1 + \frac{2(l+\alpha)}{n}$. Then ([26, Problem 3.14])

$$\|L(\,\cdot\,,x)\| = \frac{\Gamma(l+1+\alpha)\Gamma(n+1-l-\alpha)\sin\alpha\pi}{\pi} \sum_{k=0}^n \frac{1}{k!(n-k)!|l-k+\alpha|}.$$

The following asymptotic relations (see [26, Problems 3.24 and 3.30]) are true as $n \to \infty$:

$$\|L\| = \frac{2^{n+1}}{en\ln n}(1+o(1)),$$
$$\left\|L\left(\,\cdot\,,\frac{1}{2}\right)\right\| = \frac{\ln m}{\pi}(1+o(1)), \quad n = 2m+1.$$

In [27, formula (3.4)], the intermediate moments were estimated:

$$\|G_r(x,\,\cdot\,)\|_{L_1(E)} \le \min_{0 \le i_{r-1} < \cdots < i_n \le n} \frac{1}{r!} \sum_{j=r-1}^n \prod_{\substack{r-1 \le l \le n,\\ l \ne j}} \frac{|\Omega(x)|}{|x_{i_j} - x_{i_l}|}.$$

The book [28, §3.1 and 3.2] contains asymptotics for the quantities $\|G_r(x,\,\cdot\,)\|_{L_2[-1,1]}$ in the cases of interpolation with respect to the Chebyshev nodes of the first and the second kind.

A. Yu. Shadrin [29] proved that

$$\max_{x \in E}\left\|G_{n+1}^{(\alpha,0)}(x,\,\cdot\,)\right\|_{L_1(E)} = \frac{\|\Omega^{(\alpha)}\|_{C(E)}}{(n+1)!}$$

for all $\alpha \in [0:n]$. For $\alpha = n$, this result was established earlier by Howell in [30]; another proof can be found in [31].

Along with the estimates (pointwise and in norm) for the deviation $f - L(f)$ and its derivatives, it is natural to estimate the quantity $\frac{f-L(f)}{\Omega}$ and its derivatives. Suppose that a point $x \in E$ does not coincide with any of the nodes x_k. Rewriting formula (4.1) divided by the nodal polynomial, for all $r \in [\rho : n+1]$ we have

$$\frac{f(x) - L(f,x)}{\Omega(x)} = \int_E f^{(r)}(t) H_r(x,t)\,dt, \tag{4.5}$$

where

$$H_r(x,t) = \frac{1}{\Omega(x)}\Big(\frac{(x-t)_+^{r-1}}{(r-1)!} - L\Big(\frac{(\,\cdot\,-t)_+^{r-1}}{(r-1)!}, x\Big)\Big).$$

The derivative of order β of the left-hand side of (4.5) vanishes on the polynomials of degree at most $\beta + n$. By Lemma 4.1, if $\beta \in \mathbb{N}$, $\max\{\beta+1, \rho\} \le r \le \beta + n + 1$, and $f \in W_1^{(r)}(E)$, then

$$\Big(\frac{f(x) - L(f,x)}{\Omega(x)}\Big)^{(\beta)} = \int_E f^{(r)}(t) H_r^{(\beta,0)}(x,t)\,dt. \tag{4.6}$$

As usual, for $r = \beta$ we treat the kernel as a signed measure.

Formulas (4.5) and (4.6) show that if $\max\{\beta+1,\rho\} \le r \le n+\beta+1$, $p \in [1,+\infty]$, and $f \in W_p^{(r)}(E)$, then

$$\Big|\Big(\frac{f(x) - L(f,x)}{\Omega(x)}\Big)^{(\beta)}\Big| \le \big\|H_r^{(\beta,0)}(x,\,\cdot\,)\big\|_{L_{p'}(E)} \big\|f^{(r)}\big\|_{L_p(E)}. \tag{4.7}$$

Moreover, if $\max\{\beta, \rho-1\} \le r \le n+\beta+1$, $f \in C^{(r)}(E)$, then

$$\Big|\Big(\frac{f(x) - L(f,x)}{\Omega(x)}\Big)^{(\beta)}\Big| \le \big\|H_r^{(\beta,0)}(x,\,\cdot\,)\big\|_{L_1(E)} \big\|f^{(r)}\big\|_{C(E)}. \tag{4.8}$$

Inequalities (4.7) and (4.8) are sharp.

Theorem 4.2. *Assume that $s, n, \beta \in \mathbb{Z}_+$, $\rho, m \in \mathbb{N}$, L is an interpolational operator of degree n, $\max\{\beta+1,\rho\} \le s \le n+\beta+1-2m$, $x \in E$, x does not coincide with any of interpolation nodes, $h > 0$, and $p \in [1,+\infty]$. Then for every $f \in W_p^{(s)}(E)$ we have*

$$\Big|\Big(\frac{f(x) - L(f,x)}{\Omega(x)}\Big)^{(\beta)}\Big| \le \Big(\frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \big\|H_{s+2k}^{(\beta,0)}(x,\,\cdot\,)\big\|_{L_{p'}(E)} \frac{\nu_m^k}{h^{2k}} + \big\|H_{s+2m}^{(\beta,0)}(x,\,\cdot\,)\big\|_{L_{p'}(E)} \frac{\nu_m^m}{4^m h^{2m}}\Big) \lambda_{2m,E}(h)_p\, \omega_{2m}\big(f^{(s)}, h\big)_{L_p(E)}.$$

If $p = +\infty$ and $s = \max\{\beta, \rho-1\}$, then the claim is valid for every $f \in C^{(s)}(E)$. For $p = +\infty$, we can replace $\lambda_{2m,E}(h)_\infty$ by $\lambda_{2m,E}(h)_C$.

Remark 4.1. If $x = x_k$, and ρ_k is the multiplicity of the node x_k, then the quantity $\big(\frac{f(x)-L(f,x)}{\Omega(x)}\big)^{(\beta)}$ is well defined for $f \in C^{(s)}(E)$ when $\max\{\beta+\rho_k, \rho-1\} \le s$; in this case the identities for the moments remain true. In particular, if $\beta + \rho \le s$, then the above quantity is defined for all points $x \in E$. Thus, the assertions of Theorem 4.2 remain valid for the nodes provided $\max\{\beta+\rho_k+1, \rho\} \le s$ and $s = \max\{\beta+\rho_k, \rho-1\}$.

An analog of Theorem 4.2 can be stated for estimates of the norms $\big\|\big(\frac{f-L(f)}{\Omega}\big)^{(\beta)}\big\|$ in $C(E)$ and $L_p(E)$. Since we have no explicit expressions for the moments of these functionals, we shall not formulate this assertion in its general form, but obtain estimates in the case of equidistant nodes.

Lemma 4.2. *Assume that $\beta \in \mathbb{Z}_+$, $n \in \mathbb{N}$, $l \in [0,n]$, $\eta = \frac{b-a}{n}$, L is an interpolational operator of degree n with respect to the equidistant nodes $x_k = a + k\eta$ $(k \in [0,n])$, $p \in [1,+\infty]$, and $\beta + p > 1$. Then for every $f \in W_p^{(\beta+l+1)}(E)$ we have*

$$\Big\|\Big(\frac{f-L(f)}{\Omega}\Big)^{(\beta)}\Big\|_{L_p(E)} \le \frac{\mathrm{B}\big(l+1,\beta+\frac{1}{p'}\big)}{n!\eta^{n-l}}\,\omega_{n-l}\big(f^{(\beta+l+1)},\eta\big)_{L_p(E)}.$$

Proof. We use the formula (see [32])

$$\frac{f(x)-L(f,x)}{\Omega(x)} = \frac{1}{n!\eta^n}\int_0^1 \Delta_{u\eta}^n\big(f', au + x(1-u)\big)\,du.$$

Differentiating β times and denoting the derivative of the left-hand side by A, we get

$$A(x) = \Big(\frac{f(x)-L(f,x)}{\Omega(x)}\Big)^{(\beta)} = \frac{1}{n!\eta^n}\int_0^1 \Delta_{u\eta}^n\big(f^{(\beta+1)}, au + x(1-u)\big)(1-u)^\beta\,du.$$

Hence,

$$\|A\|_{L_p(E)} \le \frac{1}{n!\eta^n}\int_0^1 \big\|\Delta_{u\eta}^n\big(f^{(\beta+1)}, au + \cdot(1-u)\big)\big\|_{L_p(E)}(1-u)^\beta\,du.$$

Since $au + b(1-u) = b - \eta n u$, it is clear that

$$\begin{aligned}\big\|\Delta_{u\eta}^n(g, au + \cdot(1-u))\big\|_{L_p(E)}^p &= \int_a^b \big|\Delta_{u\eta}^n(g, au + x(1-u))\big|^p\,dx \\ &= \frac{1}{1-u}\int_a^{b-\eta n u}\big|\Delta_{u\eta}^n(g,t)\big|^p\,dt \le \frac{1}{1-u}\,\omega_n^p(g)_{L_p(E)}.\end{aligned}$$

Consequently, for all $l \in [\,0,n]$,

$$\begin{aligned}\|A\|_{L_p(E)} &\le \frac{1}{n!\eta^n}\int_0^1 \omega_n\big(f^{(\beta+1)}, u\eta\big)_{L_p(E)}(1-u)^{\beta-\frac{1}{p}}\,du \\ &\le \frac{1}{n!\eta^n}\int_0^1 (u\eta)^l\omega_{n-l}\big(f^{(\beta+l+1)}, u\eta\big)_{L_p(E)}(1-u)^{\beta-\frac{1}{p}}\,du \\ &\le \frac{\mathrm{B}\big(l+1,\beta+\frac{1}{p'}\big)}{n!\eta^{n-l}}\,\omega_{n-l}\big(f^{(\beta+l+1)},\eta\big)_{L_p(E)}.\end{aligned}$$

□

Remark 4.2. If $p = 1$ and $\beta = 0$, the inequality of Lemma 4.2 does not make sense.

Remark 4.3. If $p = \infty$, then the constant before the modulus of continuity is equal to $\frac{l!\beta!}{(\beta+l+1)!n!\eta^{n-l}}$.

We deduce two consequences of Lemma 4.2. First, it yields a simple estimate of the moments of the functional $\|A\|_{L_p(E)}$.

Corollary 4.1. *Under the conditions of Lemma* 4.2, *we have*

$$\Big\|\Big(\frac{f-L(f)}{\Omega}\Big)^{(\beta)}\Big\|_{L_p(E)} \le \frac{\mathrm{B}\big(l+1,\beta+\frac{1}{p'}\big)2^{n-l}}{n!\eta^{n-l}}\,\big\|f^{(\beta+l+1)}\big\|_{L_p(E)}.$$

Second, the estimate in Lemma 4.2 is given in terms of the modulus of continuity with a fixed step η. This yields an estimate by the modulus of continuity with an arbitrary step h.

Corollary 4.2. *Under the conditions of Lemma* 4.2, *let $h > 0$. Then*

$$\Big\|\Big(\frac{f-L(f)}{\Omega}\Big)^{(\beta)}\Big\|_{L_p(E)} \le \frac{\mathrm{B}\big(l+1,\beta+\frac{1}{p'}\big)}{n!\eta^{n-l}}\Big\lceil\frac{\eta}{h}\Big\rceil^{n-l}\omega_{n-l}\big(f^{(\beta+l+1)},h\big)_{L_p(E)}.$$

Here $\lceil x\rceil$ is the smallest integer that is not smaller than x.

Remark 4.4. Unlike Theorem 2.1, Lemma 4.2 does not allow us to estimate $\|A\|_{L_p(E)}$ by the modulus of continuity of $f^{(\beta)}$.

4.2. Estimates for error terms of numerical differentiation formulas. Assume that $\rho\in\mathbb{N}$, $\{x_k\}_{k\in\mathbb{Z}}$ is a real sequence, and $\{A_{kl}\}_{k\in\mathbb{Z},\,l\in[\,0:\rho-1]}$ is a real or complex sequence with

$$\sum_{k=-\infty}^{\infty}\sum_{l=0}^{\rho-1}|A_{kl}|<+\infty.$$

Consider a summatory operator S of the form

$$S(f,x)=\sum_{k=-\infty}^{\infty}\sum_{l=0}^{\rho-1}A_{kl}f^{(l)}(x+x_k). \tag{4.9}$$

For a function $S(f)$ to be defined on a segment $F=[c,d]$, we demand the function f to be defined on a segment E such that $c+x_k, d+x_k\in E$ for all $k\in\mathbb{Z}$.

Also, let $\alpha,n\in\mathbb{N}$, $\rho\le\alpha\le n$. Assume that the deviation $f^{(\alpha)}-S(f)$ vanishes on the polynomials of degree at most n. Then the operator S can be utilized for approximation of the derivative of order α, and the error term admits estimation in terms of the derivative of order $n+1$.

If $\rho=1$, we use the one-fold numeration of coefficients:

$$S(f,x)=\sum_{k=-\infty}^{\infty}A_kf(x+x_k).$$

In this case the conditions of vanishing of the deviation $f^{(\alpha)}-S(f)$ on the polynomials of degree at most n are written as

$$\sum_{k=-\infty}^{\infty}A_kx_k^j=0,\quad j\in[\,0:n\,]\setminus\{\alpha\};\qquad \sum_{k=-\infty}^{\infty}A_kx_k^{\alpha}=\alpha!.$$

By Lemma 4.1, if $\alpha+1\le r\le n+1$ and $f\in W_1^{(r)}(E)$, then

$$f^{(\alpha)}(x)-S(f,x)=\int_E f^{(r)}(t)M_r(x,t)\,dt, \tag{4.10}$$

where

$$M_r(x,t)=\frac{(x-t)_+^{r-\alpha-1}}{(r-\alpha-1)!}-S\Big(\frac{(\,\cdot\,-t)_+^{r-1}}{(r-1)!},x\Big).$$

As usual, for $r=\alpha$ we treat the kernel as a signed measure.

The representation (4.10) shows that if $\alpha+1\le r\le n+1$, $p\in[1,+\infty]$, and $f\in W_p^{(r)}(E)$, then

$$\big|f^{(\alpha)}(x)-S(f,x)\big|\le\big\|M_r(x,\,\cdot\,)\big\|_{L_{p'}(E)}\big\|f^{(r)}\big\|_{L_p(E)}. \tag{4.11}$$

Moreover, if $\alpha\le r\le n+1$ and $f\in C^{(r)}(E)$, then

$$\big|f^{(\alpha)}(x)-S(f,x)\big|\le\big\|M_r(x,\,\cdot\,)\big\|_{L_1(E)}\big\|f^{(r)}\big\|_{C(E)}. \tag{4.12}$$

Inequalities (4.11) and (4.12) are sharp.

Applying Theorem 2.1 to the functional Φ defined by $\Phi(f)=\big|f^{(\alpha)}(x)-S(f,x)\big|$, we arrive at the following result.

Theorem 4.3. *Assume that $s,\rho,\alpha,n,m \in \mathbb{N}$, $\rho \le \alpha \le n$, S is an operator of the form* (4.9) *that vanishes on the polynomials of degree at most n, $F=[c,d]$, $c+x_k, d+x_k \in E$ for all $k\in\mathbb{Z}$, $\alpha+1\le s\le n+1-2m$, $x\in F$, $h>0$, and $p\in[1,+\infty]$. Then for every $f\in W_p^{(s)}(E)$ we have*

$$\bigl|f^{(\alpha)}(x)-S(f,x)\bigr| \le \biggl(\frac{1}{\binom{2m}{m}}\sum_{k=0}^{m-1}\bigl\|M_{s+2k}(x,\,\cdot\,)\bigr\|_{L_{p'}(E)}\frac{\nu_m^k}{h^{2k}} + \bigl\|M_{s+2m}(x,\,\cdot\,)\bigr\|_{L_{p'}(E)}\frac{\nu_m^m}{4^m h^{2m}}\biggr)\lambda_{2m,E}(h)_p\,\omega_{2m}\bigl(f^{(s)},h\bigr)_{L_p(E)}.$$

If $p=+\infty$ and $s=\alpha$, then the claim is valid for every $f\in C^{(s)}(E)$. For $p=+\infty$, we can replace $\lambda_{2m,E}(h)_\infty$ by $\lambda_{2m,E}(h)_C$.

The possibility to replace $\lambda_{2m,E}(h)_\infty$ by $\lambda_{2m,E}(h)_C$ is provided by Lemma 2.1.

A similar assertion can be stated for estimates of the norm $\|f^{(\alpha)}-S(f)\|_{L_p(F)}$. We give two examples.

Let $N\in\mathbb{N}$, $\theta>0$. Consider the following expansion of a derivative in terms of differences:

$$f^{(\alpha)} = \frac{1}{\theta^\alpha}\sum_{\nu=0}^{N-1}\beta_\nu^{(\alpha)}\delta_\theta^{\alpha+2\nu}(f)+R_N(f). \tag{4.13}$$

For the right-hand side to be defined on a segment $F=[c,d]$, we demand the function f to be defined on the segment

$$E=\bigl[c-\bigl(\tfrac{\alpha}{2}+N-1\bigr)\theta, d+\bigl(\tfrac{\alpha}{2}+N-1\bigr)\theta\bigr].$$

The coefficients $\beta_\nu^{(\alpha)}$ (which do not depend on θ) are defined by

$$\Bigl(\frac{\ln(z/2+\sqrt{1+z^2/4}\,)}{z/2}\Bigr)^r = \sum_{\nu=0}^{\infty}\beta_\nu^{(r)}z^{2\nu}, \quad |z|<2.$$

They satisfy the following relations (see, e.g., [33, pp. 186, 61, 194]):

$$\beta_0^{(\alpha)}=1,\quad \beta_1^{(\alpha)}=-\frac{\alpha}{24},\quad \beta_\nu^{(1)}=(-1)^\nu\frac{((2\nu-1)!!)^2}{2^{2\nu}(2\nu+1)!},$$
$$\beta_\nu^{(2)}=(-1)^\nu\frac{2\,(\nu!)^2}{(2\nu+2)!},\quad \beta_\nu^{(\alpha+1)}=\sum_{\mu=0}^{\nu}\beta_\mu^{(\alpha)}\beta_{\nu-\mu}^{(1)},\quad \beta_\nu^{(\alpha)}=\frac{D^\alpha O^{[\alpha+2\nu]}}{(\alpha+2\nu)!}.$$

In the last identity, $D^\alpha O^{[\alpha+2\nu]}$ means the derivative of order α of the central factorial polynomial

$$x^{[\alpha+2\nu]}=x\Bigl(x+\frac{\alpha+2\nu}{2}-1\Bigr)\dots\Bigl(x-\frac{\alpha+2\nu}{2}+1\Bigr)$$

at the zero point.

For even α, formula (4.13) is called the *Bessel formula,* for odd α it is called the *Stirling formula* of numerical differentiation. It is exact on the polynomials of degree at most $\alpha+2N-1$.

Denote

$$E_x=\bigl[x-\bigl(\tfrac{\alpha}{2}+N-1\bigr)\theta, x+\bigl(\tfrac{\alpha}{2}+N-1\bigr)\theta\bigr],\quad F_n=[c-n\theta,d+n\theta].$$

It is known (see [34] and also [35, Chapter III, §1, p. 196]) that for real-valued functions f the remainder term $R_N(f)$ can be written in the Lagrange form

$$R_N(f,x)=\beta_N^{(\alpha)}\theta^{2N}f^{(\alpha+2N)}(\xi),\quad \xi\in E_x.$$

Hence, we obtain sharp estimates of the last moment:

$$|R_n(f,x)| \le \bigl|\beta_N^{(\alpha)}\bigr|\,\theta^{2N}\bigl\|f^{(\alpha+2N)}\bigr\|_{C(E_x)}, \tag{4.14}$$

$$\|R_n(f)\|_{C(F)} \le \bigl|\beta_N^{(\alpha)}\bigr|\,\theta^{2N}\bigl\|f^{(\alpha+2N)}\bigr\|_{C(E)}. \tag{4.15}$$

On the other hand, the remainder term can be written in the integral form:

$$R_N(f,0) = \int_{E_0} f^{(\alpha+2N)}(t)M(t)\,dt \tag{4.16}$$

(we do not reflect notationally the dependence of the function M on the parameters). Applying (4.16) to the function $f(x+\,\cdot\,)$, we get

$$R_N(f,x) = \int_{E_0} f^{(\alpha+2N)}(x+t)M(t)\,dt. \tag{4.17}$$

The possibility to write the integral (4.17) in the Lagrange form means that the kernel does not change its sign and

$$\int_{E_0} |M| = \bigl|\beta_N^{(\alpha)}\bigr|\,\theta^{2N}.$$

Moving the norm under the integral sign, we obtain the inequalities

$$\|R_N(f)\|_{L_p(F)} \le \bigl|\beta_N^{(\alpha)}\bigr|\,\theta^{2N}\bigl\|f^{(\alpha+2N)}\bigr\|_{L_p(E)}. \tag{4.18}$$

Moreover, estimates (4.14), (4.15), and (4.18) are also true for complex-valued functions, and inequality (4.18) is sharp for $p=1$.

Now we prove a lemma that gives estimates of the intermediate moments.

Lemma 4.3. *Suppose $\alpha, N \in \mathbb{N}$, $n \in [\,0:N]$, $\theta > 0$, $p \in [1,+\infty]$, and*

$$F=[c,d], \quad E=\bigl[c-\bigl(\tfrac{\alpha}{2}+N-1\bigr)\theta, d+\bigl(\tfrac{\alpha-1}{2}+N-1\bigr)\theta\bigr].$$

Then for every $f \in W_p^{(\alpha+2n)}(E)$ we have

$$\|R_N(f)\|_{L_p(F)} \le \Bigl(\bigl|\beta_n^{(\alpha)}\bigr| + \sum_{\nu=n}^{N-1}\bigl|\beta_\nu^{(\alpha)}\bigr|2^{2\nu-2n}\Bigr)\theta^{2n}\bigl\|f^{(\alpha+2n)}\bigr\|_{L_p(E)}. \tag{4.19}$$

Proof. For $n=N$, inequality (4.19) has already been proved and coincides with (4.18). For $n<N$, we represent $R_N(f)$ in the form

$$R_N(f) = R_n(f) + \frac{1}{\theta^\alpha}\sum_{\nu=n}^{N-1}\beta_\nu^{(\alpha)}\delta_\theta^{\alpha+2\nu}(f).$$

Using the estimate for the last moment, which has already been established, and the inequaltites

$$\|\delta_\theta^{\alpha+2\nu}(f)\|_{L_p(F)} \le 2^{2\nu-2n}\|\delta_\theta^{\alpha+2n}(f)\|_{L_p(F_n)} \le 2^{2\nu-2n}\theta^{\alpha+2n}\|f^{(\alpha+2n)}\|_{L_p(E)},$$

we get the required inequality. □

Substituting the estimates for the moments, we arrive at the following theorem.

Theorem 4.4. *Suppose that $q \in \mathbb{Z}_+$, $\alpha, N, m \in \mathbb{N}$, $\alpha+2q \le 2N-2m$, $\theta, h > 0$, $p \in [1,+\infty]$, and*

$$F=[c,d], \quad E=\bigl[c-\bigl(\tfrac{\alpha}{2}+N-1\bigr)\theta, d+\bigl(\tfrac{\alpha-1}{2}+N-1\bigr)\theta\bigr].$$

Then for every $f \in W_p^{(\alpha+2q)}(E)$ *we have*

$$\begin{aligned}&\Big\|f^{(\alpha)}-\frac{1}{\theta^\alpha}\sum_{\nu=0}^{N-1}\beta_\nu^{(\alpha)}\delta_\theta^{\alpha+2\nu}(f)\Big\|_{L_p(F)}\\&\quad\le\Bigg(\frac{1}{\binom{2m}{m}}\sum_{k=0}^{m-1}\Bigg(\big|\beta_{q+k}^{(\alpha)}\big|+\sum_{\nu=q+k}^{N-1}\big|\beta_\nu^{(\alpha)}\big|2^{2\nu-2q-2k}\Bigg)\frac{\nu_m^k\theta^{2k}}{h^{2k}}\\&\qquad+\Bigg(\big|\beta_{q+m}^{(\alpha)}\big|+\sum_{\nu=q+m}^{N-1}\big|\beta_\nu^{(\alpha)}\big|2^{2\nu-2q-2m}\Bigg)\frac{\nu_m^m\theta^{2m}}{4^mh^{2m}}\Bigg)\theta^{2q}\lambda_{2m,E}(h)_p\,\omega_{2m}\big(f^{(\alpha+2q)},h\big)_{L_p(E)}.\end{aligned}$$

For $p=+\infty$*, we can replace* $\lambda_{2m,E}(h)_\infty$ *by* $\lambda_{2m,E}(h)_C$*.*

The possibility to replace $\lambda_{2m,E}(h)_\infty$ by $\lambda_{2m,E}(h)_C$ is provided by Lemmas 2.1 and 2.2 and Remark 2.1.

Consider the expansion of a derivative in terms of means of differences:

$$f^{(\alpha)}=\frac{1}{\theta^\alpha}\sum_{\nu=0}^{N-1}\varepsilon_\nu^{(\alpha)}\Box_\theta\delta_\theta^{\alpha+2\nu}(f)+Q_N(f). \tag{4.20}$$

Here $\Box_\theta$ is the arithmetic mean operator, i.e.,

$$\Box_\theta(g,x)=\frac{g\big(x+\frac{\theta}{2}\big)+g\big(x-\frac{\theta}{2}\big)}{2}.$$

For the right-hand side to be defined on a segment $F=[c,d]$, we demand a function f to be defined on the segment

$$E=\big[c-\big(\tfrac{\alpha-1}{2}+N\big)\theta,d+\big(\tfrac{\alpha-1}{2}+N\big)\theta\big].$$

The coefficients $\varepsilon_\nu^{(\alpha)}$ (which do not depend on θ) are defined by

$$\Big(\frac{\ln(z/2+\sqrt{1+z^2/4})}{z/2}\Big)^\alpha\frac{1}{\sqrt{1+z^2/4}}=\sum_{\nu=0}^{\infty}\varepsilon_\nu^{(\alpha)}z^{2\nu},\quad |z|<2.$$

They satisfy the following relations (see [33, pp. 187, 195]):

$$\varepsilon_0^{(\alpha)}=1,\quad\varepsilon_1^{(\alpha)}=-\frac18-\frac{\alpha}{24},\quad\varepsilon_\nu^{(1)}=(-1)^\nu\frac{(\nu!)^2}{(2\nu+1)!},$$
$$\varepsilon_\nu^{(\alpha+1)}=\sum_{\mu=0}^{\nu}\varepsilon_\mu^{(\alpha)}\beta_{\nu-\mu}^{(1)},\quad\varepsilon_\nu^{(\alpha)}=\frac{D^{\alpha+1}O^{[\alpha+1+2\nu]}}{(\alpha+2\nu)!\,(\alpha+1)}.$$

Inversely to (4.13), for even α formula (4.20) is called the *Stirling formula,* and for odd α it is called the *Bessel formula* of numerical differentiation. It is exact on the polynomials of degree at most $\alpha+2N-1$.

For the error term in (4.20), the same moment estimates as for (4.13) hold true, with the replacement of $\beta_\nu^{(\alpha)}$ by $\varepsilon_\nu^{(\alpha)}$ and of E_x by $\big[x-\big(\frac{\alpha-1}{2}+N\big)\theta,x+\big(\frac{\alpha-1}{2}+N\big)\theta\big]$.

Substituting these moment estimates, we arrive at the following theorem.

Theorem 4.5. *Suppose that* $q\in\mathbb{Z}_+$*,* $\alpha,N,m\in\mathbb{N}$*,* $\alpha+2q\le 2N-2m$*,* $\theta,h>0$*,* $p\in[1,+\infty]$*, and*

$$F=[c,d],\quad E=\big[c-\big(\tfrac{\alpha-1}{2}+N\big)\theta,d+\big(\tfrac{\alpha-1}{2}+N\big)\theta\big].$$

Then for every $f \in W_p^{(\alpha+2q)}(E)$ *we have*

$$
\begin{aligned}
&\left\| f^{(\alpha)} - \frac{1}{\theta^\alpha} \sum_{\nu=0}^{N-1} \varepsilon_\nu^{(\alpha)} \square_\theta \delta_\theta^{\alpha+2\nu}(f) \right\|_{L_p(F)} \\
&\quad \leq \left(\frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \left(\left|\varepsilon_{q+k}^{(\alpha)}\right| + \sum_{\nu=q+k}^{N-1} \left|\varepsilon_\nu^{(\alpha)}\right| 2^{2\nu-2q-2k} \right) \frac{\nu_m^k \theta^{2k}}{h^{2k}} \right. \\
&\qquad \left. + \left(\left|\varepsilon_{q+m}^{(\alpha)}\right| + \sum_{\nu=q+m}^{N-1} \left|\varepsilon_\nu^{(\alpha)}\right| 2^{2\nu-2q-2m} \right) \frac{\nu_m^m \theta^{2m}}{4^m h^{2m}} \right) \theta^{2q} \lambda_{2m,E}(h)_p \, \omega_{2m}\big(f^{(\alpha+2q)}, h\big)_{L_p(E)}.
\end{aligned}
$$

For $p = +\infty$, *we can replace* $\lambda_{2m,E}(h)_\infty$ *by* $\lambda_{2m,E}(h)_C$.

The possibility to replace $\lambda_{2m,E}(h)_\infty$ by $\lambda_{2m,E}(h)_C$ is provided by Lemmas 2.1 and 2.2 and Remark 2.1.

4.3. Estimates for error terms of quadrature formulas. Suppose $E = [a, b]$, $\rho \in \mathbb{Z}_+$, $f \in C^{(\rho)}(E)$, $N \in \mathbb{N}$, $p_{kl} \in \mathbb{R}$, $x_k \in E$ ($k \in [0 : N-1]$, $l \in [0 : \rho]$). Consider a quadrature formula

$$
\int_a^b f = \sum_{k=0}^{N-1} \sum_{l=0}^{\rho} p_{kl} f^{(l)}(x_k) + R(f), \tag{4.21}
$$

where $R(f)$ is its error term. We do not exclude that some of the coefficients p_{kl} are equal to zero.

By Lemma 4.1, if $d \in \mathbb{Z}_+$, the quadrature formula (4.21) is exact on the polynomials of degree at most d, and $\rho \leq d$, then for every $r \in [\rho + 1 : d + 1]$, $p \in [1, +\infty]$, and $f \in W_p^{(r)}(E)$ we have the sharp inequality

$$
|R(f)| \leq \|F_r\|_{L_{p'}(E)} \big\|f^{(r)}\big\|_{L_p(E)}. \tag{4.22}
$$

where

$$
F_r(t) = \frac{(b-t)^r}{r!} - \sum_{k=0}^{N-1} \sum_{l=0}^{\rho} p_{kl} \frac{(x_k - t)_+^{r-1-l}}{(r-1-l)!}.
$$

For $r = \rho$ and $p = +\infty$, a sharp estimate of the type (4.22) is also valid, provided for $n = -1$ we treat the functions $t \mapsto \frac{(x_k - t)^n}{n!}$ as δ-measures, and the norm $\|F_\rho\|_{L_1(E)}$ as the variation of the signed measure F_ρ. Namely, we have the sharp inequality

$$
|R(f)| \leq \|F_\rho\|_{L_1(E)} \big\|f^{(\rho)}\big\|_{C(E)}, \tag{4.23}
$$

where

$$
\|F_\rho\|_{L_1(E)} = \int_a^b \left| \frac{(b-t)^\rho}{\rho!} - \sum_{k=0}^{N-1} \sum_{l=0}^{\rho-1} p_{kl} \frac{(x_k - t)_+^{\rho-1-l}}{(\rho-1-l)!} \right| dt + \sum_{k=0}^{N-1} |p_{k\rho}|.
$$

We write separately the formulas in the case where $\rho = 0$, i.e., where the quadrature sum contains no derivatives. Suppose $f \in C(E)$, $N \in \mathbb{N}$, $p_k \in \mathbb{R}$, $x_k \in E$ ($k \in [0 : N-1]$). Consider a quadrature formula

$$
\int_a^b f = \sum_{k=0}^{N-1} p_k f(x_k) + R(f), \tag{4.24}
$$

$R(f)$ being its error term.

By Lemma 4.1, if the quadrature formula (4.24) is exact on the polynomials of degree at most $d \in \mathbb{Z}_+$, then for every $r \in [1 : d+1]$, $p \in [1, +\infty]$, and $f \in W_p^{(r)}(E)$ we have the sharp inequality

$$|R(f)| \le \|F_r\|_{L_{p'}(E)} \big\|f^{(r)}\big\|_{L_p(E)},$$

where

$$F_r(t) = \frac{(b-t)^r}{r!} - \sum_{k=0}^{N-1} p_k \frac{(x_k - t)_+^{r-1}}{(r-1)!}.$$

If we denote

$$\|F_0\|_{L_1(E)} = b - a + \sum_{k=0}^{N-1} |p_k|,$$

then, moreover, we have the following sharp inequality:

$$|R(f)| \le \|F_0\|_{L_1(E)} \|f\|_{C(E)}.$$

Applying Theorem 2.1, we immediately get the following result.

Theorem 4.6. *Assume that $s, \rho, d \in \mathbb{Z}_+$, $m \in \mathbb{N}$, formula (4.21) is exact on the polynomials of degree at most d, $\rho + 1 \le s \le d + 1 - 2m$, $h > 0$, and $p \in [1, +\infty]$. Then for every $f \in W_p^{(s)}(E)$ we have*

$$|R(f)| \le \bigg(\frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \|F_{s+2k}\|_{L_{p'}(E)} \frac{\nu_m^k}{h^{2k}} + \|F_{s+2m}\|_{L_{p'}(E)} \frac{\nu_m^m}{4^m h^{2m}} \bigg) \lambda_{2m,E}(h)_p \, \omega_{2m}\big(f^{(s)}, h\big)_{L_p(E)}.$$

If $p = +\infty$ and $s = \rho$, then the claim is valid for every $f \in C^{(s)}(E)$. For $p = +\infty$, we can replace $\lambda_{2m,E}(h)_\infty$ by $\lambda_{2m,E}(h)_C$.

The possibility to replace $\lambda_{2m,E}(h)_\infty$ by $\lambda_{2m,E}(h)_C$ is provided by Lemma 2.1.

In [36, §5, pp. 27–36] we find the history of calculaton and the tables of the values of $\|F_r\|_{L_1(E)}$ for the quadrature formulas of Newton–Cotes, Chebyshev, and Gauss with a small number of nodes.

As an example, we give an estimate for the error term of the Simpson formula for the segment $E = [0, 1]$ by the fourth modulus of continuity of the integrand. In the Simpson formula we have $N = 3$, $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_2 = 1$, $p_0 = p_2 = \frac{1}{6}$, and $p_1 = \frac{2}{3}$, and the constants $c_r = \|F_r\|_{L_1[0,1]}$ that we need are as follows: $c_0 = 2$, $c_2 = \frac{1}{81}$, $c_4 = \frac{1}{2880}$. For estimation, we must also take into account that, by (2.6) and Remark 2.5, $\lambda_4(h)_C \le 15\mathcal{W}_{4,C} \le 15$, and $\nu_2 = \frac{16}{3}$.

Corollary 4.3. *For every $f \in C[0, 1]$ and $h > 0$ we have*

$$\bigg| \int_0^1 f - \bigg(\frac{1}{6} f(0) + \frac{2}{3} f\Big(\frac{1}{2}\Big) + \frac{1}{6} f(1) \bigg) \bigg| \le \Big(5 + \frac{40}{243h^2} + \frac{1}{648h^4} \Big) \omega_4(f, h)_{C[0,1]}.$$

In [7, Part II], the quantities $\|F_r\|_{L_2(E)}$ were calculated by the methods of Sobolev and Zlamal for quadrature formulas of the type (4.24), i.e., those that do not involve derivatives. We give the formula for these norms obtained by the Sobolev method, for the segment $[0,1]$ (see [7, Chapter 2, Theorem 6, p. 263]):

$$\|F_r\|^2_{L_2[0,1]} = \frac{(-1)^r}{(2r+1)!} + \sum_{j=0}^{r-1} \frac{(-1)^j}{(r-j)!(r+j+1)!}$$
$$+ \sum_{k=0}^{N-1} p_k \left(\frac{2(-1)^{r+1}}{(2r)!} x_k^{2r} + \sum_{j=0}^{r-1} \frac{(-1)^{j+1} x_k^{r+j}}{(r-j)!(r+j)!} + \frac{(-1)^r}{(2r-1)!2} \sum_{l=0}^{N-1} p_l |x_k - x_l|^{2r-1} \right).$$

For specific quadratures, these expressions were simplified and tabulated in [7]. For the formulas of rectangles and trapezoids, and for the Simpson formula, they are also contained in [36, p. 29].

As for estimates for the error terms of quadrature formulas in terms of higher order moduli of continuity, we know the result by G. I. Natanson in [37] for the Newton–Cotes formula with $2m+1$ nodes. Let $\eta = \frac{b-a}{2m}$, $f \in W^{(2)}_\infty(E)$. Then

$$|R(f)| \le C_m (b-a)^3 \int_0^1 (1-u)\omega_{2m}(f'', u\eta)_{L_\infty(E)}\, du, \tag{4.25}$$

where the constant

$$C_m = -\frac{1}{(2m)!4m^3} \int_0^m \prod_{k=0}^m (t^2 - k^2)\, dt$$

is sharp. As was mentioned in [37], for $f \in C^{(2m+2)}(E)$, inequality (4.25) refines the sharp estimate

$$|R(f)| \le \frac{C_m (b-a)^{2m+3}}{(2m)^{2m}(2m+1)(2m+2)} \|f^{(2m+2)}\|_{C(E)}.$$

An inequality similar to (4.25) is true also for for the composite Newton–Cotes formula.

In [8], the error terms of the composite quadrature formulas were estimated via the method described at the end of §2.

References

[1] O. L. Vinogradov and V. V. Zhuk, *Estimates for functionals with a known sequence of moments in terms of the deviation of moments in terms of the deviation of Steklov-type means*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **383** (2010), 5–32; English transl., J. Math. Sci. (N.Y.) **178** (2011), no. 2, 115–131. MR2749339 (2012e:41016)

[2] ———, *The rate of decrease of constants in Jackson-type inequalities depending on the order of the modulus of continuity*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **383** (2010), 33–52; English transl., J. Math. Sci. (N.Y.) **178** (2011), no. 2, 132–143. MR2749340 (2012f:41015)

[3] ———, *Estimates for functionals with a known finite set of moments in terms of the deviations of operators constructed on the basis of Steklov averages of finite differences*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **392** (2011), 32–66; English transl., J. Math. Sci. (N.Y.) **184** (2012), no. 6, 679–698. MR2870218

[4] V. V. Zhuk and G. I. Natanson, *On the question of approximation in an integral metric of functions specified on an interval*, Vestnik Leningrad. Univ. Mat. Mekh. Astronom, **1979**, vyp. 3, 16–26. (Russian) MR555973 (81d:41004)

[5] Bl. Sendov, *A modified Steklov function*, C. R. Acad. Bulgare Sci. **36** (1983), no. 3, 315–317. (Russian) MR709014 (85c:41034)

[6] V. V. Zhuk and G. I. Natanson, *Guidelines on the course "Approximation of functions". Part. 4*, Izd. St.-Petersburg. Univ., St.Petersburg, 1991. (Russian)

[7] V. V. Zhuk and V. F. Kuzyutin, *Approximation of functions and numerical integration*, Izd. St.Peterburg.-Univ., St.Petersburg, 1995. (Russian) MR1664064 (99j:41001)

[8] B. Sendov and V. A. Popov, *Averaged moduli of smoothness*, Mir, Moscow, 1988. (Russian) MR954190 (90e:41001)

[9] N. P. Korneĭchuk, *Splines in approximation theory*, Nauka, Moscow, 1984. (Russian) MR758443 (87a:41001)
[10] S. Foucart, Yu. V. Kryakin, and A. Shadrin, *On the exact constant in the Jackson–Stechkin inequality for the uniform metric*, Constr. Approx. **29** (2009), 157–179. MR2481587 (2009k:41025)
[11] Yu. V. Kryakin, *Whitney's theorem for oscillating on* $\mathbb{R}$ *functions*, 2006, `arXiv:math/0612442v1`.
[12] H. Whitney, *On functions with bounded n-th differences*, J. Math. Pures Appl. (9) **36** (1957), no. 5, 67–95. MR0084611 (18:889f)
[13] O. D. Zhelnov, *Whitney constants are bounded by* 1 *for* $k = 5, 6, 7$, East J. Approx. **8** (2002), 1–14. MR1916324 (2003d:41013)
[14] ———, *Whinney's inequalites and their generalizations*, PhD thesis, Kiev, 2004. (Russian)
[15] J. Gilewicz, Yu. V. Kryakin, and I. A. Shevchuk, *Boundedness by* 3 *of the Whitney interpolation constant*, J. Approx. Theory **119** (2002), 271–290. MR1939285 (2003h:41026)
[16] Yu. V. Kryakin and L. Kovalenko, *Whitney constants in the classes* L_p, $1 \leq p < \infty$, Izv. Vyssh. Uchebn. Zaved. Mat. (1992), no. 1, 69–77. (Russian) MR1205041 (93m:46028)
[17] V. V. Zhuk, *Guidelines the course "Approximation of functions", Part* 1, Izd. Leningrad. Univ., Leningrad, 1988. (Russian)
[18] ———, *Functions of the* Lip 1 *class and S. N. Bernstein's polynomials*, Vestnik Leningrad. Univ. Mat. Mekh. Astronom. **1989**, vyp. 1, 25–30; English transl., Vestnik Leningrad. Univ. Math. **22** (1989), no. 1, 38–44. MR1021474 (91c:41022)
[19] O. L. Vinogradov, *Some exact inequalities for the second modulus of continuity of periodic functions and functions extended from a segment*, Zap. Nauchn. Sem. S.-Petersburg. Otdel. Mat. Inst. Steklov. (POMI) **232** (1996), 33–49; English transl., J. Math. Sci. (N.Y.) **92** (1998), no. 1, 3560–3572. MR1464422 (98m:26002)
[20] V. V. Zhuk and G. I. Natanson, *On the approximation of functions defined on a segment*, Metody Vychisl. **17** (1995), 105–121. (Russian) MR1791827
[21] V. V. Zhuk, *Guidelines for the course "Approximation theory of functions and its applications". Part* 1, Izd. St.-Petersburg Univ., St.Petersburg, 1992. (Russian)
[22] O. L. Vinogradov and V. V. Zhuk, *Estimates for functionals with a known finite set of moments in terms of moduli of continuity and the behavior of constants in Jackson-type inequalities*, Algebra i Analiz **24** (2012), no. 5, 1–43; English transl., St. Petersburg Math. J. **24** (2013), no. 5, 691–721. MR3087819
[23] H. F. Sinwel, *Uniform approximation of differentiable functions by algebraic polynomials*, J. Approx. Theory **32** (1981), no. 1, 1–8. MR629577 (82i:41011)
[24] I. K. Daugavet, *Introduction to the classial theory of approximation of functions*, Izd. St.-Petersburg. Univ., St.Petersburg, 2011. (Russian)
[25] ———, *Introduction to the theory of approximation of functions*, Izd. Leningrad. Univ., Leningrad, 1977. (Russian) MR0470560 (57:10310)
[26] A. Kh. Turetskiĭ, *Theory of interpolation in problem form*, Vyšeišaja Škola, Minsk, 1968. (Russian) MR0261224 (41:5840)
[27] A. Chakrabarti and Hamsapriye, *Derivation of the errors involved in interpolation and their application to numerical quadrature formulae*, J. Comput. Appl. Math. **92** (1998), no. 1, 59–68. MR1629271 (2000j:65019)
[28] A. Kh. Turetskiĭ, *Theory of interpolation in problem form. Part* 2, Vyšeišaja Škola, Minsk, 1977. (Russian) MR0463750 (57:3690)
[29] A. Shadrin, *Error bounds for Lagrange interpolation*, J. Approx. Theory **80** (1995), no. 1, 25–49. MR1308593 (95k:41007)
[30] G. W. Howell, *Derivative error bounds for Lagrange interpolation: an extension of Cauchy's bound for the error of Lagrange interpolation*, J. Approx. Theory **67** (1991), no. 2, 164–173. MR1133057 (93b:41003)
[31] M. S. Floater, *Error formulas for divided difference expansions and numerical differentiation*, J. Approx. Theory **122** (2003), no. 3, 1–9. MR1976120 (2004b:65027)
[32] V. V. Zhuk and G. I. Natanson, *Some remarks on periodic equidistant splines*, Vestnik Leningrad. Univ. Mat. Mekh. Astronom. **1985**, vyp. 2, 12–17. (Russian) MR801697 (87a:41016)
[33] J. F. Steffensen, *Interpolation*, Chelsea Publ. Co., New York, N.Y., 1950. MR0036799 (12:164d)
[34] M. L. Brodskiĭ, *Estimation of the remainder term in formulas of numerical differentiation*, Uspehi Mat. Nauk **13** (1958), no. 6, 73–77. (Russian) MR0135725 (24:B1770)
[35] L. V. Kantorivič and V. I. Krylov, *Approximate methods of higher analysis*, Fifth corrected edition, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow–Leningrad, 1962. (Russian) MR0154389 (27:4338)
[36] S. M. Nikol′skiĭ, *Quadrature formulas*, Nauka, Moscow, 1988. (Russian) MR949691 (89f:41033)

[37] G. I. Natanson, *A remark on the Newton–Cotes formula*, Metody Vychisl. **14** (1985), 58–59. (Russian) MR1000509 (90e:65028)

Department of Mathematics and Mechanics, St. Petersburg State University, Universitetskii pr. 28, Staryi Peterhof, St. Petersburg 198504, Russia
E-mail address: olvin@math.spbu.ru

Department of Mathematics and Mechanics, St. Petersburg State University, Universitetskii pr., 28, Staryi Peterhof, St. Petersburg 198504, Russia
E-mail address: zhuk@math.spbu.ru

Received 10/JAN/2013

Translated by O. L. VINOGRADOV