MOMENTS FOR THE MULTIDIMENSIONAL MÖNKENMEYER ALGORITHM

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Abstract. Moments asymptotics is studied for the partition corresponding to the multidimensional Mönkemeyer algorithm. A multidimensional generalization of a two-dimensional result by Moshchevitin and Vielhaber is proved.

The following procedure can be employed to obtain all rational numbers in [0, 1]. Put

\[ F_0 = \left\{ \frac{0}{1}, \frac{1}{1} \right\}. \]

The sets \( F_n \) with \( n = 1, 2, 3, \ldots \) are defined by induction. If

\[ F_n = \left\{ 0 = 0 < \xi_{0,n} < \xi_{1,n} < \cdots < \xi_{j,n} < \xi_{j+1,n} < \cdots < \xi_{N_n,n} = \frac{1}{1} = 1 \right\}, \quad N_n = 2^n, \]

then \( F_{n+1} = F_n \cup Q_{n+1} \), where the set \( Q_{n+1} \) consists of all mediants of two neighboring fractions \( \xi_{j,n} \) and \( \xi_{j+1,n} \) in \( F_n \):

\[ Q_n = \left\{ \frac{a+c}{b+d}, \quad a = \xi_{j,n}, \quad c = \xi_{j+1,n}, \quad j = 0, 1, \ldots, N_n - 1 \right\}. \]

The sets \( F_n \) are called the Stern–Brocot sequences. They give rise to partitions of the unit segment:

\[ [0, 1] = \bigcup_{j=0}^{2^n-1} [\xi_{j,n}, \xi_{j+1,n}]. \]

For these partitions, we can consider the moments

\[ \sum_{j=0}^{2^n-1} |\xi_{j+1,n} - \xi_{j,n}|^\beta, \quad \beta \in \mathbb{R}. \]

The partitions (1) and the corresponding moments (2) are related to investigations devoted to the dynamics generated by the Farey map

\[ T(x) = \begin{cases} \frac{x}{1-x}, & 0 \leq x \leq \frac{1}{2}, \\ \frac{1-x}{x}, & \frac{1}{2} \leq x \leq 1. \end{cases} \]

See [7] and also [5, 9]. Also in (2), an asymptotic formula for the moments was proved; this formula was refined later in [13].

In [4], Hurwitz used the partition (1) for interpreting the expansion of a real number in a continued fraction. In the same paper [4], he suggested an idea of using multidimensional analogs of (1). It should be noted that there are many ways to generalize (1) to the multidimensional case. The simplest multidimensional analogs (Farey nets) were

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considered by Mönkemeyer [6] and Grabiner [3]. For the multidimensional Farey nets, some partitions of multidimensional simplexes serve as analogs of (1); in the present paper we deal with precisely this type of partitions. The corresponding analogs of the moments (2) were treated in [8, 12]. The construction of multidimensional Farey nets is naturally related to algorithms for expanding real vectors in multidimensional continued fractions (the corresponding general theory and the analysis of various algorithms can be found in [1 10]).

The present paper is organized as follows. In §1 we give the necessary definitions of the Farey–Brocot partitions and describe their simplest properties. §2 is devoted to the geometry of the n-dimensional Mönkemeyer algorithm [6]. In §3 we state our main result, an asymptotic formula for the partition moments for β > 1, and compare it with earlier results. §§3-4–7 contain the proof of the main result.

§1. DESCRIPTION OF GENERALIZED MULTIDIMENSIONAL FAREY–BROCOT PARTITIONS

Let \( \mathcal{E} = \{g_1, \ldots, g_{k+1}\} \) be a basis of the integral lattice \( \mathbb{Z}^{k+1} \). We introduce the cone

\[
C(\mathcal{E}) = \left\{ x \in \mathbb{R}^{k+1} : x = \sum_{j=1}^{k+1} a_j g_j, \ a_1, \ldots, a_{k+1} \geq 0 \right\}.
\]

For this basis \( \mathcal{E} = \{g_1, \ldots, g_{k+1}\} \), we can consider a natural number \( D \geq 2 \) and collections of integral vectors \( \mathcal{E}^d = \{g_1^d, \ldots, g_{k+1}^d\}, 1 \leq d \leq D \), with the following properties.

(i) Each collection \( \mathcal{E}^d \) is a basis of \( \mathbb{Z}^{k+1} \).

(ii) The collection of cones \( \mathcal{C}(\mathcal{E}^d), 1 \leq d \leq D \), forms a regular partition of the cone \( \mathcal{C}(\mathcal{E}) \). (This means that \( \mathcal{C}(\mathcal{E}) = \bigcup_{1 \leq d \leq D} \mathcal{C}(\mathcal{E}^d) \) and the intersection of any two cones \( \mathcal{C}(\mathcal{E}^{d_1}), \mathcal{C}(\mathcal{E}^{d_2}) \) is a full \( l \)-dimensional face, for some \( 0 \leq l \leq k \), of either of the cones \( \mathcal{C}(\mathcal{E}^{d_1}) \) and \( \mathcal{C}(\mathcal{E}^{d_2}) \).

We shall work in the Euclidean space \( \mathbb{R}^{k+1} \) with coordinates \( (x, y_1, \ldots, y_k) \).

Consider the unit simplex \( \Delta_0 = \{z = (x, y_1, \ldots, y_k) : x = 1, \sum y_j \leq 1, y_j \in [0, 1]\} \).

The set \( \Delta^{0,1} \) of its vertices is a basis of the integral lattice \( \mathbb{Z}^{k+1} \).

By a generalized Farey–Brocot algorithm (GFBA) we shall mean a set of rules in accordance with which, given a collection of bases \( (\mathcal{E}^{\nu-1,d}, 1 \leq d \leq D_{\nu-1}) \) obtained at the preceding step, a new collection

\[
(\mathcal{E}^{\nu,d}, 1 \leq d \leq D_{\nu} = D_{\nu}(\mathcal{E}^{\nu-1,1}, \ldots, \mathcal{E}^{\nu-1,d}_{D_{\nu-1}}))
\]

of bases can be constructed in such a way that with every basis of the preceding step we can associate a subcollection of bases of the current step so that the requirements listed in (i), (ii) are fulfilled.

A GFBA \( \mathfrak{F} \) is said to be complete if any integral vector \( (x, y_1, \ldots, y_k) \in \mathbb{Z}^{k+1}, x \geq 1, 0 \leq y_j \leq x, \text{g.c.d.}(x, y_1, \ldots, y_k) = 1 \), belongs to some basis \( \mathcal{E}^{\nu,d} \) occurring in the algorithm in question (there are GFBA’s that are not complete).

Fix a GFBA \( \mathfrak{F} \). Consider the collection of all bases \( \mathcal{E} \) at the \( \nu \)-th step of the algorithm. The number of such bases may vary with \( \mathfrak{F} \), but the corresponding cones \( \mathcal{C}(\mathcal{E}) \) form a regular partition of the unit cone \( \{z = (x, y_1, \ldots, y_k) : x \geq 0, \sum y_j \leq 1, y_j \in [0, 1]\} \).

Restricting this partition to the set \( \{z = (x, y_1, \ldots, y_k) : x = 1, \sum y_j \leq 1, y_j \in [0, 1]\} \), we get the partition \( \text{Til}_\nu(\mathfrak{F}) \) of the unit simplex \( \Delta_0 \) into the simplexes \( \Delta = \mathcal{C}(\mathcal{E}) \cap \{z = (x, y_1, \ldots, y_k) : x = 1, \sum y_j \leq 1, y_j \in [0, 1]\} \). In this paper, the main object is the sum

\[
\sigma_{\nu,\beta}(\mathfrak{F}) = \sum_{\Delta \in \text{Til}_\nu(\mathfrak{F})} (\text{meas} \Delta)^\beta,
\]

called the moment of order \( \beta \).
We define a graph $T_\nu = T_\nu(\mathfrak{F})$ as follows. The set $V_\nu = V_\nu(\mathfrak{F})$ of its vertices coincides with the set of all vectors from all bases occurring at the $\nu$th step of the algorithm $(\mathfrak{F})$. Two vertices $u$ and $v$ are connected by an edge whenever the vectors $u$ and $v$ belong to one and the same basis $E$.

The definition of a finite GFBA was given in [8, 12]. Generally speaking, the formal definition of [8, 12] is not consistent. Below we present a corrected version of that definition.

Consider a vertex $v \in V_\nu$. Clearly, $v \in V_\mu$ for all $\mu \geq \nu$. Therefore, for each $\mu \geq \nu$ we can define the degree $d_\mu(v)$ of $v$ viewed as a vertex of the graph $T_\mu$. Obviously, $d_\mu(v) \leq d_{\mu+1}(v)$. The finite or infinite limit $d(v) = \lim_{\mu \to \infty} d_\mu(v)$ will be called the $d$-degree of the vertex $v$.

An algorithm $\mathfrak{F}$ is said to be $d$-finite if there exists a constant $M = M(\mathfrak{F})$ such that $d(v) < M$ for all $v \in \bigcup_{\nu \in \mathbb{N}} V_\nu(\mathfrak{F})$.

In fact, precisely this definition was used in [8, 12]. Also in [8, 12], the Dirichlet series played the role of the leading term in the asymptotics as $n \to \infty$ of the moments of order $\beta$ for some two-dimensional algorithms. In the present paper, we obtain an asymptotic formula for the moments of order $\beta$ corresponding the Mönkemeyer algorithm (to be defined in §2) in any dimension. For this, we shall need other Dirichlet series. To define them, we modify the definition of the $d$-degree.

Consider a vertex $v \in V_\nu$. Since, obviously, $v \in V_\mu$ for all $\mu \geq \nu$, it follows that for each $\mu \geq \nu$ we can define the quantity $s_\mu(v)$ equal to the number of the bases occurring at the step $\mu$ and containing $v$. Clearly, $s_\mu(v) \leq s_{\mu+1}(v)$. The finite or infinite limit $s(v) = \lim_{\mu \to \infty} s_\mu(v)$ is called the $s$-degree of the vertex $v$.

An algorithm $\mathfrak{F}$ is said to be $s$-finite if there exists a constant $M = M(\mathfrak{F})$ such that $s(v) < M$ for all $v \in \bigcup_{\nu \in \mathbb{N}} V_\nu(\mathfrak{F})$.

The Dirichlet series that we need are defined as

$$L(\mathfrak{F}, \beta) = \sum_{v \in V(\mathfrak{F})} \frac{s(v)}{(x(v))^\beta},$$

where $x(v)$ is the first coordinate of the integral vector $v = (x, y_1, \ldots, y_k)$. For $x$ fixed, the number of integral vectors $(x, y_1, \ldots, y_k)$ satisfying $0 \leq y_i \leq 1$ and \text{g.c.d.}(x, y_1, \ldots, y_k) = 1 is at most $(x + 1)^k$; therefore, the series $L(\mathfrak{F}, \beta)$ converges for $\beta > k + 1$.

Observe that for $k = 2$ the notions of the $d$-degree and $s$-degree coincide, so that, in this case, the result obtained in the present paper coincides with that in [8, 12].

The obvious inequalities $d(v) \leq ks(v)$, $s(v) \leq C_d(v)$ show that the $d$- and $s$-finiteness are equivalent. Therefore, in what follows we shall use the term “finite” not specifying what definition is meant. Also, we shall use the notation $\deg(v)$ in place of $s(v)$.

With each GFBA, we can associate the following multidimensional continued fraction algorithm (in the terminology of Brentjes [1]). Let $\Theta = (\theta_1, \ldots, \theta_k)$ be a real vector; then at each step of the algorithm we choose a basis $E^{\nu,d}$ such that $\Theta \in C(E^{\nu,d})$. This sequence of bases $E^{\nu,d} = \{g^{\nu^1}_1, \ldots, g^{\nu^k}_{d+1}\}$ yields an expansion of the vector $\Theta$ in some multidimensional continued fraction. In accordance with Brentjes [1], a multidimensional continued fraction algorithm is said to be weakly convergent at the point $\Theta$ if all the angles $\angle(g^{\nu^j}_j, \Theta)$, where the $g^{\nu^j}_j$ are the vectors defined above, tend to zero.

We list several claims proved in [3] that will be used in what follows.
Claim 1. Let a simplex $\Delta$ correspond to a basis $E = \{g_1, \ldots, g_{k+1}\}$, and let $g_j$ have the coordinates $g_j = (x_j, y_{j,1}, \ldots, y_{j,k})$. Then
\[
\text{meas } \Delta = \frac{1}{k! x_1 \ldots x_{k+1}}.
\]

Claim 2. Suppose that the multidimensional continued fraction algorithm corresponding to a GFBA $\mathfrak{F}$ weakly converges at every rational point. Then the GFBA $\mathfrak{F}$ is complete.

Claim 3. If a GFBA $\mathfrak{F}$ is finite, then it is complete, and, generally speaking, the converse is not true.

§2. Description of the Mönkemeyer algorithm $\mathfrak{M}$

The GFBA whose construction is described below was defined in $[6]$. The collection
\[
a^0 = (1,0,\ldots,0), \quad a^0_{i+1} = (1,0,\ldots,0, 1_i, 0,\ldots,0)
\]
of the vertices of the unit simplex $\Delta_0$ is a basis for $\mathbb{Z}^{k+1}$. Let $E^{0,1}$ denote this basis. Now suppose we are given a basis
\[
E^{v,j} = (a^{v,j}_1, a^{v,j}_2, a^{v,j}_3, \ldots, a^{v,j}_{k+1}).
\]
The rule for obtaining the bases of the next step of the algorithm looks like this:

( operation “0”)
\[
E^{v+1,2j-1} = (a^{v,j}_1, a^{v,j}_2, a^{v,j}_3, \ldots, a^{v,j}_{k+1}, a^{v,j}_1 + a^{v,j}_2),
\]

( operation “1”)
\[
E^{v+1,2j} = (a^{v,j}_1, a^{v,j}_2, a^{v,j}_3, \ldots, a^{v,j}_{k+1}, a^{v,j}_1 + a^{v,j}_2).
\]

Note that the order of elements in $E^{v,j}$ is important for constructing $E^{v+1,2j-1}, E^{v+1,2j}$. Obviously, this rule satisfies the GFBA restrictions: every new collection $E^{v+1,2j-1}, E^{v+1,2j}$ is a basis of the integral lattice, and the cones $C(E^{v+1,2j-1}), C(E^{v+1,2j})$ form a regular partition of the cone $C(E^{v,j})$.

The hyperplane $\{ x = 1 \} \subset \mathbb{R}^{k+1}$ can be identified with the $k$-dimensional space $\mathbb{R}^k$. The algorithm $\mathfrak{M}$ can be presented in terms of constructing some rational points in the unit simplex in $\mathbb{R}^k$; it looks like this. With each vector $(Q, X_1, \ldots, X_k)$ satisfying $\text{g.c.d.}(Q, X_1, \ldots, X_k) = 1$ and $\sum i X_i \leq Q$, we associate the point $(\frac{X_1}{Q}, \ldots, \frac{X_k}{Q})$ of the unit simplex. Suppose that a rational point $A$ corresponds to a vector $a = (Q, X_1, \ldots, X_k)$, with $\text{g.c.d.}(Q, X_1, \ldots, X_k) = 1$ and $\sum i X_i \leq Q$, and a rational point $B$ corresponds to a vector $b = (R, Y_1, \ldots, Y_k)$ with $\text{g.c.d.}(R, Y_1, \ldots, Y_k) = 1$ and $\sum i Y_i \leq R$.

Then, whenever the sum $a + b = (Q + R, X_1 + Y_1, \ldots, X_k + Y_k)$ satisfies the conditions $\text{g.c.d.}(Q + R, X_1 + Y_1, \ldots, X_k + Y_k) = 1$ and $\sum (X_i + Y_i) \leq Q + R$ (for instance, if these two vectors can be supplemented up to a basis of $\mathbb{Z}^{k+1}$, in the unit simplex we can find the rational point corresponding to the sum $a + b$. This rational point will be called the median of the points $A$ and $B$ and will be denoted by $A \oplus B := (\frac{X_1 + Y_1}{Q + R}, \ldots, \frac{X_k + Y_k}{Q + R})$.

The partitions of the cone $C(E^{0,1})$ resulting from the algorithm give rise to the partitions $\text{Til}_{\nu}$ of the initial simplex. The partitions $\text{Til}_{\nu}$ can be obtained as follows. The starting partition $\text{Til}_0$ consists of the initial unit simplex. Its vertices $a^0_1, \ldots, a^0_{k+1}$ can be viewed as rational points $a^0_1, \ldots, a^0_{k+1} \in \mathbb{R}^k$. Then, each simplex $\Delta$ having vertices $A_1, \ldots, A_{k+1}$ that belong to the partition $\text{Til}_{\nu}$ is split into two simplexes with the vertices $A_1, A_3, A_4, \ldots, A_{k+1}, A_1 \oplus A_2$ and $A_2, A_3, A_4, \ldots, A_{k+1}, A_1 \oplus A_2$ by the hyperplane of dimension $k - 1$ passing through the points $A_3, A_4, \ldots, A_{k+1}, A_1 \oplus A_2$. The resulting simplexes form the partition $\text{Til}_{\nu+1}$.

Note that the order of vertices of $\Delta$ is important in this construction.
We prove that the algorithm $\mathcal{M}$ is finite; moreover, we estimate the degree of a vertex from above.

**Claim 4.** The algorithm $\mathcal{M}$ is complete.

*Proof.* Indeed, in [6] it was shown that the multidimensional continued fraction algorithm that corresponds to $\mathcal{M}$ is weakly convergent; hence, $\mathcal{M}$ is complete by Claim 2. \qed

**Claim 5.** Suppose that a rational point $A_0$ lies on a $d$-dimensional face $\Gamma$ of the initial simplex and fails to lie on any of its faces of smaller dimension. Then, at each step of the algorithm starting with some $N_0$, the point $A_0$ is a vertex of at most

$$2^{k+d-1} \prod_{i=0}^{d-1} \max(1, 2^{k-(d-i)-1})$$

simplexes.

For the proof, we shall need several auxiliary statements.

**Claim 6.** Let $\Gamma$ be a $d$-dimensional face of a simplex $\Delta' \in \text{Til}_n$. Then, at each step of the algorithm, there are at most $\max(1, 2^{k-d-1})$ partition simplexes contained in $\Delta'$ for which $\Gamma$ is a face.

We preface the proof of Claim 6 with a series of remarks.

For convenience, we shall work in a $(k+1)$-dimensional space, i.e., we prove a similar statement for the partition cones. Observe that a $(k-1)$-dimensional hyperplane $\alpha$ splitting a $k$-dimensional simplex $\Delta$ gives rise to a $k$-dimensional hyperplane $\beta$ passing through 0 and $\alpha$, and splitting the cone $C(\Delta)$ corresponding to $\Delta$. Moreover, if $\alpha$ is a hyperplane passing through points $A_1, \ldots, A_k$, and the point $A_k$ corresponds to a vector $a_i$, then $\beta$ is formed by all points representable as linear combinations of $a_1, \ldots, a_k$ (in more detail, a $k$-dimensional hyperplane passing through points $X_1, \ldots, X_{k+1}$ is formed by all points of the form $\sum_{i=1}^{k+1} q_i X_i$, where $q_i \in \mathbb{R}, \sum_{i=1}^{k+1} q_i = 1$. But since, for $\beta$, the point $X_1$ is the zero point, we may write $q_1 = 1 - \sum_{i=2}^{k+1} q_i$, ensuring the condition $\sum_{i=1}^{k+1} q_i = 1$).

Let $\Gamma$ be the face occurring in Claim 6, and let the vertices of $\Gamma$ correspond to vectors $a_{i_1}, \ldots, a_{i_{d+1}}, i_1 < i_2 < \cdots < i_{d+1}$. Denote $I = \{i_1, \ldots, i_{d+1}\}$; we say that the points corresponding to the vectors of the form $\sum_{i \in I} q_i a_i$ are special relative to $\Gamma$.

**Claim 7.** Suppose vectors $a'_1, a'_2$ are obtained as a result of the algorithm. Suppose also that the vector $a'_1 + a'_2$ is special. Then the vectors $a'_1$ and $a'_2$ are special.

*Proof.* In the course of the algorithm, new vectors are obtained via addition only. Therefore, such a new vector has the form $\sum_{i=1}^{k+1} q_i a_i$, with $q_i \in \mathbb{Z}, q_i \geq 0$. Suppose $a'_1$ is not special, i.e., there exists $l \notin I$ such that $q_l > 0$ in the expansion of $a'_1$. Since $q_l \geq 0$ in the expansion of $a'_2$, we see that $q_l > 0$ for the expansion of $a'_1 + a'_2$, which contradicts our assumption.

The fact that $a'_2$ is special is proved similarly. \qed

**Claim 8.** Put $t = \max(n, n + i_1 - 2)$. In the course of constructing partitions at each step of the algorithm, starting with $t+1$, none of the hyperplanes involved can contain $\Gamma$.

*Proof.* We are interested in the cones for which the hyperplane corresponding to $\Gamma$ is a face. Consider an arbitrary cone in $\text{Til}_n$ containing at least one point of $\Gamma$. Let $(a_1, \ldots, a_{k+1})$ be the corresponding basis. At a step of its partition, the number of cones can grow only if the hyperplane in question contains $\Gamma$ (if this hyperplane does not intersect $\Gamma$, then only one of the new cones can contain points of $\Gamma$; and if this hyperplane intersects but does not contain $\Gamma$, then $\Gamma$ cannot be a face for the new simplexes). This
is equivalent to the fact that among the vectors $a_3, a_4, \ldots, a_{k+1}$, $a_1 + a_2$ there are $d+1$ special ones (we cannot have a larger number of them because otherwise the vectors would be linearly dependent). But in $Til_t$, in all bases containing all vertices of $\Gamma$, one of the first two vectors is special, so that the number of special ones among the last $k-1$ vectors is at most $d$. This number cannot become larger at the further steps of the algorithm: indeed, if the new vector $a_1 + a_2$ is special, then $a_1$ and $a_2$ are special by Claim 6, and hence, among the vectors $a_3, a_4, \ldots, a_{k+1}$ there are at most $d-1$ special ones, which implies that none of the hyperplanes in question can contain $\Gamma$.

**Proof of Claim 6.** By Claim 8, starting with the partition $Til_t$, no new simplexes will arise that satisfy the condition of Claim 6. Therefore, the number of simplexes in question cannot be greater than the total number of simplexes contained in $\Delta'$ and belonging to $Til_t$, i.e., is at most $\max(1, 2^{n-2})$. Since $\Gamma$ has exactly $d+1$ vertices, we have $i_1 = \min(i : a_i \in \Gamma) \leq k + 1 - d$. Thus, the number of simplexes under consideration does not exceed $\max(1, 2^{k-d-1})$.

**Proof of Claim 5.** Applying Claim 6 to $\Gamma$, we obtain at most $2^{k-d-1}$ simplexes for which $\Gamma$ is a face. For each of them, no splitting hyperplane of a further partition can contain $\Gamma$; hence, either such a hyperplane does not contain $A_0$ (and then the number of simplexes containing $A_0$ does not change), or it contains $A_0$, and hence, intersects (but does not contain) $\Gamma$. In the latter case, the intersection of this hyperplane with $\Gamma$ is a $(d-1)$-dimensional face of two simplexes, each satisfying the condition “$A_0$ lies on a $(d-1)$-dimensional face of that simplex and does not lie on any face of smaller dimension”; in total, there are at most $2 \max(1, 2^{k-d-1})$ such simplexes. Applying a similar argument to each of the resulting simplexes, we get at most $4 \times \max(1, 2^{k-d-1}) \times \max(1, 2^{k-(d-1)-1})$ simplexes satisfying the condition “$A_0$ lies on a $(d-2)$-dimensional face of such a simplex, but not on any face of smaller dimension”. Continuing, we get at most

$$d-1 \prod_{i=0}^{d-1} (2 \max(1, 2^{k-(d-i)-1}))$$

simplexes for which $A_0$ is a vertex, i.e., a face of zero dimension. Now, for each of these simplexes, we apply Claim 6 to $A_0$ viewed as a face of zero dimension, showing that, starting with some step, $A_0$ will serve as a vertex for at most

$$d-1 \prod_{i=0}^{d-1} (2 \max(1, 2^{k-(d-i)-1})) \times 2^{k-1}$$

simplexes, as required.

**Corollary.** If a point $A_0$ lies inside the initial simplex, then it will be a vertex for at most $2^{k^2+1}$ simplexes at each step of the algorithm.

§3. **Asymptotics of moments for partitions corresponding to the Mönkemeyer algorithm**

The following theorem on an asymptotic formula for $\sigma_{n, \beta}(\mathfrak{M})$ is the main result of this paper.

**Theorem 1.** For all $\beta > 1$, as $n \to \infty$ we have the asymptotic formula

$$\sigma_{n, \beta}(\mathfrak{M}) = \left(\frac{k^k}{k! n^k}\right)^\beta L(\mathfrak{M}, (k+1)\beta) \left(1 + O\left(\frac{(\log n)^{1-\frac{1}{(k+1)\beta}}}{n^{\frac{1-\beta}{(k+1)\beta}}}\right)\right).$$
In [8, 12], the following asymptotic formula was proved for \( k = 2 \):

\[
\sigma_{n,\beta}(\mathfrak{M}) = \frac{L(\mathfrak{M}, 3\beta)}{(n^2/2)\beta} \left( 1 + O\left( \frac{\log n}{n^{\frac{3\beta}{2}}} \right) \right),
\]

which is easily seen to coincide with the result of Theorem 1 in the case where \( k = 2 \).

A similar result for \( k = 1 \) was obtained in [7].

Also, the paper [13] by Dushistova should be mentioned, where some sharper estimators for \( k = 1 \) were obtained; however, in [13] there is a mistake in the exponent of the logarithm in the remainder term.

The proof of Theorem 1 occupies §§4–7.

§4. ON THE STRUCTURE OF THE MÖNKEMEYER ALGORITHM

Now we state and prove some simple statements about simplexes arising in the Mönkemeyer algorithm, and about their vertices.

Given a rational point of the form

\[
A = \left( \frac{a_1}{q}, \ldots, \frac{a_k}{q} \right), \quad \text{where} \quad (q, a_1, \ldots, a_k) = 1,
\]

we denote \( q = q(A) \).

**Lemma 1.**

(i) \( q(A_1) + q(A_2) \geq q(A_{k+1}) \geq q(A_k) \geq \cdots \geq q(A_1) \) for all simplexes.

(ii) Suppose that a simplex \( \Delta' \) is obtained from \( \Delta \) via the operation “1”. Then \( \meas \Delta' \leq \frac{1}{2} \meas \Delta \).

(iii) Suppose that a simplex \( \Delta' \) is obtained from \( \Delta \) via \( t \) operations “0”. If \( k \) divides \( t \), we have \( \Delta' = (A_1, A_2 + hA_1, A_3 + hA_1, \ldots, A_{k+1} + hA_1) \), where \( h = t/k \). If \( k \) does not divide \( t \), then, putting \( h = \left[ \frac{t}{k} \right] \) and \( t' = t - hk \), we get

\[
\Delta' = (A_1, A_2 + v + hA_1, A_3 + v + hA_1, \ldots, A_{k+1} + hA_1, A_2 + (h+1)A_1, \ldots, A_{k+1} + (h+1)A_1).
\]

Also, in both cases we have \( q(A_i') \geq (h + 1)q(A_1) \).

(iv) Suppose \( \Delta \in \text{Til}_n \). Then

\[
q(A_{i'}^n) \leq \left( \left\lfloor \frac{n}{k} \right\rfloor + 1 \right) q(A_1^n) \quad \text{for all} \quad i.
\]

**Proof.**

(i) Induction on the step \( v \) of the algorithm. The base of induction is obvious: \( q(A^n_0) = 1 \) for all \( i \). Suppose the claim is true for a step \( v \). Since \( q(A_{k+1}^{n+1}) = q(A_{n+1}^v) + q(A_2^n) \) and \( q(A_{k+1}^v) + q(A_{2}^v) \leq q(A_{k+1}^{n+1}) + q(A_2^{n+1}) \), we have \( q(A_{1}^{n+1}) + q(A_2^{n+1}) \geq q(A_{k+1}^{n+1}) \). At the same time, \( q(A_{1}^v) + q(A_2^v) \geq q(A_{k+1}^v) \) implies \( q(A_{k+1}^{n+1}) \geq q(A_{k+1}^v) \). The remaining inequalities are obvious.

Obviously, statement (ii) follows from (i) and Claim 1.

(iii) If \( k \mid t \), the statement is obvious. If \( k \nmid t \), then there is no loss of generality in assuming that \( h = \left[ \frac{t}{k} \right] = 0 \). But then the assertion about the form of \( \Delta' \) becomes obvious, implying, by (i), the inequality \( q(A_i') \geq (h + 1)q(A_1) \), whence \( q(A_i') \geq (\frac{t+1}{k} - 1)q(A_1) \).

(iv) We use induction on \( n \). The base of induction for \( n = 0, \ldots, k \) is obvious. Suppose the claim is true for each step not exceeding \( \leq n \). Suppose that the operation “0” was applied. Then

\[
q(A_i^{n+1}) \leq q(A_{k+1}^{n+1}) = q(A_i^n) + q(A_2^n) = q(A_1^n) + q(A_{k+1}^n) \\
\leq q(A_1^n) + \left( \left\lfloor \frac{n-k}{k} \right\rfloor + 1 \right) q(A_1^{n-k}) \\
\leq q(A_1^n) + \left\lfloor \frac{n}{k} \right\rfloor q(A_1^n) = \left( \left\lfloor \frac{n}{k} \right\rfloor + 1 \right) q(A_1^{n+1}).
\]
In the case where the operation “1” was applied, we consider the auxiliary vector
\((B_1, \ldots, B_{k+1})\) obtained from the vector \((A_1^n, \ldots, A_{k+1}^n)\) by the operation “0”. Then
\[
q(A_i^{n+1}) \leq q(A_{k+1}^{n+1}) = q(B_{k+1}) \leq \left(\left\lfloor \frac{n}{k} \right\rfloor + 1\right) q(B_1) < \left(\left\lfloor \frac{n}{k} \right\rfloor + 1\right) q(A_1^{n+1}),
\]
because the required inequality for the vector \((B_1, \ldots, B_{k+1})\) has already been proved.

\[\square\]

§5. AN AUXILIARY SUM

For the further proof, we need to estimate the quantity \(\sum_{n=0}^{\infty} \sigma_{n,\beta}(\mathcal{M})\).

**Lemma 2.** For all \(\beta > 1\), we have
\[
\sum_{n=0}^{\infty} \sigma_{n,\beta}(\mathcal{M}) \leq 2^{-\frac{3\beta}{2}} \left(\frac{k^2}{k!}\right)^{\beta} \beta \left(\frac{3}{2}\right)^{k} \zeta(k\beta) \zeta((k+1)\beta - k).
\]

**Proof.** Given a simplex \(\Delta\), we denote by \(\alpha(\Delta)\) its first vertex (i.e., the point corresponding to the first vector of the basis related to \(\Delta\)). Then
\[
\sum_{n=0}^{\infty} \sigma_{n,\beta}(\mathcal{M}) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{\Delta \in \tilde{T}_{n}} (\text{meas } \Delta)^{\beta} \sum_{\alpha(\Delta) \in V_{m} \setminus V_{m-1}} (\text{meas } \Delta)^{\beta}.
\]

Fix a point \(\alpha \in V_{m} \setminus V_{m-1}\). Since it has a finite degree \(\deg(\alpha)\), there exists a step \(h\) of the algorithm at which \(\alpha\) becomes a vertex for \(\deg(\alpha)\) simplexes.

Consider one of them. Using at most \(k \times t\) operations “1”, we can obtain a simplex \(\Delta'\) for which \(\alpha\) will be the first vertex. Let \(t\) be the number of operations required. Applying \(k - t\) operations “0” to \(\Delta'\), we get a simplex \(\Delta''\) in \(\tilde{T}_{n+k}\) such that \(\alpha(\Delta'') = \alpha\). Acting similarly with the remaining \(\deg(\alpha) - 1\) simplexes in \(\tilde{T}_{n}\), we obtain \(\deg(\alpha)\) (obviously, different) simplexes \(\Delta^{(1)}, \ldots, \Delta^{(\deg(\alpha))}\) in \(\tilde{T}_{n+k}\) for which \(\alpha\) is the first vertex. Each simplex obtained at the further steps of the algorithm and having \(\alpha\) as the first vertex is a result of (possibly, repeated) application of the operation “0” to one of the simplexes \(\Delta^{(i)}\).

Indeed, suppose this is not true for a simplex \(\Delta \in \tilde{T}_{h+k+s}\), \(s > 0\). Then, applying the operation “0” \(s\) times to each of \(\Delta^{(1)}, \ldots, \Delta^{(\deg(\alpha))}\), we get \(\deg(\alpha) + 1\) different simplexes in one partition for which \(\alpha\) is a vertex, which is impossible.

Now consider an arbitrary simplex \(\Delta\) such that \(\alpha(\Delta) = \alpha\). If it is obtained by the operation “1”, then it comes from a simplex for which \(\alpha\) is not the first vertex, and then \(\Delta\) is as desired. If it is obtained by the operation “0”, then it comes from a simplex for which \(\alpha\) is the first vertex; we denote this simplex by \(\Delta\) and repeat the argument. Acting this way with all simplexes \(\Delta^{(1)}, \ldots, \Delta^{(\deg(\alpha))}\) results in a set of \(\deg(\alpha)\) simplexes such that any simplex occurring in the algorithm and having \(\alpha\) as its first vertex either belongs to this set, or can be obtained from one of the simplexes in this set by (possibly, repeated) application of the operation “0”.

Using Claim 1, Lemma 1(i),(iii), and also the upper bound for \(\deg(\alpha)\), we see that
\[
\sum_{n=m}^{\infty} \sum_{\Delta \in \tilde{T}_{n}} (\text{meas } \Delta)^{\beta} \leq \frac{2^{\frac{3\beta}{2}}}{(k! (q(A))^{k+1})^{\beta}} \sum_{j=0}^{\infty} \left(\frac{k^2}{(j+1)^k}\right)^{\beta} \leq \frac{2^{\frac{3\beta}{2}}}{((q(A))^{k+1})^{\beta}} \left(\frac{k}{k!}\right)^{\beta} \zeta(k\beta).
\]
It follows that
\[ \sum_{n=0}^{\infty} \sigma_{n,\beta}(\mathcal{M}) \leq 2^k \frac{k^k}{k!} \beta(k\beta) \left( \sum_{m=0}^{\infty} \sum_{\alpha \in V_m \setminus V_{m-1}} \frac{1}{(q(\alpha))^{(k+1)\beta}} \right). \]

To complete the proof of the lemma, it suffices to use the inequality
\[ \sum_{m=0}^{\infty} \sum_{a \in V_m \setminus V_{m-1}} \frac{1}{(q(A))^{(k+1)\beta}} \leq C \sum_{q=1}^{\infty} \frac{1}{q^{(k+1)\beta-k}}, \]
where
\[ C = \max_q \left( \frac{(q + 1)^k - 2^k}{q^k} \right) \leq \left( \frac{3}{2} \right)^k. \]

Thus,
\[ \sum_{m=0}^{\infty} \sum_{a \in V_m \setminus V_{m-1}} \frac{1}{(q(A))^{(k+1)\beta}} \leq \left( \frac{3}{2} \right)^k \zeta((k+1)\beta - k) \]
and
\[ \sum_{n=0}^{\infty} \sigma_{n,\beta}(\mathcal{M}) \leq 2^k \frac{k^k}{k!} \beta(k\beta) \left( \frac{3}{2} \right)^k \zeta((k+1)\beta - k). \]

\section{Extracting the Leading Term of $\sigma_{n,\beta}(\mathcal{M})$}

Each simplex $\Delta \in \text{Til}_n$ can be obtained from the unit simplex $\Delta_0$ by consecutive application of the operations “0” and “1”. We introduce the code $c(\Delta) = c_1 \ldots c_n$, where the $c_k \in \{0, 1\}$ determines what operation was applied at the $k$th partition. Also we define $|c(\Delta)| = \sum_{k=1}^{n} c_k$; this is the number of the operations “1” applied for obtaining the simplex $\Delta$.

Choosing the parameters
\[ \gamma = \beta^2 k(k+1) + \beta - 1 \frac{1}{(k+1)\beta(\beta-1)\log 2}, \quad \omega = (\log n)^{1 - \frac{1}{(k+1)\beta}} n^{\frac{k\beta+1}{(k+1)\beta}}, \quad \omega_1 = n - \left[ \frac{n - \omega}{k} \right] k \]
we split the sum defining $\sigma_{n,\beta}(\mathcal{M})$ into three sums:
\[ \sigma_{n,\beta}(\mathcal{M}) = \sum_{\Delta \in \text{Til}_n} (\text{meas } \Delta)^{\beta} = \sum_{(1)} + \sum_{(2)} + \sum_{(3)}, \]
where $\sum_{(1)}$ is the sum over all $\Delta$ in Til$_n$ such that
\[ |c(\Delta)| \geq \gamma \log n, \]
$\sum_{(2)}$ is the sum over all $\Delta$ in Til$_n$ such that
\[ |c(\Delta)| < \gamma \log n, \quad \exists k > \omega : c_k = 1, \]
and $\sum_{(3)}$ is the sum over all $\Delta$ in Til$_n$ such that
\[ |c(\Delta)| < \gamma \log n, \quad c_{\omega+1} = \cdots = c_n = 0. \]

\textbf{Lemma 3.}
\[ \sum_{(1)} \leq \frac{1}{k! n^{\gamma(\beta-1)\log 2}}. \]
Proof. Obviously,

$$
\sum_{(1)} \leq \max_{\Delta \in \text{Til}_n} \left( \frac{\text{meas } \Delta}{|c|(|\Delta|)^{\beta-1}} \right) \sum_{\Delta \in \text{Til}_n} \text{meas } \Delta.
$$

Applying Lemma 1 (ii) and the identity $\sum_{\Delta \in \text{Til}_n} \text{meas } \Delta = \frac{1}{k!}$, we obtain

$$
\max_{\Delta \in \text{Til}_n} \left( \frac{\text{meas } \Delta}{|c|(|\Delta|)^{\beta-1}} \right) \sum_{\Delta \in \text{Til}_n} \text{meas } \Delta \leq \frac{1}{(2|c|(|\Delta|))^{\beta-1} k!} \leq \frac{1}{2^{2\gamma-1} \left( 1 - \gamma(\beta-1) \log n \right) k!} = \frac{1}{k! n^{(\beta-1) \log 2}}.
$$

Lemma 4. Consider a simplex $\Delta = (A_1, \ldots, A_{k+1})$, and suppose that a simplex $\Delta' = (A_1', \ldots, A_{k+1}')$ is obtained from $\delta$ with an operation $\delta_0$, and $\Delta'' = (A_1'', \ldots, A_{k+1}'')$ is obtained from $\Delta'$ with $k+1$ operations $\delta_1, \ldots, \delta_{k-1}, 1, 0$ ($\delta_i \in \{0, 1\}, i = 0, \ldots, k-1$). Then $A_1'$ is not a vertex of $\Delta$, but is a vertex of $\Delta'$. 

Proof. Note that $A_{k+1}' = A_1 + A_2$. Then for $\Delta^* = (A_1^*, \ldots, A_{k+1}^*)$ obtained from $\Delta'$ with the operations $\delta_1, \ldots, \delta_{k-1}$, we have $A_2^* = A_{k+1}' = A_1 + A_2$. Consequently, for the simplex $\Delta''$ obtained from $\Delta^*$ with the operations 1, 0, we have $A_1'' = A_2^* = A_{k+1}' = A_1 + A_2$, which proves the claim. 

Lemma 5.

$$
\sum_{(2)} \leq 2 \frac{k^2 + k}{k(k+1)\beta} \left( \frac{C^k + 2^{k+1}}{k^2} \right) \frac{3}{2} \left( \frac{k}{2} \right) \zeta((k+1)\beta - k)
$$

$$
\times 2 \frac{k^2 + k}{k!} \frac{\beta}{\zeta(k\beta)} \left( \frac{3}{2} \right) \zeta((k+1)\beta - k) \left( \frac{\gamma \log n}{\omega} \right)^{(k+1)\beta - 1}.
$$

Proof. The definition of $\sum_{(2)}$ shows that $|c|(|\Delta|) < \gamma \log n$. Since the last operation “1” is applied after the first $w$ operations, it follows that, for some $t > \tau = \frac{\omega}{\gamma \log n}$, there exists $d \leq w$ such that $c_{d+1} = \ldots = c_{d+t} = 0$, $c_{d+t+1} = 1$. (Indeed, a segment of length $w$ is “split” into parts by at most $\gamma \log n$ “1”s; hence, by the Dirichlet principle, there is a segment of at least $t$ “0”s; moreover, there is at least one “1”, by the definition of $\sum_{(2)}$.) Such a segment is as desired.)

For a simplex $\Delta$ with code $c(\Delta) = c_1 \ldots c_n$, where $c_k \in \{0, 1\}$, consider the corresponding sequence of simplexes $\Delta = \Delta_n \subset \cdots \subset \Delta_1 \subset \Delta_0$; the simplexes $\Delta_{d+1} \in \text{Til}_{d+1}$ and $\Delta_{d+t} \in \text{Til}_{d+t}$ are of particular interest. We denote $\Delta_{d+1} = (A_1, \ldots, A_{k+1})$ and $\Delta_{d+t} = (A_1', \ldots, A_{k+1}')$.

By Lemma 1 (iii), the corresponding common denominators satisfy the conditions

$$
q(A_i') = q(A_i), \quad q(A_i') \geq \frac{t+1}{k} q(A_i) \quad (2 \leq i \leq k+1).
$$

Now we consider the simplex $\Delta_{d+t+1} = (A_{k+1}', \ldots, A_{k+1}''')$ obtained from $\Delta_{d+t}$ with the operation “1”. Obviously, $q(A_i'') \geq \frac{t+1}{k} q(A_i)$.

There is a natural bijection between the restriction of the partition $\text{Til}_n$ to the simplex $\Delta_{d+t+1} \in \text{Til}_{d+t+1}$ and the partition $\text{Til}_{n-d-t-1}$ (namely, let $\Delta_{d+t+1}$ correspond vertexwise to the initial simplex; then each “new” vertex in the restriction of $\text{Til}_{d+t+1}$ to $\Delta_{d+t+1}$ will correspond naturally to a “new” vertex in $\text{Til}_{n-d-t-1}$). Moreover, for each simplex $\Delta \in \text{Til}_n$ with vertices $B_1, \ldots, B_{k+1}$ and the corresponding simplex $\Delta' \in \text{Til}_{n-d-t-1}$ with vertices $B_1', \ldots, B_{k+1}'$, we have

$$
q(B_j) \geq \left( \min_{1 \leq i \leq k+1} q(A_i'') \right) q(B_j') \geq \frac{t+1}{k} q(A_1) q(B_j'),
$$
which implies that
\[
\operatorname{meas} \Delta = \frac{1}{k! q(B_1) \ldots q(B_{k+1})} \leq \frac{1}{k! q(B'_1) \ldots q(B'_{k+1}) (\frac{t+1}{k} q(A_1))^{k+1}} = \frac{\operatorname{meas} \Delta'}{(\frac{t+1}{k} q(A_1))^{k+1}}.
\]

Fix some \( t \) and \( d \). Since the above inequality is valid for the simplexes in \( \text{Til}_n \) that correspond to these \( t \) and \( d \), the \( \beta \)-sum over all such simplexes is at most
\[
\left( \sum_{\Delta \in \text{Til}_{d+1}} \left( \frac{k}{t+1} \min_{1 \leq i \leq k+1} q(A_i) \right)^{(k+1)\beta} \right) \sum_{\Delta \in \text{Til}_{n-t-d-1}} (\operatorname{meas} \Delta)^\beta.
\]

Therefore,
\[
\sum_{(2)} \leq \sum_{t=\tau}^{n} \sum_{d \geq 0} \sum_{d \leq n-t-1} \left( \left( \sum_{\Delta \in \text{Til}_{d+1}} \left( \frac{k}{t+1} \min_{1 \leq i \leq k+1} q(A_i) \right)^{(k+1)\beta} \right) \sum_{\Delta \in \text{Til}_{n-t-d-1}} (\operatorname{meas} \Delta)^\beta \right).
\]

We need more information about the simplexes involved in summation in \( \sum_{(2)} \) and, consequently, in the sums on the right above.

In the case where \( d < k \), to the sum we should add at most
\[
2^{\frac{k^2 + k}{2}} \times (#V_0 + #V_1 + \cdots + #V_{k-1}) \leq 2^{\frac{k^2 + k}{2}} 2^{k+1}
\]
terms of the form \( q(A)^{-(k+1)\beta} \leq 1 \).

But if \( d \geq k \), then the conditions of Lemma 4 are fulfilled, so that \( A_i \in V_{d-k} \setminus V_{d-k-1} \), and we may bound \( \sum_{(2)} \) by distinguishing the values \( d = 0, \ldots, k-1 \):
\[
\sum_{(2)} \leq \sum_{t=\tau}^{n} \left( \sum_{h=n-t-d-1}^{n} 2^{\frac{k^2 + k}{2}} \cdot \#V_d \cdot \left( \frac{k}{t+1} \right)^{(k+1)\beta} \sum_{\Delta \in \text{Til}_h} (\operatorname{meas} \Delta)^\beta \right)
+ \sum_{d \geq k} \left( A_i \in V_{d-k} \setminus V_{d-k-1} \right) \frac{2^{\frac{k^2 + k}{2}}}{(q(A_1))^{(k+1)\beta}} \left( \frac{k}{t+1} \right)^{(k+1)\beta} \sum_{\Delta \in \text{Til}_h} (\operatorname{meas} \Delta)^\beta
\leq \frac{2^{\frac{k^2 + k}{2}} k^{(k+1)\beta}}{\tau(k+1)\beta-1} \left( 2^{k+1} + \sum_{d \geq k} \left( \sum_{A_i \in V_{d-k} \setminus V_{d-k-1}} \left( \frac{1}{q(A_1)} \right)^{(k+1)\beta} \right) \right) \sum_{n=0}^{\infty} \sum_{\Delta \in \text{Til}_n} (\operatorname{meas} \Delta)^\beta.
\]

We have
\[
\sum_{n=0}^{\infty} \left( \sum_{\Delta \in \text{Til}_n} (\operatorname{meas} \Delta)^\beta \right) = \sum_{n=0}^{\infty} \sigma_{n,\beta}(\mathcal{M}) \leq \frac{2^{\frac{k^2 + k}{2}} k^{\beta}}{k!} \left( \frac{3}{2} \right)^k \zeta(k\beta) \zeta((k+1)\beta - k),
\]
by Lemma 2, and
\[
2^{k+1} + \sum_{d \geq k} \left( \sum_{A_i \in V_{d-k} \setminus V_{d-k-1}} \left( \frac{1}{q(A_1)} \right)^{(k+1)\beta} \right) \leq 2^{k+1} + \left( \frac{3}{2} \right)^k \zeta((k+1)\beta - k).
\]

Thus, finally,
\[
\sum_{(2)} \leq \frac{2^{\frac{k^2 + k}{2}} k^{(k+1)\beta}}{\tau(k+1)\beta-1} \left( 2^{k+1} + \left( \frac{3}{2} \right)^k \zeta((k+1)\beta - k) \right) \frac{2^{\frac{k^2 + k}{2}} k^{\beta}}{k!} \left( \frac{3}{2} \right)^k \zeta(k\beta) \zeta((k+1)\beta - k).
\]
Lemma 6.

\[ \sum_{(3)} = \left( \frac{k^{k+1}}{(k!)^2 n^{k+1}} L(M,(k+1)\beta) + O\left( \frac{1}{n^{k+1} \omega(k+1)(\beta-1)} \right) \right) \left( 1 + O\left( \frac{\omega}{n} \right) \right). \]

**Proof.** We introduce

\[ \sum'_{(3)} = \sum_{\Delta \in \text{Til}_n : \text{(meas } \Delta)^{\beta}}. \]

Obviously, \( \sum'_{(3)} = \sum_{(3)} + O\left( \sum_{(1)} \right) \).

Each simplex \( \Delta = (A_1, \ldots, A_{k+1}) \in \text{Til}_n \) with a code \( c(\Delta) = c_1 \ldots c_\omega \) with \( c_{\omega+1} = \cdots = c_n = 0 \) is included in a simplex \( \Delta' = (A_1', \ldots, A'_{k+1}) \in \text{Til}_\omega \) with the code \( c(\Delta') = c_1 \ldots c_\omega \). By Lemma 1(iii), \( \Delta = (A_1', A_2' + hA_1', A_3' + hA_1', \ldots, A'_{k+1} + hA_1') \), where \( h = \frac{n-\omega_1}{k} \). Then

\[ (h+1)q(A'_i) \leq q(A_i) = q(A'_i) + hq(A'_i) \leq \frac{n+k+1}{k} q(A'_i) \quad (2 \leq i \leq k+1). \]

(The first inequality follows from Lemma 1(i), and the second from Lemma 1(iv.).)

Using Claim 1, we get

\[ \text{meas } \Delta = \frac{1}{k! q(A'_1)(q(A'_2) + hq(A'_1)) \ldots (q(A'_{k+1}) + hq(A'_1))}. \]

Therefore,

\[ \sum_{A_1 \in V_{\omega_1}} \frac{\deg A_1}{(k!)^2 n^{k+1} (q(A_1))^{k+1}} \leq \sum_{A_1 \in V_{\omega_1}} \frac{\deg A_1}{(k!)^2 n^{k+1} (q(A_1))^{k+1}} \leq \sum_{A_1 \in V_{\omega_1}} \frac{\deg A_1}{(k!)^2 n^{k+1} (q(A_1))^{k+1}} \leq \sum_{q=1}^{\infty} \frac{1}{q^{(k+1)(\beta-1)}}. \]

Recalling our choice of the parameters, we see that

\[ \sum_{A_1 \in V_{\omega_1}} \frac{\deg A_1}{(q(A_1))^{k+1}} \left( 1 + O\left( \frac{\omega}{n} \right) \right). \]

In order to estimate this expression in terms of Dirichlet series, we introduce the quantity \( G_l(q) \) equal to the number of the rational points \( A \) in the unit simplex \( \Delta_0 \) for which \( q(A) = q \) and \( \deg(A) = l \). Obviously,

\[ \sum_{q=1}^{\omega_1-1} \frac{1}{q^{(k+1)(\beta-1)}} \leq \sum_{A_1 \in V_{\omega_1}} \frac{\deg A_1}{(q(A_1))^{k+1}} \leq \sum_{q=1}^{\infty} \frac{1}{q^{(k+1)(\beta-1)}}. \]

Since

\[ \sum_{q=\omega_1}^{\infty} \frac{1}{q^{(k+1)\beta}} \leq \sum_{q=\omega_1}^{\infty} \frac{1}{q^{(k+1)\beta-1}} \leq \omega^{-(k+1)(\beta-1)}, \]

we obtain the inequality

\[ \sum_{A_1 \in V_{\omega_1}} \frac{\deg A_1}{(q(A_1))^{k+1}} = L(M,(k+1)\beta) + O(\omega^{-(k+1)(\beta-1)}). \]

As a result,

\[ \sum_{(3)} = \left( \frac{k^{k+1}}{(k!)^2 n^{k+1}} L(M,(k+1)\beta) + O\left( \frac{1}{n^{k+1} \omega(k+1)(\beta-1)} \right) \right) \left( 1 + O\left( \frac{\omega}{n} \right) \right). \]
§7. Completing the Proof of Theorem 1

Proof. Employing the results of Lemmas 3, 5, and 6, we get

\[
\sigma_{n,\beta}(\mathfrak{M}) = \left(\frac{k^k}{k!\; n^k}\right)^\beta L(\mathfrak{M}, (k+1)\beta) \\
+ O\left(n^{-\gamma(\beta-1)}\log^2 + \left(\frac{\gamma \log n}{\omega}\right)^{(k+1)\beta-1} + \frac{\omega}{n^{k+1} \beta} + \frac{1}{n^{k^2 \beta (k+1)(\beta-1)}}\right) \\
= \left(\frac{k^k}{k!\; n^k}\right)^\beta L(\mathfrak{M}, (k+1)\beta) \left(1 + O\left(\frac{(\log n)^{1-(k+1)\beta}}{n^{(k+1)\beta}}\right)\right). \quad \square
\]

References


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