

ON BLOWUP DYNAMICS IN THE KELLER–SEGEL MODEL OF CHEMOTAXIS

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*In memory of V. S. Buslaev,
a scientist and a friend*

ABSTRACT. The (reduced) Keller–Segel equations modeling chemotaxis of bio-organisms are investigated. A formal derivation and partial rigorous results of the blowup dynamics are presented for solutions of these equations describing the chemotactic aggregation of the organisms. The results are confirmed by numerical simulations, and the formula derived coincides with the formula of Herrero and Velázquez for specially constructed solutions.

§1. INTRODUCTION

In this paper we analyze the aggregation dynamics in the (reduced) Keller–Segel model of chemotaxis. Chemotaxis is the directed movement of organisms in response to the concentration gradient of an external chemical signal and is common in biology. The chemical signals can come from external sources or they can be secreted by the organisms themselves.

Chemotaxis is believed to underlie many social activities of micro-organisms, e.g., social motility, fruiting body development, quorum sensing and biofilm formation. A classical example is the dynamics and the aggregation of *Escherichia coli* colonies under starvation conditions [16]. Another example is the *Dictyostelium* amoeba, where single cell bacterivores, when challenged by adverse conditions, form multicellular structures of $\sim 10^5$ cells [14, 22]. Also, endothelial cells of humans react to vascular endothelial growth factor to form blood vessels through aggregation [21].

Consider organisms moving and interacting in a domain $\Omega \subseteq \mathbb{R}^d$, $d = 1, 2$ or 3 . Assuming that the organism population is large and the individuals are small relative to the domain Ω , Keller and Segel derived a system of reaction-diffusion equations governing the organism density $\rho : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and chemical concentration $c : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$. The equations are of the form

$$(1) \quad \begin{aligned} \partial_t \rho &= D_\rho \Delta \rho - \nabla \cdot (f(\rho) \nabla c), \\ \partial_t c &= D_c \Delta c + \alpha \rho - \beta c. \end{aligned}$$

Here D_ρ , D_c , α , β are positive functions of x and t , ρ and c , and $f(\rho)$ is a positive function modeling chemotaxis. Assuming a closed system, one is led to impose no-flux boundary conditions on ρ and c :

$$(2) \quad \partial_\nu \rho = 0 \text{ and } \partial_\nu c = 0 \text{ on } \partial\Omega,$$

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where $\partial_\nu g$ is the normal derivative of g . With these boundary conditions, the total number of organisms in Ω is conserved. We refer the reader to [14, 16, 49, 57] for more information about chemotaxis and the Keller–Segel model.

We presently focus on the case of positive chemotaxis, where the organisms secrete the chemical and move towards areas of higher chemical concentration. This leads to aggregation of organisms. Mathematically this is expressed as a blowup (or collapse) of solutions of (1). It was first suggested by Nanjundiah in [57] that the density ρ may become infinite and form a Dirac delta singularity. One refers to this process as (chemotactic) collapse. This is, arguably, the most interesting feature of the Keller–Segel equations. As argued below, the “collapsing” profile and contraction law have a universal (close to self-similar) form, independent of particulars of initial configurations and, to a certain degree, of the equations themselves, and can be associated with chemotactic aggregation. Though the equations are rather crude and unlikely to produce patterns one observes in nature or experiments, the collapse phenomenon could be useful in verifying assumptions about biological mechanisms¹.

Phenomena of blowup and collapse in nonlinear evolution equations are hard to simulate numerically, and the rigorous theory, or at least a careful analysis, is pertinent here. The recent years witnessed a tremendous progress in the development of such theories. We can now describe the shape of blowup profile and contraction law in Yang–Mills, σ -model, nonlinear Schrödinger, and heat equations [74, 72, 50, 51, 69, 8, 55, 56, 28]². Yet, after 40 years of intensive research and important progress, we still cannot give a rigorous description of collapse in the Keller–Segel equations modeling chemotaxis. (See [16, 85, 86, 9, 10, 11, 12, 13, 4] for some recent works, [15] for a nice discussion of the subject, and [59, 44, 45, 41, 71] for reviews.)

This is not to say that the Keller–Segel equations are harder than Yang–Mills, σ -model, or nonlinear Schrödinger equations, they are not, but neither are they less important.

There are three common approximations made in the literature for system (1). Firstly, one assumes that the coefficients in (1) are constant and satisfy

$$(3) \quad \epsilon := \frac{D_\rho}{D_c} \ll 1, \quad \tilde{\alpha} := \frac{\alpha}{D_c} = O(1) \quad \text{and} \quad \tilde{\beta} := \frac{\beta}{D_c} \ll 1.$$

The first of these conditions states that the chemical diffuses much faster than the organisms do. This is the case in practically all situations. As a result of this relation, one drops the $\partial_t c$ term in (1) (after rescaling time $t \rightarrow t/D_\rho$, this term becomes $\epsilon \partial_t c$). Secondly, one takes $f(\rho)$ to be a linear function $f(\rho) = K\rho$. Thirdly, the term βc in (1) is neglected compared with $\alpha\rho$, as one expects that it would not effect the blow-up process where $\rho \gg 1$ (it is also small due to the last relation in (3)). These approximations, after rescaling, lead to the system

$$(4) \quad \begin{aligned} \frac{\partial \rho}{\partial t} &= \Delta \rho - \nabla \cdot (\rho \nabla c), \\ 0 &= \Delta c + \rho, \end{aligned}$$

with ρ and c satisfying the no-flux Neumann boundary conditions.

Equations (4) in three dimensions also appear in the context of stellar collapse (see [39, 87, 23, 75]); similar equations — the Smoluchowski or nonlinear Fokker–Planck equations — model non-Newtonian complex fluids (see [30, 52, 25, 26]). This is the equation studied in this paper.

¹There are numerous refinements of the Keller–Segel equations, e.g., taking into account the finite size of organisms [1, 2, 54] preventing complete collapse, which model chemotaxis more precisely. We believe the techniques we outline and develop here can be applied to these models as well.

²Numerical simulations for these equations failed until the compression rate was derived analytically (see [8, 69, 81]).

We emphasize that in dropping the time derivative term of c , we have made the adiabatic approximation, in which the chemical is assumed to reach its steady state given by the second equation of (4) instantaneously.

In this paper, we consider the collapse of radially symmetric solutions to the reduced Keller–Segel system (4) on the plane \mathbb{R}^2 with a smooth, positive and integrable initial condition ρ_0 and with the boundary conditions $\rho, \nabla\rho, \nabla c \rightarrow 0$ as $|x| \rightarrow \infty$. To provide a right context for the discussion below, we mention that equation (4) has the following key properties.

- It is invariant under the scaling transformations

$$(5) \quad \rho(x, t) \rightarrow \frac{1}{\lambda^2} \rho\left(\frac{1}{\lambda}x, \frac{1}{\lambda^2}t\right) \quad \text{and} \quad c(x, t) \rightarrow c\left(\frac{1}{\lambda}x, \frac{1}{\lambda^2}t\right).$$

- It has the static solution,

$$(6) \quad R(x) := \frac{8}{(1 + |x|^2)^2}, \quad C(x) := -2 \ln(1 + |x|^2).$$

- The total “mass” is conserved: $\int_{\Omega} \rho(x, t) \, dx = \int_{\Omega} \rho(x, 0) \, dx$.

We also mention that (4) (as well as (1)) is a gradient flow, $\partial_t \rho = \nabla \cdot \rho \nabla \mathcal{E}'(\rho)$, or $\partial_t \rho = -\text{grad } \mathcal{E}(\rho)$, where $\mathcal{E}'(\rho)$ is the formal L^2 -gradient of \mathcal{E} and $\text{grad } \mathcal{E}(\rho)$ is the formal gradient of \mathcal{E} in the space with metric $\langle v, w \rangle_J := -\langle v, J^{-1}w \rangle_{L^2}$. Here $J := \nabla \cdot \rho \nabla \leq 0$, whose inverse is an unbounded operator, and $\mathcal{E}(\rho)$ is the “energy” functional given by

$$(7) \quad \mathcal{E}(\rho) = \int_{\mathbb{R}^2} \left(-\frac{1}{2} \rho \Delta^{-1} \rho + \rho \ln \rho - \rho \right) dx$$

(see §6 for more details). We remark that the first term of \mathcal{E} can be thought of as the internal energy of the system and the remaining terms are the entropy. The solution (6) is a minimizer of \mathcal{E} under the constraint that $\int \rho = \text{const}$. Note that $\int_{\mathbb{R}^2} R \, dx = 8\pi$, which is the source of 8π in (8). Under the scaling (5), the total mass changes as

$$\int \frac{1}{\lambda^2} \rho\left(\frac{1}{\lambda}x, 0\right) = \lambda^{(d-2)} \int \rho(x, 0).$$

Thus one does not expect collapse for $d = 1$, and that collapse is possible for $d \geq 2$ with critical collapse for $d = 2$ and supercritical collapse for $d > 2$. (Equation (4) in $d = 2$ is said to be L^1 -critical, etc.)

Take $\rho_0 \geq 0$. One has the following criteria for blowup of solutions of (4), see [58, 6]. If the dimension d equals 2 and the total mass satisfies

$$(8) \quad M := \int_{\mathbb{R}^2} \rho_0 \, dx > 8\pi,$$

or, if $d \geq 3$ and $\frac{\int_{\mathbb{R}^d} x^2 \rho_0 \, dx}{\int_{\mathbb{R}^d} \rho_0 \, dx}$ is sufficiently small (this means that ρ_0 is concentrated at $x = 0$), then the solution to (4) blows up in finite time.

There is a fair amount of work done on equations (1) and (4) and closely related equations. We give a very brief and incomplete review of it. Childress and Percuss [24] found that collapse for (1) with f linear does not occur when $d = 1$ and can occur when $d \geq 3$. For the two-dimensional case, they advanced arguments that collapse requires a threshold number of organisms. This threshold behaviour was confirmed by Jäger and Luckhaus in [48] (see also [58, 62, 61, 64, 64, 63, 59]).

Herrero and Velázquez proved that there exist radial solutions of (4) for $d = 2$ with the threshold mass $8|\Omega|$ collapsing to a Dirac delta singularity in finite time (see [36]). Also, unlike previous results, the authors gave an explicit asymptotic expression of the developing singularity. Using matched asymptotics and a topological argument, they

proved that for $T > 0$ there exists a radial solution to (4), which blows up at $r = 0$ and $t = T$ and is of the form

$$\rho(r, t) = \frac{1}{\lambda(t)^2} R_{\lambda(t)}(1 + o(1)) + \begin{cases} 0, & r < \lambda(t), \\ O\left(\frac{e^{-\sqrt{2}|\ln(T-t)|^{\frac{1}{2}}}}{r^2}\right), & r \geq \lambda(t), \end{cases}$$

as $t \rightarrow T$, where $R_\lambda(r) := R(r/\lambda)$, $R(r)$ is the stationary solution to (4) (see (6)) and

$$\lambda(t) = C(T - t)^{\frac{1}{2}} e^{-\frac{1}{\sqrt{2}}|\ln(T-t)|^{\frac{1}{2}}} |\ln(T - t)|^{\frac{1}{4}} |\ln(T-t)|^{-\frac{1}{2} - \frac{1}{4}} (1 + o(1)).$$

They also considered collapse of solutions to (1) with linear $f(\rho)$ (see [37] and [38]). Obtaining similar results, they suggested that Jäger and Luckhaus’ adiabatic assumption does not affect the collapse mechanism.

In the papers [53, 31] Lushnikov et al derived the log-log scaling as well as corrections beyond leading order log-log scaling.

As noted in [36], the asymptotics reproduced above are not of self-similar type; that is, they are not of the form $(T - t)\Phi(r/(T - t)^{\frac{1}{2}})$ for some function Φ . In fact, as shown in [40], self-similar blowup is not possible. Lastly, we mention that similar work has been done for the three-dimensional case, where the existence of collapsing shock waves has been shown. We refer the reader to [40, 39, 15] for these results.

The above results are valid for radially symmetric domains and initial conditions. It was shown by numerous authors that the blowup threshold mentioned above decreases for the nonspherically symmetric situation. Moreover, Dirac delta singularities may develop on the boundary of the domain. We refer the reader to [5, 6, 32, 42, 46, 60] for details. In [85], Velázquez considered small radial and nonradial perturbations of a collapsing solution and concluded, using formal matched asymptotics, that they are stable to these perturbations, leading only to small shifts in the blowup time and the blowup point. Existence of blowup or bounded solutions when $f(\rho)$ is nonlinear was recently studied in [47]. We also refer to [43] for a blowup result of a related Keller–Segel model. Lastly, we refer the reader to Horstmann [44, 45] for a more complete review of the literature including results on other models of chemotaxis and on the derivation of the Keller–Segel model as a continuous limit of biased random walks (see, e.g., [66, 68, 79]).

In spite of the considerable progress, the question of whether the mass collects in isolated points, forming Dirac delta distributions, remained unanswered. Moreover, these results give no information about the dynamics of blowup. These are the questions we address.

Now, we describe the results of the present paper. Given a radially symmetric initial condition $\rho_0(r) > 0$ sufficiently close to some R_{λ_0} , for some λ_0 , and satisfying $\int \rho_0 > \int R$, we show (formally, but with some rigorous supporting results) that the solution $\rho(x, t)$ to (4) is of the form

$$\rho(x, t) = \frac{1}{\lambda^2(t)} R_{\lambda(t)}(r)(1 + o(1))$$

with $\lambda(t) \rightarrow 0$ as $t \rightarrow T$ for some $0 < T < \infty$. Thus, all the mass $\int \rho dx$ collapses to the single point $x = 0$ in finite time, or equivalently, the density ρ forms a Dirac delta singularity with weight 8π in finite time. Furthermore, we show that the compression scale, λ , has the following explicit asymptotics

$$(9) \quad \lambda(t) = c(T - t)^{\frac{1}{2}} e^{-\frac{1}{\sqrt{2}}|\ln(T-t)|^{\frac{1}{2}}} |\ln(T - t)|^{\frac{1}{4}} (1 + o(1))$$

for some constant c . In Figure 1 we compare the blowup asymptotics (9) with direct numerical simulation of (4).

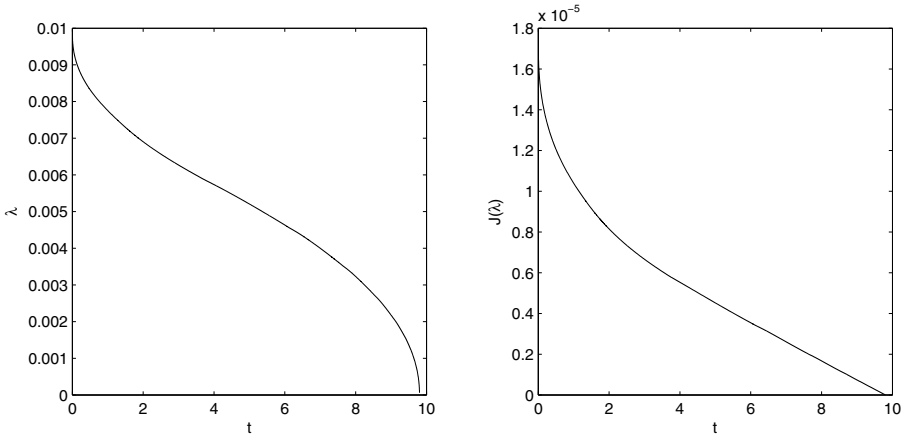


FIGURE 1. The left pane shows the scaling parameter $\lambda(t)$ obtained by numerically computing the solution to (4) with the initial condition $m_0 := 4y^2/(1 + \delta y + y^2)|_{y=r/\lambda_0}$ with $\lambda_0 = -\delta = .01$. The right pane plots the quantity $J(\lambda) := \frac{\varepsilon \sqrt{4 \ln \frac{\lambda_0}{\lambda}}}{\sqrt{\ln \frac{\lambda_0}{\lambda}}} \left(\frac{\lambda}{\lambda_0}\right)^2$ against time, which according to (9) should be linear as the blowup time is approached.

We also give an estimate of the error term, $\rho(x, t) - R_{\lambda(t)}(r)$, in the case where the nonlinear part in equation (16), given below, can be neglected. We believe that our results and our analysis can be made rigorous and can be extended to the full Keller–Segel system.

We outline the approach used in this paper. In the case of radially symmetric solutions, the system (4), which consists of coupled parabolic and elliptic PDEs, is equivalent to a single PDE. Indeed, the change of the unknown, by passing from the density $\rho(x, t)$ to the normalized mass

$$m(r, t) := \frac{1}{2\pi} \int_{|x| \leq r} \rho(x, t) dx$$

of organisms contained in a ball of radius r , discovered by [48, 15], maps two equations (4) into a single equation

$$(10) \quad \partial_t m = \Delta_r^{(0)} m + r^{-1} m \partial_r m,$$

on $(0, \infty)$ (with the initial condition $m_0(r) := \frac{1}{2\pi} \int_{|x| \leq r} \rho_0(x) dx$). Here $\Delta_r^{(n)}$ is the n -dimensional radial Laplacian, $\Delta_r^{(n)} := r^{-(n-1)} \partial_r r^{n-1} \partial_r = \partial_r^2 + \frac{n-1}{r} \partial_r$. Thus, (4) in the radially symmetric case is equivalent to (10) and therefore we concentrate on the latter equation.

The properties of equation (4) discussed above imply the following key properties of equation (10).

- It is invariant under the scaling transformations $m(r, t) \rightarrow m(\frac{1}{\lambda}r, \frac{1}{\lambda^2}t)$.
- It has the static solution (coming from the static solution $R(r) = \frac{8}{(1+r^2)^2}$ of (4)),

$$(11) \quad \chi(r) := \frac{4r^2}{1 + r^2}.$$

- The total “mass” is conserved: $2\pi \lim_{r \rightarrow \infty} m(r, t) = \int \rho(x, t) dx = \text{const.}$

Note that the stationary solution has total mass $2\pi \lim_{r \rightarrow \infty} \chi(r) = 8\pi$, which, recall, is the sharp threshold between global existence and singularity development in solutions to (4) (see (8)).

The properties above yield, as in the case of (4), the manifold of static solutions $\mathcal{M}_0 := \{\chi(r/\lambda) \mid \lambda > 0\}$ and suggest a likely scenario of collapse: sliding along \mathcal{M}_0 in the direction of $\lambda \rightarrow 0$. To analyze the collapse, we pass to the reference frame collapsing with the solution, by introducing the adaptive blowup variables,

$$m(r, t) = u(y, \tau), \quad \text{with } y = \frac{r}{\lambda} \quad \text{and } \tau = \int_0^t \frac{1}{\lambda^2(s)} ds,$$

where $\lambda : [0, T) \rightarrow [0, \infty)$, $T > 0$, is a positive differentiable function (*compression* or *dilatation* parameter) such that $\lambda(t) \rightarrow 0$ as $t \uparrow T$. The advantage of passing to blowup variables is that the function u is expected to have bounded derivatives and the blowup time is eliminated from consideration (it is mapped to ∞). Writing (10) in blowup variables, we find the equation for the rescaled mass function

$$(12) \quad \partial_\tau u = \Delta_y^{(0)} u + y^{-1} u_\lambda \partial_y u - ay \partial_y u,$$

where $a := -\dot{\lambda}\lambda$. Now, the blowup problem for (10) is mapped into the problem of asymptotic dynamics of solitons for equation (12), which was already studied in the pioneering works of [76, 77, 78, 17, 18, 82, 83, 84, 33, 34].

The boundary conditions on u are $\partial_y^\alpha u(y, \tau) \rightarrow 0$ as $y \rightarrow \infty$ for $\alpha = 1, 2$. As with the boundary conditions for (10), these imply that mass is conserved: $\lim_{y \rightarrow \infty} u(y, \tau) = \lim_{y \rightarrow \infty} u(y, 0)$. Equivalently, as a solution of (12), u depends on a , which determines λ , given $\lambda(0) = \lambda_0$, according to the formula

$$(13) \quad \lambda^2(t) = \lambda_0^2 - 2 \int_0^t a(s) ds.$$

Equation (12) has the static solution $(\chi(y), a = 0)$. It was shown in [29] that the linearized operator on this solution has one negative eigenvalue $-2a + \frac{a}{\ln \frac{1}{a}} + O(a \ln^{-2} \frac{1}{a})$ (corresponding to the scaling mode — for a fixed parabolic scaling it is related to possible variation of the blowup time)³ and one near zero eigenvalue, while the third eigenvalue, $2a + \frac{2a}{\ln \frac{1}{a}} + O(a \ln^{-2} \frac{1}{a})$, is positive, but vanishing as $a \rightarrow 0$. (It also isolates the correct perturbation (adiabatic) parameter $\frac{1}{\ln \frac{1}{a}}$.) Hence, we have to construct a one-parameter deformation of $\chi(y)$ (besides the parameter λ , or a). For technical reasons it is convenient to use a two-parameter family $\chi_{bc}(y)$,

$$(14) \quad \chi_{bc}(y) := \frac{4by^2}{c + y^2},$$

with $b > 1$ and both parameters b and c close to 1, with an extra relation between the parameters a , b , and c . The family $\chi_{bc}(y)$ gives approximate solutions to (12) (see (43)) and forms the deformation (or almost center-unstable) manifold $\mathcal{M} := \{\chi_{bc}(r/\lambda) \mid \lambda > 0, p\}$. We expect that the solution to (12) approaches this manifold as $\tau \rightarrow \infty$, and therefore we decompose the solution $u(y, \tau)$ to (12) as the leading term $\chi_{b(\tau)c(\tau)}(y)$ and the fluctuation $\phi(y, \tau)$,

$$(15) \quad u(y, \tau) = \chi_{b(\tau)c(\tau)}(y) + \phi(y, \tau),$$

³A similar analysis applies also in the subcritical case $M < 8\pi$, where the solution converges to a self-similar one as $\tau \rightarrow \infty$, which vanishes as $t \rightarrow \infty$. In this case the operator \mathcal{L}_{ab} has strictly positive spectrum.

and require that the fluctuation $\phi(y, \tau)$ be orthogonal to the tangent space of \mathcal{M} at $\chi_{b(\tau)c(\tau)}(y)$, $\langle \partial_p \chi_{p(\tau)}(\cdot), \phi(\cdot, \tau) \rangle = 0$, where $p := (b, c)$. Note that this family evolves on a different spatial scale than $\phi(y, \tau)$ in (15), as it can be rewritten as $\chi_{bc}(y) = \chi_{\frac{b}{c}, 1}\left(\frac{y}{\sqrt{c}}\right) = \chi_{bc}$.

In parametrizing solutions as above, we split the dynamics of (4) into a finite-dimensional part describing motion over the manifold \mathcal{M} , and an infinite-dimensional fluctuation (the error between the solution and the manifold approximation) which is assumed to stay small. Substituting the decomposition (15) into (12), we arrive at the equation

$$(16) \quad \partial_\tau \phi = -\mathcal{L}_{abc} \phi + \mathcal{F}_{abc} + \mathcal{N}(\phi),$$

where \mathcal{L} is a selfadjoint linear operator, \mathcal{F} is a forcing term, and \mathcal{N} is a quadratic nonlinearity. Due to the definition of \mathcal{M} , it turns out that its tangent space is very close to the subspace spanned by the negative and almost zero spectrum eigenfunctions (unstable modes) of the linearized operator \mathcal{L}_{abc} , and therefore ϕ is (approximately) orthogonal to the latter subspace.

The contraction law is obtained by using the orthogonality condition $\langle \partial_{bc} \chi_{bc}, \phi \rangle = 0$. The latter is equivalent to two conditions,

$$(17) \quad \partial_\tau \langle \partial_{bc} \chi_{b(\tau)c(\tau)}(\cdot), \phi(\cdot, \tau) \rangle = 0$$

and $\langle \partial_{bc} \chi_{b(\tau)c(\tau)}(\cdot), \phi(\cdot, \tau) \rangle|_{t=0} = 0$, which lead, to leading order, to the differential equation

$$(18) \quad a_\tau = -\frac{2a^2}{\ln\left(\frac{1}{a}\right)},$$

whose solutions, to leading order, are (9) (see §4).

We now describe the organization of this paper. In §2, solutions to (4) are parametrized by the parameters (a, b, c, ϕ) related to u by (12). In §3, we study the operator \mathcal{L}_{abc} in (16) and show that it has one negative eigenvalue and one simple eigenvalue near zero. We also give approximate eigenfunctions corresponding to these eigenvalues and prove that \mathcal{L}_{abc} is positive on the space orthogonal to these quasi-eigenfunctions. In §4, we state the relationship between the blowup parameters a, b , and c , whose proof is given in §6, and use it to obtain a dynamical equation for the blowup parameter $a = -\lambda \partial_t \lambda$ and derive the leading order behaviour of the scaling parameter λ in terms of the original time variable. In §5, we derive the lower bounds on the operator \mathcal{L}_{abc} . We use these bounds in §6 in order to control the fluctuation ϕ in the linearized equation, i.e., for (16), with the nonlinearity $\mathcal{N}(\phi)$ omitted.

In §6 we present the family of solutions to (10),

$$\chi^{(\mu)}(r) := \frac{r^{\mu-2} \mu + 4 - \mu}{r^{\mu-2} + 1}$$

with mass $2\pi\mu$, where $\mu \in (2, 4]$. These solutions describe partial collapse with $2\pi(4 - \mu)$ units of mass concentrated at the origin. In the remainder of our work we shall make no further use of these partially collapsed solutions. In §6 we provide a proof of the orthogonal splitting theorem of §2 and discuss the gradient structure of equations (1) and (4).

In the following discussion, we use the notation $f \lesssim g$ if there exists a positive constant C such that $f \leq Cg$. If the inequality $|f| \leq C|g|$ holds true, then we write $f = O(g)$. We also write $f \ll g$ or $f = o(g)$ if $f(a)/g(a) \rightarrow 0$ as $a \rightarrow 0$ and $f \sim g$ if the quotient converges to 1.

§2. PARAMETRIZATION OF SOLUTIONS

We parameterize solutions $u_\lambda(y, \tau)$ of equation (12) by the parameters a, b and c , and the fluctuation ϕ according to

$$(19) \quad u_\lambda(y, \tau) = \chi_{bc}(y) + \phi(y, \tau),$$

where $\lambda = \lambda(\tau)$, $b = b(\tau)$, and $a = a(\tau)$. Substituting (19) into equation (12) shows that the fluctuation ϕ satisfies

$$(20) \quad \partial_\tau \phi = -\mathcal{L}_{abc}\phi + \mathcal{F}_{abc} + \mathcal{N}(\phi),$$

where the linear operator, the forcing terms, and the nonlinear term are

$$(21) \quad \mathcal{L}_{abc} := -\Delta^{(4)} - \frac{8bc}{(c + y^2)^2} - \frac{4}{y} \left(b - 1 - \frac{bc}{c + y^2} \right) \partial_y + ay\partial_y,$$

$$(22) \quad \mathcal{F}_{abc} := -4b_\tau + 4 \frac{b_\tau c + bc_\tau - 2bca}{c + y^2} + 4bc \frac{8(b - 1) + 2ac - c_\tau}{(c + y^2)^2} - \frac{32bc^2(b - 1)}{(c + y^2)^3},$$

$$(23) \quad \mathcal{N}(\phi) := y^{-1} \phi \partial_y \phi.$$

Consider the weighted L^2 -space $L^2(\mathbb{R}_+, \gamma_{ab}(y)y^3 dy)$, with the weight

$$(24) \quad \gamma_{abc}^{-1/2}(y) = \frac{4y^2 e^{\frac{a}{4}y^2}}{(c + y^2)^b},$$

and the corresponding inner product

$$(25) \quad \langle f, g \rangle := \int_0^\infty f(y)g(y) \gamma_{abc}(y)y^3 dy.$$

The norm corresponding to this inner product will be denoted by $\| \cdot \|$. The significance of this space is that, as we show below, the operator \mathcal{L}_{abc} is selfadjoint on it.

Remark 2.1. Remark Another way to write \mathcal{L}_{abc} is as follows:

$$(26) \quad \mathcal{L}_{abc} = -\Delta^{(0)} - \frac{8bc}{(c + y^2)^2} - \frac{4by}{c + y^2} \partial_y + ay\partial_y;$$

we can treat it as a selfadjoint operator on $L^2(\mathbb{R}_+, \tilde{\gamma}_{ab}(y)y^3 dy)$, with the weight

$$\tilde{\gamma}_{abc}^{-1/2}(y) = \frac{e^{\frac{a}{4}y^2}}{(c + y^2)^b}$$

and the corresponding inner product

$$\langle f, g \rangle := \int_0^\infty f(y)g(y) \tilde{\gamma}_{abc}(y)y^{-1} dy.$$

The decomposition (19) is not unique and as a result we have a single equation (20) for four unknowns a, b, c and ϕ . Hence we supplement equation (20) with three additional equations. Two of these equations can be chosen as in [28] to make the parameters a, b , and c satisfy a chosen relation, say, $f(a, b, c) = 0$. In addition, we have the relations

$$(27) \quad \langle \phi, \zeta_{bci} \rangle = 0, \quad i = 0, 1,$$

in $L^2(\mathbb{R}^+, \gamma_{abc}(y)y^3 dy)$, for all times $\tau > 0$, where ζ_{bci} are the tangent vectors to the manifold $\mathcal{M} := \{ \chi_{bc}(r/\lambda) \mid \lambda > 0, b, c \}$:

$$(28) \quad \begin{aligned} \zeta_{bc0}(y) &:= \frac{1}{8bc} y \partial_y \chi_{bc}(y) = \frac{y^2}{(c + y^2)^2}, & \zeta_{bc1}(y) &:= \frac{1}{4} \partial_b \chi_{bc}(y) = \frac{y^2}{c + y^2}, \\ \zeta_{bc2}(y) &:= -\frac{1}{4b} \partial_c \chi_{bc}(y) = \frac{y^2}{(c + y^2)^2}. \end{aligned}$$

(The vectors $\zeta_{bc0}(y)$ and $\zeta_{bc2}(y)$ are seen to be multiples of each other, which confirms that one of the parameters is superfluous.)

We proceed here differently and choose

$$(29) \quad -\langle \phi, \partial_\tau \zeta_{bci} + (\partial_\tau \ln \gamma_{ab}) \zeta_{bci} \rangle = -\langle \mathcal{L}_{abc} \phi, \zeta_{bci} \rangle + \langle \mathcal{F}_{abc}, \zeta_{bci} \rangle + \langle \mathcal{N}, \zeta_{bci} \rangle, \quad i = 0, 1.$$

As we shall show, the latter vectors are approximate eigenvectors of the operator \mathcal{L}_{abc} , having the negative and almost zero eigenvalues. Moreover, we shall choose a relation between the parameters a, b, c . Equations (29) imply that

$$(30) \quad \partial_\tau \langle \phi, \zeta_{bci} \rangle = 0,$$

and therefore the inner products $\langle \phi, \zeta_{bci} \rangle, i = 0, 1$, are constant (one can think that this is a constraint on a, b , and c). The next proposition shows that a_0, b_0 , and c_0 can be taken so that $\langle \zeta_{bci}, \phi \rangle|_{\tau=0} = 0$, and hence, by (29), we have (27). To be able to formulate a precise statement, for a fixed $\delta > 0$ we introduce open neighbourhoods of \mathcal{M} ,

$$(31) \quad \mathcal{U}_\varepsilon = \left\{ f : \|e^{-\frac{\delta}{3}y^2}(f(y) - \chi_{bc}(y))\|_\infty < \varepsilon, \text{ for some } 1 \leq b \leq 2, \frac{1}{2} \leq c \leq 1 \right\}$$

and, for a fixed $\lambda > 0$,

$$(32) \quad \tilde{\mathcal{U}}_\varepsilon = \left\{ f(r) : f\left(\frac{r}{\lambda}\right) \in \mathcal{U}_\varepsilon \right\}.$$

Proposition 2.1. *Fix $\lambda_0 > 0$ and $0 < \delta \ll 1$. Then there exist $\varepsilon > 0$ and a unique C^1 -function $g : \mathcal{U}_\varepsilon \rightarrow (\delta, 1) \times (1/2, 1)$ such that for $m_0 \in \tilde{\mathcal{U}}_\varepsilon$ we have $\langle \zeta_{b_0 c_0 i}, \phi_0 \rangle = 0$, or in detail, for $i = 1, 2$,*

$$(33) \quad \langle m_0(\lambda_0 \cdot) - \chi_{b_0 c_0}, \zeta_{b_0 c_0 i} \rangle|_{(a_0, c_0)=g(m_0)} = 0.$$

For the proof of this proposition, see §6.

Equations (20) and (29) form a system of coupled, partial and ordinary differential equations for the parameters ϕ, a, b , and c . We assume that this system has a unique local solution, given initial conditions ϕ_0, a_0, b_0 and c_0 the values of which are related to the initial value of m (recall that $u_\lambda(y) = m(\lambda y)$).

§3. GENERAL PROPERTIES OF THE OPERATOR \mathcal{L}_{abc}

Before proceeding, we discuss the general properties of the operator \mathcal{L}_{abc} mentioned above and used below.

Proposition 3.1. *The operator \mathcal{L}_{abc} defined on $L^2([0, \infty), \gamma_{abc}(y)y^3 dy)$ (with the inner product (25)), is selfadjoint and has purely discrete spectrum. Moreover, we have the lower bound*

$$(34) \quad \langle \phi, \mathcal{L}_{abc} \phi \rangle \gtrsim - \left[2a + \frac{(b-1)+a}{\sqrt{\ln \frac{1}{a}}} \right] \|\phi\|^2.$$

Proof. One can check the selfadjointness of \mathcal{L}_{abc} directly or use the unitary map $\xi(y) \rightarrow \gamma_{abc}^{1/2}(y)\xi(y)$ from $L^2([0, \infty), \gamma_{abc}(y)y^3 dy)$ to $L^2([0, \infty), y^3 dy)$ to map this operator into the operator

$$(35) \quad L_{abc} := \gamma_{abc}^{1/2} \mathcal{L}_{abc} \gamma_{abc}^{-1/2}$$

acting on $L^2([0, \infty), y^3 dy)$ with the inner product $(\xi, \eta) := \int \xi \eta y^3 dy$. The latter operator can be explicitly computed:

$$(36) \quad L_{abc} := -\Delta^{(4)} + \frac{1}{4}a^2 y^2 - 2ab + \frac{2b(2(b-1)+ac)}{c+y^2} - \frac{4bc(b+1)}{(c+y^2)^2}.$$

It is of Schrödinger type with the real continuous potential tending to ∞ as $y \rightarrow \infty$ as $O(y^2)$. Hence, using standard arguments (see, e.g., [35]), one can show that L_{abc} is selfadjoint and that its spectrum, and hence the spectrum of \mathcal{L}_{abc} , is purely discrete.

Now, we investigate the bottom of the spectrum of the operator \mathcal{L}_{abc} . We begin with the operator $\mathcal{L}_{0bc} := \mathcal{L}_{abc}|_{a=0}$. In what follows we use the convenient shorthand notation $f_\lambda(r) = f(r/\lambda)$, which we apply only for the subscript λ .

Lemma 3.1. *The function $\zeta_{bc}(y) := \frac{y^2}{(c+y^2)^{2b}}$ is a zero mode of the operator $\mathcal{L}_{0bc} := \mathcal{L}_{abc}|_{a=0}$:*

$$(37) \quad \mathcal{L}_{0bc}\zeta_{bc} = 0.$$

The spectrum of \mathcal{L}_{0bc} starts with 0, which is a simple eigenvalue.

Proof. Since χ_λ is a static solution of (10), differentiating the equation

$$\Delta_r^{(0)}\chi_\lambda + r^{-1}\chi_\lambda \partial_r \chi_\lambda = 0$$

with respect to λ at $\lambda = 1$ shows that

$$(38) \quad \zeta := r \partial_r \chi = \frac{8r^2}{(1+r^2)^2}$$

is a zero mode of the linearization of (10) around χ :

$$(39) \quad \mathcal{L}_0\zeta = 0,$$

where $\mathcal{L}_0 := -\Delta_r^{(0)} + r^{-1}\partial_r \cdot \chi$. (The vector ζ_λ spans the tangent space of the manifold $\mathcal{M}_0 := \{\chi_\lambda \mid \lambda > 0\}$ at a point χ_λ .) We deform this result as (37). Consequently, by the Perron–Frobenius theorem, the spectrum of \mathcal{L}_{0bc} starts with 0, which is a simple eigenvalue. □

The results above can be carried over to the operator

$$L_{bc} := L_{abc}|_{a=0} := \gamma_{0bc}^{1/2} \mathcal{L}_{0bc} \gamma_{0bc}^{-1/2},$$

which is explicitly given by

$$(40) \quad L_{bc} = -\Delta^{(4)} - \frac{8bc}{(c+y^2)^2} + \frac{4b(1-b)y^2}{(c+y^2)^2}.$$

This is a deformation in b and c of the operator

$$(41) \quad L_0 := -\Delta_y^{(4)} - \frac{8}{(1+y^2)^2}.$$

The operator L_{bc} , defined on the Hilbert space $L^2([0, \infty), y^3 dy)$, is selfadjoint with spectrum $[0, \infty)$. The bottom of the spectrum, 0, is a simple eigenvalue,

$$(42) \quad L_{bc}\eta_{bc} = 0, \quad \text{with } \eta_{bc} := 4\gamma_{0bc}^{1/2}\zeta_{bc} = \frac{1}{(c+y^2)^b}.$$

The tangent vectors $\zeta_{bc0}(y)$ and $\zeta_{bc1}(y)$ (to the manifold \mathcal{M}) are approximate eigenfunctions of the operator \mathcal{L}_{abc} . Indeed, first observe that the functions $\chi_{bc}(y)$ are approximate solution to (12). Indeed, let $\Phi(u)$ be the map defined by the right-hand side of (12), $\Phi(u) := \Delta_y^{(0)}u + y^{-1}u \partial_y u - ay \partial_y u$; then

$$(43) \quad \Phi(\chi_{bc})(y) = -\frac{8bca}{c+y^2} + 8bc \frac{4(b-1)+ac}{(c+y^2)^2} - \frac{32bc^2(b-1)}{(c+y^2)^3}.$$

Now, differentiating $\Phi(\chi_{bc})$ with respect to c and b and using (43), we obtain

$$(44) \quad \begin{aligned} \mathcal{L}_{abc}\zeta_{bc0}(y) &= -2a\zeta_{bc0}(y) + \left[4\frac{4(b-1) + ac}{c + y^2} - 24\frac{c(b-1)}{(c + y^2)^2} \right] \zeta_{bc0}(y), \\ \mathcal{L}_{abc}\zeta_{bc1}(y) &= \left[\frac{2ca}{c + y^2} - \frac{8c(2b-1)}{(c + y^2)^2} \right] \zeta_{bc1}(y). \end{aligned}$$

Though on the first sight ζ_{bc0} and, especially, ζ_{bc1} do not seem to be approximate eigenfunctions of \mathcal{L}_{abc} , in fact they are. Indeed, assuming $b - 1 = O(a \ln \frac{1}{a})$, $c = O(1)$, we obtain

$$(45) \quad \|(\mathcal{L}_{abc} + 2a\delta_{i0})\zeta_{bci}\| \lesssim \begin{cases} (b-1) + a, & i = 0, \\ 1, & i = 1. \end{cases}$$

However, if one takes into account the normalizations

$$(46) \quad \|\zeta_{bc0}\| = \frac{1}{4\sqrt{2}} \ln^{\frac{1}{2}} \frac{1}{a} + O\left(\frac{1}{\ln^{\frac{1}{2}} \frac{1}{a}}\right), \quad \|\zeta_{bc1}(y)\| = \frac{\sqrt{2}}{a} + O\left(\ln^2 \frac{1}{a}\right),$$

then for the normalized vectors we have

$$(47) \quad \left\| (\mathcal{L}_{abc} + 2a\delta_{i0}) \frac{\zeta_{bci}}{\|\zeta_{bci}\|} \right\| \lesssim \begin{cases} \frac{(b-1)+a}{\sqrt{\ln \frac{1}{a}}}, & i = 0, \\ a, & i = 1. \end{cases}$$

This relation for $i = 0$ implies (34). □

§4. RELATION BETWEEN THE PARAMETERS a, b , AND c AND BLOWUP DYNAMICS

In this section, we state the relations between the parameters a, b , and c , which are obtained by evaluating the equations in (29) and are proved in §6. Using these relations, we find the governing equation for $a(\tau)$.

Proposition 4.1. *Let $d := b - 1$ and assume*

$$(48) \quad \|\phi\| \ll \left(\ln \frac{1}{a}\right)^{-1},$$

and, for simplicity, $d \lesssim a \ln(a^{-1})$. Then

$$(49) \quad c_\tau + S_0(\phi, a, b, c)a_\tau = 4\left(\frac{ac}{2} - \frac{d}{\ln \frac{1}{a}}\right) + O\left(\frac{a}{\ln \frac{1}{a}}\right) + \mathcal{R}_0(\phi, a, b, c),$$

$$(50) \quad d_\tau + S_1(\phi, a, b, c)a_\tau = -\frac{2da}{\ln \frac{1}{a}} + O\left(\frac{a^2}{\ln \frac{1}{a}} + a^2 d \ln \frac{1}{a}\right) + \mathcal{R}_1(\phi, a, b, c),$$

where $S_i(\phi, a, b, c)$ and $\mathcal{R}_i(\phi, a, b, c)$, $i = 0, 1$, satisfy the estimates

$$(51) \quad |S_i(\phi, a, b, c)| \lesssim \|\phi\| (a \ln(a^{-1}))^{i-1},$$

$$(52) \quad |\mathcal{R}_i(\phi, a, b, c)| \lesssim \frac{a^{i+1}}{\ln^{1-i}(a^{-1})} \|\phi\| + \frac{a^i}{\ln(a^{-1})} \|\phi\|^2.$$

Remark 4.1. From (52) we see that for the terms $\mathcal{R}_i(\phi, a, b, c)$ in (49)–(50) to be sub-leading we should have $\|\phi\| \ll (a \ln(a^{-1})^{-1})^{1/2}$.

As was mentioned above, this proposition is proved in §6. Now, we choose a relation between a, c and d , so that the leading order term on the right-hand side of equation (49) for c_τ vanishes:

$$(53) \quad d = \frac{1}{2} a \ln(a^{-1}).$$

Proposition 4.2. Assume $\|\phi\| \leq \sqrt{\frac{a}{\ln(a^{-1})}}$ and (53). Then the function $a(\tau)$ satisfies the differential equation

$$(54) \quad a_\tau = -\frac{2a^2}{\ln(a^{-1})} \left(1 + O\left(\frac{1}{\ln(a^{-1})}\right) \right),$$

which gives

$$(55) \quad a(\tau) = \frac{\ln \tau}{2\tau} \left(1 + O\left(\frac{1}{\ln^2 \tau}\right) \right).$$

Proof. Plugging relation (53) into (50) and remembering that $d = b - 1$, we obtain

$$(56) \quad \frac{1}{2} (\ln(a^{-1}) - 1 + 2S_2(\phi, a, b, c))a_\tau = -\frac{2da}{\ln \frac{1}{a}} + O\left(\frac{a^2}{\ln \frac{1}{a}}\right) + \mathcal{R}_2(\phi, a, b, c).$$

We see that to solve this equation for a_τ , we need $|S_2| \ll \ln(1/a)$, which in view of (51) with $i = 2$ requires that $\|\phi\| \ll \ln(1/a)$. Due to the conditions of the proposition and estimate (52) with $i = 2$, the high order terms in equation (54) give a small correction upon integration and the leading part can be integrated exactly yielding (55). \square

Remark 4.2. The above expression for a_τ passes a consistency test: $a_\tau < 0$ and $|a_\tau| \ll a^2$.

Proposition 4.3. For $|T - t| \ll 1$, the scaling parameter λ , with $a = -\lambda\dot{\lambda}$ satisfying (55), is asymptotic to

$$(57) \quad \lambda = ke^{-\frac{1}{4} \ln^2 \tau},$$

where τ is related to t by

$$(58) \quad k(T - t) = \frac{\tau}{\ln \tau} e^{-\frac{1}{2} \ln^2 \tau}.$$

Proof. Using the definition $a = -\lambda\dot{\lambda}$ and the relation $\partial_t \lambda = \lambda^{-2} \partial_\tau \lambda$ we arrive at $a = -\lambda^{-1} \partial_\tau \lambda$. Combining this with (55), we obtain the equation $\lambda^{-1} \partial_\tau \lambda = -\frac{\ln \tau}{2\tau} \left(1 + O\left(\frac{1}{\ln^2 \tau}\right) \right)$. Solving this differential equation gives (57). Combining (57) with $\partial_t \tau = \lambda^{-2}$ gives a differential equation for $\tau(t)$, and solving it in the leading order leads to (58). \square

Solving (58) for $\ln^2 \tau$ and substituting the result into (57) gives (9).

§5. LOWER BOUND FOR THE OPERATOR \mathcal{L}_{abc}

In this section we investigate the linear operator \mathcal{L}_{abc} defined in (21). The main result of this section is the following lower bound on the quadratic form $\langle \phi, \mathcal{L}_{abc} \phi \rangle$, $\phi \perp \zeta_{bci}$, where, recall, the vectors ζ_{bci} are defined in (28).

Theorem 5.1. For $|a| \ll 1$, $|b - 1| \ll 1$, $|a_\tau| \ll a^2$, $|b_\tau| \ll a(1 - b)$ and for any $\phi \in H^1([0, \infty), \gamma_{abc}(y)y^3 dy)$, $\phi \perp \zeta_{bci}$, $i = 0, 1$, for some absolute constant $c > 0$, we have

$$(59) \quad \langle \phi, \mathcal{L}_{abc} \phi \rangle \geq ca \|\phi\|_{H^1}^2.$$

Proof. Recall that the operator \mathcal{L}_{abc} is unitarily equivalent to the operator L_{abc} ,

$$(60) \quad L_{abc} = \gamma_{abc}^{1/2} \mathcal{L}_{abc} \gamma_{abc}^{-1/2},$$

acting on the space $L^2([0, \infty), y^3 dy)$ with the inner product $(\phi, \eta) := \int \phi \eta y^3 dy$. The latter operator has been explicitly computed in the proof of Proposition 3.1 to be

$$(61) \quad L_{abc} = L_* + W(y) - 2ab,$$

where

$$(62) \quad L_* := -\Delta_y^{(4)} - \frac{4bc(b+1)}{(c+y^2)^2} + \frac{4b(b-1)}{c+y^2} + \frac{1}{4}a^2y^2$$

and

$$W(y) := \frac{2bac}{c+y^2} \geq 0.$$

Since the lower bound of L_{abc} is equal to the lower bound of \mathcal{L}_{abc} , we estimate the former lower bound. We observe that, like L_{abc} , the operator L_* is self-adjoint on $L^2([0, \infty), y^3 dy)$ and its spectrum is purely discrete, provided $a > 0$. The last property follows from the fact that the potential in (62) goes to infinity as $y \rightarrow \infty$. Moreover, $L_* \geq 0$. Indeed, write $L_* = L_{0bc} + \frac{1}{4}a^2y^2$, where

$$L_{0bc} := -\Delta^{(4)} - \frac{4bc(b+1)}{(c+y^2)^2} + \frac{4b(b-1)}{c+y^2}.$$

Define $\eta_1(y) := \frac{1}{2\chi(y)}\zeta(y)$, where ζ is defined in (38), so that $L_{011}\eta_1 = 0$. We compute

$$(63) \quad \eta_1(y) := \frac{1}{2\chi(y)}\zeta(y) = \frac{1}{1+y^2}.$$

An extension of relation (63) leads to the equation

$$(64) \quad L_{0bc}\eta_{bc} = 0,$$

where η_{bc} is a deformation in b and c of η_1 given by

$$(65) \quad \eta_{bc} := \frac{1}{(c+y^2)^b}.$$

Since $\eta_{bc} > 0$ we conclude, as in Lemma 3.1, that $L_{0bc} \geq 0$ (with the zero being a resonance of L_{bc}). Together with $L_* = L_{0bc} + \frac{1}{4}a^2y^2$, this implies that $L_* \geq 0$.

Lemma 5.2. *For $\phi \in H^2([0, \infty), y^3 dy)$, $\phi \perp \zeta_{bci}$, $i = 0, 1$, and $a \ll 1$, we have*

$$(66) \quad \langle L_*\phi, \phi \rangle \geq \left(4 - O\left(\frac{1}{\sqrt{\ln \frac{1}{a}}}\right)\right) a \|\phi\|_*^2,$$

where $(\phi, \eta) := \int \phi \eta y^3 dy$ is the inner product in $L^2([0, \infty), y^3 dy)$ and $\|\cdot\|_*$ is the corresponding norm.

Proof. To this end we shall use the minimax principle for selfadjoint operators (see [73]), which states that for the third eigenvalue λ_3 of L_* we have

$$(67) \quad \lambda_3 = \inf_{\dim V=3} \max_{\phi \in V} \frac{\langle L_*\phi, \phi \rangle}{\|\phi\|_*^2},$$

where V is an arbitrary subspace of $H^1(\mathbb{R}^4)$; also, we need the estimate from [29] of λ_3 :

$$(68) \quad \lambda_3 = 4a + \frac{Ca}{\ln \frac{1}{a}}(1 + o(1))$$

for some constant C . Now, let η be the minimizer of $\langle L_*\phi, \phi \rangle$ over

$$\{\phi \in H^2([0, \infty), y^3 dy) \mid \phi \perp \zeta_{bci}, i = 0, 1, \|\phi\|_* = 1\}.$$

Since L_* is selfadjoint, η can be chosen to be real. Since the spectrum of L_* is discrete, this minimizer exists. By the linear independence of ζ_{bci} , $i = 0, 1$, and the orthogonality

of η to ζ_{bci} , $i = 0, 1$, the three vectors η , ζ_{bci} , $i = 0, 1$, span a three-dimensional space. The minimax principle then asserts that

$$(69) \quad \lambda_3 \leq \max_{\phi \in W} \frac{(L_*\phi, \phi)}{\|\phi\|_*^2},$$

where $W := \text{span}\{\eta, \zeta_{bci}, i = 0, 1\}$. Let ϕ_{na} , $n = 0, 1$, be an appropriate orthonormal basis in $\text{span}\{\zeta_{bci}, i = 0, 1\}$:

$$(70) \quad \phi_{na} := \eta_{bc}\psi_{na},$$

with

$$(71) \quad \begin{aligned} \psi_{0a} &:= \sqrt{\frac{2}{\ln \frac{1}{a}}} \left[1 + O\left(\frac{1}{\ln \frac{1}{a}}\right) \right] e^{-\frac{a}{4}y^2}, \\ \psi_{1a} &:= \left[1 + O\left(\frac{1}{\ln \frac{1}{a}}\right) \right] \left(\frac{c_1}{\ln^{\frac{3}{2}} \frac{1}{a}} - c_2ay^2 \right) e^{-\frac{a}{4}y^2}, \end{aligned}$$

for some positive constants c_1 and c_2 . We write $\phi = \gamma_1\eta + \gamma_2\phi_{0a} + \gamma_3\phi_{1a}$ in the inner product $(L_*\phi, \phi)$, where

$$(72) \quad |\gamma_1|^2 + |\gamma_2|^2 + |\gamma_3|^2 = 1,$$

and use the selfadjointness of L_* to obtain

$$(73) \quad \begin{aligned} (L_*\phi, \phi) &= |\gamma_1|^2(L_*\eta, \eta) + |\gamma_2|^2(L_*\phi_{0a}, \phi_{0a}) + |\gamma_3|^2(L_*\phi_{1a}, \phi_{1a}) \\ &\quad + 2\text{Re}(\gamma_1\gamma_2^*)(\eta, L_*\phi_{0a}) + 2\text{Re}(\gamma_1\gamma_3^*)(\eta, L_*\phi_{1a}) + 2\text{Re}(\gamma_2\gamma_3^*)(L_*\phi_{0a}, \phi_{1a}). \end{aligned}$$

We compute the various matrix elements on the right-hand side of equation (73). Using the relation

$$\Delta_y^{(4)}\phi_{na} = \psi_{na}\Delta_y^{(4)}\eta_{bc} + \eta_{bc}\Delta_y^{(4)}\psi_{na} + 2(\partial_y\psi_{na})(\partial_y\eta_{bc}),$$

we find

$$L_*\phi_{na} = \psi_{na}L_{0bc}\eta_{bc} + \eta_{bc}H_a\psi_{na} - 2(\partial_y\eta_{bc})(\partial_y\psi_{na}).$$

Using the facts that $L_{0bc}\eta_{bc} = 0$, $H_a\psi_{0a} = 2a\psi_{0a}$, and $H_a\psi_{1a} = 4a\psi_{1a} + (8c_2a\sqrt{\frac{\ln \frac{1}{a}}{2}} - \sqrt{2}c_1a)\psi_{0a}$ and computing $2(\partial_y\eta_{bc})(\partial_y\psi_{na})$, we see that

$$(L_* - 2an)\phi_{na} = S_n,$$

with

$$S_0 := -\left(2a(b-1) + \frac{2abc}{c+y^2}\right)\phi_{0a},$$

and

$$S_1 := -2a\left((b-1) - \frac{bc}{c+y^2}\right)\phi_{1a} - \sqrt{2}a\left(4c_2\sqrt{\ln \frac{1}{a}}(b-1) - \frac{4c_2bc}{c+y^2}\sqrt{\ln \frac{1}{a}} + 2c_1\right)\phi_{0a}.$$

This and the fact that the functions ϕ_{ia} are normalized allows us to estimate

$$(74) \quad (L_*\phi_{0a}, \phi_{0a}) = -\frac{a}{2} + O\left(\frac{a}{\ln \frac{1}{a}}\right),$$

$$(75) \quad (L_*\phi_{0a}, \phi_{1a}) = -\frac{c_1a}{\sqrt{2}\ln \frac{1}{a}} + O\left(\frac{a}{\ln^{\frac{3}{2}} \frac{1}{a}}\right),$$

$$(76) \quad (L_*\phi_{1a}, \phi_{1a}) = \frac{a}{\ln^{\frac{1}{2}} \frac{1}{a}} \left(2 + \frac{c_1}{\sqrt{2}}\right) + O\left(\frac{a}{\ln^2 \frac{1}{a}}\right).$$

Let P^\perp be the orthogonal projection onto the orthogonal complement of the two vectors ϕ_{0a} and ϕ_{1a} . We compute

$$(77) \quad \|P^\perp L_* \phi_{ia}\|_* = O\left(\frac{a}{\ln^{\frac{1}{2}} \frac{1}{a}}\right), \quad i = 0, 1.$$

Using estimates (74)–(77) together with (69) and (73), we find

$$\lambda_3 \leq \max_{\gamma_i} \left\{ |\gamma_1|^2 (L_* \eta, \eta) - \frac{a}{2} |\gamma_2|^2 + O\left(\frac{a}{\ln^{\frac{1}{2}} \frac{1}{a}}\right) \right\}.$$

Now, since $L_* \geq 0$, we know that $(L_* \eta, \eta) \geq 0$. Then the above relation implies

$$(78) \quad \lambda_3 \leq (L_* \eta, \eta) + O\left(\frac{a}{\ln^{\frac{1}{2}} \frac{1}{a}}\right).$$

Using expression (68) for λ_3 in the last inequality, we conclude that

$$(L_* \eta, \eta) \geq 4 \left(1 - O\left(\frac{1}{\sqrt{\ln \frac{1}{a}}}\right) \right) a,$$

which gives inequality (66). □

Because of the decomposition (61) and since $W(y) \geq 0$ and $0 < b - 1 \lesssim \frac{1}{\sqrt{\ln \frac{1}{a}}}$, we arrive at

$$(79) \quad (\phi, L_{abc} \phi) \geq \frac{3}{2} a \|\phi\|_*^2,$$

or, by the unitary map $\phi \mapsto \gamma_{abc}^{1/2} \phi$,

$$(80) \quad \langle \phi, \mathcal{L}_{abc} \phi \rangle \geq \frac{3}{2} a \|\phi\|^2.$$

To pass from this bound to (59), we decompose $\langle \phi, \mathcal{L}_{abc} \phi \rangle = (1 - \delta) \langle \phi, \mathcal{L}_{abc} \phi \rangle + \delta \langle \phi, \mathcal{L}_{abc} \phi \rangle$ and use (80) for the first term and $\mathcal{L}_{abc} \geq -\Delta^{(4)} - C$, for some $C > 0$, for the second term. Optimizing with respect to δ produces (59). □

§6. ANALYSIS OF FLUCTUATIONS

In this section, neglecting the nonlinearity $N(\phi)$, we find a bound $\|\phi\| \lesssim |1 - b|$ on the fluctuation ϕ . Given that we expect, from (53), that $1 - b \sim \frac{a}{2} \ln \frac{1}{a}$, this is sufficient to close the estimates. Neglecting the nonlinearity $N(\phi)$ in (20), we arrive at the linear equation

$$(81) \quad \partial_\tau \phi = -\mathcal{L}_{ab} \phi + \mathcal{F}_{ab}.$$

More precisely, we have the following proposition,

Proposition 6.1. *Assume a, b , and ϕ solve (81) and (29) (with $\mathcal{N} = 0$), which is equivalent to (27), and are such that $1 - b = O(a \ln \frac{1}{a})$ and $b_\tau = O(a)$, and assume (55). Then, for $\tau \gg 1$, ϕ satisfies the estimate*

$$\|\phi\| \lesssim \|\phi(0)\| \left(\frac{2a}{\ln \frac{1}{a}}\right)^{\ln \frac{1}{a}} + a \ln \frac{1}{a}.$$

Proof. We use a Lyapunov argument with the Lyapunov functional $\phi \mapsto \|\phi\|^2$. The time derivative of this functional on solutions ϕ of (81) is

$$(82) \quad \partial_\tau \|\phi\|^2 = -2 \langle \phi, \mathcal{L}_{abc}\phi \rangle + 2 \langle \mathcal{F}_{abc}, \phi \rangle + \langle \phi, (\partial_\tau \ln \gamma_{abc})\phi \rangle.$$

We estimate the right-hand side of this relation. Let $\chi(y), \bar{\chi}(y) \geq 0$ be a smooth partition of unity, $\chi^2 + \bar{\chi}^2 = 1$, such that $\chi(y)$ is a cutoff function that equals 1 on the set $\{ay^2 \leq \kappa\}$, for some convenient large constant $\kappa > 0$, and is supported on $\{ay^2 \leq 2\kappa\}$.

Proposition 6.2. *For any $\phi \in H^1([0, \infty), \gamma_{abc}(y)y^3 dy), \phi \perp \zeta_{bci}, i = 0, 1$, for some absolute constants $k_1, k_2 > 0$ we have*

$$(83) \quad 2 \langle \phi, \mathcal{L}_{abc}\phi \rangle \geq a \|\phi\|_{L^2}^2 + k_1 a \|\phi\|_{H^1}^2 + k_2 \langle \bar{\chi}\phi, a^2 y^2 \bar{\chi}\phi \rangle.$$

Proof. Since ϕ is orthogonal to the vectors $\zeta_{bci}, i = 0, 1$, and since a and b satisfy the conditions of Theorem 5.1, we have estimates (59) and (80), which imply

$$(84) \quad 2 \langle \phi, \mathcal{L}_{abc}\phi \rangle \geq \frac{3}{2} a \|\phi\|_{L^2}^2 + ka \|\phi\|_{H^1}^2.$$

Next, we estimate $\langle \phi, \mathcal{L}_{abc}\phi \rangle$ in a different way. For the partition of unity defined after (82), we have the IMS formula (see, e.g., [27])

$$(85) \quad \mathcal{L}_{abc} = \chi \mathcal{L}_{abc} \chi + \bar{\chi} \mathcal{L}_{abc} \bar{\chi} - |\nabla \chi|^2 - |\nabla \bar{\chi}|^2.$$

By (34), we have $\chi \mathcal{L}_{abc} \chi \gtrsim -a\chi^2$. Using the inner product $(\xi, \eta) := \int \xi \eta dy$ and the notation $\bar{\phi} := \bar{\chi}\phi$, we obtain

$$\langle \bar{\phi}, \mathcal{L}_{abc}\bar{\phi} \rangle = \langle \bar{\phi}, \gamma_{abc}^{1/2} L_{abc} \gamma_{abc}^{1/2} \bar{\phi} \rangle,$$

which, together with (36) gives, for κ sufficiently large,

$$\langle \bar{\phi}, \mathcal{L}_{abc}\bar{\phi} \rangle \geq \left\langle \bar{\phi}, \gamma_{abc}^{1/2} \left[\frac{1}{4} a^2 y^2 - 2ab - \frac{4bc(b+1)}{(c+y^2)^2} \right] \gamma_{abc}^{1/2} \bar{\phi} \right\rangle \geq \left\langle \bar{\phi}, \left(\frac{1}{8} a^2 y^2 + \frac{1}{9} \kappa a \right) \bar{\phi} \right\rangle.$$

Next, using the fact that $|\nabla \chi|$ and $|\nabla \bar{\chi}|$ are of the form $\sqrt{\frac{a}{\kappa}} \tilde{\chi}$, where $\tilde{\chi}$ is supported between $ay^2 = \kappa$ and $ay^2 = 2\kappa$, we compute $|\nabla \chi|^2 + |\nabla \bar{\chi}|^2 \simeq \frac{a}{\kappa} \tilde{\chi}$, which leads to

$$(86) \quad \langle \phi, (|\nabla \chi|^2 + |\nabla \bar{\chi}|^2)\phi \rangle \lesssim \frac{a}{\kappa} \|\tilde{\chi}\phi\|_{L^2}^2.$$

Using the IMS formula (85) and the estimates above, we find

$$(87) \quad \langle \phi, \mathcal{L}_{abc}\phi \rangle \geq -ca \|\chi\phi\|_{L^2}^2 + \frac{1}{9} \langle \bar{\chi}\phi, (a^2 y^2 + \kappa a) \bar{\chi}\phi \rangle - C \frac{a}{\kappa} \|\phi\|_{L^2}^2$$

for positive constants c, C . Now, write $\langle \phi, \mathcal{L}_{abc}\phi \rangle = (1 - \delta) \langle \phi, \mathcal{L}_{abc}\phi \rangle + \delta \langle \phi, \mathcal{L}_{abc}\phi \rangle$, and use (84) for the first term on the right hand side and (87) for the second term, and choose δ sufficiently small to arrive at (83). \square

We substitute the expression (22) for \mathcal{F}_{abc} and observe that the orthogonality of ϕ to $\zeta_{bci}, i = 0, 1$, implies

$$\langle 1, \phi \rangle = c \left\langle \frac{1}{c+y^2}, \phi \right\rangle = c^2 \left\langle \frac{1}{(c+y^2)^2}, \phi \right\rangle,$$

to obtain

$$\langle \mathcal{F}_{abc}, \phi \rangle = 32bc(b-1) \left[\left\langle \frac{1}{(c+y^2)^2}, \phi \right\rangle - \left\langle \frac{1}{(c+y^2)^3}, \phi \right\rangle \right].$$

Hölder’s inequality yields

$$(88) \quad \langle \mathcal{F}_{abc}, \phi \rangle = O((b-1) \|\langle y \rangle^{4-\varepsilon} \phi\|).$$

Next, we estimate $\langle \phi, (\partial_\tau \ln \gamma_{abc})\phi \rangle$. Since $\partial_\tau \ln \gamma_{abc} = -a_\tau y^2/2 + 2b_\tau \ln(c + y^2) + 2bc_\tau/(c + y^2)$, we obtain

$$(89) \quad \langle \phi, (\partial_\tau \ln \gamma_{abc})\phi \rangle = -\frac{1}{2}a_\tau \|y\phi\|_{L^2}^2 + 2b_\tau \|(\ln(c + y^2))^{1/2}\phi\|_{L^2}^2 + 2bc_\tau \left\| \frac{1}{\sqrt{c + y}}\phi \right\|_{L^2}^2.$$

By (50) and (54), we have $a_\tau < 0$, $b_\tau < 0$, and assuming $c < 1$, we see that $c_\tau < 0$ by (49) and (53). Hence,

$$(90) \quad \langle \phi, (\partial_\tau \ln \gamma_{abc})\phi \rangle \leq -\frac{1}{2}a_\tau \|y\phi\|_{L^2}^2.$$

Now, $y^2 \leq \kappa/a$ on $\text{supp } \chi$, which implies $\langle \chi\phi, (\partial_\tau \ln \gamma_{abc})\chi\phi \rangle \leq -\frac{a_\tau \kappa}{2a} \|\chi\phi\|_{L^2}^2$. Together with the relation

$$\langle \chi\phi, (\partial_\tau \ln \gamma_{abc})\chi\phi \rangle = \langle \chi\phi, (\partial_\tau \ln \gamma_{abc})\chi\phi \rangle + \langle \bar{\chi}\phi, (\partial_\tau \ln \gamma_{abc})\bar{\chi}\phi \rangle,$$

this gives

$$(91) \quad \langle \phi, (\partial_\tau \ln \gamma_{abc})\phi \rangle \leq -\frac{a_\tau \kappa}{2a} \|\chi\phi\|_{L^2}^2 - \frac{1}{2}a_\tau \|y\bar{\chi}\phi\|_{L^2}^2.$$

Combining the last estimate with (82), (83), (88), (91), (53), and (55), we obtain, for some absolute constants $k_1, k_2, C > 0$,

$$\partial_\tau \|\phi\|_{L^2}^2 \leq -a\|\phi\|_{L^2}^2 - k_1 a \|\phi\|_{H^1}^2 - k_2 \langle \bar{\chi}\phi, (a^2 y^2 + \kappa a)\bar{\chi}\phi \rangle + C \left(a \ln \frac{1}{a} \right) \|\langle y \rangle^{-\frac{7}{4} + \varepsilon} \phi\|_{L^2}^2.$$

Using $\partial_\tau \|\phi\|^2 = 2\|\phi\|\partial_\tau \|\phi\|$, dropping the second and third terms (these terms can be used to control the nonlinearity), and dividing the resulting inequality by $\|\phi\|$, we obtain

$$(92) \quad \partial_\tau \|\phi\| \leq -\frac{a}{2}\|\phi\| + Ca \ln \frac{1}{a}.$$

Now, integrating the last inequality shows that

$$(93) \quad \|\phi\| \lesssim e^{-\int_0^\tau a(s) ds} \|\phi(0)\| + \int_0^\tau e^{-\int_\sigma^\tau a(s) ds} \left(a \ln \frac{1}{a} \right) (\sigma) d\sigma.$$

We have computed that, in the sense of asymptotic equivalence, $a(\tau) \sim \frac{\ln \tau}{2\tau}$, as $\tau \rightarrow \infty$ (see equation (55)). Consequently, as $\sigma \rightarrow \infty$,

$$\int_\sigma^\tau a(s) ds \sim \frac{1}{4} \ln(\tau\sigma) \ln \left(\frac{\tau}{\sigma} \right) = \ln \left(\frac{\tau}{\sigma} \right)^{\frac{1}{4} \ln(\tau\sigma)}.$$

Using the identity

$$e^{-\int_0^\tau a(s) ds} = e^{-\int_0^{\sqrt{\tau}} a(s) ds} e^{-\int_{\sqrt{\tau}}^\tau a(s) ds}$$

and noting that the first term on the right is uniformly bounded, and the second term is $\sim \tau^{-\frac{3}{16} \ln \tau}$, we obtain $e^{-\int_0^\tau a(s) ds} = O(\tau^{-\frac{3}{16} \ln \tau})$. Using now the relation

$$\tau = \frac{\ln \frac{1}{a}}{2a} \left(1 - O\left(\frac{1}{\ln \frac{1}{a}} \right) \right)$$

we see that the term involving the initial condition in (93) is bounded as

$$(94) \quad e^{-\int_0^t a(s) ds} \|\phi(0)\| \lesssim \|\phi(0)\| \left(\frac{2a}{\ln \frac{1}{a}} \right)^{\ln \frac{1}{a}}.$$

To bound the integral term in (93), we begin by splitting the domain of integration into $[0, \alpha\tau]$ and $[\alpha\tau, \tau]$ for some $0 < \alpha < 1$ to be chosen later:

$$\int_0^{\alpha\tau} e^{-\int_\sigma^\tau a(s) ds} \left(a \ln \frac{1}{a} \right) (\sigma) d\sigma + \int_{\alpha\tau}^\tau e^{-\int_\sigma^\tau a(s) ds} \left(a \ln \frac{1}{a} \right) (\sigma) d\sigma.$$

Since $(a \ln \frac{1}{a})(\sigma)$ is monotone decreasing and $\sigma \mapsto e^{-\int_{\sigma}^{\tau} a(s) ds}$ is monotone increasing and both are positive, we can bound these terms from above by

$$e^{-\int_{\alpha\tau}^{\tau} a(s) ds} \int_0^{\alpha\tau} \left(a \ln \frac{1}{a}\right)(\sigma) d\sigma + \left(a \ln \frac{1}{a}\right)(\alpha\tau) \int_{\alpha\tau}^{\tau} e^{-\int_{\sigma}^{\tau} a(s) ds} d\sigma.$$

Since $e^{-\int_{\alpha\tau}^{\tau} a(s) ds} \sim C_{\alpha} \tau^{\frac{1}{2} \ln \alpha}$, the first term is bounded from above by the quantity $C_{\alpha} (a \ln \frac{1}{a})(0) \tau^{1+\frac{1}{2} \ln \alpha}$. If α is such that $\ln(\alpha)/2 < -2$, then the first term is $\lesssim \tau^{-1}$ and the second is bounded by $(a \ln \frac{1}{a})(\alpha\tau) \sim C \ln^2 \tau / \tau$. So, we find

$$(95) \quad \int_0^{\tau} e^{-\int_{\sigma}^{\tau} a(s) ds} \left(a \ln \frac{1}{a}\right)(\sigma) d\sigma \lesssim \left(a \ln \frac{1}{a}\right)(\tau).$$

Using the bounds (94) and (95) in (93) completes the proof. □

A. COMPLETE SET OF STATIC SOLUTIONS FOR THE RADIAL RKS

The static solutions of equation (10) satisfy the second order differential equation

$$(96) \quad \partial_r^2 \chi + \frac{1}{r}(\chi - 1)\partial_r \chi = 0$$

and hence form a two-dimensional manifold.

Proposition A.1. *Equation (96) admits the one-parameter family of static solutions*

$$\chi^{(\mu)}(r) := \frac{r^{\mu-2}\mu + 4 - \mu}{1 + r^{\mu-2}}, \quad \mu \in [2, \infty),$$

(and therefore, also the two-parameter family $\chi_{\lambda}^{(\mu)}(r) := \chi^{(\mu)}(r/\lambda)$). The mass at infinity of $\chi_{\lambda}^{(\mu)}$ is μ .

Remark A.2. If $\mu < 4$, then the mass at the origin is nonzero, i.e., blowup has already occurred. If $\mu > 4$, then the mass at the origin is negative, so that the static solution is not physical.

Proof. In (96), we use the transformation

$$\psi(\chi) = r \frac{\partial_r \chi}{\chi}$$

under the assumption that the right-hand side is indeed a function of χ alone. After this transformation, equation (96) becomes

$$\chi \partial_{\chi} \psi + \psi = 2 - \chi.$$

Integrating this equation, we get

$$\psi = 2 - \frac{1}{2}\chi + \frac{\mu}{2} \frac{1}{\chi},$$

and hence, upon substituting this into the definition of ψ and integrating over r , we obtain the general solution

$$\chi = \frac{\left(\frac{r}{\lambda}\right)^{\sqrt{4+\nu}} r_+ + r_-}{1 + \left(\frac{r}{\lambda}\right)^{\sqrt{4+\nu}}},$$

where $r_{\pm} = 2 \pm \sqrt{4 + \nu}$ are the roots of $\chi^2 - 4\chi - \nu = 0$. The total mass at infinity of these solutions is r_+ ; therefore, it is natural to define a new parameter $\mu = r_+ \in [2, \infty)$. The static solution in terms of the parameters λ and ν is

$$\chi = \frac{\left(\frac{r}{\lambda}\right)^{\mu-2} \mu + 4 - \mu}{1 + \left(\frac{r}{\lambda}\right)^{\mu-2}}.$$

The constant λ is positive because it is the exponential of the constant obtained in the last integration. \square

The tangent space of the manifold $M_{\lambda,\mu}$ is spanned by the functions

$$\zeta_{\lambda,\mu}^0 := \partial_\lambda \chi_{\lambda,\mu} = -\frac{2(\mu-2)^2}{\lambda} \frac{y^{\mu-2}}{(1+y^{\mu-2})^2}$$

and

$$\zeta_{\lambda,\mu}^1 := \partial_\mu \chi_{\lambda,\mu} = \frac{y^{2(\mu-2)} - 1 - 2y^{\mu-2} \ln y}{(1+y^{\mu-2})^2},$$

where $y = \frac{r}{\lambda}$.

We return to the situation of $\kappa = 1$ and $n = 4$. After the gauge transformation, the zero modes $\zeta_{\lambda,\mu}^0$ and $\zeta_{\lambda,\mu}^1$ transform to

$$\eta_{\lambda,\mu}^\lambda := \frac{1}{\lambda^2} \frac{1+y^2}{y^2} \zeta_{\lambda,\mu}^0 \quad \text{and} \quad \eta_{\lambda,\mu}^\mu := \frac{1}{\lambda^2} \frac{1+y^2}{y^2} \zeta_{\lambda,\mu}^1$$

(neither of which is in $L^2(r^3 dr)$), and hence, are generalized eigenfunctions of \mathcal{L} (without the $\dot{\lambda}$ term). By the ODE theory, the above functions are the only linearly independent solutions of the equation $\mathcal{L}_0 \phi = 0$. The Perron–Frobenius theory shows that 0 is the lowest point of the spectrum of \mathcal{L} .

B. PROOF OF PROPOSITION 2.1

Both existence and uniqueness follow from a standard implicit function theorem argument. Fix $0 < \delta \ll 1$ and let $Z := e^{\frac{\delta}{3}y^2} L^\infty([0, \infty))$. Recall that $b = 1 + 1/2a \log(1/a)$ and define the vector-valued function

$$(97) \quad G(f, a, c) := (\langle f - \chi_{bc}, \zeta_{bci} \rangle, \quad i = 0, 1)$$

$$(98) \quad = \left(\frac{1}{16} \int_0^\infty (f(y) - \chi_{bc}(y))(c + y^2)^{2b-2+i} y e^{-\frac{\delta}{2}y^2} dy, \quad i = 0, 1 \right).$$

This function maps $Z \times \mathbb{R}_+ \times \mathbb{R}_+$ into \mathbb{R}^2 . It is a C^1 -function and $G(\chi_{bc}, a, c) = 0$. Moreover, the derivative of G with respect to (a, c) at $f = \chi_{bc}$ is

$$(99) \quad A := \begin{pmatrix} \Gamma_{0a} & \Gamma_{0c} \\ \Gamma_{1a} & \Gamma_{1c} \end{pmatrix},$$

where

$$(100) \quad \Gamma_{ia} := \frac{\partial_a b}{4} \int_0^\infty y^3 (c + y^2)^{2b-3+i} e^{-\frac{\delta}{2}y^2} dy,$$

$$(101) \quad \Gamma_{ic} := -\frac{b}{4} \int_0^\infty y^3 (c + y^2)^{2b-4+i} e^{-\frac{\delta}{2}y^2} dy.$$

For the determinant of A we have

$$|\det A| = \frac{1}{64a^2} \left(\ln^2 \frac{1}{a} + O(1) \right)$$

as $a \rightarrow 0$, whence $|\det A| \geq C > 0$ for some constant C , for $(a, c) \in (0, \delta) \times (1, 2)$ with δ sufficiently small. Thus, by the implicit function theorem, for any $a_* \in (0, \delta)$ and $c_* \in (1, 2)$ there exist open sets $U_{a_*c_*} \subset Z$ and $V_{a_*c_*} \subset (0, \delta) \times (1, 2)$ containing $\chi_{b_*c_*}$ and (a_*, c_*) , respectively, and a unique function $g_{a_*c_*} : U_{a_*c_*} \rightarrow V_{a_*c_*}$. To determine the size of the neighbourhoods $U_{a_*c_*}$ we look more closely into a proof of the implicit function theorem. Write $\mu = (a, c)$ and expand

$$(102) \quad G(f, \mu) = G(f, \mu_*) + \partial_\mu G(f, \mu_*)(\mu - \mu_*) + R_f(\mu),$$

where $R_f(\mu) = O(|\mu - \mu_*|^2)$ uniformly in $f \in B_C(\chi_{b_*c_*})$ and $(a_*, c_*) \in (\delta/2, \delta) \times (1, 2)$, for any fixed constant C . By continuity and the above computations, there is $\varepsilon > 0$ such that $\det \partial_\mu G(f, \mu_*)$ is bounded away from zero uniformly for $f \in B_\varepsilon(\chi_{b_*c_*})$ and $(a_*, c_*) \in (\delta/2, \delta) \times (1, 2)$. From (102) we find a fixed-point equation for $\mu - \mu_*$:

$$\mu - \mu_* = \Phi_f(\mu - \mu_*),$$

where

$$\Phi_f(\mu) = -(\partial_\mu G(f, \mu_*))^{-1}(G(f, \mu_*) + R_f(\mu)).$$

The above observations imply that there exists $\varepsilon_1 > 0$ such that Φ_f is a contraction on $B_{\varepsilon_1}(\mu_*)$ for any $f \in B_\varepsilon(\chi_{b_*c_*}) =: U_{a_*c_*}$. Taking the union of U_{ac} over $a \in (\delta, 1)$ and $c \in (1/2, 1)$ gives the open set \mathcal{U}_ε . Patching together the functions g_{ac} gives g .

C. GRADIENT FORMULATION

The Keller–Segel models (1) and (4) are gradient systems. We begin with formulating a normalized version of (1),

$$(103) \quad \begin{aligned} \partial_t \rho &= \Delta \rho - \nabla \cdot (f(\rho) \nabla c), \\ \varepsilon \partial_t c &= \Delta c + \rho - \gamma c, \end{aligned}$$

as a gradient system. This system is obtained from (1) by setting unimportant constants to 1.

Define the energy (or Lyapunov) functional

$$(104) \quad \mathcal{E}_f(\rho, c) := \int_\Omega \frac{1}{2} |\nabla c|^2 - \rho c + \frac{\gamma}{2} c^2 + G(\rho) \, dx,$$

where $G(\rho) := \int^\rho g(s) \, ds$ and $g(\rho) := \int^\rho \frac{1}{f(s)} \, ds$. The L^2 -gradient of $\mathcal{E}_f(\rho, c)$ is

$$\text{grad}_{L^2} \mathcal{E}_f(\rho, c) = \begin{pmatrix} -c + g(\rho) \\ -\Delta c - \rho + \gamma c \end{pmatrix},$$

and hence, if we define $U = (\rho, c)$, then (103) can be written in the form $\partial_t U = I \mathcal{E}'_f(U)$, where

$$I = \begin{pmatrix} \nabla \cdot f(\rho) \nabla & 0 \\ 0 & -\frac{1}{\varepsilon} \end{pmatrix}.$$

The operator I is nonpositive and may be degenerate; however, assuming it is invertible, the operator I gives rise to the metric $\langle v, w \rangle_I := -\langle v, I^{-1} w \rangle_{L^2 \oplus L^2}$. In this metric $\text{grad} \mathcal{E}(U) = -I \mathcal{E}'(U)$, whence

$$\partial_t U = -\text{grad} \mathcal{E}_f(U).$$

This shows that (103) has the structure of a gradient system. A consequence of this is that the energy decreases on solutions of the KS system. Indeed, if $f > 0$, then

$$\partial_t \mathcal{E}_f(\rho, c) = -\|f(\rho)^{\frac{1}{2}} \nabla(c - g(\rho))\|^2 - \frac{1}{\varepsilon} \|\Delta c + \rho - c\|^2.$$

The gradient formulation for (4) is similar to that for (103). Instead of (104), one uses the energy (7). The latter is obtained from (104) by dropping the quadratic term $\frac{1}{2}c^2$, replacing c with $-\Delta^{-1}\rho$ in the remaining terms, and using the fact that $f(\rho) = \rho$. The formal Gâteaux derivative of \mathcal{E} is $\partial_\rho \mathcal{E}(\rho)\phi = \int(\Delta^{-1}\rho + \ln \rho)\phi$. Therefore, the gradient in the metric $\langle v, w \rangle_J := -\langle v, J^{-1}w \rangle_{L^2}$, where $J := \nabla \cdot \rho \nabla < 0$, is

$$\text{grad} \mathcal{E}(\rho) = -\nabla \cdot \rho \nabla(\Delta^{-1} \rho + \ln \rho) = -\nabla \cdot \rho \nabla \Delta^{-1} \rho - \Delta \rho,$$

which is the negative of the right-hand side of the first equation in (4) with $c = -\Delta^{-1}\rho$. Hence, equation (4) can be written as $\partial_t \rho = -\text{grad } \mathcal{E}(\rho)$ in the space with the metric $\langle v, w \rangle_J := -\langle v, J^{-1}w \rangle_{L^2}$. Again, the energy \mathcal{E} decreases on solutions of (4):

$$(105) \quad \partial_t \mathcal{E} = \langle \mathcal{E}', I\mathcal{E}' \rangle = -\|\rho^{\frac{1}{2}} \nabla \mathcal{E}'\|^2.$$

This can be thought of as an entropy monotonicity formula.

The stationary solutions of (4) are critical points of the energy functional \mathcal{E} given in (7), under the constraint that $\int \rho = \text{const}$. Thus, they satisfy $\mathcal{E}'(\rho) = C$, where C is a constant. Explicitly, $\mathcal{E}'(\rho) = C$ reads

$$(106) \quad \log(\rho) + \frac{1}{\Delta} \rho = C \iff \Delta \log(\rho) + \rho = 0 \iff \Delta u + e^u = 0,$$

where $u = \log(\rho)$. Solutions of (106) can be written in the form of the ‘‘Gibbs states’’ $\rho = M \frac{e^c}{\int e^c}$ (see [32]), with the concentration c viewed as a negative potential (remember that $\Delta c = -\rho$). In two dimensions, this equation has the solution $R = \frac{8}{(1+|x|^2)^2}$. This solution is a minimizer of \mathcal{E} under the constraint $\int \rho = 8\pi$.

D. PROOF OF PROPOSITION 4.1

In this Appendix, we prove Proposition 4.1, relating the parameters a, b and c , by evaluating the equations in (29).

Proof of Proposition 4.1. Let $R_i(\phi) := \langle \mathcal{L}_{abc} \phi, \zeta_{bci} \rangle - \langle \mathcal{N}, \zeta_{bci} \rangle$. Here and in what follows $i = 0, 1$. Equations (29) can be rewritten as

$$(107) \quad \langle \mathcal{F}_{abc}, \zeta_{bci} \rangle + \langle \phi, \partial_\tau \zeta_{bci} + (\partial_\tau \ln \gamma_{ab}) \zeta_{bci} \rangle = R_i(\phi).$$

We begin with evaluating $\langle \mathcal{F}_{abc}, \zeta_{bci} \rangle$ to *leading order*. To this end, we begin with the elementary computation

$$(108) \quad \langle 1, \zeta_{bci} \rangle = 2^{i-4} a^{-i-1} + O\left(\frac{1}{a^i} \ln^2 \frac{1}{a}\right),$$

$$(109) \quad \left\langle \frac{1}{c+y^2}, \zeta_{bci} \right\rangle = 2^{i-5} a^{-i} \ln^{1-i} \frac{1}{a} + O\left(\ln^{2i} \frac{1}{a}\right),$$

$$(110) \quad \left\langle \frac{1}{(c+y^2)^2}, \zeta_{bci} \right\rangle = 2^{-5} c^{i-1} \ln^i \frac{1}{a} + O\left(a^{1-i} \ln^{1-i} \frac{1}{a}\right),$$

$$(111) \quad \left\langle \frac{1}{(c+y^2)^3}, \zeta_{bci} \right\rangle = 2^{i-6} c^{i-2} + O\left(a \ln \frac{1}{a}\right).$$

These estimates are proved at the end of this appendix. Using these estimates in (22), we arrive at

$$(112) \quad \begin{aligned} \langle \mathcal{F}_{abc}, \zeta_{bci} \rangle = & -b_\tau 2^{i-2} a^{-i-1} \left(1 - c 2^{-1} a \ln^{1-i} \frac{1}{a}\right) \\ & + c_\tau \left(b 2^{i-3} a^{-i} \ln^{1-i} \frac{1}{a} - b c^i 2^{-3} \ln^i \frac{1}{a}\right) \\ & - 2^{i-2} b c a^{1-i} \ln^{1-i} \frac{1}{a} + (b c^i d + a b c^{i+1} 2^{-2}) \ln^i \frac{1}{a} - b c^i d 2^{i-1} \\ & + O\left(b_\tau a^{-i} \ln^2 \frac{1}{a}\right) + O\left((a_\tau + a) \ln^{2i} \frac{1}{a}\right) \\ & + O\left((d + a + c_\tau) a^{1-i} \ln^{1-i} \frac{1}{a}\right) + O\left(d a \ln \frac{1}{a}\right). \end{aligned}$$

Next, we compute the term $\langle \phi, \partial_\tau \zeta_{bci} + (\partial_\tau \ln \gamma_{abc}) \zeta_{bci} \rangle$. Differentiating ζ_{bci} and $\partial_\tau \ln \gamma_{ab}$ with respect to τ , we obtain

$$(113) \quad \partial_\tau \zeta_{bci} + (\partial_\tau \ln \gamma_{abc}) \zeta_{bci} = \left[2b_\tau \ln(c + y^2) + (2b - 2^{1-i}) \frac{c_\tau}{c + y^2} - \frac{a_\tau}{2} y^2 \right] \zeta_{bci}.$$

Using the fact that ϕ is orthogonal to ζ_{bc0} in the case of $i = 1$, we find

$$(114) \quad \langle \phi, \partial_\tau \zeta_{bci} + (\partial_\tau \ln \gamma_{ab}) \zeta_{bci} \rangle = b_\tau S_{i1}(\phi) + c_\tau S_{i2}(\phi) - a_\tau S_{i3}(\phi),$$

where

$$\begin{aligned} S_{i1}(\phi) &:= 2 \langle \phi, \ln(c + y^2) \zeta_{bci} \rangle, \\ S_{i2}(\phi) &:= 2\delta_{i0}(b - 1) \left\langle \phi, \frac{1}{c + y^2} \zeta_{bci} \right\rangle, \\ S_{i3}(\phi) &:= \frac{1}{2} \langle \phi, y^2 \zeta_{bci} \rangle. \end{aligned}$$

Collecting (112) and (114), we see that, for $i = 0$,

$$(115) \quad \begin{aligned} (S_{03}(\phi) + O(1))a_\tau + \left[\frac{1}{4a} - \frac{c}{8} \ln(a^{-1}) - S_{01}(\phi) + O\left(\ln^2 \frac{1}{a}\right) \right] b_\tau \\ - \left[\frac{b}{8} (\ln(a^{-1}) - 1) + S_{02}(\phi) + O\left(a \ln \frac{1}{a}\right) \right] c_\tau \\ = -\frac{b}{2} \left[\frac{1}{2} ac \ln(a^{-1}) - d - ac \right] - R_0(\phi), \end{aligned}$$

and, for $i = 1$,

$$(116) \quad \begin{aligned} (S_{13}(\phi)a + O\left(\ln^2 \frac{1}{a}\right))a_\tau + \left[\frac{1}{2a} - \frac{c}{4} - aS_{11}(\phi) + O\left(a^{-1} \ln^2 \frac{1}{a}\right) \right] b_\tau \\ - \left[\frac{b}{4} - \frac{1}{8} abc \ln(a^{-1}) + aS_{12}(\phi) + O(1) \right] c_\tau \\ = -abc \left[\frac{1}{2} - d \ln(a^{-1}) - \frac{ac}{4} \ln(a^{-1}) + d \right] - aR_1(\phi). \end{aligned}$$

We manipulate equations (115) and (116) and solve them for b_τ and c_τ to obtain

$$(117) \quad -f_0 a_\tau + g c_\tau = \frac{1}{8} bc \ln(a^{-1}) - \frac{bd}{4a} + v_0 + r_0, \quad -f_1 a_\tau + g b_\tau = -\frac{b^2 d}{8} + v_1 + r_1,$$

where

$$\begin{aligned} f_0 &:= \left(\frac{1}{2a} - \frac{c}{4} - aS_{11} \right) S_{03} + \left(\frac{1}{4} + \frac{c}{8} a \ln(a^{-1}) - aS_{01} \right) S_{13}, \\ g &:= \frac{b}{16} \frac{\ln(a^{-1})}{a} - \frac{1}{4} \left(\frac{b}{2a} - \frac{bc}{8} - \frac{1}{16} bc^2 a \ln(a^{-1})^2 - \frac{1}{8} bc \ln(a^{-1}) \right) \\ &\quad - \left(\frac{2}{a} - c \right) S_{02} - bS_{01} + S_{12} \\ &\quad - \frac{a}{8} \ln(a^{-1}) (bcS_{01} + bS_{11} - cS_{12}) + a \left(\frac{1}{8} bS_{11} - S_{02}S_{11} + S_{01}S_{12} \right), \\ v_0 &:= \frac{1}{8} bc \left[2d \ln(a^{-1}) + 3 + d + ca \ln(a^{-1}) \left(d \ln(a^{-1}) - \frac{1}{2} - d \right) - ca - \frac{1}{4} a^2 c^2 \ln(a^{-1})^2 \right], \\ r_0 &:= \left(\frac{1}{2a} - \frac{1}{4} c \right) R_0 - \frac{1}{4} R_1 + ca \ln(a^{-1}) \left(bdS_{01} - \frac{1}{8} R_1 \right) \\ &\quad + a \left(\frac{1}{2} bcS_{01} + \frac{1}{2} bdS_{11} + R_0S_{11} + S_{01}R_1 \right) + \frac{1}{2} a^2 bc \left(S_{11} - \frac{1}{2} \ln(a^{-1}) (cS_{01} + S_{11}) \right), \\ f_1 &:= a \left(\frac{b}{8} \ln(a^{-1}) - \frac{b}{8} + S_{02} \right) S_{13} - \left(\frac{b}{4} - \frac{bc}{8} a \ln(a^{-1}) + aS_{12} \right) S_{03}, \end{aligned}$$

$$\begin{aligned}
 v_1 &:= -\frac{1}{8}b^2c \left[da \ln^2(a^{-1}) - \frac{3}{2}da \ln(a^{-1}) - a \left(\frac{1}{2} - d \right) + \frac{1}{4}ca^2 \ln(a^{-1}) \right], \\
 r_1 &:= -\frac{b}{4}R_0 + b \left(\frac{1}{8}cR_0 + -dcS_{02} + \frac{1}{8}R_1 \right) a \ln(a^{-1}) \\
 &\quad + a \left(\frac{bcS_{02}}{2} + bcdS_{02} + \frac{bdS_{12}}{2} - R_0S_{12} - \frac{1}{8}bR_1 + R_1S_{02} \right) \\
 &\quad + \frac{1}{2}bca^2 \left(S_{12} + \frac{1}{2}cS_{02} \ln(a^{-1}) + \frac{1}{2}S_{12} \ln(a^{-1}) \right).
 \end{aligned}$$

Next, we derive estimates on $S_{i1}(\phi)$, $S_{i2}(\phi)$, $R_0(\phi)$, and $R_1(\phi)$. Using the Cauchy–Schwarz inequality and simple modifications of estimates (46), we arrive at the estimates

$$\begin{aligned}
 (118) \quad |S_{i1}(\phi)| &\lesssim \|\phi\| a^{-i} \ln(a^{-1})^{(3-i)/2}, \\
 |S_{i2}(\phi)| &\lesssim \|\phi\| \ln(a^{-1})^{\frac{i}{2}}, \\
 |S_{i3}(\phi)| &\lesssim \|\phi\| a^{-(i+1)}.
 \end{aligned}$$

As was shown above, the operator \mathcal{L}_{abc} is selfadjoint in the inner product (25), whence $\langle \mathcal{L}_{abc}\phi, \zeta_{bci} \rangle = \langle \phi, \mathcal{L}_{abc}\zeta_{bci} \rangle$. Using (45) and the fact that ζ_{bc0} is orthogonal to ϕ , we obtain the estimate

$$(119) \quad |\langle \mathcal{L}_{abc}\phi, \zeta_{bci} \rangle| \lesssim \|\phi\| (d+a)^{1-i}.$$

Finally, we estimate $\langle \mathcal{N}, \zeta_{bci} \rangle$, which can be written, using integration by parts, in the form

$$\langle \mathcal{N}, \zeta_{bci} \rangle = -\frac{1}{2} \int_0^\infty \phi^2 \partial_y (\gamma_{abc} y^2 \zeta_{bci}) dy,$$

where, recall, γ_{abc} is the gauge function (see (24)). Here we have used the fact that $y\phi \rightarrow 0$ as $y \rightarrow \infty$, so that the boundary terms vanish. Using the identity

$$\begin{aligned}
 \partial_y (\gamma_{abc} y^2 \zeta_{bci}) &= \partial_y \left(\frac{(c+y^2)^{2b}}{16y^2} e^{-\frac{a}{2}y^2} \zeta_{bci} \right) \\
 &= \left[\left(\frac{4b}{c+y^2} - \frac{2}{y^2} - a \right) \zeta_{bci} + y^{-1} \partial_y \zeta_{bci} \right] \gamma_{abc} y^3,
 \end{aligned}$$

we find

$$(120) \quad |\langle \mathcal{N}, \zeta_{bci} \rangle| \lesssim \|(c+y^2)^{-\frac{2-i}{2}} \phi\|^2 + a \|(c+y^2)^{-\frac{1-i}{2}} \phi\|^2.$$

Estimates (119) and (120) give

$$(121) \quad |R_i(\phi)| \lesssim (d+a)^{1-i} \|\phi\| + \|(c+y^2)^{-\frac{2-i}{2}} \phi\|^2.$$

Since we have assumed that $d \lesssim a \ln(a^{-1})$, the above estimates imply the following inequalities for f_i and r_i :

$$\begin{aligned}
 (122) \quad |f_i| &\lesssim a^{i-2} \left(\ln \frac{1}{a} \right)^i \|\phi\|, \\
 |r_i| &\lesssim a^{i-1} |R_0(\phi)| + (a \ln(a^{-1}))^i |R_1(\phi)| \\
 (123) \quad &\lesssim a^{i-1} \left[(d+a) \|\phi\| + \|(c+y^2)^{-1} \phi\|^2 \right] + (a \ln(a^{-1}))^i \left[\|\phi\| + \|(c+y^2)^{-\frac{1}{2}} \phi\|^2 \right] \\
 &\lesssim (a \ln(a^{-1}))^i \|\phi\| + a^{i-1} \|\phi\|^2.
 \end{aligned}$$

Estimates (118) show that $g = \frac{\ln(a^{-1})}{a} (1 + o(1))$, provided

$$\begin{aligned}
 (124) \quad a^{-1}|S_{02}|, |S_{01}|, |S_{02}|, |S_{12}|, a \ln(a^{-1})|S_{11}|, a|S_{02}||S_{11}|, a|S_{01}||S_{12}| \\
 \ll a^{-1} \ln(a^{-1}),
 \end{aligned}$$

which holds true whenever $\|\phi\| \ll 1$. Therefore, g is invertible and its inverse is of the form $g^{-1} = \frac{a}{\ln(a-1)}(1 - o(1))$. Hence, equations (117) can be rewritten as (49), (50), with $S_i(\phi, a, b, c) = \frac{f_i}{g}$ and $\mathcal{R}_i(\phi, a, b, c) = \frac{r_i}{g}$. Then the estimates of f_i , r_i and g given above, imply (51), (52). \square

Proof of estimates (108)–(111). Since $e^{-ay^2/2} = -\frac{1}{ay}\partial_y e^{-ay^2/2}$, integration by parts yields

$$\langle 1, \zeta_{bc0} \rangle = \frac{1}{16} \int_0^\infty y(c+y^2)^{2d} e^{-a\frac{y^2}{2}} dy = \frac{1}{16a} \left(c^{2d} + 4d \int_0^\infty (c+y^2)^{2d-1} y e^{-a\frac{y^2}{2}} dy \right).$$

To extract the leading part in the last integral above, we rescale $y \rightarrow \sqrt{ay}$ to obtain

$$\int_0^\infty (c+y^2)^{2d-1} y e^{-a\frac{y^2}{2}} dy = a^{-2d} \int_0^\infty (ac+y^2)^{2d-1} y e^{-\frac{y^2}{2}} dy.$$

Next, we split the integral up as follows:

$$\int_0^\infty (ac+y^2)^{2d-1} y e^{-\frac{y^2}{2}} dy = \int_0^1 (ac+y^2)^{2d-1} y e^{-\frac{y^2}{2}} dy + \int_1^\infty (ac+y^2)^{2d-1} y e^{-\frac{y^2}{2}} dy.$$

Since the second term on the left is uniformly bounded in a , d small, it suffices to investigate the first term. Write

$$\int_0^1 (ac+y^2)^{2d-1} y e^{-\frac{y^2}{2}} dy = \int_0^1 (ac+y^2)^{2d-1} y dy + \int_0^1 (ac+y^2)^{2d-1} y (e^{-\frac{y^2}{2}} - 1) dy,$$

where again the second term is uniformly bounded in a , d small. Explicit integration in the first term yields

$$\int_0^1 (ac+y^2)^{2d-1} y dy = \frac{1}{4d} ((1+ac)^{2d} - (ac)^{2d}).$$

By assumption, there exists $\varepsilon > 0$ such that $d(a) \leq a^\varepsilon$. In particular, $a^d \rightarrow 1$ as $a \rightarrow 0$, whence

$$(1+ac)^{2d} - (ac)^{2d} = (1+2dO(ac)) - (1+2dO(\ln ac)) = O\left(d \ln \frac{1}{a}\right),$$

yielding

$$\langle 1, \zeta_{bc0} \rangle = \frac{1}{16a} \left(1 + O\left(d \ln \frac{1}{a}\right) \right).$$

The remaining terms are estimated similarly. \square

Addition. While the present paper has been in press, we became aware of the recent remarkable preprint:

P. Raphael and R. Schweyer, *On the stability of critical chemotactic aggregation*, [arXiv:1209.2517](https://arxiv.org/abs/1209.2517), which presents a fairly complete rigorous analysis of singularity formation in the Keller–Segel equations. The analysis in this paper is quite different from ours.

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