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BOUNDARY BEHAVIOR AND THE DIRICHLET PROBLEM FOR BELTRAMI EQUATIONS

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Abstract. It is shown that a homeomorphic solution of the Beltrami equation $\bar{\partial} f = \mu\, \partial f$ in the Sobolev class $W^{1,1}_{\mathrm{loc}}$ is a so-called ring and, simultaneously, lower Q-homeomorphism with $Q(z) = K_\mu(z)$, where $K_\mu(z)$ is the dilatation ratio of this equation. On this basis, the theory of the boundary behavior of such solutions is developed and, under certain conditions on $K_\mu(z)$, the existence of regular solutions is established for the Dirichlet problem for degenerate Beltrami equations in arbitrary Jordan domains. Also, the existence of pseudoregular as well as many-valued solutions is proved in the case of arbitrary finitely connected domains bounded by mutually disjoint Jordan curves.

§1. Introduction

In this paper, the recent results of the Donetsk school in the theory of the so-called ring and lower Q-homeomorphisms are applied to the study of the boundary behavior of arbitrary homeomorphic solutions with distributional derivatives and of the Dirichlet problem for the Beltrami equations with degeneration, see, e.g., the monographs [1, 2] and also the papers [3, 4, 5, 6, 7, 8]. The corresponding existence theorems for homeomorphic solutions in the class $W^{1,1}_{\mathrm{loc}}$ have recently been proved for many degenerate Beltrami equations, see, e.g., [1, 2] and also the surveys [9, 10] with subsequent references therein.

Let D be a domain of the complex plane $\mathbb{C}$, i.e., an open connected subset of $\mathbb{C}$, and let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. (almost everywhere) in D. Equations of the form

$$f_{\bar z} = \mu(z) f_z \tag{1}$$

are called the *Beltrami equations*. Here $f_{\bar z} = \bar{\partial} f = (f_x + i f_y)/2$, $f_z = \partial f = (f_x - i f_y)/2$, $z = x + iy$, f_x and f_y are the partial derivatives of a mapping f in x and y, respectively. The function μ is called the *complex coefficient*, and

$$K_\mu(z) := \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \tag{2}$$

is the *dilatation ratio* of equation (1). The Beltrami equation (1) is said to be *degenerate* if K_μ is not essentially bounded, i.e., $K_\mu \notin L^\infty(D)$.

Any analytic function f in the domain D satisfies the simplest Beltrami equation

$$f_{\bar z} = 0, \tag{3}$$

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in which $\mu(z) \equiv 0$. If a function f is analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and continuous in its closure, then by the Schwarz formula we have

$$f(z) = i \operatorname{Im} f(0) + \frac{1}{2\pi i} \int_{|\zeta|=1} \operatorname{Re} f(\zeta) \cdot \frac{\zeta + z}{\zeta - z} \cdot \frac{d\zeta}{\zeta}, \tag{4}$$

and, thus, an analytic function f in the unit disk $\mathbb{D}$ is determined uniquely up to a purely imaginary number ic, $c = \operatorname{Im} f(0)$, by its real part $\varphi(\zeta) = \operatorname{Re} f(\zeta)$ on the boundary of the unit disk. Obviously, if f solves the Dirichlet problem, then the function $F(z) = f(z)+ic$ also solves it for every constant $c \in \mathbb{R}$.

Hence, the *Dirichlet problem* for the Beltrami equation (1) in a domain D is the problem on the existence of a continuous function $f : D \to \mathbb{C}$ having partial derivatives of the first order a.e. and satisfying equation (1) a.e. and also the boundary condition

$$\lim_{z \to \zeta} \operatorname{Re} f(z) = \varphi(\zeta), \qquad \zeta \in \partial D, \tag{5}$$

for a prescribed continuous function $\varphi : \partial D \to \mathbb{R}$.

For the first time, boundary value problems for Beltrami equations were studied in the famous dissertation of Riemann, who considered the partial case where $\mu(z) \equiv 0$; then, such problems were investigated by Hilbert (1904, 1924), who treated the corresponding Cauchy–Riemann system for the real and imaginary parts of analytic functions $f = u+iv$, and also by Poincaré (1910) in his work on influxes. The Dirichlet problem has been well studied for uniformly elliptic systems of equations, see, e.g., [11] and [12]. The Dirichlet problem for degenerate Beltrami equations in the unit disk was studied in [13]. However, the solvability criteria for the Dirichlet problem formulated in [13] are not invariant with respect to Riemann mappings. Therefore, here we give theorems on the existence of regular solutions of the Dirichlet problem in simply connected domains as well as theorems on the existence of pseudoregular and many-valued solutions in multiply connected domains.

§2. Relationship between $W^{1,1}_{\mathrm{loc}}$ and lower Q-homeomorphisms

A continuous mapping γ acting from an open subset Δ of the real axis $\mathbb{R}$ or a circle into D is called a *dashed line* in D, see, e.g., [2, Section 6.3]. Recall that any open set Δ in $\mathbb{R}$ consists of a countable collection of mutually disjoint intervals. That gives a motivation for the term "dashed line".

Given a family Γ of dashed lines γ in the complex plane $\mathbb{C}$, we say that a Borel function $\varrho : \mathbb{C} \to [0, \infty]$ is *admissible* for Γ, and write $\varrho \in \operatorname{adm} \Gamma$, if

$$\int_\gamma \varrho \, ds \geq 1 \quad \text{for any} \quad \gamma \in \Gamma. \tag{6}$$

The *conformal modulus* of the family Γ is the quantity

$$M(\Gamma) = \inf_{\varrho \in \operatorname{adm} \Gamma} \int_{\mathbb{C}} \varrho^2(z) \, dm(z), \tag{7}$$

where $dm(z)$ stands for the Lebesgue measure in $\mathbb{C}$. We say that a property P holds for *a.e.* (almost every) $\gamma \in \Gamma$ if the subfamily of all lines in Γ for which P fails has modulus zero, cf. [14]. Also, a (Lebesgue) measurable function $\varrho : \mathbb{C} \to [0, \infty]$ is said to be *extensively admissible* for Γ (we write $\varrho \in \operatorname{ext} \operatorname{adm} \Gamma$) if (6) is true for a.e. $\gamma \in \Gamma$, see, e.g., [2, §9.2].

The following notion was motivated by the ring definition of quasiconformal mappings given by Gehring, see, e.g., [15]. For domains D and D' in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $z_0 \in \overline{D} \setminus \{\infty\}$, and a measurable function $Q : D \to (0,\infty)$, a homeomorphism $f : D \to D'$ is said to be a *lower Q-homeomorphism at the point z_0* if

$$M(f\Sigma_\varepsilon) \ge \inf_{\varrho\in \mathrm{ext\,adm}\,\Sigma_\varepsilon} \int_{D\cap R_\varepsilon} \frac{\varrho^2(z)}{Q(z)}\, dm(z) \tag{8}$$

for every ring

$$R_\varepsilon = \{z \in \bar{\mathbb{C}} : \varepsilon < |z - z_0| < \varepsilon_0\}, \quad \varepsilon \in (0, \varepsilon_0),\ \varepsilon_0 \in (0, d_0),$$

where

$$d_0 = \sup_{z\in D} |z - z_0|,$$

and Σ_ε denotes the family of dashed lines formed by all intersections of the circles

$$S(r) = S(z_0, r) = \{z \in \mathbb{C} : |z - z_0| = r\}, \quad r \in (\varepsilon, \varepsilon_0),$$

with the domain D, see [5] or [2, Chapter 9].

This notion can be extended to the case of $z_0 = \infty \in \bar{D}$ in a standard way, by applying the inversion T with respect to the unit circle in $\bar{\mathbb{C}}$, $T(z) = z/|z|^2$, $T(\infty) = 0$, $T(0) = \infty$. A homeomorphism $f : D \to D'$ is called a *lower Q-homeomorphism at $\infty \in \bar{D}$* if the mapping $F = f \circ T$ is a lower Q_*-homeomorphism with $Q_* = Q \circ T$ at 0. Also, a homeomorphism $f : D \to \bar{\mathbb{C}}$ is said to be a *lower Q-homeomorphism on ∂D* if f is a lower Q-homeomorphism at every point $z_0 \in \partial D$.

We recall a criterion for a homeomorphism to be a lower Q-homeomorphism, see Theorem 2.1 in [5] or Theorem 9.2 in [2].

Proposition 1. *Let D and D' be domains in $\mathbb{C}$, let $z_0 \in \bar{D} \setminus \{\infty\}$, and let $Q : D \to (0,\infty)$ be a Lebesgue measurable function. A homeomorphism $f : D \to D'$ is a lower Q-homeomorphism at a point z_0 if and only if*

$$M\big(f(\Sigma_\varepsilon)\big) \ge \int_\varepsilon^{\varepsilon_0} \frac{dr}{\| Q\|_1(r)} \quad \textit{for all } \ \varepsilon \in (0, \varepsilon_0),\ \varepsilon_0 \in (0, d_0), \tag{9}$$

where

$$\|Q\|_1(r) = \int_{D(z_0,r)} Q(z)\, ds \tag{10}$$

is the L_1-norm of the function Q over the set

$$D(z_0, r) = D \cap S(z_0, r) = \{z \in D : |z - z_0| = r\}.$$

Later on, we shall show that every homeomorphic solution of the Beltrami equation (1) belonging to the Sobolev class $W^{1,1}_{\mathrm{loc}}$ is a lower Q-homeomorphism with $Q(z) = K_\mu(z)$ and, thus, the entire theory of boundary behavior in [5], see also Chapter 9 in [2], can be applied to such solutions. This fact is important for the study of boundary value problems for Beltrami equations with degeneration, see, e.g., [13].

Theorem 1. *Let f be a homeomorphic solution of class $W^{1,1}_{\mathrm{loc}}$ for the Beltrami equation* (1)*. Then f is a lower Q-homeomorphism at every point $z_0 \in \bar{D}$ with $Q(z) = K_\mu(z)$.*

Proof. Let B be the Borel set of all points z in D at which f has a total differential with $J_f(z) \ne 0$. It is known that B can be represented as a union of countably many Borel sets B_l, $l = 1, 2, \dots$, such that the restrictions $f_l = f|_{B_l}$ are bi-Lipschitz homeomorphisms, see, e.g., Lemma 3.2.2 in [25]. Without loss of generality, we may assume that the B_l are mutually disjoint. Let B_* be a set of all points $z \in D$ where f has a total differential with $f_z = 0 = f_{\bar{z}}$.

Note that, by the well-known Gehring–Lehto–Menshov theorem, $B_0 = D \setminus (B \cup B_*)$ has Lebesgue measure zero in $\mathbb{C}$, see [26, 27]. Consequently, by Theorem 2.11 in [28], see also Lemma 9.1 in [2], $l(\gamma \cap B_0) = 0$ for a.e. dashed lines γ in D. We show that $l(f(\gamma) \cap f(B_0)) = 0$ for a.e. circle γ with center at the point z_0.

This follows from the absolute continuity of f on the closed subarcs of $\gamma \cap D$ for a.e. γ. Indeed, the class $W^{1,1}_{\text{loc}}$ is invariant with respect to locally quasiisommetric transformations of an independent variable, see, e.g., Theorem 1.1.7 in [29], and the functions in $W^{1,1}_{\text{loc}}$ are absolutely continuous on lines, see, e.g., Theorem 1.1.3 in [29]. Using, for instance, the coordinate transformation $\log(z - z_0)$, we arrive at absolute continuity on a.e. circle γ with center at z_0.

Thus, $l(\gamma_* \cap f(B_0)) = 0$, where $\gamma_* = f(\gamma)$, for a.e. circle γ centered at z_0. Now, suppose $\varrho_* \in \operatorname{adm} f(\Gamma)$, $\varrho_* \equiv 0$ outside of $f(D)$, where Γ is the collection of all dashed lines formed by the intersections of all circles γ centered at z_0 with D. Let $\varrho \equiv 0$ outside of D, and let

$$\varrho(z) := \varrho_*(f(z))\big(|f_z| + |f_{\bar z}|\big) \quad \text{for a.e.} \quad z \in D.$$

Arguing piecewise on B_l and using [25, Theorem 3.2.5] (for $m = 1$), we obtain

$$\int_\gamma \varrho\, ds \geq \int_{\gamma_*} \varrho_*\, ds_* \geq 1 \quad \text{for a.e.} \quad \gamma \in \Gamma$$

because $l(f(\gamma) \cap f(B_0)) = 0$, and also $l(f(\gamma) \cap f(B_*)) = 0$ for a.e. $\gamma \in \Gamma$ by the absolute continuity of f on a.e. $\gamma \in \Gamma$. Consequently, $\varrho \in \operatorname{ext} \operatorname{adm} \Gamma$.

On the other hand, again arguing piecewise on B_l, we get

$$\int_D \frac{\varrho^2(x)}{K_\mu(z)}\, dm(z) \leq \int_{f(D)} \varrho_*^2(w)\, dm(w),$$

because $\varrho(z) = 0$ on B_*. Consequently,

$$M(f\Gamma) \geq \inf_{\varrho \in \operatorname{ext} \operatorname{adm} \Gamma} \int_D \frac{\varrho^2(z)}{K_\mu(z)}\, dm(z),$$

i.e., f is indeed a lower Q-homeomorphism with $Q(z) = K_\mu(z)$. □

§3. Relationship with ring Q-homeomorphisms

Let D be a domain in $\mathbb{C}$, and let $Q : D \to [0, \infty]$ be a Lebesgue measurable function. Set

$$A(z_0, r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\},$$
$$S_i = S(z_0, r_i) = \{z \in \mathbb{C} : |z - z_0| = r_i\}, \quad i = 1, 2.$$

As usual, $\Delta(E, F; D)$ denotes the family of all curves $\gamma : [a, b] \to \overline{\mathbb{C}}$ connecting E and F in D, i.e, $\gamma(a) \in E$, $\gamma(b) \in F$, and $\gamma(t) \in D$ for $a < t < b$.

The following notion is motivated by the ring definition of quasiconformal mappings given by Gehring, see, e.g., [15], and is closely related to solutions of degenerate Beltrami type equations in the plane.

We say that a homeomorphism $f : D \to \overline{\mathbb{C}}$ is a *ring Q-homeomorphism at a point* $z_0 \in D$ if f satisfies the inequality

$$M\big(\Delta(fS_1, fS_2; fD)\big) \leq \int_A Q(z) \cdot \eta^2(|z - z_0|)\, dm(z) \tag{11}$$

for any ring $A = A(z_0, r_1, r_2)$, $0 < r_1 < r_2 < d(z_0) = \operatorname{dist}(z_0, \partial D)$, and any Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int_{r_1}^{r_2} \eta(r)\, dr \geq 1. \tag{12}$$

We say that a homeomorphism f from D into $\overline{\mathbb{C}}$ is a *ring Q-homeomorphism* in D if condition (11) is fulfilled for all points $z_0 \in D$.

The above notion was first introduced in [30] in connection with investigations of the Beltrami equations in the plane, and later it was extended to the space case, see [8] and also [2].

For the first time, the ring Q-homeomorphisms at boundary points of a domain D were considered in the papers [32] and [33]. We say that a homeomorphism $f : D \to \overline{\mathbb{C}}$ is a *ring Q-homeomorphism at a boundary point* $z_0 \in \partial D$ if

$$M\big(\Delta(fC_1, fC_2; fD)\big) \le \int_{A\cap D} Q(z) \cdot \eta^2(|z - z_0|)\, dm(z) \tag{13}$$

for any ring $A = A(z_0, r_1, r_2)$ and arbitrary continua C_1 and C_2 in D belonging to two different components of the complement of the ring A in $\overline{\mathbb{C}}$ (which contain z_0 and ∞), and for any measurable function $\eta : (r_1, r_2) \to [0, \infty]$ satisfying (12). We say that a homeomorphism $f : D \to \overline{\mathbb{C}}$ is a *ring Q-homeomorphism in $\overline{D}$* if condition (13) is fulfilled for all points $z_0 \in \overline{D}$.

We present a criterion for a homeomorphism to be a ring Q-homeomorphism at inner points of a domain, see Theorem 2.1 in [8] and also Theorem 7.2 in [2]. Below, we use the following standard agreements: $a/\infty = 0$ for $a \ne \infty$, $a/0 = \infty$ for $a > 0$ and $0 \cdot \infty = 0$, see, e.g., [35, I.3].

Proposition 2. *Let $Q : D \to [0, \infty]$ be a Lebesgue measurable function. A homeomorphism $f : D \to \mathbb{C}$ is a ring Q-homeomorphism at a point $z_0 \in D$ if and only if*

$$M\big(\Delta(fS_1, fS_2; fD)\big) \le \frac{2\pi}{I} \tag{14}$$

for all $0 < r_1 < r_2 < d(z_0) = \operatorname{dist}(z_0, \partial D)$, where $S_i = S(z_0, r_i)$, $i = 1, 2$,

$$I = I(r_1, r_2) = \int_{r_1}^{r_2} \frac{dr}{r q_{z_0}(r)},$$

and $q_{z_0}(r)$ is the average of the function Q over the circle $|z - z_0| = r$.

Lemma 1. *Let D and D' be domains in $\mathbb{C}$, $Q : D \to (0, \infty)$ a Lebesgue measurable function, and $f : D \to D'$ a lower Q-homeomorphism at a point $z_0 \in \overline{D}$. Then*

$$M\big(\Delta(fS_1, fS_2; fD)\big) \le \frac{1}{I} \tag{15}$$

where $S_i = S(z_0, r_i) = \{z \in \mathbb{C} : |z - z_0| = r_i\}$, $i = 1, 2$, $0 < r_1 < r_2$,

$$I = I(z_0, r_1, r_2) = \int_{r_1}^{r_2} \frac{dr}{\|Q\|_1(z_0, r)}, \tag{16}$$

and

$$\|Q\|_1(z_0, r) = \int_{D(z_0, r)} Q(z)\, ds \tag{17}$$

is the L^1-norm of Q on $D(z_0, r) = \{z \in D : |z - z_0| = r\} = D \cap S(z_0, r)$.

Lemma 2. *Suppose D is a domain in $\mathbb{C}$, $z_0 \in \overline{D}$, $0 < r_1 < r_2 < d(z_0) := \sup_{z\in D} |z - z_0|$, $A = A(z_0, r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$ is a ring, and $Q : \mathbb{C} \to (0, \infty)$ is a locally integrable function. Set*

$$\eta_0(t) = \frac{1}{I \cdot \|Q\|_1(z_0, t)}, \tag{18}$$

where $I = I(x_0, r_1, r_2)$ *and* $\|Q\|_1(z_0, r)$, $r \in (r_1, r_2)$, *are defined as in* (16) *and* (17), *respectively. Then*

$$\frac{1}{I} = \int_{A\cap D} Q(z)\cdot \eta_0^2(|z-z_0|)\, dm(z) \le \int_{A\cap D} Q(z)\cdot \eta^2(|z-z_0|)\, dm(z) \tag{19}$$

for any measurable function $\eta : (r_1, r_2) \to [0, \infty]$ *such that*

$$\int_{r_1}^{r_2} \eta(r) dr = 1. \tag{20}$$

Proof. If $I = \infty$, then the left-hand side of (19) is equal to zero, and the inequality is obvious. If $I = 0$, then $\|Q\|_1(z_0, r) = \infty$ for a.e. $r \in (r_1, r_2)$, and the two sides of (19) are infinite by the Fubini theorem. Now, let $0 < I < \infty$. Then $\|Q\|_1(z_0, r) \neq 0$ and $\eta_0(r) \neq \infty$ a.e. in (r_1, r_2). Setting

$$\alpha(r) = \eta(r)\cdot \|Q\|_1(z_0, r),$$
$$\omega(r) = \frac{1}{\|Q\|_1(z_0, r)},$$

and using the standard agreements, we see that $\eta(r) = \alpha(r)\omega(r)$ a.e. in (r_1, r_2) and that

$$C := \int_{A\cap D} Q(z)\cdot \eta^2(|z-z_0|)\, dm(z) = \int_{r_1}^{r_2} \alpha^2(r)\cdot \omega(r)\, dr.$$

Applying the weighted Jensen inequality, see, e.g., Theorem 2.6.2 in [36], to the convex function $\varphi(t) = t^2$ on the interval $\Omega = (r_1, r_2)$ with the probability measure

$$\nu(E) = \frac{1}{I}\int_E \omega(r)\, dr,$$

we obtain

$$\left(⨍ \alpha^2(r)\omega(r) dr\right)^{1/2} \ge ⨍ \alpha(r)\omega(r)\, dr = \frac{1}{I};$$

here we have also applied the fact that $\eta(r) = \alpha(r)\omega(r)$ satisfies (20). Thus,

$$C \ge \frac{1}{I},$$

which implies (19). ☐

Combining Lemmas 1 and 2, we obtain the following statement.

Corollary 1. *Under the hypotheses and in the notation of Lemmas* 1 *and* 2, *we have*

$$M\big(\Delta(fS_1, fS_2; fD)\big) \le \int_{A\cap D} Q(z)\cdot \eta^2(|z-z_0|)\, dm(z). \tag{21}$$

Theorem 2. *Let* D *and* D' *be domains in* $\mathbb{C}$, *and let* $Q : \mathbb{C} \to (0, \infty)$ *be a locally integrable function. If* $f : D \to D'$ *is a lower* Q*-homeomorphism at a point* $z_0 \in \bar{D}$, *then* f *is a ring* Q*-homeomorphism at* z_0.

Proof. The family $\Delta(fC_1, fC_2; fD)$ is minorized by the family $\Delta(fS_1, fS_2; fD)$, where $S_1 = S(z_0, r_1)$ and $S_2 = S(z_0, r_2)$. Therefore, we have

$$M\big(\Delta(fC_1, fC_2; fD)\big) \le M\big(\Delta(fS_1, fS_2; fD)\big),$$

and the conclusion of Theorem 2 is obtained with the help of Corollary 1. ☐

Theorems 1 and 2 show that the following conclusion is true.

Theorem 3. *Let* f *be a homeomorphic solution of the Beltrami equation* (1) *with* $K_\mu \in L^1(D)$ *of class* $W^{1,1}_{\text{loc}}$. *Then* f *is a ring* Q*-homeomorphism at every point* $z_0 \in \bar{D}$ *with* $Q(z) = K_\mu(z)$.

Thus, the theory of the boundary behavior of ring Q-homeomorphisms, as presented in [1] and [7], can also be applied to the study of arbitrary homeomorphic solutions for the Beltrami equation with generalized derivatives.

§4. On domains with regular boundary

First, we recall the following topological notion. A domain $D \subset \mathbb{C}$ is said to be *locally connected at a point* $z_0 \in \partial D$ if, for every neighborhood U of the point z_0, there exists a neighborhood $V \subseteq U$ of z_0 such that $V \cap D$ is connected. Note that any Jordan domain D in $\mathbb{C}$ is locally connected at every point in ∂D, see, e.g., [16, p. 66].

We say that ∂D is *weakly flat at a point* $z_0 \in \partial D$ if, for every neighborhood U of the point z_0 and every number $P > 0$, there exists a neighborhood $V \subset U$ of z_0 such that

$$M\big(\Delta(E, F; D)\big) \geq P \tag{22}$$

for all continua E and F in D intersecting ∂U and ∂V. We also say that the boundary ∂D is *weakly flat* if it is weakly flat at every point of ∂D.

A point $z_0 \in \partial D$ is said to be *strongly accessible* if, for every neighborhood U of the point z_0, there exists a compact set E in D, a neighborhood $V \subset U$ of z_0, and a number $\delta > 0$ such that

$$M\big(\Delta(E, F; D)\big) \geq \delta \tag{23}$$

for every continuum F in D intersecting ∂U and ∂V. We say that the boundary ∂D is *strongly accessible* if every point $z_0 \in \partial D$ is strongly accessible.

Here, in the definitions of strongly accessible and weakly flat boundaries, for the role of the neighborhoods U and V of z_0, we can take either only disks (closed or open) centered at z_0, or only neighborhoods of z_0 in another fundamental system of the point z_0. These notions can be extended naturally to the case of $\overline{\mathbb{C}}$ and to the case where $z_0 = \infty$. It is necessary to use the corresponding neighborhoods of ∞ in this case.

It is easily seen that if a domain D in $\mathbb{C}$ is weakly flat at a point $z_0 \in \partial D$, then the point z_0 is strongly accessible from D. Moreover, it was proved that if a domain D in $\mathbb{C}$ is weakly flat at a point $z_0 \in \partial D$, then D is locally connected at z_0, see, e.g., Lemma 5.1 in [5] or Lemma 3.15 in [2].

The notions of strongly accessible and weakly flat points of a domain in $\mathbb{C}$ were suggested in [17]; they localize and generalize the corresponding notions introduced in [18, 19], compare with the properties P_1 and P_2 introduced by Väisälä in [20] and also with quasiconformal accessibility and quasiconformal flatness introduced by Näkki in [21]. Many theorems on homeomorphic extension to the boundary for quasiconformal mappings and their generalizations hold true under the condition that boundaries are weakly flat. The strong accessibility condition plays a similar role in the theory of continuous extension of mappings to the boundary. In particular, recently we proved the following statements, see Theorem 10.1 (Lemma 6.1) in [5] or Theorem 9.8 (Lemma 9.4) in [2].

Proposition 3. *Let D and D' be bounded domains in $\mathbb{C}$, let $Q : D \to (0, \infty)$ be a measurable function, and let $f : D \to D'$ be a lower Q-homeomorphism at ∂D. Suppose that the domain D is locally connected at ∂D, and that $\partial D'$ is* (*strongly accessible*) *weakly flat. If*

$$\int_0^{\delta(z_0)} \frac{dr}{\|Q\|_1(z_0, r)} = \infty \quad \textit{for all} \quad z_0 \in \partial D \tag{24}$$

and for some $\delta(z_0) \in (0, d(z_0))$, *where* $d(z_0) = \sup_{z\in D} |z - z_0|$ *and*

$$\|Q\|_1(z_0, r) = \int_{D\cap S(z_0,r)} Q(z)\, ds,$$

then f *has a (continuous) homeomorphic extension* $\bar{f}$ *to* $\bar{D}$ *that maps* $\bar{D}$ *onto* $\overline{D'}$.

Here, as usual, $S(z_0, r)$ denotes the circle $|z - z_0| = r$.

A domain $D \subset \mathbb{C}$ is called a *quasiextremal distance domain* (a QED-*domain*, see [22]), if

$$M\big(\Delta(E, F; \bar{\mathbb{C}})\big) \le K \cdot M\big(\Delta(E, F; D)\big) \tag{25}$$

for some $K \ge 1$ and any two disjoint continua E and F in D.

It is well known (see, e.g., Theorem 10.12 in [20]) that

$$M(\Delta(E, F; \mathbb{C})) \ge \frac{2}{\pi} \log \frac{R}{r} \tag{26}$$

for any sets E and F in $\mathbb{C}$ intersecting all circles $S(z_0, \rho)$, $\rho \in (r, R)$. Consequently, the QED-domains have weakly flat boundaries. The example presented in [2, Subsection 3.8] shows that the converse conclusion fails even for simply connected domains in the plane.

A domain $D \subset \mathbb{C}$ is called a *uniform domain* if any two points z_1 and $z_2 \in D$ can be connected by a rectifiable curve γ in D such that

$$s(\gamma) \le a \cdot |z_1 - z_2| \tag{27}$$

and

$$\min_{i=1,2} s(\gamma(z_i, z)) \le b \cdot d(z, \partial D) \tag{28}$$

for all $z \in \gamma$, where $\gamma(z_i, z)$ is a part of the curve γ with the ends z_i and z, see [23]. It is known that every uniform domain is a QED-domain, but there exist QED-domains that are not uniform, see [22]. Bounded convex domains and domains with smooth boundaries give simple examples of uniform domains and, consequently, QED-domains as well as domains with weakly flat boundaries.

Lipschitz domains are often met in the theory of mappings and differential equations. A domain D in $\mathbb{C}$ is said to be *Lipschitz* if every point $z_0 \in \partial D$ has a neighborhood U that is transformed under some bi-Lipschitz mapping onto the unit disk $\mathbb{D}$ in $\mathbb{C}$ in such a way that $\partial D \cap U$ is transformed onto the intersection of $\mathbb{D}$ and the real axis.

In this connection, we recall that a mapping $f : X \to X'$ between metric spaces (X, d) and (X', d') is said to be *Lipschitz* if $d'(f(x_1), f(x_2)) \le C \cdot d(x_1, x_2)$ for all $x_1, x_2 \in X$ and some finite constant C. If, moreover, $d(x_1, x_2) \le c \cdot d'(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$, we say that the mapping f is *bi-Lipschitz*. Note that the bi-Lipschitz mappings f in $\mathbb{C}$ are quasiconformal for which the modulus is quasiinvariant. Hence, the boundaries of Lipschitz domains are weakly flat.

In what follows, $C(X, f)$ denotes the *cluster set* of the mapping $f : D \to \bar{\mathbb{C}}$ on the set $X \subset \bar{D}$,

$$C(X, f) := \big\{ w \in \bar{\mathbb{C}} : w = \lim_{k\to\infty} f(z_k),\ \ z_k \to z_0 \in X,\ \ z_k \in D \big\}. \tag{29}$$

Note that $C(\partial D, f) \subseteq \partial D'$ for any homeomorphism $f : D \to D'$, see, e.g., Proposition 13.5 in [2].

§5. Boundary behavior of homeomorphic solutions

By Theorem 1, the following statement is a consequence of Theorem 6.1 in [5] or Lemma 9.4 in [2].

Lemma 3. *Let D and D' be domains in $\mathbb{C}$, let $z_0 \in \partial D$, and let $f : D \to D'$ be a homeomorphic solution of the Beltrami equation* (1) *of class $W^{1,1}_{\mathrm{loc}}$. Suppose that the domain D is locally connected at $z_0 \in \partial D$ and $\partial D'$ is strongly accessible at least at one point of the cluster set $C(z_0, f)$. If*

$$\int_0^{\varepsilon_0} \frac{dr}{\|K_\mu\|_1(r)} = \infty, \tag{30}$$

where $0 < \varepsilon_0 < d_0 = \sup_{z\in D} |z - z_0|$ and

$$\|K_\mu\|_1(r) = \int_{D\cap S(z_0,r)} K_\mu \, ds, \tag{31}$$

then f is extended to the point z_0 by continuity in $\overline{\mathbb{C}}$.

The following lemma on cluster sets serves as a basis for the proof of extendability of inverse mappings for homeomorphic solutions of class $W^{1,1}_{\mathrm{loc}}$ for the Beltrami equation (1).

Lemma 4. *Let D and D' be domains in $\mathbb{C}$, z_1 and z_2 different points of ∂D, $z_1 \neq \infty$, and $f : D \to D'$ a homeomorphic solution of the Beltrami equation* (1) *of class $W^{1,1}_{\mathrm{loc}}$. Suppose that the function K_μ is integrable on the dashed lines*

$$D(r) = \{z \in D : |z - z_1| = r\} = D \cap S(z_1, r) \tag{32}$$

for a set E of numbers $r < |z_1 - z_2|$ of positive linear measure. If D is locally connected at z_1 and z_2 and $\partial D'$ is weakly flat, then

$$C(z_1, f) \cap C(z_2, f) = \varnothing. \tag{33}$$

By Theorem 1, Lemma 4 follows from Lemma 9.1 in [5] or Lemma 9.5 in [2].

As an immediate consequence of Lemma 4, we obtain the following statement.

Theorem 4. *Suppose D and D' are domains in $\mathbb{C}$, D is locally connected at ∂D, and $\partial D'$ is weakly flat. If $f : D \to D'$ is a homeomorphic solution of the Beltrami equation* (1) *of class $W^{1,1}_{\mathrm{loc}}$ with $K_\mu \in L^1(D)$, then f^{-1} admits extension to $\overline{D'}$ by continuity in $\overline{\mathbb{C}}$.*

Proof. Indeed, by the Fubini theorem, the set

$$E = \{r \in (0, d) : K_\mu|_{D(r)} \in L^1(D(r))\} \tag{34}$$

has positive linear measure because $K_\mu \in L^1(D)$. □

Remark 1. In Theorem 4 it suffices to assume that K_μ is integrable only in a neighborhood of ∂D.

Moreover, using Theorem 1 and Theorem 9.2 in [5] (or Theorem 9.7 in [2]), we get the following result.

Theorem 5. *Suppose D and D' are domains in $\mathbb{C}$, D is locally connected at ∂D, and $\partial D'$ is weakly flat, and let $f : D \to D'$ be a homeomorphic solution of the Beltrami equation* (1) *of class $W^{1,1}_{\mathrm{loc}}$ with a coefficient μ such that*

$$\int_0^{\delta(z_0)} \frac{dr}{\|K_\mu\|_1(z_0, r)} = \infty \quad \text{for all } z_0 \in \partial D, \tag{35}$$

where $\delta(z_0) \in (0, d(z_0))$, $d(z_0) = \sup_{z\in D} |z - z_0|$, *and*

$$\|K_\mu\|_1(z_0, r) = \int_{D(z_0,r)} K_\mu \, ds \tag{36}$$

is the L_1*-norm of* K_μ *over* $D(z_0, r) = \{z \in D : |z - z_0| = r\} = D \cap S(z_0, r)$. *Then there exists an extension of* f^{-1} *to* $\overline{D'}$ *by continuity in* $\overline{\mathbb{C}}$.

Combining Lemma 3 and Theorem 5, we arrive at the following statement.

Theorem 6. *Let* D *and* D' *be bounded domains in* $\mathbb{C}$, *and let* $f : D \to D'$ *be a homeomorphic solution of the Beltrami equation* (1) *of class* $W^{1,1}_{\mathrm{loc}}$. *Suppose that the domain* D *is locally connected at* ∂D *and that the boundary* $\partial D'$ *is weakly flat. If*

$$\int_0^{\delta(z_0)} \frac{dr}{\|K_\mu\|_1(z_0, r)} = \infty \quad \textit{for all} \quad z_0 \in \partial D \tag{37}$$

for some $\delta(z_0) \in (0, d(z_0))$, *where* $d(z_0) = \sup_{z\in D} |z - z_0|$ *and*

$$\|K_\mu\|_1(z_0, r) = \int_{D\cap S(z_0,r)} K_\mu \, ds, \tag{38}$$

then f *admits extension to a homeomorphism of* $\bar{D}$ *onto* $\overline{D'}$.

In particular, Theorem 6 implies the following generalization to the plane case of the well-known Gehring–Martio theorem on a homeomorphic extension to the boundary of quasiconformal mappings between QED-domains, cf. [22].

Corollary 2. *Let* D *and* D' *be domains in* $\mathbb{C}$ *with weakly flat boundaries, and let* $f : D \to D'$ *be a homeomorphic solution of the Beltrami equation* (1) *of class* $W^{1,1}_{\mathrm{loc}}$. *If condition* (37) *is fulfilled at every point* $z_0 \in \partial D$, *then* f *has a homeomorphic extension* $f : \bar{D} \to \overline{D'}$.

§6. Regular solutions of the Dirichlet problem in simply connected domains

Under the condition $\varphi(\zeta) \not\equiv \mathrm{const}$, a *regular solution* of such a problem is a continuous in $\mathbb{C}$, discrete and open mapping $f : D \to \mathbb{C}$ of class $W^{1,1}_{\mathrm{loc}}$ with the Jacobian

$$J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2 \neq 0 \quad \text{a.e.} \tag{39}$$

that satisfies condition (5) and solves equation (1) almost everywhere. In the case where $\varphi(\zeta) \equiv c$, $\zeta \in \partial D$, a regular solution of the Dirichlet problem (5) for the Beltrami equation (1) is any function $f(z) \equiv c + ic'$ with $c' \in \mathbb{R}$.

Recall that a mapping $f : D \to \mathbb{C}$ is said to be *discrete* if the preimage $f^{-1}(y)$ of every point $y \in \mathbb{C}$ consists of isolated points, and *open* if the image $f(U)$ of any open set $U \subseteq D$ is an open set in $\mathbb{C}$.

Lemma 5. *Let* $\mu : D \to \mathbb{C}$ *be a measurable function in a Jordan domain* D *such that* $|\mu(z)| < 1$ *a.e., and let* $K_\mu \in L^1(D)$. *Suppose that for every* $z_0 \in \bar{D}$ *there exists* $\varepsilon_0 < d(z_0) := \sup_{z\in D} |z - z_0|$ *and a one-parameter family of measurable functions* $\psi_{z_0,\varepsilon} : (0, \infty) \to (0, \infty)$, $\varepsilon \in (0, \varepsilon_0)$, *such that*

$$0 < I_{z_0}(\varepsilon) := \int_\varepsilon^{\varepsilon_0} \psi_{z_0,\varepsilon}(t) \, dt < \infty \quad \textit{for all} \quad \varepsilon \in (0, \varepsilon_0) \tag{40}$$

and

$$\int_{D(z_0,\varepsilon,\varepsilon_0)} K_\mu(z) \cdot \psi^2_{z_0,\varepsilon}(|z - z_0|) \, dm(z) = o(I^2_{z_0}(\varepsilon)) \tag{41}$$

as $\varepsilon \to 0$*, where* $D(z_0, \varepsilon, \varepsilon_0) = \{z \in D : \varepsilon < |z - z_0| < \varepsilon_0\}$*. Then the Dirichlet problem* (5) *for the Beltrami equation* (1) *has a regular solution* f *for any continuous function* $\varphi : \partial D \to \mathbb{R}$*.*

Proof. Let F be a regular homeomorphic solution of the Beltrami equation (1) of class $W^{1,1}_{\mathrm{loc}}$ that is a ring Q-homeomorphism in $\bar{D}$ with $Q = K_\mu$; such a solution exists by Lemma 4.1 in [33], in view of condition (41). Note that the set $\bar{\mathbb{C}} \setminus D^*$, where $D^* = F(D)$, cannot consist of only one point ∞, because otherwise the boundary of D^* would be weakly flat and, by Lemma 1 and [7, Theorem 3], F would admit a homeomorphic extension to $\bar{D}$, which is impossible because the boundary of D consists of more than one point. Moreover, the domain D^* is simply connected, see, e.g., Lemma 5.3 in [4] or Lemma 6.5 in [2]. Thus, by the Riemann theorem (see, e.g., Theorem II.2.1 in [38]), D^* can be mapped onto the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with a conformal mapping R. Since the modulus is invariant under conformal mappings, the function $g := R \circ F$ is also a regular homeomorphic solution of the Beltrami equation (1) that is a ring Q-homeomorphism in $\bar{D}$ with $Q = K_\mu$, and it maps D onto $\mathbb{D}$. Furthermore, by Lemma 1 and Theorem 3 in [7], g admits extension to a homeomorphism $g_* : \bar{D} \to \bar{\mathbb{D}}$, because $\mathbb{D}$ has a weakly flat boundary and the Jordan domain D is locally connected at its boundary.

We find a solution of the initial Dirichlet problem (5) in the form $f = h \circ g$, where h is an analytic function in $\mathbb{D}$ with the boundary condition

$$\lim_{z \to \zeta} \operatorname{Re} h(z) = \varphi(g_*^{-1}(\zeta)), \quad \text{for all } \ \zeta \in \partial\mathbb{D}.$$

As is known, the analytic function h in $\mathbb{D}$ can be recovered via the Schwarz formula, see, e.g., [39, §8, Chapter III, Part 3], by its real part on the boundary, up to a purely imaginary additive constant,

$$h(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \varphi \circ g_*^{-1}(\zeta) \cdot \frac{\zeta + z}{\zeta - z} \cdot \frac{d\zeta}{\zeta}.$$

It is easily seen that the function $f = h \circ g$ gives the desired regular solution of the Dirichlet problem (5) for the Beltrami equation (1). The proof is complete. ☐

A function $\psi : D \to \mathbb{R}$, $\psi \in L^1_{\mathrm{loc}}(D)$, is said to have *bounded mean oscillation* in the sense of John and Nirenberg (we write $\psi \in \mathrm{BMO}$) if

$$\|\psi\|_* = \sup_{B \subset D} \frac{1}{|B|} \int_B |\psi(z) - \psi_B| \, dm(z) < \infty, \tag{42}$$

where the supremum is taken over all disks $B \subset D$ and ψ_B is the average of the function ψ in the disk B. We write $\psi \in \mathrm{BMO}(\bar{D})$ if $\psi \in \mathrm{BMO}(G)$, where G is a domain in $\mathbb{C}$ including $\bar{D}$.

As in [4], we say that a function $\psi : D \to \mathbb{R}$ has *finite mean oscillation* at a point $z_0 \in D$ (and write $\psi \in \mathrm{FMO}(z_0)$) if

$$\limsup_{\varepsilon \to 0} \fint_{B(z_0, \varepsilon)} |\psi(z) - \widetilde{\psi}_\varepsilon| \, dm(z) < \infty, \tag{43}$$

where $B(z_0, \varepsilon) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$ and $\widetilde{\psi}_\varepsilon$ is the average of ψ in $B(z_0, \varepsilon)$. We write $\psi \in \mathrm{FMO}(D)$ if (43) holds true for every point $z_0 \in D$. We also write $\psi \in \mathrm{FMO}(\bar{D})$ if $\psi \in \mathrm{FMO}(G)$, where G is a domain in $\mathbb{C}$ including $\bar{D}$.

As is known, $L^\infty(D) \subset \mathrm{BMO}(D) \subset L^p_{\mathrm{loc}}(D)$ for all $p \in [1, \infty)$. However, $\mathrm{FMO}(D)$ is not a subclass of $L^p_{\mathrm{loc}}(D)$ for any $p > 1$, although $\mathrm{FMO}(D) \subset L^1_{\mathrm{loc}}(D)$, see, e.g., [2, p. 211]. Thus, FMO is substantially wider than $\mathrm{BMO}_{\mathrm{loc}}$.

A point $z_0 \in D$ is called a *Lebesgue point* of a function $\psi : D \to \mathbb{R}$ if ψ is integrable in a neighborhood of z_0 and

$$\lim_{\varepsilon \to 0} \fint_{B(z_0,\varepsilon)} |\psi(z) - \psi(z_0)| \, dm(z) = 0. \tag{44}$$

It is known that for $\psi \in L^1(D)$ almost every point of D is its Lebesgue point.

Using Lemma 5 with the choice $\psi_{z_0,\,\varepsilon}(t) \equiv 1/t \log \frac{1}{t}$, see Corollary 2.3 in [4], we obtain the following result.

Theorem 7. *Let $\mu : D \to \mathbb{C}$ be a measurable function in a Jordan domain D such that $|\mu(z)| < 1$ a.e., and let*

$$K_\mu(z) \le Q(z) \in \mathrm{FMO}(\bar{D}). \tag{45}$$

Then for any continuous function $\varphi : \partial D \to \mathbb{R}$, the Dirichlet problem (5) *for the Beltrami equation* (1) *has a regular solution f.*

Corollary 3. *In particular, the conclusion of Theorem* 7 *is valid if $K_\mu(z) \le Q(z) \in \mathrm{BMO}(\bar{D})$.*

Using [4, Corollary 2.1], we deduce the following statement from Theorem 7.

Corollary 4. *The conclusion of Theorem* 7 *is valid if*

$$\limsup_{\varepsilon \to 0} \fint_{B(z_0,\varepsilon)} K_\mu(z) \, dm(z) < \infty \quad \text{for all} \quad z_0 \in \bar{D}.$$

Here and in what follows, we assume that K_μ is extended by zero outside of the domain D.

Theorem 8. *Let $\mu : D \to \mathbb{C}$ be a measurable function in a Jordan domain D with $|\mu(z)| < 1$ a.e. and such that $K_\mu \in L^1(D)$, and let*

$$\int_0^{\delta(z_0)} \frac{dr}{\|K_\mu\|_1(z_0,\, r)} = \infty \quad \text{for all} \quad z_0 \in \bar{D}, \tag{46}$$

where $\|K_\mu\|_1(z_0, r) = \int_{S(z_0,r)} K_\mu(z)\,|dz|$ is the L^1-norm of the function K_μ on the circles $S(z_0,\, r) = \{z \in \mathbb{C} : |z - z_0| = r\}$, $0 < r < \delta(z_0) < d(z_0)$, $d(z_0) = \sup_{z \in D} |z - z_0|$. Then the Dirichlet problem (5) *for the Beltrami equation* (1) *has a regular solution f for any continuous function $\varphi : \partial D \to \mathbb{R}$.*

Proof. Theorem 8 is implied by Lemma 5 under the special choice

$$\psi_{z_0,\varepsilon}(t) \equiv \psi_z(t) = \begin{cases} 1/[t k_{z_0}(t)] & \text{if } \ t \in (0, \varepsilon_0), \\ 0 & \text{if } \ t \in [\varepsilon_0, \infty), \end{cases}$$

where $\varepsilon_0 = \varepsilon(z_0)$ and $k_{z_0}(t)$ is the average of $K_\mu(z)$ on the circle $S(z_0, t)$. □

Corollary 5. *In particular, the conclusion of Theorem* 8 *is valid if*

$$k_{z_0}(\varepsilon) = O\Big(\log \frac{1}{\varepsilon}\Big)$$

as $\varepsilon \to 0$ for all $z_0 \in \bar{D}$, where $k_{z_0}(\varepsilon)$ is the average of the function K_μ on the circle $S(z_0, \varepsilon)$.

Using Theorem 8 and [31, Theorem 3.1], we have the following result.

Theorem 9. *Let $\mu : D \to \mathbb{C}$ be a measurable function in a Jordan domain D with $|\mu(z)| < 1$ a.e. and such that*

$$\int_D \Phi(K_\mu(z))\, dm(z) < \infty, \tag{47}$$

where $\Phi : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is a monotone nondecreasing convex function with the condition

$$\int_\delta^\infty \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty \tag{48}$$

for some $\delta > \Phi(0)$. Then the Dirichlet problem (5) for the Beltrami equation (1) has a regular solution for any continuous function $\varphi : \partial D \to \mathbb{R}$.

Corollary 6. *In particular, the conclusion of Theorem 9 is valid if for some $\alpha > 0$ we have*

$$\int_D e^{\alpha K_\mu(z)}\, dm(z) < \infty. \tag{49}$$

Remark 2. By the Stoilow theorem on factorization (see, e.g., [37]), any regular solution f of the Dirichlet problem for the Beltrami equation (1) with $K_\mu \in L^1_{\mathrm{loc}}(D)$ can be represented in the form of the composition $f = h \circ F$, where h is an analytic function and F is a homeomorphic regular solution of (1) in the class $W^{1,1}_{\mathrm{loc}}$. Thus, by Theorem 5.1 in [34], condition (48) is not only sufficient but also necessary for the existence of a regular solution of the Dirichlet problem (5) with any continuous function $\varphi : \partial D \to \mathbb{R}$ for any Beltrami equation with integral restrictions of the form (47).

Setting $H(t) = \log \Phi(t)$, we observe that, by Theorem 2.1 in [31], condition (48) is equivalent to each of the following conditions:

$$\int_\Delta^\infty H'(t)\, \frac{dt}{t} = \infty, \tag{50}$$

$$\int_\Delta^\infty \frac{dH(t)}{t} = \infty, \tag{51}$$

$$\int_\Delta^\infty H(t)\, \frac{dt}{t^2} = \infty \tag{52}$$

for some $\Delta > 0$,

$$\int_0^\delta H\left(\frac{1}{t}\right)\, dt = \infty \tag{53}$$

for some $\delta > 0$, and

$$\int_{\Delta_*}^\infty \frac{d\eta}{H^{-1}(\eta)} = \infty \tag{54}$$

for some $\Delta_* > H(+0)$.

Here the integral in (51) is understood as a Lebesgue–Stieltjes integral, and the integrals in (49), (52)–(54) are understood in the Lebesgue sense.

§7. Pseudoregular solutions in multiply connected domains

As it was first noted by Bojarski (see, e.g., §6 of Chapter 4 in [12]), in the case of multiply connected domains, generally speaking, the Dirichlet problem for the Beltrami equation has no solutions in the class of continuous (single-valued) functions in $\mathbb{C}$.

Hence the following question is natural: does there exist a solution of the Dirichlet problem in a wider class? The answer turns out to be positive if we search a solution in the class of functions having some number of prescribed isolated poles in the domain D.

Namely, under the condition $\varphi(\zeta) \not\equiv \mathrm{const}$, a *pseudoregular solution* of such a problem is a continuous in $\overline{\mathbb{C}}$, discrete and open mapping $f : D \to \overline{\mathbb{C}}$ in the Sobolev class $W^{1,1}_{\mathrm{loc}}$ (outside of the poles) with the Jacobian $J_f(z) = |f_z|^2 - |f_{\bar z}|^2 \neq 0$ a.e., satisfying condition (5), and solving equation (1) a.e.

Arguing as in to the case of simply connected domains and applying Theorem V.6.2 in [38] about conformal mappings of multiply connected domains onto circular domains and also [12, Theorem 4.14], we obtain the following results.

Theorem 10. *Let D be a bounded domain in $\mathbb{C}$ whose boundary consists of $n \geq 2$ mutually disjoint Jordan curves, and let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and such that*

$$K_\mu(z) \leq Q(z) \in \mathrm{FMO}(\bar D). \tag{55}$$

Then the Dirichlet problem (5) *for the Beltrami equation* (1) *has a pseudoregular solution for any continuous function $\varphi : \partial D \to \mathbb{R}$, $\varphi(\zeta) \not\equiv \mathrm{const}$, with poles at $n-1$ prescribed inner points of D.*

Corollary 7. *In particular, the conclusion of Theorem* 10 *is valid if $K_\mu(z) \leq Q(z) \in \mathrm{BMO}(\bar D)$.*

Corollary 8. *The conclusion of Theorem* 10 *is also valid if*

$$\limsup_{\varepsilon\to 0} \fint_{B(z_0,\varepsilon)} K_\mu(z)\, dm(z) < \infty \quad \text{for all} \quad z_0 \in \bar D.$$

Theorem 11. *Let D be a bounded domain in $\mathbb{C}$ whose boundary consists of $n \geq 2$ mutually disjoint Jordan curves, and let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and such that $K_\mu \in L^1(D)$ and*

$$\int_0^{\delta(z_0)} \frac{dr}{\|K_\mu(z_0,r)\|_1(r)} = \infty \quad \text{for all} \quad z_0 \in \bar D, \tag{56}$$

where $\|K_\mu\|_1(z_0,r) = \int_{\gamma_r} K_\mu(z)|dz|$ are the L^1-norms of the function K_μ on the circles $S(z_0,r) = \{z \in \mathbb{C} : |z - z_0| = r\}$, $0 < r < \delta(z_0) < d(z_0)$, $d(z_0) = \sup_{z\in D}|z - z_0|$. Then the Dirichlet problem (5) *for the Beltrami equation* (1) *has a pseudoregular solution for any continuous function $\varphi : \partial D \to \mathbb{R}$, $\varphi(\zeta) \not\equiv \mathrm{const}$, with poles at $n-1$ prescribed inner points of D.*

Corollary 9. *In particular, the conclusion of Theorem* 11 *is valid if*

$$k_{z_0}(\varepsilon) = O\left(\log\frac{1}{\varepsilon}\right)$$

as $\varepsilon \to 0$ for all $z_0 \in \bar D$, where $k_{z_0}(\varepsilon)$ is the average of the function K_μ on the circle $S(z_0,\varepsilon)$.

Theorem 12. *Let D be a bounded domain in $\mathbb{C}$ whose boundary consists of $n \geq 2$ mutually disjoint Jordan curves, and let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and such that*

$$\int_D \Phi(K_\mu(z))\, dm(z) < \infty, \tag{57}$$

where $\Phi : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is a monotone nondecreasing convex function satisfying

$$\int_\delta^\infty \frac{d\tau}{\tau\Phi^{-1}(\tau)} = \infty \tag{58}$$

for some $\delta > \Phi(0)$. *Then the Dirichlet problem* (5) *for the Beltrami equation* (1) *has a pseudoregular solution for any continuous function* $\varphi : \partial D \to \mathbb{R}$, $\varphi(\zeta) \not\equiv \mathrm{const}$, *with poles at* $n-1$ *prescribed inner points of* D.

Corollary 10. *In particular, the conclusion of Theorem* 12 *is valid if for some* $\alpha > 0$ *we have*

$$\int_D e^{\alpha K_\mu(z)}\, dm(z) < \infty. \tag{59}$$

§8. Multivalued solutions in multiply connected domains

In multiply connected domains $D \subset \mathbb{C}$, in addition to pseudoregular solutions, the Dirichlet problem (5) for the Beltrami equation (1) admits multivalued solutions in the spirit of the theory of multivalued analytic functions. We say that a discrete and open mapping $f : B(z_0, \varepsilon_0) \to \mathbb{C}$, where $B(z_0, \varepsilon_0) \subset D$, is a *local regular solution* of equation (1) if $f \in W^{1,1}_{\mathrm{loc}}$, $J_f \neq 0$, and f satisfies (1) a.e. We say that two local regular solutions $f_0 : B(z_0, \varepsilon_0) \to \mathbb{C}$ and $f_* : B(z_*, \varepsilon_*) \to \mathbb{C}$ of equation (1) are *an extension of each other* if there exists a finite chain of local regular solutions $f_i : B(z_i, \varepsilon_i) \to \mathbb{C}$, $i = 1, \ldots, m$, such that $f_1 = f_0$, $f_m = f_*$ and $f_i(z) \equiv f_{i+1}(z)$ for $z \in E_i := B(z_i, \varepsilon_i) \cap B(z_{i+1}, \varepsilon_{i+1}) \neq \varnothing$, $i = 1, \ldots, m$. A collection of local regular solutions $f_j : B(z_j, \varepsilon_j) \to \mathbb{C}$, $j \in J$, is called a *multivalued solution* of equation (1) in D if the disks $B(z_j, \varepsilon_j)$ cover the domain D and the f_j are pairwise extensions of each other. A multivalued solution of (1) is called a *multivalued solution of the Dirichlet problem* (5) if $u(z) = \operatorname{Re} f(z) = \operatorname{Re} f_j(z)$, $z \in B(z_j, \varepsilon_j)$, $j \in J$, is a single-valued function in D satisfying condition (5).

As in the preceding sections, the proof of the existence of multivalued solutions of the Dirichlet problem (5) for the Beltrami equation (1) in multiply connected domains reduces to the Dirichlet problem for harmonic functions in circular domains; see, e.g., §3 of Chapter VI in [38].

Theorem 13. *Let* D *be a bounded domain in* $\mathbb{C}$ *whose boundary consists of a finite collection of mutually disjoint Jordan curves, and let* $\mu : D \to \mathbb{C}$ *be a measurable function with* $|\mu(z)| < 1$ *a.e. and satisfying the hypotheses of either Theorems* 10–12 *or Corollaries* 7–9. *Then the Dirichlet problem* (5) *for the Beltrami equation* (1) *has a multivalued solution for any continuous function* $\varphi : \partial D \to \mathbb{R}$.

Remark 3. It is possible to show that an analog of the well-known monodromy theorem is valid, saying that any multivalued solution of the Beltrami equation (1) in a simply connected domain D is its regular single-valued solution.

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