CWIKEL TYPE ESTIMATE AS A CONSEQUENCE OF CERTAIN PROPERTIES OF THE HEAT KERNEL

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Abstract. Estimates for the singular values of the operator \( T_{fg} := f(H)g(x) \) are investigated for suitable functions \( f(\lambda), \lambda \in \mathbb{R}, g(x), x \in \mathbb{R}^d \), and a selfadjoint operator \( H \) in \( L_2(\mathbb{R}^d) \). It is assumed that the kernel of the semigroup \( e^{-tH} \) satisfies special conditions. Power-like estimates for the singular values of the operator \( T_{fg} \) are obtained, in particular, in the case where \( T_{fg} \in \mathcal{S}_2 \). Conditions for the operator \( T_{fg} \) to belong to the trace class are established. Neither any smoothness conditions for the kernel of the operator \( T_{fg} \), nor any knowledge of the (partial) diagonalization of the operator \( H \) are required. The results admit further refinement under additional conditions imposed on the generalized eigenfunctions of the operator \( H \).

§0. Introduction

1. In the present paper we study compactness conditions as well as singular values estimates for a certain class of operators in \( L_2(\mathbb{R}^d) \). The simplest (and most important) member of this class is the operator \( f(i\nabla)g(x) := \Phi^* f\Phi g \). Here \( \Phi \) is the Fourier transform in \( L_2(\mathbb{R}^d) \), and \( f, g : \mathbb{R}^d \to \mathbb{C} \) are suitable measurable functions. Operators of this type arise naturally in the spectral theory of differential operators and have been studied thoroughly.

In [1], Cwikel obtained the following estimate for the singular values of the operator \( f(i\nabla)g(x) \):

\[
s_n(f(i\nabla)g(x)) \leq C(p, d)\|f\|_{L_p}\|g\|_{L_{p,\infty}} n^{-1/p}, \quad n \in \mathbb{N}, \quad p > 2.
\]

The restriction \( p > 2 \) in Cwikel’s work is essential, and his results give no conditions sufficient for the singular values of the operator \( f(i\nabla)g(x) \) to decay as \( O(n^{-1/p}), \ n \to +\infty, \ p \in (0, 2) \). An important generalization of Cwikel’s estimate was obtained in [2] (see Proposition 1.4 below), where integral operators bordered by scalar weights were considered. Namely, let \((\mathcal{X}, dp)\) and \((\mathcal{Y}, d\tau)\) be separable spaces with \( \sigma \)-finite measures, let \( T : L_2(\mathcal{Y}, d\tau) \to L_2(\mathcal{X}, dp) \) be an arbitrary bounded integral operator with bounded kernel \( t(x, y) \), and let \( f : \mathcal{X} \to \mathbb{C}, g : \mathcal{Y} \to \mathbb{C} \) be suitable measurable functions. In [2], integral operators \( \hat{T}_{fg} := fTg \) with kernels \( f(x)t(x, y)g(y) \) were considered, and the estimate \( s_n(\hat{T}_{fg}) \leq C(p, T)\|f\|_{L_p}\|g\|_{L_{p,\infty}} n^{-1/p}, \ n \in \mathbb{N}, \ p > 2 \) was obtained. In the papers [3] [4], it was suggested to estimate the singular values of the operators \( \hat{T}_{fg} \) by using the norm of the function \( (fg)(x, y) := f(x)g(y) \) in a suitable Lorentz class \( L_{p,q} \) on \( \mathcal{X} \times \mathcal{Y} \). We remark that conditions for integral operators to belong to \( \mathcal{S}_{p,q} \) were studied in the papers [1] [2] [3] [4] only for \( p > 2 \). The results of those papers give no conditions under which the singular values decay as \( O(n^{-1/p}), \ n \to +\infty, \ p \in (0, 2) \). Conditions

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for integral operators to belong to the classes $\mathcal{S}_{p,q}$, $p < 2$, which are contained in the Hilbert–Schmidt class, have been studied to a lesser extent. To my best knowledge, without additional assumptions about the smoothness of the kernel, such results have been available only for the operator $f(i\nabla)g(x)$ (see [5, 6] and [7]) and also for the Dirac operator (see [8]). An overview of some of the results mentioned above can be found in [7] and [4]; the paper [7] covers some applications.

In the present paper we deal with estimates for the singular values of operators of two similar types. First, consider an operator of the form $T_{fg} := f(H)g(x)$ for suitable functions $f(\lambda), \lambda \in \mathbb{R}, x \in \mathbb{R}^d$, and a selfadjoint operator $H$ in $L_2(\mathbb{R}^d)$. It is assumed that the semigroup $e^{-tH}$ satisfies the so-called Nash–Aronson estimate also known as the upper Gaussian bound (see Subsection 2 of the Introduction); in particular, the semigroup generated by a selfadjoint uniformly elliptic second order differential operator with uniformly bounded coefficients fulfills this requirement. We require neither any smoothness conditions for the coefficients of the operator $H$, nor any knowledge of its (partial) diagonalization. In Theorem 1.3 we obtain conditions for the operator $T_{fg}$ to be in the classes $\mathcal{S}_{p,q}, p \in (0, 2), q \in (0, +\infty]$. We also give conditions (see Theorem 1.2) for the operator $T_{fg}$ to belong to the classes $\mathcal{S}_{p,q}, p \in (2, +\infty), q \in (0, +\infty]$.

Moreover, in the present paper we consider certain integral operators bordered by suitable scalar weights. Namely, let $(\mathcal{K}, d\mu)$ be a separable measurable space with $\sigma$-finite measure, let $t: (\cdot, \cdot) \in L_\infty(\mathcal{K} \times \mathbb{R}^d, d\mu \cdot dx)$ be the kernel of a bounded integral operator $T: L_2(\mathbb{R}^d) \to L_2(\mathcal{K}, d\mu)$ and let $f: \mathcal{K} \to \mathbb{C}, g: \mathbb{R}^d \to \mathbb{C}$ be suitable measurable functions. It is assumed that the operator $T$ (partially) diagonalizes the selfadjoint operator $H$ acting in $L_2(\mathbb{R}^d)$ and satisfying the conditions described above. In Theorem 1.5 we give estimates for the singular values of the integral operator $\hat{T}_{fg} = fTg$ with the kernel $f(k)t(k, y)g(y)$. Conditions for the operator $\hat{T}_{fg}$ to belong to the classes $\mathcal{S}_{p,q}, p \in (0, 2)$, $q \in (0, +\infty]$, included in the Hilbert–Schmidt class, are formulated.

Denote the singular values of the operator $T_{fg}$ by $s_n(T_{fg}), n \in \mathbb{N}$. In the power scale, the results of the paper can be illustrated as follows (see Proposition 3.2 below): Suppose that, for some $l > 0, \varepsilon > 0$, either the condition

\begin{equation}
|f(\lambda)| \leq C_1(1 + \lambda^2)^{-l/4 - \varepsilon}, \quad \lambda \in \mathbb{R}, \quad |g(x)| \leq C_2(1 + |x|^2)^{-l/2}, \quad x \in \mathbb{R}^d;
\end{equation}

or the condition

\begin{equation}
|f(\lambda)| \leq C_1(1 + \lambda^2)^{-l/4}, \quad \lambda \in \mathbb{R}, \quad |g(x)| \leq C_2(1 + |x|^2)^{-l/2 - \varepsilon}, \quad x \in \mathbb{R}^d
\end{equation}

is fulfilled. Then the singular values of the operator $T_{fg}$ satisfy the estimate

\begin{equation}
s_n(T_{fg}) \leq Cn^{-l/d}, \quad n \in \mathbb{N}.
\end{equation}

Here $C = C(d, H, l, \varepsilon) \cdot C_1 \cdot C_2$; observe that, for $l > d/2$, from (1.3) it follows that $T_{fg}$ is a Hilbert–Schmidt operator.

Our arguments are based on combination of ideas of [3, 6, 7] with certain considerations related to pointwise estimates of the kernel of the operator $e^{-tH}, t > 0$. Applications of our estimates to the study of differential operators’ spectra will appear elsewhere.

The paper consists of the Introduction and four sections. In the Introduction, we discuss the Nash–Aronson estimate and introduce the necessary classes of functions and operators. The main results (Theorems 1.2, 1.3, and 1.5) are contained in §1; §2 is devoted to corollaries to the main results, convenient for applications. In §3 the results obtained are illustrated by an example of functions $f$ and $g$ admitting power estimates. The proofs of the main results are contained in §4.

For a selfadjoint operator $H$, the symbols $\sigma(H), \text{Dom } H$, and $E_H(\cdot)$ denote the spectrum, the domain, and the spectral measure, respectively. For a measurable function $f: \mathcal{X} \to \mathbb{C}$, the symbol $[f(\cdot)]$ denotes the operator of multiplication by $f$. 
The Lebesgue measure of a set $\delta \subset \mathbb{R}^d$ is denoted by $\text{meas}(\delta)$. The norm in a (quasi)normed space $X$ is denoted by $\|\cdot\|_X$; if spaces $X$, $Y$ are (quasi)normed, the standard norm of a linear bounded operator $T: X \to Y$ is denoted either by $\|T\|_{X \to Y}$ or by $\|T\|$ (with no index), provided that this causes no confusion. The symbol $\mathcal{S}_\infty$ denotes the class of compact operators.

2. **Nash–Aronson estimate.** Throughout the present paper, the following condition is assumed.

**Condition 1.** Let $H$ be a lower bounded selfadjoint operator in $L_2(\mathbb{R}^d)$, $d \geq 1$. Suppose that the operator $e^{-tH}$, $t > 0$, is an integral operator in $L_2(\mathbb{R}^d)$ with kernel $K(t, x, y)$, $x, y \in \mathbb{R}^d$, $t > 0$, such that

$$
|K(t, x, y)| \leq M_i t^{-d/2} e^{M_d t - \frac{|x-y|^2}{M_2 t^2}}, \quad x, y \in \mathbb{R}^d, \quad t > 0,
$$

where $M_i > 0$, $i = 1, 2, 3$, do not depend on $x, y \in \mathbb{R}^d$, $t > 0$.

Inequality (0.4), called the Nash–Aronson estimate, was obtained in [9] for the kernel of the semigroup generated by an elliptic second order differential operator. Namely, the results of [9] (see also [10] Chapter 3, §2) imply that the uniformly elliptic selfadjoint differential operator $H = -\text{div} a(x) \text{grad} + V(x)$ in the entire space satisfies Condition 1. More precisely, it is assumed that $a(x)$ is a real-valued matrix-valued function of size $(d \times d)$ and $V(x)$ is a real-valued potential, such that

$$
(a = a^t \in L_\infty(\mathbb{R}^d, \text{Matr}(\mathbb{R}, d \times d)),$$

$V = \nabla \in L_\infty(\mathbb{R}^d), \quad c_01 \leq a(x), \quad 0 < c_0.$

It is easily seen that, under condition (0.5), the operator $H$ is well defined by its quadratic form, is selfadjoint in $L_2(\mathbb{R}^d)$, and is lower bounded. The results of [9] imply the following statement.

**Theorem 0.1.** If condition (0.5) is fulfilled, then the operator $H$ satisfies Condition 1.

3. We introduce the required function spaces and classes of compact operators.

**Classes of functions.** Let $(Z, d\nu)$ be a separable measurable space with $\sigma$-finite measure. Alongside with the standard classes $L_p(Z, d\nu)$, we shall use the Lorentz classes $L_{p,q}(Z, d\nu)$, $p \in (0, +\infty)$, $q \in (0, +\infty)$ (see, e.g., [5, 7]). Namely, for a $\nu$-measurable function $f: Z \to \mathbb{C}$ put $O_f(s) := \{z \in Z : |f(z)| > s\}$, $\nu_f(s) := \nu(O_f(s)), s > 0$. The class $L_{p,q}$ is singled out by the requirement that the following functional be finite:

$$
\|f\|_{L_{p,q}} := \begin{cases} 
\left( \sup_{s>0} s^{q-1} \nu_f^{p/q}(s) ds \right)^{1/q}, & 0 < q < +\infty; \\
\sup_{s>0} s^{1/q} \nu_f^{1/p}(s), & q = +\infty.
\end{cases}
$$

The space $L_{p,q}$ is complete with respect to the quasinorm (0.6) and is separable for $q \in (0, +\infty)$; the space $L_{p,\infty}$ is in general nonseparable and contains the separable subspace

$$
L_{p,\infty}^0 := \{f \in L_{p,\infty} : \nu_f(s) = o(s^{-p}), \quad s \to +0, \quad s \to +\infty\}.
$$

Observe that $L_{p,p} = L_p$, $\|f\|_{L_{p,p}} = \|f\|_{L_p}$. If $Z$ is a countable set, we shall use the notation $L_{p,q}(Z, d\nu) := \ell_{p,q}(Z, d\nu)$; in case $d\nu$ is the counting measure, we write $\ell_{p,q}(Z, d\nu) := \ell_{p,q}(Z).

Also, we shall use the standard norm in the space $L_2(Z, d\nu) + L_\infty(Z, d\nu)$:

$$
\|f\|_{L_2 + L_\infty} := \inf_{a, b} \{ \|a\|_{L_\infty} + \|b\|_{L_2}, \quad f = a + b, \quad a \in L_\infty, \quad b \in L_2 \}.
$$
**Operator classes.** For an arbitrary compact operator \( T \) acting from a Hilbert space \( H_1 \) to a Hilbert space \( H_2 \), we denote by \( s_n(T) \), \( n \in \mathbb{N} \), the singular values of the operator \( T \) (i.e., the monotone nonincreasing sequence of eigenvalues of the operator \((T^*T)^{1/2}\)) and by \( n(s, T) := \#\{n \in \mathbb{N} : s_n(T) > s\} \) the distribution function of the singular values. We mention the following inequality to be used in the sequel (see [11, 11.1, Subsection 3]):

\[
(n(s + t, S + T) - n(s, S) - n(t, T)) \leq n(s, T) + n(t, T), \quad s, t > 0, \quad S, T \in \mathcal{S}_\infty.
\]

The class \( \mathcal{S}_{p,q}(H_1, H_2) \) (see, e.g., [5]) is singled out by the condition \( \{s_n(T)\}_{n \in \mathbb{N}} \in \ell_{p,q}(\mathbb{N}) \), which is equivalent to the requirement of finiteness for the functional

\[
(0.7) \quad \|T\|_{\mathcal{S}_{p,q}} := \left\{ \left. \begin{array}{ll}
(q \int_0^{+\infty} s^{-1}n^{-q/p}(s, T) ds \right)^{1/q}, & 0 < q < +\infty; \\
\sup_{s>0} sn^{1/p}(s, T), & q = +\infty.
\end{array} \right\}
\]

The space \( \mathcal{S}_{p,q} \) is complete relative to the quasinorm \( (0.7) \) and is separable when \( q \in (0, +\infty) \). The space \( \mathcal{S}_{p,\infty} \) is nonseparable and contains the separable subspace \( \mathcal{S}_{0,\infty}^0 := \{T \in \mathcal{S}_{p,\infty} : n(s, T) = o(s^{-p}), s \to +0\} \), in which the set of finite-rank operators is dense.

For an arbitrary compact operator \( T \), the fact that \( T \in \mathcal{S}_{p,\infty} \) is equivalent to the condition \( s_n(T) = O(n^{-1/p}), n \to +\infty \), and we have \( \|T\|_{\mathcal{S}_{p,\infty}} = \sup_{n \in \mathbb{N}} n^{1/p}s_n(T) \). An operator \( T \) belongs to \( \mathcal{S}_{p,\infty}^0 \) if and only if \( s_n(T) = o(n^{-1/p}), n \to +\infty \). The class \( \mathcal{S}_{p,\infty} \) coincides with \( \mathcal{S}_p \), where \( T \in \mathcal{S}_p \) means that \( \{s_n(T)\}_{n \in \mathbb{N}} \in \ell_p(\mathbb{N}) \). In this case, the functional \( (0.8) \) coincides with the standard (quasi)norm in \( \mathcal{S}_p \): \( \|T\|_{\mathcal{S}_p} = (\sum_{n \in \mathbb{N}} s_n^p(T))^{1/p} \). The following inclusions are valid (see, e.g., [5]): \( \mathcal{S}_{p_1,q_1} \subset \mathcal{S}_{p_2,q_2}, p_1 < p_2, q_1, q_2 \in (0, +\infty] \).

We shall need the following two assertions (see, e.g., [11] §11.5, Subsection 4, §11.6, Subsection 3).

**Proposition 0.2.** If \( T_1 \in \mathcal{S}_p \) and \( T_2 \in \mathcal{S}_q \), then \( T_1T_2 \in \mathcal{S}_r, \) where \( r^{-1} = p^{-1} + q^{-1} \); moreover, we have

\[
(0.9) \quad \|T_1T_2\|_{\mathcal{S}_r} \leq \|T_1\|_{\mathcal{S}_p}\|T_2\|_{\mathcal{S}_q}, \quad r^{-1} = p^{-1} + q^{-1}.
\]

**Proposition 0.3.** If \( T_1 \in \mathcal{S}_{p,\infty} \) and \( T_2 \in \mathcal{S}_{q,\infty} \), then \( T_1T_2 \in \mathcal{S}_{r,\infty} \), where \( r^{-1} = p^{-1} + q^{-1} \); we have

\[
\|T_1T_2\|_{\mathcal{S}_{r,\infty}} \leq C(p, q)\|T_1\|_{\mathcal{S}_{p,\infty}}\|T_2\|_{\mathcal{S}_{q,\infty}}, \quad r^{-1} = p^{-1} + q^{-1}.
\]

If, moreover, either \( T_1 \in \mathcal{S}_{p,\infty}^0 \) or \( T_2 \in \mathcal{S}_{q,\infty}^0 \), then \( T_1T_2 \in \mathcal{S}_{r,\infty}^0 \).

§1. **Main results**

Here we formulate conditions for boundedness and compactness. We also provide estimates for the singular values of the operators \( T_{fg} \) and \( \tilde{T}_{fg} \) in terms of \( f \) and \( g \).

1. **Conditions for the operator** \( T_{fg} \) **to belong to the classes** \( \mathcal{S}_{p,q}, \) \( p > 2, q \in (0, +\infty] \). For any selfadjoint, lower bounded operator \( H \) in \( L_2(\mathbb{R}^d) \), any bounded Borel function \( f(\lambda), \lambda \in \mathbb{R} \), and any \( g \in L_2,\text{loc}(\mathbb{R}^d) \), the operator \( T_{fg} := f(H)g(x) \) is well defined on the linear set \( \mathcal{F} \) of measurable, bounded, compactly supported functions from \( \mathbb{R}^d \) to \( \mathbb{C} \). For brevity, denote \( a_0 := \min\{\inf \sigma(H), 0\}, a_j := 2^j, j \in \mathbb{N} \). With any bounded Borel function \( f \) : \( \mathbb{R} \to \mathbb{C} \) we associate the sequence

\[
(1.1) \quad u(f) := \{u_j(f)\}_{j=0}^{\infty}, \quad u_j(f) := \sup_{\lambda} \{ |f(\lambda)|, \lambda \in [a_j, a_{j+1}) \}, \quad j \in \mathbb{Z}_+.
\]

For any bounded Borel function \( f(\lambda), \lambda \in \mathbb{R} \), and any \( g \in L_2,\text{loc}(\mathbb{R}^d) \), we define the function

\[
(u(f)g)(j, x) := u_j(f)g(x), \quad (j, x) \in Z := \mathbb{Z}_+ \times \mathbb{R}^d.
\]
We introduce the measure \( dv(j,x) := a_{j+1}^{d/2} dj dx \) on the set \( \mathcal{Z} := \mathbb{Z}_+ \times \mathbb{R}^d \); here \( dj \) is the counting measure on \( \mathbb{Z}_+ \), and \( dx \) is the Lebesgue measure on \( \mathbb{R}^d \).

Let the operator \( H \) satisfy Condition \([1]\). Suppose that, for some bounded Borel function \( f \) on the line and for some \( g \in L_{2, \text{loc}}(\mathbb{R}^d) \), we have

\[
(1.6) \\
|u(f)g| \in L_2(\mathbb{Z}, dv) + L_{\infty}(\mathbb{Z}, dv).
\]

**Proposition 1.1.** Suppose that the operator \( H \) satisfies Condition \([1]\) and that condition \((1.2)\) is fulfilled. Then the operator \( T_{fg} \) admits extension from the linear set \( \mathcal{F} \) to a bounded operator on \( L_2(\mathbb{R}^d) \). We have

\[
(1.3) \\
\|T_{fg}\| \leq C(M_1, M_2, M_3, d)\| u(f)g\|_{L_2 + L_{\infty}}.
\]

If, moreover, \( \nu_{ug}(s) < \infty \) for all \( s > 0 \), then the operator \( T_{fg} \) is compact.

**Theorem 1.2.** Suppose that the operator \( H \) satisfies Condition \([1]\) and that \( u(f)g \in L_{p,q}(\mathbb{Z}, dv) \) for \( p > 2 \), \( q \in (0, +\infty) \), or for \( p = q = 2 \). Then \( T_{fg} \in \mathcal{S}_{p,q} \) and

\[
(1.4) \\
\|T_{fg}\|_{\mathcal{S}_{p,q}} \leq C(M_1, M_2, M_3, d, p, q)\| u(f)g\|_{L_{p,q}}.
\]

If, moreover, \( q = +\infty \) and \( \nu_{ug}(s) = o(s^{-p}) \), \( s \to +0 \), then \( T_{fg} \in \mathcal{S}_{p,\infty}^0 \).

The proofs of Proposition \([1.1]\) and Theorem \([1.2]\) are presented in §4.

**Comments.** Conditions for integral operators to belong to the classes \( \mathcal{S}_{p,q} \) for \( p > 2 \) were studied thoroughly in the past. In the paper \([1]\) by Cwikel, the estimate

\[
\|f(H)g(x)\|_{\mathcal{S}_{p,\infty}} \leq C_p\|f\|_{L_p}\|g\|_{L_{p,\infty}}, \quad p > 2
\]

was established for the operator \( H = i\nabla \). Various generalizations of Cwikel’s estimate were obtained, e.g., in \([2, 7, 3, 4]\). In all the papers mentioned above, the diagonalization of \( H \) was assumed to be known, which is not always convenient in applications. Theorem \([1.2]\) requires no diagonalization of the operator \( H \). Unlike the standard Cwikel estimate, Theorem \([1.2]\) requires that the function \( f \) be bounded. In §2, certain corollaries to Theorem \([1.2]\) are proved with separate conditions on \( f \) and \( g \) that are more convenient for applications.

2. **Conditions for the operator \( T_{fg} \) to belong to the classes \( \mathcal{S}_{p,q} \), \( p \in (0,2) \), \( q \in (0, +\infty) \).** As before, with any bounded Borel function \( f(\lambda), \lambda \in \mathbb{R} \), we associate the sequence \((1.1)\). Putting \( \Omega := [0,1]^d, \Omega_n := \Omega + n, n \in \mathbb{Z}^d \), for each function \( g \in L_{2, \text{loc}}(\mathbb{R}^d) \) we introduce the sequence

\[
(1.5) \\
v(g) := \{v_n(g)\}_{n \in \mathbb{Z}^d}, \quad v_n(g) := \left( \int_{\Omega_n} |g(x)|^2 \, dx \right)^{1/2}, \quad n \in \mathbb{Z}^d.
\]

Denote by \( u(f)v(g) \) the sequence \( \{u_j(f)v_n(g)\}_{(j,n) \in \mathbb{Z}_+ \times \mathbb{Z}^d} \). We introduce the measure \( dv := a_{j+1}^{d/2} dj \) on the set \( \mathbb{Z}_+ \times \mathbb{Z}^d \), where \( dj \) and \( dn \) are the counting measures on \( \mathbb{Z}_+ \) and \( \mathbb{Z}^d \), respectively.

Let the operator \( H \) satisfy Condition \([1]\). Suppose that for some bounded Borel function \( f \) on the line and for some \( g \in L_{2, \text{loc}}(\mathbb{R}^d) \) we have

\[
(1.6) \\
\|u(f)v(g)\|_{\ell_{p,q}(\mathbb{Z}_+ \times \mathbb{Z}^d, dv)} = \|u(f)v(g)\|_{\ell_{p,q}}.
\]

**Theorem 1.3.** Suppose \( H \) satisfies Condition \([1]\) and \( f \) and \( g \) satisfy condition \((1.6)\). Then the operator \( T_{fg} \) admits extension from the linear set \( \mathcal{F} \) to a bounded operator in \( L_2(\mathbb{R}^d) \), and we have \( T_{fg} \in \mathcal{S}_{p,q}(L_2(\mathbb{R}^d)) \) and

\[
\|T_{fg}\|_{\mathcal{S}_{p,q}} \leq C(M_1, M_2, M_3, d, p, q)\| u(f) v(g)\|_{\ell_{p,q}}.
\]

Under the additional condition that \( q = +\infty \), \( u(f)v(g) \in \ell_{p,\infty}^0 \), we have \( T_{fg} \in \mathcal{S}_{p,\infty}^0 \).

The proof of Theorem \([1.3]\) is presented in §4.
Comments. Under the assumptions of Theorem 1.3, the operator \( T_{fg} \) is an integral Hilbert–Schmidt operator on \( L_2(\mathbb{R}^d) \). Unlike the conditions for integral operators to belong to \( \mathcal{G}_{p,q} \), \( p > 2 \), the conditions for integral operators to be in \( \mathcal{G}_{p,q} \) for \( p \in (0, 2) \) have been studied to a much lesser extent. The corresponding results requiring no additional conditions on the smoothness of the kernel of the operator \( f(H)g(x) \) have been apparently obtained only in the case of \( H = i\nabla \) in [5, 6] and [7] (see also [8], where estimates of this type were obtained for the Dirac operator with nonconstant coefficients). In \( \S 2 \) below, we present some corollaries to Theorem 1.3 with separate conditions on the functions \( f \) and \( g \).

3. Estimates for singular values of bordered integral operators. First, we formulate a generalization of Cwikel’s estimate that was obtained in [2]. Let \((\mathcal{X}, dp), (\mathcal{Y}, d\tau)\) be separable measure spaces with \( \sigma \)-finite measures; let \( t(\cdot, \cdot) \in L_\infty(\mathcal{X} \times \mathcal{Y}, dp d\tau) \) be the kernel of a bounded integral operator \( T: L_2(\mathcal{Y}, d\tau) \to L_2(\mathcal{X}, dp) \); let \( f: \mathcal{X} \to \mathbb{C} \), \( g: \mathcal{Y} \to \mathbb{C} \) be measurable functions.

**Proposition 1.4.** If \( f \in L_{p,\infty}(\mathcal{X}, dp) \) and \( g \in L_p(\mathcal{Y}, d\tau) \), where \( p > 2 \), then the kernel \( f(x)t(x,y)g(y) \) gives rise to a bounded integral operator \( \widehat{T}_{fg} \in \mathcal{G}_{p,\infty} \), and the following estimate is true:

\[
\|\widehat{T}_{fg}\|_{\mathcal{G}_{p,\infty}} \leq C(p)\|t(\cdot, \cdot)\|^{2/p}\|T\|^{1-2/p}\|f\|_{L_{p,\infty}}\|g\|_{L_p}.
\]

If, moreover, \( \rho_f(s) = o(s^{-p}) \), \( s \to +0 \), then \( \widehat{T}_{fg} \in \mathcal{G}_{0,\infty}^0 \). Surely, the conditions on the functions \( f \) and \( g \) can be interchanged.

Conditions for the operator \( \widehat{T}_{fg} \) to belong to \( \mathcal{G}_{p,q}, p \in (2, +\infty), q \in (0, +\infty) \), are also known (see [7, 3, 4]), but we do not give these here. Below, we formulate conditions for the operator \( \widehat{T}_{fg} \) to belong to the classes \( \mathcal{G}_{p,q}, p \in (0, 2), q \in (0, +\infty) \), for a fairly special choice of the space \((\mathcal{Y}, d\tau)\) and the operator \( T \). Let the operator \( H \) satisfy Condition 1 let \((\mathcal{K}, d\mu)\) be a separable measurable space with \( \sigma \)-finite measure, and let \( T: L_2(\mathbb{R}^d) \to L_2(\mathcal{K}, d\mu) \) be a bounded linear integral operator with bounded kernel \( t(\cdot, \cdot) \in L_\infty(\mathcal{K} \times \mathbb{R}^d, d\mu dx) \) (\( dx \) is the Lebesgue measure in \( \mathbb{R}^d \)). Suppose there exists a measurable function \( h: \mathcal{K} \to \mathbb{C} \) such that, whenever \( u \in \text{Dom} e^{sH}, \) we have \( Tu \in \text{Dom} [e^{sh(k)}], \) \( s > 0, \) and

\[
(1.7) \quad Te^{sH}u = [e^{sh(k)}]Tu, \quad u \in \text{Dom} e^{sH}, \quad s > 0.
\]

As before, each function \( g \in L_{2,\text{loc}}(\mathbb{R}^d) \) is associated with the sequence \( [1.5] \); \( a_0 := \min\{0, \inf \sigma(H)\} \), \( a_j := 2^j, j \in \mathbb{N} \). Put \( \mathcal{K}_j := \{k \in \mathcal{K}: |h(k)| \in [a_j, a_{j+1})\}, j \in \mathbb{Z}_+ \) and associate the sequence

\[
(1.8) \quad \widehat{u}(f) := \{\widehat{u}_j(f)\}_{j=0}^\infty, \quad \widehat{u}_j(f) := a_{j+1}^{-d/4}\|f\|_{L_2(\mathcal{K}_j)}, \quad j \in \mathbb{Z}_+
\]

with each measurable function \( f: \mathcal{K} \to \mathbb{C} \). The sequence \( \{\widehat{u}_j(f)v_n(g)\}_{(j,n) \in \mathbb{Z}_+ \times \mathbb{Z}^d} \) will be denoted by \( \widehat{u}(f)v(g) \) and, as before, the measure \( a_{j+1}^{d/2}djdn \) on \( \mathbb{Z}_+ \times \mathbb{Z}^d \) will be denoted by \( d\widehat{v} \).

Assume that

\[
(1.9) \quad \widehat{u}(f)v(g) \in \ell_{p,q}(\mathbb{Z}_+ \times \mathbb{Z}^d, d\widehat{v}), \quad p \in (0, 2), \quad q \in (0, +\infty), \ \text{or} \ p = q = 2,
\]

for some measurable function \( f: \mathcal{K} \to \mathbb{C} \) and some \( g \in L_{2,\text{loc}}(\mathbb{R}^d) \).

**Theorem 1.5.** Suppose that the operator \( H \) satisfies Condition 1 and that conditions \( [1.7] \) and \( [1.9] \) are fulfilled. Then the kernel \( f(k)t(k,y)g(y), (k, y) \in \mathcal{K} \times \mathbb{R}^d \) gives rise to a Hilbert–Schmidt operator \( \widehat{T}_{fg} \) from \( L_2(\mathbb{R}^d) \) to \( L_2(\mathcal{K}, d\mu) \), we have \( \widehat{T}_{fg} \in \mathcal{G}_{p,q} \), and also

\[
\|\widehat{T}_{fg}\|_{\mathcal{G}_{p,q}} \leq C(M_1, M_2, M_3, d, p, q)\|t(\cdot, \cdot)\|_{L_\infty} \|\widehat{u}(f)v(g)\|_{\ell_{p,q}}.
\]
If, moreover, \( g = +\infty \), \( \hat{u}(f) v(g) \in L_{p,\infty}^0 \), then \( \hat{T} f g \in \mathfrak{S}_{p,\infty}^0 \).

The proof of Theorem 1.5 is presented in §4.

**Comments.** The assumptions of Theorem 1.5 do not require that the function \( f \) be bounded. If we take the Fourier transform in \( L_2(\mathbb{R}^d) \) as the operator \( T \) and \(-\Delta \) (\( \Delta \) being the Laplacian in \( L_2(\mathbb{R}^d) \)) as the operator \( H \), then Theorem 1.5 becomes equivalent (in the power scale) to the corresponding results of \([5, 6, 7]\). If the operator \( T \) diagonalizes the operator \( H = T^*|h(k)| T \), then Theorems 1.3 and 1.5 are applicable to the operator \( T f g = T^* T(f h) g \); on the other hand, Theorem 1.3 does require that the function \( f \) be bounded, whereas Theorem 1.5 does not. §2 contains some corollaries to Theorem 1.5 with separate conditions on the functions \( f \) and \( g \).

§2. Corollaries to main results with separate conditions on \( f \) and \( g \)

Here we give some consequences of the results of the preceding section coupled with two assertions (see [3]) on Lorentz spaces. Namely, assume that \((\mathcal{X}, d\rho)\) and \((\mathcal{Y}, d\tau)\) are separable measurable spaces with \( \sigma \)-finite measures.

**Proposition 2.1.** Suppose \( f \in L_{p,\infty}(\mathcal{X}, d\rho), \ g \in L_p(\mathcal{Y}, d\tau), \ p > 0 \). Then \( f g \in L_{p,\infty}(\mathcal{X} \times \mathcal{Y}, d\rho d\tau) \), \( \|fg\|_{L_{p,\infty}} \leq C(p)\|f\|_{L_p,\infty}\|g\|_{L_p} \). The additional condition \( \rho_f(s) = o(s^{-p}), \ s \to +0 \), implies \( \rho_{fg}(s) = o(s^{-p}), \ s \to +0 \).

**Proposition 2.2.** Suppose that \( f \in L_{p,q_1}(\mathcal{X}, d\rho), \ g \in L_{p,q_2}(\mathcal{Y}, d\tau) \) for \( p > 0, \ q_1, q_2 \in [p, +\infty) \), \( 1/q := 1/q_1 + 1/q_2 - 1/p \geq 0 \). Then
\[
fg \in L_{p,q}(\mathcal{X} \times \mathcal{Y}, d\rho d\tau), \quad \|fg\|_{L_{p,q}} \leq C(p, q_1, q_2)\|f\|_{L_{p,q_1}}\|g\|_{L_{p,q_2}}.
\]
If, moreover, \( q = +\infty \), then \( fg \in L_{0,p,\infty}^0 \).

Strictly speaking, Propositions 2.1 and 2.2 were discussed in [3] only under the assumption \( p > 2 \); however, the argument is valid for all \( p > 0 \).

1. Estimates for the singular values of the operator \( T f g \). Here we provide corollaries to Theorems 1.2 and 1.3 with separate conditions on the functions \( f \) and \( g \). As before, the operator \( H \) satisfies Condition 1: \( a_0 := \min\{0, \inf \sigma(H)\}, \ a_j := 2^j, \ j \in \mathbb{N} \); and with each bounded Borel function \( f(\lambda), \ \lambda \in \mathbb{R} \), we associate the sequence (1.1). Denote by \( dp(j) \) the measure \( a^{d/2}_{j+1} \) \( dj \) on \( \mathbb{Z}_+ \) (\( dj \) being the counting measure on \( \mathbb{Z}_+ \)). The measure \( dv = a^{d/2}_{j+1} \) \( dj \) \( dx \) is the product of the measure \( dp(j) \) on \( \mathbb{Z}_+ \) by \( dx \), the Lebesgue measure on \( \mathbb{R}^d \). Theorem 1.2 and Proposition 2.1 yield the following two assertions.

**Corollary 2.3.** Suppose that the operator \( H \) satisfies Condition 1 and \( u(f) \in L_\ell_p(\mathbb{Z}_+, dp), \ g \in L_{p,\infty}(\mathbb{R}^d), \ p > 2 \). Then
\[
T f g \in \mathfrak{S}_{p,\infty}, \quad \|T\|_{\mathfrak{S}_{p,\infty}} \leq C(M_1, M_2, M_3, d, p)\|u(f)\|_{L_\ell_p}\|g\|_{L_{p,\infty}}.
\]
If, moreover, means \( \{x \in \mathbb{R}^d : |g(x)| > s\} = o(s^{-p}), \ s \to +0 \), then \( T f g \in \mathfrak{S}_{p,\infty}^0 \).

**Corollary 2.4.** Suppose the operator \( H \) satisfies Condition 1 and \( u(f) \in L_{p,\infty}(\mathbb{Z}_+, dp), \ g \in L_{\ell_0}(\mathbb{R}^d), \ p > 2 \). Then
\[
T f g \in \mathfrak{S}_{p,\infty}, \quad \|T\|_{\mathfrak{S}_{p,\infty}} \leq C(M_1, M_2, M_3, d, p)\|u(f)\|_{L_{\ell_0,\infty}}\|g\|_{L_p}.
\]
If, moreover, \( u(f) \in L_{p,\infty}^0(\mathbb{Z}_+, dp) \), then \( T f g \in \mathfrak{S}_{p,\infty}^0 \).

Theorem 1.2 and Proposition 2.2 imply the following.
Corollary 2.5. Suppose the operator $H$ satisfies Condition 1 and $u(f) \in \ell_{p,q_1}(\mathbb{Z}^+, dp)$, $g \in L_{p,q_2}(\mathbb{R}^d)$, for $p > 2$, $q_1, q_2 \in [p, +\infty)$, $1/q := 1/q_1 + 1/q_2 - 1/p \geq 0$. Then
\[ T_{fg} \in \mathcal{G}_{p,q}, \quad \|T\|_{\mathcal{G}_{p,q}} \leq C(M_1, M_2, M_3, d, p, q_1, q_2)\|u(f)\|_{\ell_{p,q_1}}\|g\|_{L_{p,q_2}}. \]
If, moreover, $q = +\infty$, then $T_{fg} \in \mathcal{G}_{0,\infty}^1$.

Next, each bounded Borel function $f(\lambda), \lambda \in \mathbb{R}$, is associated with the sequence (1.1), whereas each function $g \in L_{2,loc}(\mathbb{R}^d)$ is associated with the sequence (1.5). The measure $\hat{d}V = d\hat{\nu}_{j+1} dj dn$ is the product of the measure $dp(j)$ on $\mathbb{Z}^+$ and the counting measure $dn$ on $\mathbb{Z}^d$. Theorem 1.3 and Proposition 2.2 imply the following two assertions.

Corollary 2.6. Suppose that the operator $H$ satisfies Condition 1 and $u(f) \in \ell_{p}(\mathbb{Z}^+, dp)$, $v(g) \in \ell_{p}(\mathbb{Z}^d), p \in (0, 2)$. Then
\[ T_{fg} \in \mathcal{G}_{p,\infty}, \quad \|T\|_{\mathcal{G}_{p,\infty}} \leq C(M_1, M_2, M_3, d, p)\|u(f)\|_{\ell_{p}}\|v(g)\|_{\ell_{p}}. \]
If, moreover, $v(g) \in \ell_{p,\infty}(\mathbb{Z}^d)$, then $T_{fg} \in \mathcal{G}_{p,\infty}$.

Corollary 2.7. Suppose the operator $H$ satisfies Condition 1 and $u(f) \in \ell_{p,\infty}(\mathbb{Z}^+, dp)$, $v(g) \in \ell_{p}(\mathbb{Z}^d), p \in (0, 2)$. Then
\[ T_{fg} \in \mathcal{G}_{p,\infty}, \quad \|T\|_{\mathcal{G}_{p,\infty}} \leq C(M_1, M_2, M_3, d, p, q_1, q_2)\|u(f)\|_{\ell_{p,\infty}}\|v(g)\|_{\ell_{p}}. \]
If, moreover, $u(f) \in \ell_{0}(\mathbb{Z}^+, \rho)$, then $T_{fg} \in \mathcal{G}_{p,\infty}$.

Theorem 1.3 and Proposition 2.2 imply the following statement.

Corollary 2.8. Suppose the operator $H$ satisfies Condition 1 and $u(f) \in \ell_{p,q_1}(\mathbb{Z}^+, dp)$, $v(g) \in \ell_{p,q_2}(\mathbb{Z}^d), p \in (0, 2), q_1, q_2 \in [p, +\infty)$, $1/q := 1/q_1 + 1/q_2 - 1/p \geq 0$. Then
\[ T_{fg} \in \mathcal{G}_{p,q}, \quad \|T\|_{\mathcal{G}_{p,q}} \leq C(M_1, M_2, M_3, d, p, q_1, q_2)\|u(f)\|_{\ell_{p,q_1}}\|v(g)\|_{\ell_{p,q_2}}. \]
If, moreover, $q = +\infty$, then $T_{fg} \in \mathcal{G}_{p,\infty}$.

In the “borderline” situation where $p = 2$, $q = +\infty$, estimates for singular values of $T_{fg}$ can be deduced from Proposition 1.3 and Corollaries 2.6, 2.7, 2.8, and 2.9. Namely, the relations $T_{fg} \in \mathcal{G}_{2,\infty}, \quad T_{fg} T_{fg}^* \in \mathcal{G}_{1,\infty},$ and $T_{fg}^* T_{fg} \in \mathcal{G}_{1,\infty}$ are equivalent, and
\[ (2.1) \quad \|T_{fg}\|_{\mathcal{G}_{2,\infty}} = \|T_{fg} T_{fg}^*\|_{\mathcal{G}_{1,\infty}} = \|T_{fg}^* T_{fg}\|_{\mathcal{G}_{1,\infty}}. \]

Since $T_{fg} T_{fg}^* = f(H)|g|^{2/\alpha}|g|^{2(1-1/\alpha)\hat{T}(H)} = T_{fg}|g|^{2/\alpha} T_{fg}^*|g|^{2(1-1/\alpha)}, \quad \alpha \in (1, 2)$, we can use formula (2.1), Proposition 0.3 and Corollaries 2.6 and 2.7 (applied to the operators $T_{fg}|g|^{2(1-1/\alpha)}$ and $T_{fg}|g|^{2/\alpha}$, respectively) to prove the following statement.

Corollary 2.9. Suppose the operator $H$ satisfies Condition 1 and, for some $\alpha \in (1, 2)$, the following conditions are fulfilled: $u(f) \in \ell_{\alpha}(\mathbb{Z}^+, dp)$, $v(|g|^{2/\alpha}) \in \ell_{\alpha}(\mathbb{Z}^d)$. Then $T_{fg} \in \mathcal{G}_{2,\infty}$ and
\[ \|T_{fg}\|_{\mathcal{G}_{2,\infty}} \leq C(M_1, M_2, M_3, d, \alpha)\|u(f)\|_{\ell_{\alpha}}\|g\|_{L_{2,\infty}}^{1-1/\alpha}\|v(|g|^{2/\alpha})\|_{L_{2,\infty}}^{1/2}. \]
If, moreover, $v(|g|^{2/\alpha}) \in \ell_{0,\alpha}(\mathbb{Z}^d)$ or $\operatorname{meas}\{x \in \mathbb{R}^d : |g(x)| > s\} = o(s^{-2}), \quad s \to +0$, then $T_{fg} \in \mathcal{G}_{0,\infty}$.

Similarly, the decomposition $T_{fg}^* T_{fg} = T_{fg}^*|g|^{2/\alpha} T_{fg}|g|^{2(1-1/\alpha)}$, $\alpha \in (1, 2)$, identity (2.1), Proposition 0.3 and Corollaries 2.6 and 2.7 (applied to the operators $T_{fg}|g|^{2(1-1/\alpha)}$ and $T_{fg}|g|^{2/\alpha}$, respectively) lead to the following statement.
Corollary 2.10. Suppose that the operator $H$ satisfies Condition $\mathbb{H}$ $u(f) \in \ell_{2,\infty}(\mathbb{Z}^+, dp)$, and, for some $\alpha \in (1, 2)$, the following conditions is fulfilled: $v(g) \in \ell_{\alpha}(\mathbb{Z}^d)$, $g \in L_{\beta}(\mathbb{R}^d)$, $1/\alpha + 1/\beta = 1$. Then $T_{fg} \in \mathcal{G}_{2,\infty}$ and
\[
\|T_{fg}\|_{\mathcal{G}_{2,\infty}} \leq C(M_1, M_2, M_3, d, \alpha)\|u(f)\|_{\ell_{2,\infty}}\|v(g)\|_{\ell_{\alpha}}^{1/2}\|g\|_{L_\beta}^{1/2}.
\]
If, moreover, $u(f) \in \ell_{2,\infty}(\mathbb{Z}^+, dp)$, then $T_{fg} \in \mathcal{G}_{0,2,\infty}$.

2. Estimates for singular values of the operator $\hat{T}_{fg}$. Here we collect corollaries to Proposition 1.4 and Theorem 1.5. As before, we assume that the operator $H$ satisfies Condition $\mathbb{H}$ $(\mathcal{K}, d\mu)$ is a separable measurable space with $\sigma$-finite measure, and $T: L_2(\mathbb{R}^d) \to L_2(\mathcal{K}, d\mu)$ is a bounded linear integral operator having bounded kernel and satisfying condition (1.7). With each measurable function $f: \mathcal{K} \to \mathbb{C}$ we associate the sequence (1.8), and with each function $g \in L_{2,\text{loc}}(\mathbb{R}^d)$ we associate the sequence (1.5). As has already been mentioned, the measure $d\nu := \frac{d}{d\mu} d\nu$ is the product of the measure $d\nu(j)$ on $\mathbb{Z}^+$ and the counting measure $dn$ on $\mathbb{Z}^d$. Theorem 1.5 and Proposition 2.1 yield the following two assertions.

Corollary 2.11. Suppose the operator $H$ satisfies Condition $\mathbb{H}$ and (1.7) is valid. If $\hat{u}(f) \in \ell_p(\mathbb{Z}^+, d\mu)$, $v(g) \in \ell_{p,\infty}(\mathbb{Z}^d)$, $p \in (0, 2)$, then
\[
\hat{T}_{fg} \in \mathcal{G}_{p,\infty}, \quad \|\hat{T}_{fg}\|_{\mathcal{G}_{p,\infty}} \leq C(M_1, M_2, M_3, d, p)\|t(\cdot, \cdots, \cdot)\|_{L_\infty}\|\hat{u}(f)\|_{\ell_p}\|v(g)\|_{\ell_{p,\infty}}.
\]
If, moreover, $v(g) \in \ell_{p,\infty}(\mathbb{Z}^d)$, then $\hat{T}_{fg} \in \mathcal{G}_{0, p,\infty}$.

Corollary 2.12. Suppose the operator $H$ satisfies Condition $\mathbb{H}$ and (1.7) is valid. If $\hat{u}(f) \in \ell_{p,\infty}(\mathbb{Z}^+, d\mu)$, $v(g) \in \ell_p(\mathbb{Z}^d)$, $p \in (0, 2)$, then
\[
\hat{T}_{fg} \in \mathcal{G}_{p,\infty}, \quad \|\hat{T}_{fg}\|_{\mathcal{G}_{p,\infty}} \leq C(M_1, M_2, M_3, d, p)\|t(\cdot, \cdots, \cdot)\|_{L_\infty}\|\hat{u}(f)\|_{\ell_{p,\infty}}\|v(g)\|_{\ell_p}.
\]
If, moreover, $\hat{u}(f) \in \ell_{p,\infty}(\mathbb{Z}^+, d\mu)$, then $\hat{T}_{fg} \in \mathcal{G}_{0, p,\infty}$.

Theorem 1.5 and Proposition 2.2 imply the following.

Corollary 2.13. Suppose the operator $H$ satisfies Condition $\mathbb{H}$ and (1.7) is valid. If $\hat{u}(f) \in \ell_{p,q_1}(\mathbb{Z}^+, d\mu)$, $v(g) \in \ell_{p,q_2}(\mathbb{Z}^d)$, $p \in (0, 2)$, $q_1, q_2 \in [p, +\infty)$, $1/q := 1/q_1 + 1/q_2 - 1/p \geq 0$, then
\[
\hat{T}_{fg} \in \mathcal{G}_{p,q}, \quad \|\hat{T}_{fg}\|_{\mathcal{G}_{p,q}} \leq C(M_1, M_2, M_3, d, p, q_1, q_2)\|t(\cdot, \cdots, \cdot)\|_{L_\infty}\|\hat{u}(f)\|_{\ell_{p,q_1}}\|g\|_{\ell_{p,q_2}}.
\]
If, moreover, $q = +\infty$, then $\hat{T}_{fg} \in \mathcal{G}_{0, p,q}$.

Estimates for the singular values of the operator $\hat{T}_{fg}$ in the “borderline” situation where $p = 2, q = +\infty$ can be deduced from Corollaries 2.11, 2.12 and Propositions 0.3 and 1.4. Namely, the relations $\hat{T}_{fg} \in \mathcal{G}_{2,\infty}$, $\hat{T}_{fg} \hat{T}_{fg}^* \in \mathcal{G}_{1,\infty}$, and $\hat{T}_{fg} \hat{T}_{fg}^* \in \mathcal{G}_{1,\infty}$ are equivalent, and
\[
\|\hat{T}_{fg}\|_{\mathcal{G}_{2,\infty}} = \|\hat{T}_{fg} \hat{T}_{fg}^*\|_{\mathcal{G}_{1,\infty}} = \|\hat{T}_{fg} \hat{T}_{fg}^*\|_{\mathcal{G}_{1,\infty}}.
\]
We use the identity $\hat{T}_{fg} \hat{T}_{fg}^* = \hat{T}_{fg} |g|^{2/\alpha} \hat{T}_{fg} |g|^{2(1-1/\alpha)}$, $\alpha \in (1, 2)$. Applying Proposition 1.4 to the operator $\hat{T}_{fg} |g|^{2(1-1/\alpha)}$ and Corollary 2.11 to the operator $\hat{T}_{fg} |g|^{2/\alpha}$, and using Proposition 0.3 we obtain the following statement.

Corollary 2.14. Suppose the operator $H$ satisfies Condition $\mathbb{H}$ and (1.7) is valid. Assume that $g \in L_{2,\infty}(\mathbb{R}^d)$ and that, for some $\alpha \in (1, 2)$, we have $v(|g|^{2/\alpha}) \in \ell_{\alpha,\infty}(\mathbb{Z}^d)$, $\hat{u}(f) \in \ell_{2,\infty}(\mathbb{Z}^+, dp)$
\[\ell_\alpha(Z_+, d\rho), f \in L_\beta(K, d\mu), 1/\alpha + 1/\beta = 1.\] Then \(\hat{T}_{fg} \in \mathfrak{S}_{2,\infty}\) and
\[
\|\hat{T}_{fg}\|_{\mathfrak{S}_{2,\infty}} \leq C(M_1, M_2, M_3, d, \alpha)\|t(\cdot, \cdot)\|_{L_\infty}^{1/2 + 1/\beta}\|T\|^{1/\alpha - 1/2} \\
\times \|f\|_{L_\beta}^{1/2}\|\hat{u}(f)\|_{\ell_\alpha}^{1/\beta}\|g\|_{L_\infty}^{1/2}\|v(g)^{2/\alpha}\|_{\ell_\alpha}^{1/2}.
\]
If, moreover, \(v(|g|^{2/\alpha}) \in \ell^0_{\alpha,\infty}(\mathbb{Z}^d)\) or \(\|x\| > s\) we have \(o(s^{-2}), s \to +0,\) then \(\hat{T}_{fg} \in \mathfrak{S}_{2,\infty}^0,\) for \(\alpha \in (1, 2),\) Proposition 3.2.

Proposition 3.3. Proposition 1.4 and Corollary 2.12 (applied to the operators \(\hat{T}_{fg}^{(1-\alpha)\infty}\) and \(\hat{T}_{fg}^{(2-\alpha)\infty}\), respectively) lead to the following statement.

Corollary 2.15. Suppose the operator \(H\) satisfies Condition [1] and \(1.1\) is valid. Assume that \(f \in L_{2,\infty}(K, d\mu)\) and that, for some \(\alpha \in (1, 2),\) we have \(v(g) \in \ell_{\alpha,\infty}(\mathbb{Z}^d),\) \(\hat{u}(|f|^{2/\alpha}) \in \ell_{\alpha,\infty}(Z_+, d\rho),\) \(g \in L_\beta(\mathbb{R}^d),\) \(1/\alpha + 1/\beta = 1.\) Then \(\hat{T}_{fg} \in \mathfrak{S}_{2,\infty}\) and
\[
\|\hat{T}_{fg}\|_{\mathfrak{S}_{2,\infty}} \leq C(M_1, M_2, M_3, d, \alpha)\|t(\cdot, \cdot)\|_{L_\infty}^{1/2 + 1/\beta}\|T\|^{1/\alpha - 1/2} \\
\times \|f\|_{L_\beta}^{1/2}\|\hat{u}(|f|^{2/\alpha})\|_{\ell_\alpha}^{1/\beta}\|g\|_{L_\infty}^{1/2}\|v(g)^{2/\alpha}\|_{\ell_\alpha}^{1/2}.
\]
If, moreover, \(\hat{u}(|f|^{2/\alpha}) \in \ell^0_{\alpha,\infty}(Z_+, d\rho)\) or \(\mu\{k \in K : |f(k)| > s\} = o(s^{-2}), s \to +0,\) then \(\hat{T}_{fg} \in \mathfrak{S}_{2,\infty}^0.\)

§3. ESTIMATES FOR THE SINGULAR VALUES OF THE OPERATORS \(T_{fg}\) AND \(\hat{T}_{fg}\)
IN THE POWER SCALE

1. Here we illustrate the results of §2 for the operator \(T_{fg}\) by an example of bounded functions \(f: \mathbb{R} \to \mathbb{C}\) and \(g: \mathbb{R}^d \to \mathbb{C}\) admitting power estimates. Recall that by \(d\rho\) we denote the measure \(2^{j+1}\mu(dj)\) on the set \(Z_+\) (\(d\mu\) is the counting measure on \(Z_+\)). The following remark is obvious.

Remark 3.1.

- For a bounded function \(f(\lambda), \lambda \in \mathbb{R},\) the estimate \(f(\lambda) = O(\lambda^{-d/2p}), \lambda \to +\infty,\) \(p > 0,\) implies
  \[u(f) \in \ell_{p,\infty}(Z_+, d\rho), \ell_\alpha(Z_+, d\rho), \alpha > p,\]
and the estimate \(f(\lambda) = o(\lambda^{-d/2p}), \lambda \to +\infty,\) yields \(u(f) \in \ell^0_{p,\infty}(Z_+, d\rho)\).

- For a bounded function \(g(x), x \in \mathbb{R}^d,\) the estimate \(g(x) = O(|x|^{-d/p}), |x| \to +\infty,\) \(p > 0,\) implies
  \[g \in L_{p,\infty}(\mathbb{R}^d) \cap L_\alpha(\mathbb{R}^d), \quad v(g) \in \ell_{p,\infty}(\mathbb{Z}^d) \cap \ell_\alpha(\mathbb{Z}^d), \quad \alpha > p,\]
and the estimate \(g(x) = o(|x|^{-d/p}), |x| \to +\infty,\) yields
  \[g \in L_{p,\infty}(\mathbb{R}^d), \quad v(g) \in \ell^0_{p,\infty}(\mathbb{Z}^d)\].

Remark 3.1 and Corollaries 2.3 [2.4 [2.6 [2.7 [2.9] and 2.10] yield the following assertion.

Proposition 3.2. Let the operator \(H\) satisfy Condition 1.1 let \(f(\lambda), \lambda \in \mathbb{R},\) be a bounded Borel function, and let \(g \in L_{2, loc}(\mathbb{R}^d)\). Assume that, for some \(l > 0\) and \(\varepsilon > 0,\) either condition 0.11 or condition 0.12 is fulfilled. Then estimate 1.3 holds true. Suppose that either condition 0.11 is fulfilled and, additionally, \(g(x) = o(|x|^{-1}), |x| \to +\infty,\) or condition 0.12 is fulfilled and \(f(\lambda) = o(|\lambda|^{-1/2}), \lambda \to +\infty.\) Then \(s_n(T_{fg}) = o(n^{-1/d}), n \to +\infty,\) i.e., \(T_{fg} \in \mathfrak{S}_{d/1,\infty}^0.\)
2. Now let $(\mathcal{K}, d\mu)$ be a measurable space with $\sigma$-finite measure, and let $T : L_2(\mathbb{R}^d) \to L_2(\mathcal{K}, d\mu)$ be a bounded linear integral operator with a bounded kernel $t(\cdot, \cdot) \in L_\infty(\mathcal{K} \times \mathbb{R}^d, d\mu \, dx)$. Assume that the operator $H$ satisfies Condition [1] and that $[1.7]$ is true. Remark 3.1, Proposition 1.4, and Corollaries 2.11, 2.14 yield the following.

**Proposition 3.3.** Suppose that $g \in L_\infty(\mathbb{R}^d)$ is such that $g(x) = O(|x|^{-d/p})$, $|x| \to +\infty$, $p > 0$, and that a measurable function $f : \mathcal{K} \to \mathbb{C}$ satisfies one of the following three conditions:

a) if $p > 2$, then $f \in L_p(\mathcal{K}, d\mu)$;
b) if $p \in (0, 2)$, then $\hat{u}(f) \in \ell_p(\mathbb{Z}^+, dp)$;
c) if $p = 2$, then, for some $\alpha \in (1, 2)$,

$$\hat{u}(f) \in \ell_\alpha(\mathbb{Z}^+, dp), \quad f \in L_\beta(\mathcal{K}, d\mu), \quad 1/\alpha + 1/\beta = 1.$$ 

Then $\hat{T}_{fg} \in \mathcal{S}_{p, \infty}$ and $\|\hat{T}_{fg}\|_{\mathcal{S}_{p, \infty}} \leq C\|f\|_p$. Here the following is implied:

$$\|f\| := \|f\|_{L_p}, \quad C := C(M_1, M_2, M_3, d, p, T, g), \quad p > 2;$$
$$\|f\| := \|\hat{u}(f)\|_{\ell_p}, \quad C := C(M_1, M_2, M_3, d, p, T, g), \quad p \in (0, 2);$$
$$\|f\| := \|f\|_{\ell_\alpha}^{1/2} \|\hat{u}(f)\|_{\ell_\alpha}^{1/2}, \quad C := C(M_1, M_2, M_3, d, \alpha, T, g), \quad p = 2.$$ 

Under the additional condition $g(x) = o(|x|^{-d/p})$, $|x| \to +\infty$, we have $\hat{T}_{fg} \in \mathcal{S}_{p, \infty}^0$.

3. Estimates for the singular values of the operators $T_{fg}$ and $\hat{T}_{fg}$ in the case of an anisotropic-homogeneous function $g(x)$. In the present subsection, we provide power estimates for the singular values of the operators $T_{fg}$ and $\hat{T}_{fg}$ for the function $g(x)$, $x \in \mathbb{R}^d$, defined by the formula

$$(3.1) \quad g(x) = \zeta(|x|)|x|^{-d/p}\omega(x/|x|), \quad x \in \mathbb{R}^d, \quad p > 0.$$ 

Here $\omega(\theta), \theta \in \mathbb{S}^{d-1}$, is a suitable measurable function on the unit sphere, $\zeta$ is a continuous cut-off function at zero, i.e., $\zeta \in C(\mathbb{R})$ is monotone nondecreasing and $\zeta(t) = 0$ for $t < 1$, $\z(0) = 1$ for $t > 2$. The following assertion can be verified.

**Proposition 3.4.** Let $g(x), x \in \mathbb{R}^d$, be the function defined by (3.1). Then:

a) if $\omega \in L_p(\mathbb{S}^{d-1}), p > 0$, then $g \in L_{p, \infty}(\mathbb{R}^d)$ and

$$\|g\|_{L_{p, \infty}} \leq C(p, d)\|\omega\|_{L_p};$$
b) if $p \in (0, 2)$, $\omega \in L_2(\mathbb{S}^{d-1})$, then $v(g) \in \ell_{p, \infty}(\mathbb{Z}^d)$ and

$$\|v(g)\|_{\ell_{p, \infty}} \leq C(p, d, \zeta)\|\omega\|_{L_2}.$$ 

Proposition 3.1 and Corollaries 2.4, 2.6, and 2.9 have the following consequence.

**Proposition 3.5.** Suppose the operator $H$ satisfies Condition [1]. Let $f(\lambda), \lambda \in \mathbb{R}$, be a bounded Borel function, and let $g(x), x \in \mathbb{R}^d$, be the function defined by (3.1). Assume that one of the following three conditions is fulfilled:

a) if $p > 2$, then $u(f) \in \ell_p(\mathbb{Z}^+, dp), \omega \in L_p(\mathbb{S}^{d-1})$;
b) if $p \in (0, 2), u(f) \in \ell_p(\mathbb{Z}^+, dp), \omega \in L_2(\mathbb{S}^{d-1})$;
c) if $p = 2$, then, for some $\alpha \in (1, 2)$, $u(f) \in \ell_\alpha(\mathbb{Z}^+, dp), \omega \in L_{4/\alpha}(\mathbb{S}^{d-1})$.

Then $T_{fg} \in \mathcal{S}_{p, \infty}$ and $\|T_{fg}\|_{\mathcal{S}_{p, \infty}} \leq C\|u(f)\|\|\omega\|$. Here the following is implied:

$$\|u(f)\| := \|u(f)\|_{\ell_p}, \quad \|\omega\| = \|\omega\|_{L_p}, \quad C := C(M_1, M_2, M_3, d, p), \quad p > 2;$$
$$\|u(f)\| := \|u(f)\|_{\ell_p}, \quad \|\omega\| = \|\omega\|_{L_2}, \quad C := C(M_1, M_2, M_3, d, p, \zeta), \quad p \in (0, 2);$$
$$\|u(f)\| := \|u(f)\|_{\ell_{4/\alpha}}, \quad \|\omega\| = \|\omega\|_{L_{4/\alpha}}, \quad C := C(M_1, M_2, M_3, d, \alpha, \z), \quad p = 2.$$
Now, let \((K, d\mu)\) be a measurable space with \(\sigma\)-finite measure, and let \(T: L_2(R^d) \to L_2(K, d\mu)\) be a bounded linear integral operator with a bounded kernel \(t(\cdot, \cdot) \in L_\infty(K \times R^d, d\mu \, dx)\). Assume that the operator \(H\) satisfies Condition \([1]\) and that \([1.7]\) is fulfilled. Propositions \([1.4, 3.4]\) and Corollaries \([2.1, 2.14]\) yield the following statement.

**Proposition 3.6.** Suppose the operator \(H\) satisfies Condition \([1]\) and \([1.7]\) is valid. Let the function \(f: K \to \mathbb{C}\) be measurable, and let the function \(g(x), x \in R^d\), be defined by \([3.1]\). Assume that one of the following three conditions is fulfilled:

a) if \(p > 2\), then \(f \in L_p(K, d\mu), \omega \in L_p(S^{d-1});\)
b) if \(p \in (0, 2), then \(\|u\| \in \ell_p(Z_+, dp), \omega \in L_2(S^{d-1});\)
c) if \(p = 2\), then, for some \(\alpha \in (1, 2),\)
\[\omega \in L_{4/\alpha}(S^{d-1}), \quad \widehat{u}(f) \in \ell_\alpha(Z_+, d\rho), \quad f \in L_\beta(K, d\mu), \quad 1/\alpha + 1/\beta = 1.\]

Then \(\hat{T}_{fg} \in S_{p, \infty}\) and \(\|\hat{T}_{fg}\|_{S_{p, \infty}} \leq C\|f\| \|\omega\|\). Here the following is implied:

\[C := C(p, d)|t(\cdot, \cdot)|^{2/p}\|T\|^{1-2/p}, \quad \|f\| := \|f\|_{L_p}, \quad \|\omega\| := \|\omega\|_{L_p}, \quad p > 2;\]
\[C := C(M_2, M_3, d, p, \alpha, \beta, \gamma)|t(\cdot, \cdot)|^{(1/2+1/\alpha)}\|T\|^{1/\alpha-1/2}, \quad \|f\| := \|f\|_{L_\beta}, \quad \|\omega\| := \|\omega\|_{L_4/\alpha}, \quad p = 2.\]

§4. Proofs of the main results

**Lemma 4.1.** Let \(\nu_1: (0, +\infty) \to [0, +\infty)\) be a monotone nonincreasing function. Then for all \(\alpha > -1, \beta > -1, \gamma > 0\) we have
\[
\int_0^{+\infty} d\zeta \zeta^\alpha \left( \int_0^{+\infty} \nu_1(\sigma) \sigma^\beta d\sigma \right)^\gamma \leq C(\alpha, \beta, \gamma) \int_0^{+\infty} \zeta^{\alpha+\beta+1} \nu_1^\gamma(\zeta) d\zeta.
\]

Lemma \([4.1]\) was proved in \([4]\) (strictly speaking, in \([4]\) the case of \(\beta > 0\) was considered, but the argument remains valid for \(\beta \in (-1, 0]\) as well).

**Lemma 4.2.** Let \(\nu_1: (0, +\infty) \to [0, +\infty)\) be a monotone nonincreasing function. Then for all \(\alpha < -1, \beta > -1, \gamma > 0\) we have
\[
\int_0^{+\infty} d\zeta \zeta^\alpha \left( \int_0^{\zeta} \nu_1(\sigma) \sigma^\beta d\sigma \right)^\gamma \leq C(\alpha, \beta, \gamma) \int_0^{+\infty} \zeta^{\alpha+\beta+1} \nu_1^\gamma(\zeta) d\zeta
\]
Proof. For \(\gamma \geq 1\), inequality \([4.1]\) follows from the well-known estimate
\[
\int_0^{+\infty} d\zeta \zeta^{-r} \left( \int_0^{\zeta} d\sigma f(\sigma) \right)^\gamma \leq \gamma/(r-1)^{-\gamma} \int_0^{+\infty} \zeta^{-r}(\zeta f(\zeta))^\gamma d\zeta, \quad \gamma \geq 1, \quad r > 1
\]
(see, e.g., \([12\) Section 330]). In this case, \(C(\alpha, \beta, \gamma) = \gamma/(1-\alpha)^{-\gamma}\).

Now assume that \(\gamma \in (0, 1)\). For brevity, denote \(F(\zeta) := \int_0^{\zeta} \nu_1(\sigma) \sigma^\beta d\sigma\). Since the function \(F(\zeta)\) is monotone, we have
\[
\int_0^{+\infty} \zeta^\alpha F^\gamma(\zeta) d\zeta = \sum_{j \in \mathbb{Z}} \int_0^{2^j} F^\gamma(\zeta) d\zeta \frac{\zeta^{\alpha+1}}{\alpha+1} \leq C(\alpha) \sum_{j \in \mathbb{Z}} F^\gamma(2^j) 2^{j(\alpha+1)}.
\]
By the monotonicity of the function $\nu_1$, the quantity $F(2^j)$ admits a similar estimate:

$$
(4.3) \quad F(2^j) = \sum_{l=-\infty}^{j-1} \int_{2^l}^{2^{l+1}} \nu_1(\sigma) \, d\sigma \leq C(\beta) \sum_{l=-\infty}^{j-1} \nu_1(2^l)2^{l(\beta+1)}.
$$

Substituting (4.3) in (4.2), using the inequality $(\sum_k a_k)^\gamma \leq \sum_k a_k^\gamma$, $\gamma \in (0,1)$, $a_k \geq 0$, and changing of the order of summation, we get

$$
(4.4) \quad \int_0^{+\infty} \zeta^\alpha F^\gamma(\zeta) \, d\zeta \leq C_1(\alpha, \beta, \gamma) \sum_{l \in \mathbb{Z}} \nu_1^\gamma(2^l)2^{l(\gamma(\beta+1)+\alpha+1)}.
$$

In the same way, the monotonicity of $\nu_1$ yields a lower estimate:

$$
(4.5) \quad \int_0^{+\infty} \zeta^{-\alpha-(\beta+1)\gamma} \nu_1(\zeta) \, d\zeta \geq C_2(\alpha, \beta, \gamma) \sum_{l \in \mathbb{Z}} \nu_1^\gamma(2^l)2^{l(\gamma(\beta+1)+\alpha+1)},
$$

where $C_2(\alpha, \beta, \gamma) > 0$. Inequalities (4.4) and (4.5) yield (4.1). \hfill \square

2. A few preliminary operator-theoretical remarks are in order. Let $\mathcal{H}, \mathcal{G}$ be Hilbert spaces, let $\mathcal{H} = \bigoplus_n \mathcal{H}_n$, $\mathcal{G} = \bigoplus_n \mathcal{G}_m$, and let $P_n$ be the orthogonal projection onto $\mathcal{H}_n$, and $Q_m$ the orthogonal projection onto $\mathcal{G}_m$. Assume that $T_{mn}: \mathcal{H}_n \to \mathcal{G}_m$ are operators belonging to the Hilbert-Schmidt class. Since the system $\{Q_m T_{mn} P_n\}$ is orthogonal in the class $\mathcal{G}_2$, the following statement is true.

**Lemma 4.3.** The series $\sum_{m,n} Q_m T_{mn} P_n$ converges in $\mathcal{G}_2$ if and only if the series $\sum_{m,n} ||T_{mn}||_{\mathcal{G}_2}^2$ converges. Moreover, we have

$$
||\sum_{m,n} Q_m T_{mn} P_n||_{\mathcal{G}_2}^2 = \sum_{m,n} ||T_{mn}||_{\mathcal{G}_2}^2;
$$

the series $\sum_{m,n} (Q_m T_{mn} P_n u, v)$ converges absolutely for all $u \in \mathcal{H}, v \in \mathcal{G}$.

The results of [13] (see also [11, §11.5, Subsection 4]) imply the following assertion.

**Lemma 4.4.** If $T_1, T_2 \in \mathcal{S}_p$, $p \in (0,1)$, then $||T_1 + T_2||_{\mathcal{S}_p}^p \leq ||T_1||_{\mathcal{S}_p}^p + ||T_2||_{\mathcal{S}_p}^p$.

**Corollary 4.5.** Let $\{T_n\}_{n \in \mathbb{N}} \subset \mathcal{S}_p$, $p \in (0,1)$, and suppose that the series $\sum_n ||T_n||_{\mathcal{S}_p}^p$ converges. Then the series $\sum_n T_n$ converges in $\mathcal{S}_p$, and $||\sum_n T_n||_{\mathcal{S}_p}^p \leq \sum_n ||T_n||_{\mathcal{S}_p}^p$.

3. **Proof of Proposition 1.1.** Denote the “smoothed” modulus $(1 + |x|^2)^{1/2}$ by $\langle x \rangle$, $x \in \mathbb{R}^d$. As before, $\mathcal{F}$ denotes the set of bounded measurable compactly supported functions from $\mathbb{R}^d$ to $\mathbb{C}$. For any bounded Borel function $f(\lambda)$, $\lambda \in \mathbb{R}$, and any $g \in L_{2,loc}(\mathbb{R}^d)$, the operator $T_{fg} := f(H)g$ is well defined on the linear set $\mathcal{F}$. Assume that the operator $H$ satisfies Condition 1. It is not hard to check the following assertion by using (4.4).

**Lemma 4.6.** Let a measurable function $g: \mathbb{R}^d \to \mathbb{C}$ satisfy the condition

$$
\int_{\mathbb{R}^d} |g(x)|^2 \langle x \rangle^{2\alpha} \, dx < +\infty, \quad \alpha \geq 0.
$$

Then for all $m \in \mathbb{Z}^d$ and $t > 0$, we have $|\langle x - m \rangle^\alpha e^{-tH} g| \in \mathcal{G}_2$ and

$$
(4.6) \quad ||(x - m)^\alpha e^{-tH} [g(x)]||_{\mathcal{G}_2}^2 \leq C(M_1, M_3, d, \alpha) t^{-d/2} (1 + M_3 t)^\alpha e^{2M_3 t} \int_{\mathbb{R}^d} |g(x)|^2 \langle x - m \rangle^{2\alpha} \, dx.
$$

For brevity, we denote the spectral projection $E_H([a_k, a_{k+1}))$, $k \in \mathbb{Z}_+$, by $E_k$. 
Lemma 4.7. For any bounded Borel function \(f(\lambda), \lambda \in \mathbb{R}\), and any \(g \in L_2(\mathbb{R}^d)\), the operator \(E_k R f_g, k \in \mathbb{Z}_+\), admits extension from the linear set \(\mathcal{F}\) up to a Hilbert–Schmidt operator in \(L_2(\mathbb{R}^d)\), and the following estimate is true:

\[
\|E_k T f_g\|^2_{L^2_{\mathcal{F}}} \leq C(M_1, M_2, M_3, d) a_{k+1}^{d/2} u_k^2(f) \|g\|^2_{L^2_{2(\mathbb{R}^d)}}, \quad k \in \mathbb{Z}_+.
\]  

Proof. All assertions of Lemma 4.7 follow from the identity

\[
E_k T f_g u = E_k e^{a_{k+1}^{-1} H} f(H) e^{-a_{k+1}^{-1} H} g u, \quad u \in \mathcal{F},
\]

and from estimate (4.6) applied to the operator \(e^{-a_{k+1}^{-1} H} g\) (for \(\alpha = 0, t = a_{k+1}^{-1}\)). □

We introduce the operator \(P_j\) of multiplication by the characteristic function \(\chi_j\) of the set \(E_j(g) := \{x \in \mathbb{R}^d : |g(x)| \in (2^j, 2^{j+1}]\}, j \in \mathbb{Z}\). Denoting for brevity \(E_j(u(f)) := \{k \in \mathbb{Z}_+: u_k(f) \in (2^j, 2^{j+1}]\}, i \in \mathbb{Z}\), we define the orthogonal projections \(Q_i := \sum_{k \in E_i(u)} E_k, i \in \mathbb{Z}\). Surely, for each \(\varphi \in \mathcal{F}\) the function \(P_j \varphi, j \in \mathbb{Z}\), also belongs to \(\mathcal{F}\).

Lemma 4.8. For any \(\varphi \in \mathcal{F}\) and any \(\psi \in L_2(\mathbb{R}^d)\), the series

\[
\sum_{i+j \leq k_0} (Q_i T f_g P_j \varphi, \psi) =: (T^{(1)} f_g (k_0) \varphi, \psi), \quad k_0 \in \mathbb{Z},
\]

converges absolutely and determines a bounded operator \(T^{(1)} f_g (k_0)\). Moreover, the following estimate holds true:

\[
\|T^{(1)} f_g (k_0)\| \leq 2^{k_0+3}, \quad k_0 \in \mathbb{Z}.
\]  

Proof. Using the definition of \(Q_i\), \(P_j\) and the Cauchy inequality, we see that, for all \(\varphi \in \mathcal{F}\) and \(\psi \in L_2(\mathbb{R}^d)\),

\[
\sum_{i+j \leq k_0} |(Q_i T f_g P_j \varphi, \psi)| \leq \sum_{i+j \leq k_0} 2^{i+j+2} ||P_j \varphi|| ||Q_i \psi|| \leq \left( \sum_{i+j \leq k_0} 2^{i+j+2} ||P_j \varphi||^2 \right)^{1/2} \left( \sum_{i+j \leq k_0} 2^{i+j+2} ||Q_i \psi||^2 \right)^{1/2} = \left( \sum_{j \in \mathbb{Z}} 2^{j+2} ||P_j \varphi||^2 \sum_{i \leq k_0-j} 2^i \right)^{1/2} \left( \sum_{j \leq k_0-i} 2^{j+2} ||Q_i \psi||^2 \sum_{j \leq k_0-i} 2^j \right)^{1/2} \leq 2^{k_0+3} ||\varphi|| \|\psi||.
\]

This proves the lemma. □

As before, let \(d\nu(j, x) := a_{j+1}^{d/2} \, dj \, dx\) be the measure on \(\mathcal{Z} := \mathbb{Z} \times \mathbb{R}^d\). Denote by \(\chi_{O_{u_0}(2k_0)}\) the characteristic function of the set \(O_{u_0}(2k_0) := \{(j, x) \in \mathcal{Z} : |u_j(f)g(x)| > 2^{k_0}\}, k_0 \in \mathbb{Z}\). We assume that

\[
u(f)g \cdot \chi_{O_{u_0}(2k_0)} \in L_2(\mathcal{Z}, d\nu)
\]

for some \(k_0 \in \mathbb{Z}\).

Lemma 4.9. Under condition (4.3), for any \(\varphi \in \mathcal{F}, \psi \in L_2(\mathbb{R}^d)\) the series

\[
\sum_{i+j > k_0} (Q_i T f_g P_j \varphi, \psi) =: (T^{(2)} f_g (k_0) \varphi, \psi)
\]

is absolutely convergent and determines an operator \(T^{(2)} f_g (k_0)\) in \(\mathcal{G}_2\). Moreover, we have the following estimate:

\[
\|T^{(2)} f_g (k_0)\|^2_{\mathcal{G}_2} \leq C(M_1, M_2, M_3, d) \|u(f)g \cdot \chi_{O_{u_0}(2k_0)}\|^2_{L^2}.
\]
Proof. By Lemma 4.3 it suffices to check the estimate
\begin{equation}
\sum_{i+j>k_0} \|Q_i T_{fg} P_j\|_{L^2}^2 \leq C(M_1, M_2, M_3, d) \|u(f)g \cdot \chi_{O_{ug}(2k_0)}\|_{L^2}^2.
\end{equation}

Using the identity $Q_i T_{fg} P_j = \sum_{k \in E_i(u)} E_k f(H) g \chi_j$, Lemma 4.3 and estimate (4.7), we get
\[ \|Q_i T_{fg} P_j\|_{L^2}^2 \leq C(M_1, M_2, M_3, d) \sum_{k \in E_i(u)} a_{k+1}^d u_k^2(f) \|g \chi_j\|_{L^2}. \]

Hence,
\[ \sum_{i+j>k_0} \|Q_i T_{fg} P_j\|_{L^2}^2 \leq C(M_1, M_2, M_3, d) \sum_{i+j>k_0} \sum_{k \in E_i(u)} a_{k+1}^d u_k^2(f) \|g \chi_j\|_{L^2}^2 = C(M_1, M_2, M_3, d) \sum_{i+j>k_0} \int_{E_i(u) \times E_j(g)} |u g|^2 \, d\nu. \]

Since $\bigcup_{i+j>k_0} E_i(u) \times E_j(g) \subset O_{ug}(2k_0)$, this yields (4.11). \hfill \Box

Proof of Proposition 1.1. By condition (1.2), we may find $k_0 \in \mathbb{Z}$ such that (1.3) is valid. From Lemmas 4.8, 4.9 and the definition of the operators $P_i, Q_i$, $i \in \mathbb{Z}$, it follows that for $u(f)g \in L_2(\mathbb{Z}, d\nu) + L_\infty(\mathbb{Z}, d\nu)$ we have $(T_{fg}\varphi, \psi) = (T_{fg}^{(1)}(k_0)\varphi, \psi) + (T_{fg}^{(2)}(k_0)\varphi, \psi)$, $\varphi \in \mathcal{F}$, $\psi \in L_2(\mathbb{R}^d)$. Therefore, the operator $T_{fg}$ admits extension from the linear set $\mathcal{F}$ up to a bounded operator on $L_2(\mathbb{R}^d)$, and
\begin{equation}
\|T_{fg}\| \leq C(M_1, M_2, M_3, d) \|u(f)g \cdot \chi_{O_{ug}(2k_0)}\|_{L^2}.
\end{equation}

In order to verify inequality (1.3), it suffices to show that for an arbitrary decomposition $(u(f)g)(j, x) = a(j, x) + b(j, x)$, $a \in L_\infty(\mathbb{Z}, d\nu)$, $b \in L_2(\mathbb{Z}, d\nu)$ one has the estimate
\begin{equation}
\|T_{fg}\| \leq \widetilde{C}(M_1, M_2, M_3, d) \|a\|_{L_\infty} + \|b\|_{L_2}. \end{equation}

If $a = 0$, then (1.13) is obvious. If $a \neq 0$, choose $k_0 \in \mathbb{Z}$ satisfying the condition
\begin{equation}
2^{k_0-2} \leq \|a\|_{L_\infty} < 2^{k_0-1}.
\end{equation}

Using (4.14) and observing that $\|b(j, x)| \geq \|(u(f)g)(j, x)| - |a(j, x)|$, we see that
\begin{equation}
O_{u(f)g}(2k_0) \subset O_b(2^{k_0} - \|a\|_{L_\infty}) \subset O_b(2^{k_0} - 2^{k_0-1}) = O_b(2^{k_0-1}) \subset O_b(\|a\|_{L_\infty}),
\end{equation}

implying condition (1.9). Next, relations (1.12), (1.14), and (1.15) yield the inequalities
\[ \|T_{fg}\| \leq 2^{k_0+3} + C(M_1, M_2, M_3, d) \|u(f)g \cdot \chi_{O_b(\|a\|_{L_\infty})}\|_{L^2} \]
\[ \leq 2^{k_0}\|a\|_{L_\infty} + C(M_1, M_2, M_3, d) \|b\|_{L_2} + \|a \chi_{O_b(\|a\|_{L_\infty})}\|_{L^2} \]
\[ \leq 2^{k_0}\|a\|_{L_\infty} + C(M_1, M_2, M_3, d) \|b\|_{L_2} + \|a\|_{L_\infty} \nu_b^{1/2}(\|a\|_{L_\infty}) \]
\[ \leq 2^{k_0}\|a\|_{L_\infty} + 2C(M_1, M_2, M_3, d) \|b\|_{L_2}, \]

and (1.13) follows immediately.

If, moreover, $\nu_{u(f)g}(s) < \infty$ for $s > 0$, then condition (1.9) and, with it, relation $T_{fg}^{(2)}(k_0) \in \mathcal{G}_2$ are valid for all $k_0 \in \mathbb{Z}$. Combined with the estimate $\|T_{fg}^{(1)}(k_0)\| \leq 2^{k_0+3}$ and the identity $T_{fg} = T_{fg}^{(1)} + T_{fg}^{(2)}$, $k_0 \in \mathbb{Z}$, this proves that $T_{fg} \in \mathcal{G}_\infty$. \hfill \Box

Proof of Theorem 1.2. The assumptions of Theorem 1.2 imply (1.2) and the fact that $\nu_{ug}(s) < \infty$ for all $s > 0$. Therefore, by Proposition 1.1 the operator $T_{fg} = T_{fg}^{(1)}(k_0) + T_{fg}^{(2)}(k_0)$, $k_0 \in \mathbb{Z}$, is compact.
Lemma 4.10. The next claim is an immediate consequence of Lemma 4.6.

4. Proof of Theorem 1.3. As before, here we assume that Condition 1 is satisfied.

Next, for any \( s > 0 \) pick a number \( k_0 \in \mathbb{Z} \) such that \( 2^{k_0+3} < s \leq 2^{k_0+4} \); then, by estimates (4.8), (4.10) and inequality (4.7), we have

\[
\begin{align*}
\nu_{u,s} (s, T_{fg}) &= n(s, T_{fg}) \leq n(s, T_{fg} (k_0)) \leq 4s^{-2} \nu_{u,s} (k_0) \leq C_1 s^{-2} \int_{O_{us} (2^{-5}s)} |u(f)g|^2 \, dv \\
&= C_2 \nu_{u,s} (2^{-5}s) + C_1 s^{-2} \int_{2^{-5}s}^{+\infty} \nu_{u,s} (\sigma) \, d\sigma^2, \quad s > 0.
\end{align*}
\]

Here \( C_1, C_2 \) depend on \( M_1, M_2, M_3, \) and \( d \) only.

If \( p \in (2, +\infty), q = +\infty \), then the relation \( T_{fg} \in S_{p,\infty} \) and estimate (4.5) follow directly from (4.16). Assume additionally that \( \nu_{u,s} (s) = o(s^{-p}), s \to +0, \) i.e., for any \( \varepsilon > 0 \) we have \( \nu_{u,s} (s) \leq \varepsilon s^{-p} \) for \( s \in (0, s_0), s_0 = s_0 (\varepsilon) > 0 \). Splitting the interval of integration in (4.16) into the intervals \( (2^{-5}s, s_0) \) and \( (s_0, +\infty) \) and employing the estimates \( \nu_{u,s} (s) \leq \varepsilon s^{-p}, s \in (0, s_0), \) and \( \nu_{u,s} (s) \leq \|u\|_{L_p,\infty}^p s^{-p}, s \in (s_0, +\infty), \) we arrive at the inequality

\[
(4.17) \quad n(s, T_{fg}) \leq C_3 \varepsilon s^{-p} + C_4 (\varepsilon) \|u\|_{L_p,\infty}^p s^{-2}, \quad s \in (0, s_0).
\]

Here \( C_3 \) depends on \( M_1, M_2, M_3, d, \) and \( p \) only, and \( C_4 \) depends on \( M_1, M_2, M_3, d, p, \varepsilon \). Inequality (4.17) implies the relation \( \limsup_{s \to +0} n(s, T_{fg}) \leq C_3 \varepsilon \) for all \( \varepsilon > 0 \), which shows that \( T_{fg} \in S_{p,\infty} \).

In the case where \( p \in (2, +\infty), q \in (0, +\infty), \) from (4.8) and (4.16) we deduce the estimate

\[
\|T_{fg}\|_{S_{p,q}} \leq q \int_0^{+\infty} ds \, s^{q-1} \left( C_2 \nu_{u,s} (2^{-5}s) + C_1 s^{-2} \int_{2^{-5}s}^{+\infty} \nu_{u,s} (\sigma) \, d\sigma^2 \right)^{q/p}.
\]

Then we use the elementary inequality \((a + b)^\alpha \leq C_\alpha (a^\alpha + b^\alpha), a, b \geq 0, \alpha > 0, C_\alpha = \max\{1, 2^{\alpha-1}\},\) and make the substitution \( \zeta = 2^{-5}s \) in the outer integral, obtaining

\[
(4.18) \quad \|T_{fg}\|_{S_{p,q}} \leq C_5 \|u\|_{L_p,q}^q + C_6 \int_0^{+\infty} d\zeta \, \zeta^{(q/p)(p-2)-1} \left( \int_\zeta^{+\infty} d\sigma \, \sigma \nu_{u,s} (\sigma) \right)^{q/p},
\]

where \( C_i = C_i (M_1, M_2, M_3, d, p, q), \) \( i = 5, 6 \). Applying Lemma 4.1 to the second summand in (4.18), we arrive at (4.4).

4. Proof of Theorem 1.3. As before, here we assume that Condition 1 is satisfied. The next claim is an immediate consequence of Lemma 4.6.

Lemma 4.10. For all \( \alpha \geq 0 \) and \( \beta > \alpha + d/2, \) the operator \( [(x - m)^\alpha] e^{-tH} [(x - m)^{-\beta}], \) \( m \in \mathbb{Z}^d, \) \( t > 0, \) belongs to \( S_2, \) and

\[
\|(x - m)^\alpha] e^{-tH} [(x - m)^{-\beta}]\|_{S_2}^2 \leq C(M_1, M_3, d, \alpha, \beta) t^{-d/2}(1 + M_4 t)^\alpha e^{2M_2 t}, \quad m \in \mathbb{Z}^d, \quad t > 0.
\]
Lemma 4.11. For any bounded Borel function \( f(\lambda), \lambda \in \mathbb{R} \), and any \( g \in L_{2, \text{loc}}(\mathbb{R}^d) \), the operator \( \mathcal{E} f g \mathcal{X}_m, k \in \mathbb{Z}_+, m \in \mathbb{Z}^d \), admits extension from the linear set \( \mathcal{F} \) up to a Hilbert–Schmidt operator on \( L_2(\mathbb{R}^d) \). For all \( l \in \mathbb{N} \), the operator \( E_k \mathcal{T} f g \mathcal{X}_m, k \in \mathbb{Z}_+, m \in \mathbb{Z}^d \), belongs to the class \( \mathcal{G}_{2/l} \), and

\[
\| E_k \mathcal{T} f g \mathcal{X}_m \|_{\mathcal{G}_{2/l}}^{2/l} \leq C(M_1, M_2, M_3, d, l) a_{k+l}^{d/l} u_k (f) v_m^{2/l}(g), \quad k \in \mathbb{Z}_+, m \in \mathbb{Z}^d.
\]

Proof. For \( l = 1 \), Lemma 4.11 follows from Lemma 4.11. For an arbitrary \( l = 2, 3, \ldots \), we have

\[
E_k \mathcal{T} f g \mathcal{X}_m = E_k f(H) e^{l a_{k+l}^{1/H}} T_1 \cdot T_2 \cdots \cdot T_l,
\]

where

\[
T_j := [(x - m)^{\alpha_j - 1}] e^{-a_{k+l}^{1/H}} [x - m]^{-\alpha_j}, \quad j = 1, \ldots, l - 1,
\]

\[
T_l := [(x - m)^{\alpha_l - 1}] e^{-a_{k+l}^{1/H}} g \mathcal{X}_m;
\]

the parameters \( \alpha_0, \alpha_1, \ldots, \alpha_{l-1} \) satisfy \( \alpha_0 = 0, \alpha_j > \alpha_{j-1} + d/2, j = 1, \ldots, l - 1 \). By Lemmas 4.6 and 4.10, we have \( T_j \in \mathcal{G}_2, j = 1, \ldots, l \); hence, by Proposition 0.2, the operator \( E_k \mathcal{T} f g \mathcal{X}_m \) belongs to the class \( \mathcal{G}_{2/l} \). The inequality \( \| E_k f(H) e^{l a_{k+l}^{1/H}} \| \leq u_k (f) e^l \), estimate (4.19) applied to the operators \( T_j, j = 1, \ldots, l - 1 \), estimate (4.6) applied to the operator \( T_l \), and estimate (0.9) yield (4.20). \( \square \)

Lemma 4.3 Corollary 4.5 and Lemma 4.11 imply the following statement.

Corollary 4.12. Theorem 1.3 is valid for all \( p = q = 2/l, l \in \mathbb{N} \).

Lemma 4.13. For any bounded Borel function \( f \) on the line and any function \( g \in L_{2, \text{loc}}(\mathbb{R}^d) \) such that \( u(f)v(g) \in \ell_2(\mathbb{Z}_+ \times \mathbb{Z}_+, d\nu) \), we have \( \mathcal{C} f g = \mathcal{T}^{(1)} f g (t) + \mathcal{T}^{(2)} f g (t), t > 0 \), where \( \mathcal{T}^{(1)} f g (t) \in \mathcal{G}_2, \mathcal{T}^{(2)} f g (t) \in \mathcal{G}_p \) (for all \( p \in (0, 2) \)), \( t > 0 \); moreover, the following inequalities are valid:

\[
\| \mathcal{T}^{(1)} f g (t) \|_{\mathcal{G}_2}^{2/l} \leq C_1(M_1, M_2, M_3, d) \int_0^t \tilde{\nu}_{uv}(s) ds^2, \quad t > 0;
\]

\[
\| \mathcal{T}^{(2)} f g (t) \|_{\mathcal{G}_{2/l}}^{2/l} \leq C_2(M_1, M_2, M_3, d, l) \left[ \tilde{\nu}_{uv}(t)^{2/l} + \int_t^{+\infty} \tilde{\nu}_{uv}(s) ds^{2/l} \right], \quad t > 0, \quad l \in \mathbb{N}.
\]

Proof. We can write

\[
\mathcal{T} f g = \sum_{k=0}^{\infty} \sum_{m \in \mathbb{Z}^d} E_k f(H) g \mathcal{X}_m.
\]

Let operators \( \mathcal{T}^{(1)} f g (t), \mathcal{T}^{(2)} f g (t), t > 0 \), be defined by the formulas

\[
\mathcal{T}^{(1)} f g (t) = \sum_{u_k(f)v_m(g) \leq t} E_k f(H) g \mathcal{X}_m; \quad \mathcal{T}^{(2)} f g (t) = \sum_{u_k(f)v_m(g) > t} E_k f(H) g \mathcal{X}_m, \quad t > 0.
\]

By Lemmas 4.3 and 4.11 these series are convergent in \( \mathcal{G}_2 \); the number of terms in the second sum is at most \( \tilde{\nu}_{uv}(t), t > 0 \), and is thus finite. Moreover, by Lemma 4.11 each
term of the second sum belongs to the class $S_p$ for all $p > 0$. Therefore, $T_{fg}^{(1)}(t) \in S_p$ and $T_{fg}^{(2)}(t) \in S_p$, $p > 0$. Lemma 4.11 and Corollary 4.5 yield the estimates

$$
\|T_{fg}^{(1)}(t)\|_{S_p}^2 \leq C(M_1, M_2, M_3, d) \sum_{u_k(f)v_m(g) \leq t} a_k^{d/2}u_k(f)v_m(g)^2;
$$

(4.23)

$$
\|T_{fg}^{(2)}(t)\|_{S_p/2}^{2/l} \leq C(M_1, M_2, M_3, d, l) \sum_{u_k(f)v_m(g) > t} a_k^{d/2}u_k(f)v_m(g)^{2/l}, \; l \in \mathbb{N}, \; t > 0.
$$

(4.24)

Now, estimates (4.21), (4.22) follow from inequalities (4.23) and identity (0.6).

Proof of Theorem 1.3. In accordance with Corollary 4.12, Theorem 1.3 is valid for $p = q = 2$. For arbitrary $p \in (0, 2)$ and $q \in (0, +\infty]$, relation (1.6) implies $u(f)v(g) \in L_2(\mathbb{Z}_+ \times \mathbb{Z}^d, d\nu)$. Pick $p_0 = 2/l < p$; inequality (0.7) and Lemma 4.13 show that

$$
n(s, T_{fg}) \leq n(s/2, T_{fg}^{(1)}(s)) + n(s/2, T_{fg}^{(2)}(s))
$$

\begin{align*}
& \leq 4s^{-2} \|T_{fg}^{(1)}(s)\|_{S_p}^2 + 2p_0 s^{-p_0} \|T_{fg}^{(2)}(s)\|_{S_p}^2, \\
& \leq C(M_1, M_2, M_3, d)s^{-2} \int_0^s \tilde{\nu}_{uv}(\sigma) \, d\sigma^2 \\
& + C_2(M_1, M_2, M_3, d, p_0)s^{-p_0} \int_0^s \tilde{\nu}_{uv}(\sigma) \, d\sigma^{p_0}.
\end{align*}

(4.24)

The claims of Theorem 1.3 for $q = +\infty$ are easy consequences of (4.24). Using (0.8) and (4.24), for all $p \in (0, 2)$ and $q \in (0, +\infty)$ ($p_0 = 2/l \in (0, p)$) we get

$$
\|T_{fg}^{\eta}_{p, q}\|_{S_p}^q \leq S_1 \int_0^{+\infty} ds^q \left[ s^{-2} \int_0^s d\sigma^2 \tilde{\nu}_{uv}(\sigma) \right]^{q/p} + S_2 \int_0^{+\infty} ds^q \tilde{\nu}_{uv}^{q/p}(s) + S_3 \int_0^{+\infty} ds^q \left[ s^{-p_0} \int_0^s d\sigma^{p_0} \tilde{\nu}_{uv}(\sigma) \right]^{q/p}.
$$

(4.25)

Here $S_i = S_i(M_1, M_2, M_3, p, q, p_0), i = 1, 2, 3$. Applying Lemma 4.2 to the first integral in (4.25) and Lemma 4.1 to the third integral, we arrive at the required assertions.

5. Proof of Theorem 1.5. As before, we assume that the operator $H$ satisfies Condition 1. We also assume condition (1.7). For the most part, the proof of Theorem 1.5 resembles that of Theorem 1.3. Instead of Lemma 4.1, we use the following obvious assertion.

Lemma 4.14. If $\hat{u}(f)v(g) \in L_2(\mathbb{Z}_+ \times \mathbb{Z}^d, d\nu)$, then the kernel $f(k)t(k,y)g(y)$, $(k,y) \in K \times \mathbb{R}^d$, determines a Hilbert–Schmidt operator $\hat{T}_{fg}$ from $L_2(\mathbb{R}^d)$ to $L_2(K, dp)$, and

$$
\|\hat{T}_{fg}\|_{L_2(\mathbb{R}^d)} \leq \|t(\cdot, \cdot)\|_{L_2(\mathbb{R}^d)}^2 \|\hat{u}(f)v(g)\|_{L_2}^2.
$$

(4.26)

Note that under the assumptions of Theorem 1.5, we have $\hat{u}(f)v(g) \in L_2(\mathbb{Z}_+ \times \mathbb{Z}^d, d\nu)$. Denote by $\hat{E}_j$ the operator of multiplication by the characteristic function of the set $K_j, j \in \mathbb{Z}_+$. As before, $\hat{\chi}_m$ is the characteristic function of the set $\Omega_m, m \in \mathbb{Z}^d$. Below, instead of Lemma 4.11, we employ the following assertion.

Lemma 4.15. For any measurable function $f: K \to \mathbb{C}$ and any function $g \in L_{2,\text{loc}}(\mathbb{R}^d)$ such that $\hat{u}(f)v(g) \in L_2(\mathbb{Z}_+ \times \mathbb{Z}^d, d\nu)$, for all $l \in \mathbb{N}$ the operator $\hat{E}_j \hat{T}_{fg} \hat{\chi}_m, j \in \mathbb{Z}_+, m \in \mathbb{Z}^d$, belongs to $S_{2/l}$, and

$$
\|\hat{E}_j \hat{T}_{fg} \hat{\chi}_m\|_{S_{2/l}}^{2/l} \leq C(M_1, M_2, M_3, d, l)\|t(\cdot, \cdot)\|_{L_\infty}^{2/l} a_{j+1}^{d/2} \tilde{u}_m^{2/l}(f)v_m^{2/l}(g),
$$

(4.27)

\begin{align*}
& j \in \mathbb{Z}_+, \; m \in \mathbb{Z}^d.
\end{align*}
Proof. For \( l = 1 \), Lemma 4.15 follows from Lemma 4.14. For an arbitrary \( l = 2, 3, \ldots \) we have
\[
\hat{E}_j f g \chi_m = T_1 \cdot T_2 \cdot \ldots \cdot T_i,
\]
where \( T_1 = \hat{E}_j f e^{(l-1)a_{j+1}hT}[(x - m)^{-\alpha_i}], \ T_i := [(x - m)^{-\alpha_i}e^{-a_{j+1}hT}g \chi_m] \); the parameters \( \alpha_1, \ldots, \alpha_{j-1} \) satisfy \( \alpha_1 > d/2, \alpha_j > \alpha_{j-1} + d/2, \ i = 2, \ldots, l - 1 \). By Lemmas 4.6 and 4.10 and 4.14 we have \( T_i \in \mathcal{S}_2, i = 1, \ldots, l \); hence, by Proposition 0.2 the operator \( \hat{E}_j f g \chi_m \) belongs to \( \mathcal{S}_2/l \). Estimate (4.26) applied to the operator \( T_1 \), estimate (4.19) applied to the operators \( T_i \), \( i = 2, \ldots, l - 1 \), and estimate (4.6) applied to the operator \( T_l \), and estimate (0.9) imply inequality (4.27). \( \square \)

Lemma 4.13 is replaced by the following statement.

Lemma 4.16. For any measurable function \( f : K \to \mathbb{C} \) and any function \( g \in L_{2,0_{\operatorname{loc}}}(\mathbb{R}^d) \) such that \( \hat{u}(f)v(g) \in \ell_2(\mathbb{Z}^d, \mathbb{Z}^d, \hat{d}) \), there exists a decomposition \( \hat{T}_{fg} = \hat{T}_{fg}(1) + \hat{T}_{fg}(2) \), \( t > 0 \), where \( \hat{T}_{fg}(1) (t) \in \mathcal{S}_2 \) and \( \hat{T}_{fg}(2) (t) \in \mathcal{S}_p \) (for all \( p \in (0, 2) \)), \( t > 0 \); moreover,
\[
\begin{align*}
(4.28) & \quad \left\| \hat{T}_{fg}(1) (t) \right\|_{\mathcal{S}_2}^2 \leq C_1(M_1, M_2, M_3, d) \left\| t (\cdot, \cdot) \right\|_{L_\infty}^2 \int_0^t \hat{\nu}_{uv} (s) \, ds, \quad t > 0; \\
(4.29) & \quad \left\| \hat{T}_{fg}(2) (t) \right\|_{\mathcal{S}_2/l}^{2/l} \leq C_2(M_1, M_2, M_3, d, l) \left\| t (\cdot, \cdot) \right\|_{L_\infty}^{2/l} \left[ \hat{\nu}_{uv} (t)^{2/l} + \int_t^{+\infty} \hat{\nu}_{uv} (s) \, ds \right], \quad t > 0, \quad l \in \mathbb{N}.
\end{align*}
\]

Proof. The proof of Lemma 4.16 is quite similar to that of Lemma 4.13. We have
\[
\hat{T}_{fg} = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^d} \hat{E}_j f Tg \chi_m.
\]
Let operators \( \hat{T}_{fg}(1) (t), \hat{T}_{fg}(2) (t) \), \( t > 0 \), be defined by the formulas
\[
\begin{align*}
\hat{T}_{fg}(1) (t) & := \sum_{\hat{u} \langle f \rangle \nu_m (g) \leq t} \hat{E}_j f Tg \chi_m; \\
\hat{T}_{fg}(2) (t) & := \sum_{\hat{u} \langle f \rangle \nu_m (g) > t} \hat{E}_j f Tg \chi_m, \quad t > 0.
\end{align*}
\]
By Lemmas 4.13 and 4.14 these series are convergent in \( \mathcal{S}_2 \); the number of terms in the second sum is at most \( \hat{\nu}_{uv} (t) \), \( t > 0 \), and is thus finite. Moreover, by Lemma 4.15 every term of the second sum belongs to \( \mathcal{S}_p \) for all \( p > 0 \). Therefore, \( \hat{T}_{fg}(1) (t) \in \mathcal{S}_2 \) and \( \hat{T}_{fg}(2) (t) \in \mathcal{S}_p, p > 0 \). Combining Lemma 4.15 with Lemma 4.13 and Corollary 4.5 we get
\[
\begin{align*}
(4.30) & \quad \left\| \hat{T}_{fg}(1) (t) \right\|_{\mathcal{S}_2}^2 \leq C_1(M_1, M_2, M_3, d) \left\| t (\cdot, \cdot) \right\|_{L_\infty}^2 \sum_{\hat{u} \langle f \rangle \nu_m (g) \leq t} a_{j+1}^{d/2} \left| \hat{u} \langle f \rangle v_m (g) \right| ^2; \\
(4.31) & \quad \left\| \hat{T}_{fg}(2) (t) \right\|_{\mathcal{S}_2/l}^{2/l} \leq C_2(M_1, M_2, M_3, d, l) \left\| t (\cdot, \cdot) \right\|_{L_\infty}^{2/l} \sum_{\hat{u} \langle f \rangle \nu_m (g) > t} a_{j+1}^{d/2} \left| \hat{u} \langle f \rangle v_m (g) \right| ^{2/l}, \quad l \in \mathbb{N}, \quad t > 0.
\end{align*}
\]
Estimates (4.28) and (4.29) follow from inequalities (4.30) and identity (0.6). \( \square \)

Proof of Theorem 1.5. For the most part, the proof repeats that of Theorem 1.3. Instead of Corollary 4.12 we employ Lemma 4.14 whereas instead of Lemma 4.13 we use Lemma 4.16. All considerations should first be carried out under the assumption \( \left\| t (\cdot, \cdot) \right\|_{L_\infty} = 1 \), and then we can use the homogeneity of the quasinorm \( \left\| \hat{T}_{fg} \right\|_{\mathcal{S}_p, q} \). \( \square \)
References


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