Approximation of the Resolvent of a Two-Parametric Quadratic Operator Pencil Near the Bottom of the Spectrum

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Abstract. A two-parametric pencil of selfadjoint operators $B(t, \varepsilon) = X(t)^*X(t) + \varepsilon(Y_2^*Y(t) + Y(t)^*Y_2) + \varepsilon^2Q$ in a Hilbert space is considered, where $X(t) = X_0 + tX_1$, $Y(t) = Y_0 + tY_1$. It is assumed that the point $\lambda_0 = 0$ is an isolated eigenvalue of finite multiplicity for the operator $X_0^*X_0$, and that the operators $Y(t)$, $Y_2$, and $Q$ are subordinate to $X(t)$ in a certain sense. The object of study is the generalized resolvent $(B(t, \varepsilon) + \lambda \varepsilon^2 Q_0)^{-1}$, where the operator $Q_0$ is bounded and positive definite. Approximation of this resolvent is obtained for small $\tau = (t^2 + \varepsilon^2)^{1/2}$ with an error term of $O(1)$. This approximation is given in terms of some finite rank operators and is the sum of the principal term and the corrector. The results are aimed at applications to homogenization problems for periodic differential operators in the small period limit.

Introduction

0.1. In the papers [BSu1, BSu2, BSu3, BSu4, Su1, Su2], M. Sh. Birman and the author developed an operator-theoretic approach to problems of homogenization theory for periodic differential operators in the small period limit. We studied matrix elliptic second order differential operators in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ whose coefficients were periodic and depended on $x/\varepsilon$. Here $\varepsilon > 0$ is the small parameter. The scaling transformation and the Floquet–Bloch decomposition leads to a family of differential operators acting in $L_2(\Omega; \mathbb{C}^n)$ and depending quadratically on the parameters $k \in \mathbb{R}^d$ (the quasimomentum) and $\varepsilon$. Here $\Omega \subset \mathbb{R}^d$ is the cell of the lattice of periods. This family is an analytic family of operators with discrete spectrum. It is convenient to study the corresponding operator pencil in the framework of an abstract operator-theoretic approach. Under this approach, we take into account the dependence of operators on two one-dimensional parameters $t = |k|$ and $\varepsilon$ only abstracting from the dependence on the parameter $k/t \in S^{d-1}$.

The present paper is a further development of the corresponding operator-theoretic method and a direct continuation of the considerations in [Su2, Chapter 1]. Applications of the results of the present paper to homogenization of periodic differential operators will be published elsewhere. The content of the paper is of independent interest in the framework of the analytic perturbation theory for quadratic operator pencils.

0.2. In a Hilbert space $H$, we consider a family of selfadjoint operators of the form

\[(0.1) \quad B(t, \varepsilon) = X(t)^*X(t) + \varepsilon(Y_2^*Y(t) + Y(t)^*Y_2) + \varepsilon^2Q,\]

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\end{array}\]

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depending on two parameters \( t \in \mathbb{R} \) and \( \varepsilon > 0 \). Here \( X(t) = X_0 + tX_1 \) and \( Y(t) = Y_0 + tY_1 \). The role of the “unperturbed” operator is played by \( B(0,0) = X_0^*X_0 \). It is assumed that the point \( \lambda_0 = 0 \) is an isolated eigenvalue of finite multiplicity for the operator \( X_0^*X_0 \). Denote \( \mathcal{R} = \text{Ker} \, X_0, \, n = \dim \mathcal{R} \). It is also assumed that the operators \( Y(t), Y_2, \) and \( Q \) are subordinate to \( X(t) \) in a certain sense. We study the generalized resolvent \( (B(t,\varepsilon) + \lambda \varepsilon^2 Q_0)^{-1} \), where the operator \( Q_0 \) is bounded and positive definite. The parameter \( \lambda \) is subject to a condition ensuring that the operator \( B(t,\varepsilon) + \lambda \varepsilon^2 Q_0 \) is positive definite. We find approximation of the generalized resolvent in the operator norm in \( \mathcal{H} \) for small \( t \) and \( \varepsilon \) with an error term of \( O(1) \).

Previously, in [BSu1, Chapter 1] and [BSu2], a similar question for a simpler operator \( A(t) = X(t)^*X(t) \) was studied. In [BSu1, Chapter 1], the principal term was obtained for approximation of the generalized resolvent \( (A(t) + \lambda \varepsilon^2 Q_0)^{-1} \) in the operator norm in \( \mathcal{H} \) with an error term of \( O(\varepsilon^{-1}) \) for \( |t| \leq t_0 \). The principal term of approximation is the generalized resolvent of the operator \( t^2 S \), where \( S \) is the so-called spectral germ of the operator family \( A(t) \) at \( t = 0 \). The spectral germ is a selfadjoint operator acting in the subspace \( \mathcal{R} \) and defined in terms of the threshold characteristics (i.e., the spectral characteristics of the operator family \( A(t) \) near the bottom of the spectrum). In [BSu2], approximation of the same resolvent with an error term of \( O(1) \) was found; in this approximation a corrector was taken into account. Herewith, in [BSu1, BSu2], methods of the analytic perturbation theory with respect to the one-dimensional parameter \( t \) were applied.

The more general operator family (0.1) was studied previously in [Su1, Chapter 1], [Su2, Chapter 1] (in [Su1], under more restrictive assumptions). The principal term was obtained for the approximation of the generalized resolvent. The main technical difference from [BSu1, BSu2] was a different choice of the main parameter: the analytic perturbation theory was applied with respect to the one-dimensional parameter \( \tau = (t^2 + \varepsilon^2)^{1/2} \). It was this choice that made it possible to include the terms \( \varepsilon(Y_2 Y(t) + Y(t)^*Y_2) \) in the operator (0.1) in applications to differential operators, this allows one to include first order terms). In constructions and estimates, it is necessary to trace the dependence on the additional parameter \( \vartheta = (t/\tau, \varepsilon/\tau) \). The principal term of approximation is the resolvent of the operator \( \tau^2 S(\vartheta) \), where \( S(\vartheta) \) is the spectral germ of the operator family \( B(t,\varepsilon) + \lambda \varepsilon^2 Q_0 \) acting in the subspace \( \mathcal{R} \). The error estimate in the corresponding approximation was \( O(\tau^{-1}) \).

In the present paper, we obtain a more accurate approximation for the operator \( (B(t,\varepsilon) + \lambda \varepsilon^2 Q_0)^{-1} \) in the operator norm in \( \mathcal{H} \) with an error term of \( O(1) \). This approximation is given by the sum of the principal term and the corrector; the corrector is also a finite rank operator. This is the main result (Theorem 4.3). In the proof of this theorem, we lean upon the so-called threshold approximations. For sufficiently small \( \delta \) and \( \tau_0 \) (these numbers are controlled explicitly) and for \( |\tau| \leq \tau_0 \), precisely \( n \) eigenvalues of the perturbed operator \( B(t,\varepsilon) + \lambda \varepsilon^2 Q_0 \) lie on the interval \([0,\delta]\), while the rest of the spectrum is separated away from this interval. Let \( F(t,\varepsilon) \) be the spectral projection of the operator \( B(t,\varepsilon) + \lambda \varepsilon^2 Q_0 \) for the interval \([0,\delta]\). We obtain threshold approximations in the operator norm in \( \mathcal{H} \), namely, approximation for the spectral projection \( F(t,\varepsilon) \) with an error term \( O(\tau^2) \) (Theorem 3.1), and approximation for the operator \( (B(t,\varepsilon) + \lambda \varepsilon^2 Q_0)F(t,\varepsilon) \) with an error term \( O(\tau^4) \) (Theorem 3.3). Besides principal terms, these approximations contain the corresponding correctors. For the proof of Theorems 3.1 and 3.3, we combine the method of power series expansions for analytic (with respect to \( \tau \)) branches of eigenvalues and eigenvectors and the method of integration of the resolvent over an appropriate contour in the complex plane.
All error estimates in approximations obtained in the present paper are order sharp, and the constants in estimates are controlled explicitly in terms of the problem data.

0.3. Organization of the paper. The paper consists of four sections. In §1, we introduce the main object of study, the operator family (0.1), and pass to new variables \( t, \varepsilon \). Next, we define some auxiliary finite rank operators arising in considerations related to analytic perturbation theory. Then the analytic branches of eigenvalues and eigenvectors are analyzed, and the spectral germ is defined. In §2, we consider the difference of resolvents of the perturbed and unperturbed operators on an appropriate contour in the complex plane. Here, the main result (Theorem 2.1) is of technical nature: for the difference of resolvents we get power expansions of the required order with error estimates. In §3, using Theorem 2.1 and integrating the resolvent over a contour, we prove the required threshold approximations for the spectral projection \( F(t, \varepsilon) \) and the operator \((B(t, \varepsilon) + \lambda \varepsilon^2 Q_0)F(t, \varepsilon)\) (Theorems 3.1, 3.2, 3.3). With the help of power expansions, we calculate the invariant form of the corrector in the approximation for \((B(t, \varepsilon) + \lambda \varepsilon^2 Q_0)F(t, \varepsilon)\); this calculation is rather bulky. In §4, we apply Theorems of §3 to prove the main result of the paper (Theorem 4.3), obtaining approximation for the generalized resolvent \((B(t, \varepsilon) + \lambda \varepsilon^2 Q_0)^{-1}\) with an error of \(O(1)\).

0.4. Notation. Let \( \mathcal{H} \) and \( \mathcal{H}_s \) be separable Hilbert spaces. The symbols \((\cdot, \cdot)_{\mathcal{H}}\) and \(\| \cdot \|_{\mathcal{H}}\) stand for the inner product and the norm in \(\mathcal{H}\), respectively; the symbol \(\| \cdot \|_{\mathcal{H}_s} \rightarrow \mathcal{H}_s\) denotes the norm of a continuous linear operator acting from \(\mathcal{H}\) to \(\mathcal{H}_s\). Sometimes we omit the indices if this does not lead to confusion. By \(I = I_{\mathcal{H}}\) we denote the identity operator on \(\mathcal{H}\). If \(A : \mathcal{H} \rightarrow \mathcal{H}_s\) is a linear operator, its domain is denoted by \(\text{Dom} \ A\), and its kernel by \(\text{Ker} \ A\). If \(\mathcal{N}\) is a subspace in \(\mathcal{H}\), then \(\mathcal{N}^\perp\) is its orthogonal complement. If \(P\) is an orthogonal projection in \(\mathcal{H}\), then \(P^\perp = I - P\).

Various constants in estimates are denoted by \(c, C, \mathcal{C}\) (possibly, with indices and marks).

§1. Quadratic two-parametric operator pencils

We study a family of operators \(B(t, \varepsilon)\) depending on two real-valued parameters \(\varepsilon\) and \(t\), where \(0 \leq \varepsilon \leq 1\) and \(t \in \mathbb{R}\).

1.1. The operators \(X(t)\) and \(A(t)\). Let \(\mathcal{H}\) and \(\mathcal{H}_s\) be separable complex Hilbert spaces. Suppose \(X_0 : \mathcal{H} \rightarrow \mathcal{H}_s\) is a densely defined and closed linear operator and \(X_1 : \mathcal{H} \rightarrow \mathcal{H}_s\) is a bounded linear operator. On the domain \(\text{Dom} \ X(t) = \text{Dom} \ X_0\), we introduce the operator \(X(t) := X_0 + tX_1\), \(t \in \mathbb{R}\). Consider the family of selfadjoint positive operators

\[
A(t) := X(t)^*X(t), \quad t \in \mathbb{R},
\]

in \(\mathcal{H}\). The operator (1.1) is generated in \(\mathcal{H}\) by the closed quadratic form \(\|X(t)u\|_{\mathcal{H}_s}^2\), \(u \in \text{Dom} \ X_0\). We put \(A(0) = X_0^*X_0 =: A_0\) and \(\mathcal{N} := \text{Ker} A_0 = \text{Ker} X_0\). Suppose that the following condition is satisfied.

**Condition 1.1.** The point \(\lambda_0 = 0\) is an isolated point of the spectrum of \(A_0\), and \(0 < n := \text{dim} \mathcal{N} < \infty\).

Let \(d^0\) denote the distance from the point \(\lambda_0 = 0\) to the rest of the spectrum of \(A_0\). Denote \(\mathcal{N}_s := \text{Ker} X_0^*\), \(n_s := \text{dim} \mathcal{N}_s\). Suppose that \(n \leq n_s \leq \infty\). Let \(P\) be the orthogonal projection of \(\mathcal{H}\) onto \(\mathcal{N}\), and let \(P_s\) be the orthogonal projection of \(\mathcal{H}_s\) onto \(\mathcal{N}_s\).

The operator family \(A(t)\) was studied in detail in [BSu1, Chapter 1], [BSu2], [BSu4, Chapter 1].
1.2. The operators $Y(t)$ and $Y_2$. Let $\tilde{\mathfrak{H}}$ be yet another Hilbert space. Suppose $Y_0 : \mathfrak{H} \to \tilde{\mathfrak{H}}$ is a densely defined linear operator such that $\text{Dom} \ Y_0 \subset \text{Dom} \ Y_2$, and $Y_1 : \mathfrak{H} \to \tilde{\mathfrak{H}}$ is a bounded linear operator. We put $Y(t) = Y_0 + tY_1$, $\text{Dom} \ Y(t) = \text{Dom} \ Y_0$, and impose the following condition.

**Condition 1.2.** There exists a constant $c_1 > 0$ such that
\[
\|Y(t)u\|_{\tilde{\mathfrak{H}}} \leq c_1 \|X(t)u\|_{\mathfrak{H}}, \quad u \in \text{Dom} \ X_0, \quad t \in \mathbb{R}.
\]
From (1.2) with $t = 0$ it follows that $\text{Ker} \ X_0 \subset \text{Ker} \ Y_0$, i.e., $Y_0 P = 0$.

Let $Y_2 : \mathfrak{H} \to \tilde{\mathfrak{H}}$ be a densely defined linear operator such that $\text{Dom} \ X_0 \subset \text{Dom} \ Y_2$. We impose the following condition.

**Condition 1.3.** For any $\nu > 0$ there exists a constant $C(\nu) > 0$ such that
\[
\|Y_2u\|_{\tilde{\mathfrak{H}}}^2 \leq \nu \|X(t)u\|_{\mathfrak{H}}^2 + C(\nu)\|u\|_{\mathfrak{H}}^2, \quad u \in \text{Dom} \ X_0, \quad t \in \mathbb{R}.
\]

1.3. The form $q$. Let $q[u, v]$ be a densely defined Hermitian sesquilinear form in $\mathfrak{H}$. Assume that $\text{Dom} \ X_0 \subset \text{Dom} \ q$, and impose the following condition.

**Condition 1.4.** 1°. There exist constants $c_2 \geq 0$ and $c_3 \geq 0$ such that
\[
|q[u, v]| \leq (c_2 \|X(t)u\|_{\mathfrak{H}}^2 + c_3 \|u\|_{\mathfrak{H}}^2)^{1/2} (c_2 \|X(t)v\|_{\mathfrak{H}}^2 + c_3 \|v\|_{\mathfrak{H}}^2)^{1/2},
\]
\[
u > 0 \text{ such that } q[u, u] \geq - (1 - \nu) \|X(t)u\|_{\mathfrak{H}}^2 - c_0 \|u\|_{\mathfrak{H}}^2, \quad u \in \text{Dom} \ X_0, \quad t \in \mathbb{R}.
\]

1.4. The operator $B(t, \varepsilon)$. In the space $\mathfrak{H}$, consider the following Hermitian sesquilinear form
\[
b(t, \varepsilon)[u, v] = (X(t)u, X(t)v), + \varepsilon((Y(t)u, Y_2v) + (Y_2u, Y(t)v), + \varepsilon^2 q[u, v],
\]
\[
u > 0 \text{ such that } q[u, u] \geq - (1 - \nu) \|X(t)u\|_{\mathfrak{H}}^2 - c_0 \|u\|_{\mathfrak{H}}^2, \quad u \in \text{Dom} \ X_0, \quad t \in \mathbb{R}.
\]

By using Conditions 1.2, 1.3, and 1.4, it is easy to check the following inequalities:
\[
b(t, \varepsilon)[u, u] \leq (2 + c_1^2 + c_2^2) \|X(t)u\|_{\mathfrak{H}}^2 + (C(1) + c_3)\varepsilon^2 \|u\|_{\mathfrak{H}}^2, \quad u \in \text{Dom} \ X_0,
\]
\[
b(t, \varepsilon)[u, u] \geq \frac{\kappa}{2} \|X(t)u\|_{\mathfrak{H}}^2 - (c_0 + c_4)\varepsilon^2 \|u\|_{\mathfrak{H}}^2, \quad u \in \text{Dom} \ X_0,
\]
where
\[
c_4 := 4\kappa^{-1}c_1^2 C(\nu) \quad \text{for} \quad \nu = \kappa^2(16c_1^2)^{-1}.
\]
Details can be found in [Su2] Subsection 1.4. Combining (1.4) and (1.5), and taking into account the fact that $X(t)$ is closed, we see that the form $b(t, \varepsilon)[u, u]$ is closed and lower semibounded.

Our main object is the selfadjoint operator $B(t, \varepsilon)$ in $\mathfrak{H}$ generated by the form (1.3). Formally, we have
\[
B(t, \varepsilon) = A(t) + \varepsilon(Y_0 + Y(t)^*Y_2) + \varepsilon^2 Q.
\]
(Here $Q$ is a formal object that corresponds to the form $q$. Of course, if $q$ is bounded, then $Q$ is defined rigorously as the operator generated by $q$.)

Our goal is to study the resolvent $(B(t, \varepsilon) + \lambda \varepsilon^2 I)^{-1}$ or the generalized resolvent $(B(t, \varepsilon) + \lambda \varepsilon^2 Q_0)^{-1}$ of the operator (1.6). Here $Q_0 : \mathfrak{H} \to \mathfrak{H}$ is a bounded positive definite operator. It is convenient to study the generalized resolvent at once, then the case of $Q_0 = I$ corresponds to the usual resolvent. Denote
\[
B_\lambda(t, \varepsilon) := B(t, \varepsilon) + \lambda \varepsilon^2 Q_0,
\]
\[
b_\lambda(t, \varepsilon)[u, v] := b(t, \varepsilon)[u, v] + \lambda \varepsilon^2 (Q_0 u, v), \quad u, v \in \text{Dom} \ X_0.
\]
The parameter $\lambda$ is subject to the following restriction:

\begin{align}
\lambda &> \|Q_0^{-1}\|(c_0 + c_4) & \text{if } \lambda \geq 0, \\
\lambda &> \|Q_0\|^{-1}(c_0 + c_4) & \text{if } \lambda < 0 \text{ (and } c_0 + c_4 < 0). 
\end{align}

Condition (1.9) implies that

\begin{align}
\lambda(Q_0 u, u)_{\mathcal{D}} \geq (c_0 + c_4 + \beta)\|u\|^2_{\mathcal{D}}, & \quad u \in \mathcal{D}, \\
\end{align}

where $\beta > 0$ is defined in terms of $\lambda$ as follows:

\begin{align}
\beta = \lambda\|Q_0^{-1}\|^{-1} - c_0 - c_4 & \quad \text{if } \lambda \geq 0, \\
\beta = \lambda\|Q_0\| - c_0 - c_4 & \quad \text{if } \lambda < 0 \text{ (and } c_0 + c_4 < 0). 
\end{align}

By (1.5) and (1.10), the form (1.8) satisfies

\begin{align}
b_{\lambda}(t, \varepsilon)[u, u] \geq \frac{k}{2}\|X(t)u\|^2_{\mathcal{D}} + \beta \varepsilon^2\|u\|^2_{\mathcal{D}}, & \quad u \in \operatorname{Dom} X_0. 
\end{align}

Thus, under the above assumptions the operator (1.7) is positive definite. By (1.4) and (1.8), for $u \in \operatorname{Dom} X_0$ we have

\begin{align}
b_{\lambda}(t, \varepsilon)[u, u] \leq (2 + c_1^2 + c_2)\|X(t)u\|^2_{\mathcal{D}} + (C(1) + c_3 + |\lambda||Q_0||\varepsilon^2\|u\|^2_{\mathcal{D}}.
\end{align}

1.5. Passage to the parameters $\tau$, $\vartheta$. The family (1.7) is an analytic operator family with respect to the parameters $t$ and $\varepsilon$. For $t = \varepsilon = 0$ the operator $B_{\lambda}(t, 0)$ coincides with $A_0$. Hence, by Condition 1.1, it has an isolated eigenvalue $\lambda_0 = 0$ of multiplicity $n$. We wish to apply analytic perturbation theory. However, if $n > 1$, analytic perturbation theory cannot be applied directly because we deal with a multiple eigenvalue and a multidimensional parameter. Therefore, we introduce the one-dimensional parameter $\tau = (t^2 + \varepsilon^2)^{1/2}$ and also the additional parameters $\vartheta_1 = t\tau^{-1}$, $\vartheta_2 = \varepsilon\tau^{-1}$, $\vartheta = (\vartheta_1, \vartheta_2)$. Then $\vartheta$ belongs to the unit circle. Now, the operator $B_{\lambda}(t, \varepsilon)$ will be denoted by $B_{\lambda}(\tau; \vartheta)$, and the corresponding form $b_{\lambda}(t, \varepsilon)$ by $b_{\lambda}(\tau; \vartheta)$. By (1.3) and (1.8), this form can be written as

\begin{align}
b_{\lambda}(\tau; \vartheta)[u, v] = (X_0 u, X_0 v)_{\mathcal{D}} + \tau \vartheta_1((X_0 u, X_1 v)_{\mathcal{D}} + (X_1 u, X_0 v)_{\mathcal{D}}) \\
+ \tau^2 \vartheta_1^2((X_1 u, X_1 v)_{\mathcal{D}} + (X_1 v, X_1 u)_{\mathcal{D}}) + \tau \vartheta_2((Y_0 u, Y_2 v)_{\mathcal{D}} + (Y_2 u, Y_0 v)_{\mathcal{D}}) \\
+ \tau^2 \vartheta_2^2((Y_1 u, Y_1 v)_{\mathcal{D}} + (Y_1 v, Y_1 u)_{\mathcal{D}}) + \tau^2 \vartheta_3^2(q[u, v] + \lambda(Q_0 u, v)_{\mathcal{D}}), \quad u, v \in \operatorname{Dom} X_0.
\end{align}

The formal structure of the operator $B_{\lambda}(\tau; \vartheta)$ is given by

\begin{align}
B_{\lambda}(\tau; \vartheta) = X_0^* X_0 + \tau \vartheta_1(X_0^* X_1 + X_1^* X_0) + \tau^2 \vartheta_1^2 X_1^* X_1 + \tau \vartheta_2(Y_2^* Y_0 + Y_0^* Y_2) \\
+ \tau^2 \vartheta_1^2 Y_1^* Y_1 + \tau^2 \vartheta_3^2(Q + \lambda Q_0).
\end{align}

Note that, “in its origin”, the parameter $\tau$ is nonnegative, but the form (1.13) and the corresponding operator can be considered for any $\tau \in \mathbb{R}$, which is kept in mind below. We study the operator family $B_{\lambda}(\tau; \vartheta)$ by methods of analytic perturbation theory with respect to the one-dimensional parameter $\tau$. Herewith, we should make constructions and estimates uniform with respect to the additional parameter $\vartheta$, taking into account that $\vartheta_1^2 + \vartheta_2^2 = 1$.

Let $F(\tau; \vartheta; s)$ be the spectral projection of the operator $B_{\lambda}(\tau; \vartheta)$ for the interval $[0, s]$. Fix a number $\delta \in (0, \kappa d^0/13)$, and then fix a number $\tau_0 > 0$ such that

\begin{align}
\tau_0 \leq \delta^{1/2} \left((2 + c_1^2 + c_2)\|X_1\|^2 + C(1) + c_3 + |\lambda||Q_0||\right)^{-1/2}.
\end{align}

As was checked in [Su2] Proposition 1.5, for $|\tau| \leq \tau_0$ we have

\begin{align}
F(\tau; \vartheta; \delta) = F(\tau; \vartheta; 3\delta), \quad \text{rank } F(\tau; \vartheta; \delta) = n, \quad |\tau| \leq \tau_0.
\end{align}
We shall often use the notation $F(\tau; \vartheta)$ instead of $F(\tau; \vartheta; \delta)$.

1.6. The operators $Z$ and $\tilde{Z}$. Now we introduce some operators appearing in the analytic perturbation theory considerations. Denote $\mathcal{D} = \text{Dom} X_0 \cap \mathfrak{N}^\perp$. Since $\lambda_0 = 0$ is an isolated point of the spectrum of $A_0$, the form $(X_0 \varphi, X_0 \zeta)$, $\varphi, \zeta \in \mathcal{D}$, determines an inner product in $\mathcal{D}$ converting $\mathcal{D}$ into the Hilbert space. Let $\omega \in \mathfrak{N}$, and let $\phi \in \mathcal{D}$ be a (weak) solution of the equation

$$X_0^*(X_0 \phi + X_1 \omega) = 0,$$

i.e.,

$$(X_0 \phi, X_0 \zeta)_{\mathfrak{N}^\perp} = -(X_1 \omega, X_0 \zeta)_{\mathfrak{N}^\perp}, \quad \zeta \in \mathcal{D}.$$  

Since the right-hand side is an antilinear continuous functional of $\zeta \in \mathcal{D}$, the Riesz theorem shows that there exists a unique solution denoted by $\phi(\omega)$. Obviously,

$$\|X_0 \phi(\omega)\|_{\mathfrak{N}^\perp} \leq \|X_1 \omega\|_{\mathfrak{N}^\perp}. \tag{1.17}$$

Since

$$\|X_0 \zeta\|_{\mathfrak{N}^\perp}^2 \geq d^0 \|\zeta\|_{\mathfrak{N}^\perp}^2 \geq 13\delta\kappa^{-1}\|\zeta\|_{\mathfrak{N}^\perp}^2 \quad \text{for all} \quad \zeta \in \mathcal{D},$$

by (1.17) we have

$$\|\phi(\omega)\|_{\mathfrak{N}^\perp} \leq \kappa^{1/2}(13\delta)^{-1/2}\|X_1 \omega\|_{\mathfrak{N}^\perp}. \tag{1.19}$$

We define a bounded operator $Z : \mathfrak{N} \to \mathfrak{N}$ by the relation

$$Zu = \phi(Pu), \quad u \in \mathfrak{N}.$$  

Note that $Z$ takes $\mathfrak{N}$ into $\mathfrak{N}^\perp$ and $\mathfrak{N}^\perp$ into $\{0\}$. From (1.17) and (1.19) it follows directly that

$$\|X_0 Z\|_{\mathfrak{N} \to \mathfrak{N}^\perp} \leq \|X_1\|, \tag{1.20}$$

$$\|Z\|_{\mathfrak{N}^\perp \to \mathfrak{N}} \leq \kappa^{1/2}(13\delta)^{-1/2}\|X_1\|. \tag{1.21}$$

Similarly, given $\omega \in \mathfrak{N}$, we consider the equation

$$X_0^* X_0 \psi + Y_0^* Y_2 \omega = 0$$

for an element $\psi \in \mathcal{D}$, which is understood as the identity

$$(X_0 \psi, X_0 \zeta)_{\mathfrak{N}^\perp} = -(Y_2 \omega, Y_0 \zeta)_{\mathfrak{N}^\perp}, \quad \zeta \in \mathcal{D}.$$  

By Condition 1.2, the right-hand side is an antilinear continuous functional of $\zeta \in \mathcal{D}$. By the Riesz theorem, there exists a unique solution $\psi(\omega)$. We have

$$\|X_0 \psi(\omega)\|_{\mathfrak{N}^\perp} \leq c_1\|Y_2 \omega\|_{\mathfrak{N}^\perp} \leq c_1 C(1)^{1/2}\|\omega\|_{\mathfrak{N}^\perp}. \tag{1.22}$$

The last inequality follows from Condition 1.3 with $t = 0$ and $\nu = 1$. Taking (1.18) into account, we obtain

$$\|\psi(\omega)\|_{\mathfrak{N}^\perp} \leq c_1(\kappa C(1))^{1/2}(13\delta)^{-1/2}\|\omega\|_{\mathfrak{N}^\perp}. \tag{1.23}$$

We introduce a bounded operator $\tilde{Z} : \mathfrak{N} \to \mathfrak{N}$ by the relation

$$\tilde{Z}u = \psi(Pu), \quad u \in \mathfrak{N}.$$  

Note that $\tilde{Z}$ takes $\mathfrak{N}$ into $\mathfrak{N}^\perp$ and $\mathfrak{N}^\perp$ into $\{0\}$. From (1.22) and (1.23) it follows that

$$\|X_0 \tilde{Z}\|_{\mathfrak{N} \to \mathfrak{N}^\perp} \leq c_1 C(1)^{1/2}, \tag{1.24}$$

$$\|\tilde{Z}\|_{\mathfrak{N}^\perp \to \mathfrak{N}} \leq c_1(\kappa C(1))^{1/2}(13\delta)^{-1/2}. \tag{1.25}$$
1.7. The operators $R$ and $S$. Next, we define a linear operator $R : \mathfrak{N} \to \mathfrak{N}_*$ by the relation

$$ R\omega = X_0 \phi(\omega) + X_1 \omega = (X_0 Z + X_1) \omega, \quad \omega \in \mathfrak{N}. $$

Another description of $R$ is given by $R = P_1 X_1 |_{\mathfrak{N}}.

In accordance with [BSu1], Chapter 1, Subsection 1.3], the operator $S := R^* R : \mathfrak{N} \to \mathfrak{N}$ is called the spectral germ of the operator family $A(t)$ at $t = 0$. The germ $S$ can be also written as $S = PX_1^* P_1 X_1 |_{\mathfrak{N}}$. Obviously,

$$ \|R\| \leq \|X_1\|, \quad \|S\| \leq \|X_1\|^2. $$

The spectral germ plays an important role in the study of the spectral properties of $A(t)$ near the bottom of the spectrum. The properties of $S$ were studied in [BSu1], Chapter 1, Subsections 1.3, 1.4, 1.6] in detail. In the next subsection we shall introduce the notion of a germ for the more general operator family (1.14).

1.8. Analytic branches of eigenvalues and eigenvectors of the operator $B_\lambda(\tau; \vartheta)$. By the analytic perturbation theory (see [K]), for $|\tau| \leq \tau_0$ there exist real-analytic (in $\tau$) functions $\lambda_l(\tau; \vartheta)$ (the branches of eigenvalues) and real-analytic $\mathfrak{N}_*$-valued functions $\varphi_l(\tau; \vartheta)$ (the branches of eigenvectors) such that

$$ B_\lambda(\tau; \vartheta) \varphi_l(\tau; \vartheta) = \lambda_l(\tau; \vartheta) \varphi_l(\tau; \vartheta), \quad |\tau| \leq \tau_0, \quad l = 1, \ldots, n, $$

and the elements $\varphi_l(\tau; \vartheta), \ l = 1, \ldots, n,$ form an orthonormal basis in the eigenspace $F(\tau; \vartheta)\mathfrak{N}_*$. Moreover, for sufficiently small $\tau_*$ (of course, $\tau_* \leq \tau_0$) and $|\tau| \leq \tau_*$, we have the following converging power series expansions:

$$ \lambda_l(\tau; \vartheta) = \gamma_l(\vartheta) \tau^2 + \mu_l(\vartheta) \tau^3 + \ldots, \quad \gamma_l(\vartheta) \geq 0, \quad l = 1, \ldots, n, $$

$$ \varphi_l(\tau; \vartheta) = \omega_l(\vartheta) + \tau \varphi_l^{(1)}(\vartheta) + \tau^2 \varphi_l^{(2)}(\vartheta) + \ldots, \quad l = 1, \ldots, n. $$

The elements $\omega_l(\vartheta), \ l = 1, \ldots, n,$ form an orthonormal basis in $\mathfrak{N}$, whence

$$ P = \sum_{l=1}^{n} (\cdot, \omega_l(\vartheta))_{\mathfrak{N}_*} \omega_l(\vartheta). $$

Comparing the coefficients of the first power of $\tau$ in (1.27) (written as an identity for forms), we see that the element $\varphi_l^{(1)}(\vartheta)$ is a weak solution of the equation

$$ X_0^* (X_0 \varphi_l^{(1)}(\vartheta) + \vartheta_1 X_1 \omega_l(\vartheta)) + \vartheta_2 Y_0^* Y_2 \omega_l(\vartheta) = 0. $$

This allows us to represent $\varphi_l^{(1)}(\vartheta)$ in terms of the operators $Z$ and $\tilde{Z}$:

$$ \varphi_l^{(1)}(\vartheta) = \vartheta_1 Z \omega_l(\vartheta) + \vartheta_2 \tilde{Z} \omega_l(\vartheta) + \tilde{\omega}_l(\vartheta), \quad \tilde{\omega}_l(\vartheta) \in \mathfrak{N}. $$

See [Su2], Subsection 1.8] for the details. At the present step, the elements $\tilde{\omega}_l(\vartheta) \in \mathfrak{N}$ remain undefined, but they satisfy the identity

$$ (\tilde{\omega}_k(\vartheta), \omega_l(\vartheta))_{\mathfrak{N}_*} + (\omega_k(\vartheta), \tilde{\omega}_l(\vartheta))_{\mathfrak{N}_*} = 0, \quad k, l = 1, \ldots, n. $$

At the next step (when we compare coefficients of $\tau^2$ in (1.27)), a new object arises. It is the selfadjoint operator $S(\vartheta) : \mathfrak{N} \to \mathfrak{N}$ corresponding to the form

$$ s(\vartheta)[\omega, \zeta] = \vartheta_1^2 (S\omega, \zeta)_{\mathfrak{N}_*} + \vartheta_2 (X_0 \tilde{Z} \omega, X_0 \zeta)_{\mathfrak{N}_*} - \vartheta_2^2 (X_1 \tilde{Z} \omega, X_0 \zeta)_{\mathfrak{N}_*} + \vartheta_1 \vartheta_2 (Y_1 \omega, Y_2 \zeta)_{\mathfrak{N}_*} + (Y_2 \omega, Y_1 \zeta)_{\mathfrak{N}_*} + \vartheta_2^2 (q[\omega, \zeta] + \lambda(Q_0 \omega, \zeta)_{\mathfrak{N}_*}), \quad \omega, \zeta \in \mathfrak{N}. $$
In accordance with [Su2, Subsection 1.8], the operator $S(\vartheta)$ is called the spectral germ of the operator family $\mathcal{B}_\lambda(\tau; \vartheta)$ at $\tau = 0$. This germ can be written as

$$S(\vartheta) = \vartheta^2 S + \vartheta \vartheta_1 \vartheta_2 \left( -(X_0 Z)^* X_0 \tilde{Z} - (X_0 \tilde{Z})^* X_0 Z + P(Y_2^* Y_1 + Y_1^* Y_2) \right)_{|\mathfrak{r}}$$

and

$$\vartheta_1 \vartheta_2 \left( -(X_0 \tilde{Z})^* X_0 \tilde{Z} \right)_{|\mathfrak{r}} + Q_{\mathfrak{r}} + \lambda Q_{\mathfrak{om}}.$$ 

Here $Q_{\mathfrak{r}}$ is the selfadjoint operator in $\mathfrak{r}$ generated by the form $q[\omega, \omega]$, $\omega \in \mathfrak{r}$, and $Q_{\mathfrak{om}} := PQQ_{\mathfrak{om}}$.

The following statement was proved in [Su2, Proposition 1.6].

**Proposition 1.5.** The numbers $\gamma_l(\vartheta)$ and the elements $\omega_l(\vartheta)$ defined by the expansions (1.28) and (1.29) are eigenvalues and eigenvectors of the spectral germ $S(\vartheta)$:

$$(1.35) \quad S(\vartheta) \omega_l(\vartheta) = \gamma_l(\vartheta) \omega_l(\vartheta), \quad l = 1, \ldots, n.$$ 

Relations (1.30) and (1.35) show that

$$(1.36) \quad S(\vartheta) P = \sum_{l=1}^n \gamma_l(\vartheta) (\cdot, \omega_l(\vartheta))_\mathfrak{b} \omega_l(\vartheta).$$

The following estimate for the norm of the operator (1.34) is a consequence of (1.20), (1.24), (1.26), Condition 1.3 (with $t = 0$ and $\nu = 1$), and Condition 1.4 (with $t = 0$):

$$(1.37) \quad ||S(\vartheta)||_{\mathfrak{r} \rightarrow \mathfrak{r}} \leq ||X_1||^2 + 2c_1 C(1)^{1/2} ||X_1|| + c_2^2 C(1) + 2C(1)^{1/2} ||Y_1|| + c_3 + ||\lambda|| Q_0.$$ 

**1.9. The operator-valued functions $F(\tau; \vartheta)$ and $B_\lambda(\tau; \vartheta) F(\tau; \vartheta)$.** These operator-valued functions are real-analytic with respect to $\tau$ for $|\tau| \leq \tau_0$. From (1.16) and (1.27) it follows that

$$(1.38) \quad F(\tau; \vartheta) = \sum_{l=1}^n (\cdot, \varphi_l(\tau; \vartheta))_\mathfrak{b} \varphi_l(\tau; \vartheta), \quad |\tau| \leq \tau_0.$$ 

By (1.29), (1.30), and (1.38), for sufficiently small $\tau_*$ we have

$$(1.39) \quad F(\tau; \vartheta) = P + \tau F_1(\vartheta) + \ldots, \quad |\tau| \leq \tau_*,$$

where

$$(1.40) \quad F_1(\vartheta) = \sum_{l=1}^n \left( (\cdot, \omega_l(\vartheta))_\mathfrak{b} \varphi_l^{(1)}(\vartheta) + (\cdot, \varphi_l^{(1)}(\vartheta))_\mathfrak{b} \omega_l(\vartheta). \right).$$

In [Su2, Subsection 1.9], it was proved that the operator (1.40) admits the following invariant representation in terms of $Z$ and $\tilde{Z}$:

$$(1.41) \quad F_1(\vartheta) = \vartheta_1 (Z + Z^*) + \vartheta_2 (\tilde{Z} + \tilde{Z}^*).$$

Next, relations (1.16) and (1.27) imply that

$$(1.42) \quad B_\lambda(\tau; \vartheta) F(\tau; \vartheta) = \sum_{l=1}^n \lambda_l(\tau; \vartheta) (\cdot, \varphi_l(\tau; \vartheta))_\mathfrak{b} \varphi_l(\tau; \vartheta), \quad |\tau| \leq \tau_0.$$ 

Combining (1.42) with (1.28), (1.29), and (1.36), we obtain the power series expansion

$$(1.43) \quad B_\lambda(\tau; \vartheta) F(\tau; \vartheta) = \tau^2 S(\vartheta) P + \ldots, \quad |\tau| \leq \tau_*.$$ 

The expansions (1.39) and (1.43) are not quite appropriate for our goals. We need only finite-term expansions for the operators $F(\tau; \vartheta)$ and $B_\lambda(\tau; \vartheta) F(\tau; \vartheta)$, but with error estimates valid on the entire interval $|\tau| \leq \tau_0$. Next, for $F(\tau; \vartheta)$ it suffices to obtain a first order expansion with an estimate for the remainder $F(\tau; \vartheta) - P - \tau F_1(\vartheta)$ of order of $\tau^2$, while for $B_\lambda(\tau; \vartheta) F(\tau; \vartheta)$, besides the principal term $\tau^2 S(\vartheta) P$, we need to extract the term of order of $\tau^3$ and to prove error estimate of order of $\tau^4$. To obtain such estimates,
below (in §3) we apply integration of the resolvent \((B_\lambda(\tau; \vartheta) - zI)^{-1}\) over an appropriate contour in the complex plane.

§2. THE DIFFERENCE OF RESOLVENTS ON A CONTOUR IN THE COMPLEX PLANE

2.1. The resolvent identity. Denote

\[
R_z(0) = (A_0 - zI)^{-1}, \quad R_z(\tau; \vartheta) = (B_\lambda(\tau; \vartheta) - zI)^{-1},
\]

where \(z \in \mathbb{C}\) is a common regular point for the operators \(A_0\) and \(B_\lambda(\tau; \vartheta)\). We need a version of the resolvent identity for operators defined in terms of quadratic forms with a common domain (see [BSu1, Chapter 1, §2]).

Let \(\mathfrak{d}\) be the space \(\text{Dom} \ X_0\) with the metric form

\[
\|u\|_\mathfrak{d}^2 = \|X_0u\|_{\mathfrak{d}}^2 + \delta \|u\|_\mathfrak{d}^2.
\]

Let \(T(\tau; \vartheta)\) be the operator in \(\mathfrak{d}\) defined by the quadratic form

\[
(2.1) \quad t(\tau; \vartheta)[u, u] = \mathfrak{b}_\lambda(\tau; \vartheta)[u, u] - \|X_0u\|_{\mathfrak{d}}^2, \quad u \in \mathfrak{d}.
\]

We put

\[
(2.2) \quad \Omega_z(0) = I + (z + \delta)R_z(0), \quad \Omega_z(\tau; \vartheta) = I + (z + \delta)R_z(\tau; \vartheta).
\]

In accordance with [BSu1, Chapter 1, (2.13)], we have the following resolvent identity:

\[
(2.3) \quad R_z(\tau; \vartheta) - R_z(0) = -\Omega_z(0)T(\tau; \vartheta)R_z(\tau; \vartheta).
\]

Denote

\[
\alpha^2 = \alpha^2(\tau; \vartheta) = \sup_{0 \neq u \in \mathfrak{d}} \frac{\|X_0u\|_{\mathfrak{d}}^2 + \delta \|u\|_{\mathfrak{d}}^2}{\mathfrak{b}_\lambda(\tau; \vartheta)[u, u] + \delta \|u\|_{\mathfrak{d}}^2}.
\]

By using (1.12) and (1.15), it is easy to check that

\[
(2.4) \quad \alpha = \alpha(\tau; \vartheta) \leq 2\kappa^{-1/2}, \quad |\tau| \leq \tau_0,
\]

see [Su2, Subsection 2.1].

Let \(L\) and \(L_j\) be bounded operators on \(\mathfrak{d}\). In the sequel, the following estimates will be needed (see [BSu1, Chapter 1, (2.15)–(2.17)]):

\[
(2.5) \quad \|LR_z(0)\|_{\mathfrak{d} \to \mathfrak{d}} \leq \delta^{-1}\|\Omega_z(0)\|\|L\|_{\mathfrak{d} \to \mathfrak{d}},
\]

\[
(2.6) \quad \|LR_z(\tau; \vartheta)\|_{\mathfrak{d} \to \mathfrak{d}} \leq \alpha \delta^{-1}\|\Omega_z(\tau; \vartheta)\|\|L\|_{\mathfrak{d} \to \mathfrak{d}},
\]

\[
\|L_1\Omega_z(0)L_2R_z(\tau; \vartheta)\|_{\mathfrak{d} \to \mathfrak{d}} \leq \alpha \delta^{-1}\|\Omega_z(\tau; \vartheta)\|\|L_1\|_{\mathfrak{d} \to \mathfrak{d}}\|L_2\|_{\mathfrak{d} \to \mathfrak{d}},
\]

where \(\|\Omega_z(\tau; \vartheta)\| = \|\Omega_z(\tau; \vartheta)\|_{\mathfrak{d} \to \mathfrak{d}}\). By iteration with the help of (2.2), (2.5), and (2.7), it is easy to obtain the following inequality, which is more general than (2.7):

\[
(2.8) \quad \|L_1\Omega_z(0)L_2\Omega_z(0)\ldots L_p-1\Omega_z(0)L_pR_z(\tau; \vartheta)\|_{\mathfrak{d} \to \mathfrak{d}} \leq \alpha \delta^{-1}\|\Omega_z(\tau; \vartheta)\|\left(1 + |z + \delta|\delta^{-1}\|\Omega_z(0)\|\right)^{p-1} \prod_{j=1}^{p} \|L_j\|_{\mathfrak{d} \to \mathfrak{d}}
\]

for any natural \(p\).
2.2. The operator $T(\tau; \vartheta)$. By (1.13), the form (2.1) can be written as
\[ t(\tau; \vartheta)[u, u] = \tau t_1(\vartheta)[u, u] + \tau^2 t_2(\vartheta)[u, u], \]
where
\begin{align*}
(2.9) & \quad t_1(\vartheta)[u, u] = 2\vartheta_1 \Re(X_0u, X_1u)_{\vartheta_1} + 2\vartheta_2 \Re(Y_0u, Y_2u)_{\vartheta_2}, \quad u \in \vartheta_1, \\
(2.10) & \quad t_2(\vartheta)[u, u] = \vartheta_1^2 \|X_1u\|^2_{\vartheta_1} + 2\vartheta_1\vartheta_2 \Re(Y_1u, Y_2u)_{\vartheta_2} + \vartheta_2^2(q[u, u] + \lambda(Q_0u, u)_{\vartheta_2}), \quad u \in \vartheta_1.
\end{align*}
Then the operator $T(\tau; \vartheta)$ generated by the form (2.1) in the space $\vartheta$ can be written as
\[ T(\tau; \vartheta) = \tau T_1(\vartheta) + \tau^2 T_2(\vartheta), \]
where the operators $T_1(\vartheta)$ and $T_2(\vartheta)$ correspond to the forms (2.9) and (2.10), respectively.

Using Conditions 1.2 and 1.3 (with $\nu = 1$), we easily check that
\[ \|T_1(\vartheta)\|_{\vartheta \to \vartheta} \leq C_T^{(1)} = \max\{2 + c_1^2, (\|X_1\|^2 + C(1))\vartheta^{-1}\}. \]
Similarly, Conditions 1.3 and 1.4 imply
\[ \|T_2(\vartheta)\|_{\vartheta \to \vartheta} \leq C_T^{(2)} = \max\{c_2 + 1, (\|X_1\|^2 + \|Y_1\|^2 + C(1) + c_3 + |\lambda| \|Q_0\|\)\vartheta^{-1}\}. \]
(See [Su2] Subsection 2.2 for the details.) From (2.12) and (2.13) we obtain the following estimate for the norm of the operator (2.11):
\begin{align*}
(2.14) & \quad \|T(\tau; \vartheta)\|_{\vartheta \to \vartheta} \leq C_T|\tau|, \quad |\tau| \leq \tau_0, \\
(2.15) & \quad C_T = C_T^{(1)} + \tau_0 C_T^{(2)}.
\end{align*}

2.3. The contour $\Gamma_\delta$. Representations and estimates for the difference of resolvents. Let $\Gamma_\delta$ be the contour in the complex plane that envelopes the interval $[0, \delta]$ equidistantly at the distance $\delta$. By (1.16), for $|\tau| \leq \tau_0$ the distance from $\Gamma_\delta$ to the spectrum of $B_\lambda(\tau; \vartheta)$ is at least $\delta$, whence
\[ \|R_z(\tau; \vartheta)\|_{\vartheta \to \vartheta} \leq \delta^{-1}, \quad z \in \Gamma_\delta, \quad |\tau| \leq \tau_0. \]
Since
\[ |z| \leq 2\delta, \quad z \in \Gamma_\delta, \]
we see that the operator $\Omega_z(\tau; \vartheta)$ satisfies
\[ \|\Omega_z(\tau; \vartheta)\|_{\vartheta \to \vartheta} \leq 4, \quad z \in \Gamma_\delta, \quad |\tau| \leq \tau_0. \]
If $\tau = 0$, estimates (2.16) and (2.18) remain true (for the operators $R_z(0)$ and $\Omega_z(0)$, respectively).

By (2.3), (2.4), (2.6), (2.14), and (2.18), we have
\[ \|R_z(\tau; \vartheta) - R_z(0)\|_{\vartheta \to \vartheta} \leq 32\kappa^{-1/2}\delta^{-1}C_T|\tau|, \quad z \in \Gamma_\delta, \quad |\tau| \leq \tau_0. \]
Besides estimate (2.19), we need representations for the difference of the resolvents, in which finitely many power terms in $\tau$ are distinguished, and the error term satisfies a higher order estimate.

Theorem 2.1. For $|\tau| \leq \tau_0$ and $z \in \Gamma_\delta$ the following statements are true.

1°. We have:
\begin{align*}
(2.20) & \quad R_z(\tau; \vartheta) - R_z(0) = \tau I_1(z; \vartheta) + \Psi_1(z; \tau; \vartheta), \\
(2.21) & \quad \|\Psi_1(z; \tau; \vartheta)\|_{\vartheta \to \vartheta} \leq C_1\tau^2, \quad z \in \Gamma_\delta, \quad |\tau| \leq \tau_0.
\end{align*}
2°. We have:

\begin{align}
(2.22) \quad R_z(\tau; \vartheta) - R_z(0) &= \tau I_1(z; \vartheta) + \tau^2 I_2(z; \vartheta) + \Psi_2(z; \tau; \vartheta), \\
(2.23) \quad \|\Psi_2(z; \tau; \vartheta)\|_{\delta \rightarrow \delta} &\leq C_2|\tau|^3, \quad z \in \Gamma_{\delta}, \quad |\tau| \leq \tau_0.
\end{align}

3°. We have:

\begin{align}
(2.24) \quad R_z(\tau; \vartheta) - R_z(0) &= \tau I_1(z; \vartheta) + \tau^2 I_2(z; \vartheta) + \tau^3 I_3(z; \vartheta) + \Psi_3(z; \tau; \vartheta), \\
(2.25) \quad \|\Psi_3(z; \tau; \vartheta)\|_{\delta \rightarrow \delta} &\leq C_3\tau^4, \quad z \in \Gamma_{\delta}, \quad |\tau| \leq \tau_0.
\end{align}

The operators $I_j(z; \vartheta), \; j = 1, 2, 3,$ are bounded and independent of $\tau$; they are defined below by the expressions (2.26), (2.30), and (2.34), respectively. The constants $C_j, \; j = 1, 2, 3,$ are given below in formulas (2.28), (2.32), and (2.36).

Proof. In the proofs we often simplify the notation and write $\Omega_z, T_j, T(\tau), R_z(\tau)$ instead of $\Omega_z(0), T_j(\vartheta), T(\tau; \vartheta), R_z(\tau; \vartheta)$, respectively. Iterating, from (2.3) and (2.11) we obtain

\begin{align}
R_z(\tau; \vartheta) - R_z(0) &= -\Omega_z(\tau T_1 + \tau^2 T_2) R_z(\tau) \\
&= -\tau \Omega_z T_1 (R_z(0) - \Omega_z T(\tau) R_z(\tau)) - \tau^2 \Omega_z T_2 R_z(\tau).
\end{align}

This implies (2.20) with

\begin{align}
(2.26) \quad I_1(z; \vartheta) &= -\Omega_z(0)T_1(\vartheta)R_z(0), \\
(2.27) \quad I_3(z; \vartheta) &= \tau \Omega_z T_1 \Omega_z T(\tau) R_z(\tau) - \tau^2 \Omega_z T_2 R_z(\tau).
\end{align}

Combining (2.4), (2.6), (2.7), (2.12)–(2.14), (2.17), and (2.18), we arrive at estimate (2.21) for the norm of the operator (2.27), with the constant

\begin{align}
C_1 &= 32\delta^{-1}\kappa^{-1/2}(13C_T^{(1)} C_T + C_T^{(2)}).
\end{align}

Statement 1° is proved.

Applying (2.3) and (2.11) again, we rewrite the operator (2.27) as

\begin{align}
\Psi_1(z; \tau; \vartheta) &= \tau\Omega_z T_1 \Omega_z (\tau T_1 + \tau^2 T_2) R_z(\tau) - \tau^2 \Omega_z T_2 R_z(\tau) \\
&= \tau^2 (\Omega_z T_1)^2 (R_z(0) - \Omega_z T(\tau) R_z(\tau)) \\
&\quad + \tau^3 \Omega_z T_1 \Omega_z T_2 R_z(\tau) - \tau^2 \Omega_z T_2 (R_z(0) - \Omega_z T(\tau) R_z(\tau)).
\end{align}

It follows that

\begin{align}
(2.29) \quad \Psi_1(z; \tau; \vartheta) &= \tau^2 I_2(z; \vartheta) + \Psi_2(z; \tau; \vartheta),
\end{align}

where

\begin{align}
(2.30) \quad I_2(z; \vartheta) &= (\Omega_z(0)T_1(\vartheta))^2 R_z(0) - \Omega_z(0)T_2(\vartheta) R_z(0), \\
(2.31) \quad \Psi_2(z; \tau; \vartheta) &= -\tau^2 (\Omega_z T_1)^2 \Omega_z T(\tau) R_z(\tau) \\
&\quad + \tau^3 \Omega_z T_1 \Omega_z T_2 R_z(\tau) + \tau^2 \Omega_z T_2 \Omega_z T(\tau) R_z(\tau).
\end{align}

Combining (2.4), (2.7), (2.8), (2.12)–(2.14), (2.17), and (2.18), we obtain estimate (2.23) for the norm of the operator (2.31), with the constant

\begin{align}
C_2 &= 416\delta^{-1}\kappa^{-1/2}(13(C_T^{(1)})^2 C_T + C_T^{(1)} C_T^{(2)} + C_T^{(2)} C_T).
\end{align}

Relations (2.20) and (2.29) imply (2.22). This proves statement 2°.
Iterating once again, we reshape the operator (2.31) to the form
\[
\Psi_2(z; \tau; \vartheta) = -\tau^2(\Omega_zT_1)^2\Omega_z(\tau T_1 + \tau^2 T_2)R_z(\tau)
+ \tau^3\Omega_zT_1\Omega_zT_2R_z(\tau)
+ \tau^2\Omega_zT_2\Omega_zz(\tau T_1 + \tau^2 T_2)R_z(\tau)
= -\tau^3(\Omega_zT_1)^3(R_z(0) - \Omega_zT(\tau)R_z(\tau))
- \tau^4(\Omega_zT_1)^2\Omega_zT_2R_z(\tau)
+ \tau^3\Omega_zT_1\Omega_zT_2(R_z(0) - \Omega_zT(\tau)R_z(\tau))
+ \tau^3\Omega_zT_2\Omega_zT_1(R_z(0) - \Omega_zT(\tau)R_z(\tau))
+ \tau^4(\Omega_zT_2)^2R_z(\tau),
\]
whence
\[
(2.33) \quad \Psi_2(z; \tau; \vartheta) = \tau^3\mathcal{I}_3(z; \vartheta) + \Psi_3(z; \tau; \vartheta),
\]
where
\[
(2.34) \quad \mathcal{I}_3(z; \vartheta) = -(\Omega_z(0)T_1(\vartheta))^3R_z(0) + (\Omega_z(0)T_1(\vartheta)\Omega_z(0)T_2(\vartheta)R_z(0)
+ (\Omega_z(0)T_2(\vartheta)\Omega_z(0)T_1(\vartheta)R_z(0)),
\]
\[
(2.35) \quad \Psi_3(z; \tau; \vartheta) = \tau^3(\Omega_zT_1)^3\Omega_zT(\tau)R_z(\tau)
- \tau^4(\Omega_zT_1)^2\Omega_zT_2R_z(\tau)
- \tau^3\Omega_zT_1\Omega_zT_2\Omega_zT(\tau)R_z(\tau)
+ \tau^3\Omega_zT_2\Omega_zT_1\Omega_zT(\tau)R_z(\tau)
+ \tau^4(\Omega_zT_2)^2R_z(\tau).
\]
Applying (2.4), (2.7), (2.8), (2.12)–(2.14), (2.17), and (2.18), we obtain estimate (2.25) for the norm of the operator (2.35), with the constant
\[
(2.36) \quad C_3 = 416\delta^{-1}\kappa^{-1/2}(169(C_1^{(1)})^3C_T + 13(C_1^{(1)})^2C_T^2 + 26C_1^{(1)}C_2^{(2)}C_T + (C_2^{(2)})^2).
\]
Relations (2.22) and (2.33) imply (2.24). Statement 3° is proved.

\section{3. Threshold approximations}

\subsection{} We have (cf. [BS11], Chapter 1, (4.2), (4.4))
\[
(3.1) \quad F(\tau; \vartheta) - P = -\frac{1}{2\pi i} \int_{\Gamma_\delta}(R_z(\tau; \vartheta) - R_z(0)) \, dz,
\]
\[
(3.2) \quad B_\lambda(\tau; \vartheta)F(\tau; \vartheta) = -\frac{1}{2\pi i} \int_{\Gamma_\delta} z(R_z(\tau; \vartheta) - R_z(0)) \, dz.
\]
On the basis of these representations and the results of \S2, it is possible to find approximations for the operators $F(\tau; \vartheta)$ and $B_\lambda(\tau; \vartheta)F(\tau; \vartheta)$ with small $\tau$; they are called threshold approximations.

\subsection{Approximation of the projection $F(\tau; \vartheta)$.}

\textbf{Theorem 3.1.} 1°. We have
\[
(3.3) \quad \|F(\tau; \vartheta) - P\|_{\mathcal{B}_1} \leq C_1|\tau|, \quad |\tau| \leq \tau_0.
\]
Here the number $\tau_0$ is subject to the restriction (1.15), and the constant $C_1$ is given by
\[
(3.4) \quad C_1 = 32(1 + \pi^{-1})\kappa^{-1/2}C_T,
\]
where $C_T$ is defined as in (2.12), (2.13), (2.15).

2°. We have
\[
(3.5) \quad F(\tau; \vartheta) = P + \tau F_1(\vartheta) + F_2(\tau; \vartheta),
\]
where the operator $F_1(\vartheta)$ is defined by (1.41), and the operator $F_2(\tau; \vartheta)$ satisfies
\[
(3.6) \quad \|F_2(\tau; \vartheta)\|_{\mathcal{B}_1} \leq C_2\tau^2, \quad |\tau| \leq \tau_0.
\]
The constant $C_2$ is given by
\begin{equation}
C_2 = 32(1 + \pi^{-1})\kappa^{-1/2}(C_T^{(2)} + 13C_T^{(1)}C_T),
\end{equation}
where $C_T^{(1)}$, $C_T^{(2)}$, and $C_T$ are defined as in (2.12), (2.13), (2.15).

\textbf{Proof.} Estimate (3.3) follows directly from (2.19), (3.1), and the fact that the length of the contour $\Gamma_\delta$ is equal to $2\delta + 2\pi\delta$.

Substituting (2.20) in (3.1), we obtain
\begin{equation}
F(\tau; \vartheta) = -\tau \cdot \frac{1}{2\pi i} \int_{\Gamma_\delta} I_1(z; \vartheta) \, dz - \frac{1}{2\pi i} \int_{\Gamma_\delta} \Psi_1(z; \tau; \vartheta) \, dz.
\end{equation}
The second term on the right is denoted by $F_2(\tau; \vartheta)$. By (2.21) and (2.28), this term satisfies (3.6). Now, combining (3.8) and (1.42), (1.43), we conclude that the operator $- (2\pi i)^{-1} \int_{\Gamma_\delta} I_1(z; \vartheta) \, dz$ coincides with the operator $F_1(\vartheta)$ defined by (1.43). \hfill \square

\section{3.3. Approximation of the operator $B_\lambda(\tau; \vartheta)F(\tau; \vartheta)$}

The next theorem was proved in [Su2, Theorem 2.2]; it is convenient for us to repeat the proof.

\textbf{Theorem 3.2.} For $|\tau| \leq \tau_0$, we have
\begin{equation}
B_\lambda(\tau; \vartheta)F(\tau; \vartheta) = \tau^2 S(\vartheta)P + \Phi_1(\tau; \vartheta)
\end{equation}
and
\begin{equation}
\|\Phi_1(\tau; \vartheta)\|_{S \to S} \leq C_3|\tau|^3, \quad |\tau| \leq \tau_0.
\end{equation}
The number $\tau_0$ is subject to the restriction (1.15), and the constant $C_3$ is given by
\begin{equation}
C_3 = 832(1 + \pi^{-1})\delta\kappa^{-1/2}(13(C_T^{(1)})^2C_T + C_T^{(1)}C_T^{(2)} + C_T^{(2)}C_T),
\end{equation}
where $C_T^{(1)}$, $C_T^{(2)}$, and $C_T$ are defined as in (2.12), (2.13), (2.15).

\textbf{Proof.} Substituting (2.22) in (3.2), we obtain the representation
\begin{equation}
B_\lambda(\tau; \vartheta)F(\tau; \vartheta) = -\tau \cdot \frac{1}{2\pi i} \int_{\Gamma_\delta} zI_1(z; \vartheta) \, dz - \tau^2 \cdot \frac{1}{2\pi i} \int_{\Gamma_\delta} zI_2(z; \vartheta) \, dz
\end{equation}
\begin{equation}
- \frac{1}{2\pi i} \int_{\Gamma_\delta} z\Psi_2(z; \tau; \vartheta) \, dz.
\end{equation}
The last term on the right is denoted by $\Phi_1(\tau; \vartheta)$; this term can be estimated with the help of (2.17), (2.23), and (2.32):
\begin{equation}
\|\Phi_1(\tau; \vartheta)\|_{S \to S} \leq 2\delta^2(1 + \pi^{-1})C_2|\tau|^3 = C_3|\tau|^3, \quad |\tau| \leq \tau_0.
\end{equation}
Now, comparing (3.12) and (1.43), we conclude that
\begin{equation}
- \frac{1}{2\pi i} \int_{\Gamma_\delta} zI_1(z; \vartheta) \, dz = 0, \quad - \frac{1}{2\pi i} \int_{\Gamma_\delta} zI_2(z; \vartheta) \, dz = S(\vartheta)P. \hfill \square
\end{equation}

As in the proof of Theorem 3.2, we can use (3.2) and (2.24), (2.25), (2.36) to obtain the following result.

\textbf{Theorem 3.3.} For $|\tau| \leq \tau_0$, we have
\begin{equation}
B_\lambda(\tau; \vartheta)F(\tau; \vartheta) = \tau^2 S(\vartheta)P + \tau^3 K(\vartheta) + \Phi_2(\tau; \vartheta)
\end{equation}
and
\begin{equation}
\|\Phi_2(\tau; \vartheta)\|_{S \to S} \leq C_4|\tau|^4, \quad |\tau| \leq \tau_0.
\end{equation}
The operator $K(\vartheta)$ is defined by
\begin{equation}
K(\vartheta) = - \frac{1}{2\pi i} \int_{\Gamma_\delta} zI_3(z; \vartheta) \, dz.
\end{equation}
where \( T_3(z; \vartheta) \) is the operator (2.34). The number \( \tau_0 \) is subject to the restriction (1.15), and the constant \( C_4 \) is given by

\[
C_4 = 832(1 + \pi^{-1})5\kappa^{-1/2}(169(C_T^{(1)})^3C_T + 13(C_T^{(1)})^2C_T^{(2)} + 26C_T^{(1)}C_T^{(2)}C_T + (C_T^{(2)})^2),
\]

where \( C_T^{(1)}, C_T^{(2)}, C_T \) are defined as in (2.12), (2.13), (2.15).

In order to calculate \( K(\vartheta) \), instead of using (3.15), it is more convenient to employ power series expansions.

### 3.4. Calculation of the operator \( K(\vartheta) \)

In accordance with (1.28), (1.29), (1.36), and (1.42), for sufficiently small \( \tau_* \) and \(|\tau| \leq \tau_* \) we have

\[
B_\lambda(\tau; \vartheta)F(\tau; \vartheta) = \sum_{l=1}^{n} \left( \gamma_l(\vartheta)\tau^2 + \mu_l(\vartheta)\tau^3 \right) \left( \cdot, \omega_l(\vartheta) + \tau \varphi_l^{(1)}(\vartheta) \right) K(\vartheta) + O(\tau^4).
\]

It follows that the operator \( K(\vartheta) \) in (3.13) admits the representation

\[
K(\vartheta) = \sum_{l=1}^{n} \mu_l(\vartheta)\left( \cdot, \omega_l(\vartheta) \right) K(\vartheta) + N(\vartheta),
\]

where

\[
K_0(\vartheta) = \sum_{l=1}^{n} \gamma_l(\vartheta)\left( \cdot, \omega_l(\vartheta) \right) K_0(\vartheta) + \vartheta_1 Z \omega_l(\vartheta)
\]

\[
+ \left( \cdot, \vartheta_1 Z \omega_l(\vartheta) + \vartheta_2 Z \omega_l(\vartheta) \right) \delta \omega_l(\vartheta),
\]

\[
N(\vartheta) = \sum_{l=1}^{n} \mu_l(\vartheta)\left( \cdot, \omega_l(\vartheta) \right) N(\vartheta)
\]

\[
+ \sum_{l=1}^{n} \gamma_l(\vartheta)\left( \cdot, \omega_l(\vartheta) \right) N(\vartheta) + \left( \cdot, \omega_l(\vartheta) \right) N(\vartheta).
\]

Our next goal is to find an invariant representation for \( K(\vartheta) \). By using (1.36), it is easy to obtain such a representation for the operator (3.19):

\[
K_0(\vartheta) = \vartheta_1 (ZS(\vartheta)P + S(\vartheta)PZ^*) + \vartheta_2 \left( \tilde{Z}S(\vartheta)P + S(\vartheta)P\tilde{Z}^* \right).
\]

Note that the operator \( K_0(\vartheta) \) takes \( \mathcal{R} \) into \( \mathcal{R} \) and \( \mathcal{R} \) into \( \mathcal{R} \), whence

\[
PK_0(\vartheta)P = 0.
\]

On the contrary, the operator \( N(\vartheta) \) acts from \( \mathcal{R} \) to \( \mathcal{R} \) and takes the elements of \( \mathcal{R} \) to zero. Calculation of its invariant form is rather bulky. Let us calculate the matrix entries of \( N(\vartheta) \) in the basis \( \omega_l(\vartheta), l = 1, \ldots, n \). By (2.20), we have

\[
(N(\vartheta)\omega_j(\vartheta), \omega_k(\vartheta))_{\delta} = \mu_j(\vartheta)\delta_{jk} + \gamma_k(\vartheta)(\omega_j(\vartheta), \omega_k(\vartheta))_{\delta} + \gamma_j(\vartheta)(\omega_j(\vartheta), \omega_k(\vartheta))_{\delta}
\]

\[
= \mu_j(\vartheta)\delta_{jk} - \gamma_k(\vartheta)(\omega_j(\vartheta), \omega_k(\vartheta))_{\delta} - \gamma_j(\vartheta)(\omega_j(\vartheta), \omega_k(\vartheta))_{\delta}.
\]
We have used (1.33) at the last step.

The further calculations are based on the relations

\[
(3.24) \quad (B_\lambda(\tau; \vartheta)\varphi_j(\tau; \vartheta), \varphi_k(\tau; \vartheta))_{\delta} = \delta_{jk} \lambda_j(\tau; \vartheta), \quad j, k = 1, \ldots, n.
\]

To simplify the notation, we often omit the argument \( \vartheta \). By (1.29), the left-hand side of (3.24) can be rewritten as

\[
(3.25) \quad b_\lambda(\tau; \vartheta)[\omega_j + \tau \varphi_j^{(1)} + \tau^2 \varphi_j^{(2)} + \ldots, \omega_k + \tau \varphi_k^{(1)} + \tau^2 \varphi_k^{(2)} + \ldots].
\]

Using the expression (1.13) for the form \( b_\lambda(\tau; \vartheta) \), we find the total coefficient \( \Delta_{jk}(\vartheta) \) of \( \tau^3 \) in (3.25):

\[
\Delta_{jk}(\vartheta) = (X_0 \varphi_j^{(1)} + \vartheta_1 X_1 \omega_j, X_0 \varphi_k^{(2)})_{\delta*} + (X_0 \varphi_j^{(2)}, X_0 \varphi_k^{(1)} + \vartheta_1 X_1 \omega_k)_{\delta*},
\]

\[
+ \vartheta_1 (X_0 \varphi_j^{(1)} + \vartheta_1 X_1 \omega_j, X_1 \varphi_k^{(1)})_{\delta*} + \vartheta_1 (X_1 \varphi_j^{(1)}, X_0 \varphi_k^{(1)} + \vartheta_1 X_1 \omega_k)_{\delta*},
\]

\[
+ \vartheta_2 (Y_0 \varphi_j^{(1)} + \vartheta_1 Y_1 \omega_j, Y_2 \varphi_k^{(1)})_{\delta*} + \vartheta_2 (Y_0 \varphi_j^{(1)}, Y_2 \varphi_k^{(1)})_{\delta*} + \vartheta_1 \vartheta_2 (Y_1 \varphi_j^{(1)}, Y_2 \omega_k)_{\delta*},
\]

\[
+ \vartheta_2 (Y_2 \varphi_j^{(1)}, Y_0 \varphi_k^{(1)} + \vartheta_1 Y_1 \omega_k)_{\delta*} + \vartheta_2 (Y_2 \omega_j, Y_0 \varphi_k^{(1)} + \vartheta_1 Y_1 \omega_k)_{\delta*} + \vartheta_1 \vartheta_2 (Y_2 \omega_j, Y_1 \varphi_k^{(1)})_{\delta*}.
\]

\[
(3.26) \quad \Delta_{jk}(\vartheta) = \vartheta_1 (X_0 \varphi_j^{(1)} + \vartheta_1 X_1 \omega_j + \vartheta_2 \tilde{Z} \omega_j + \tilde{\omega}_j) + \vartheta_1 X_1 \omega_j, X_1(\vartheta_1 Z \omega_k + \vartheta_2 \tilde{Z} \omega_k + \tilde{\omega}_k)_{\delta*} + \vartheta_1 X_1 \omega_k)_{\delta*},
\]

\[
+ \vartheta_1 X_1(\vartheta_1 Z \omega_j + \vartheta_2 \tilde{Z} \omega_j + \tilde{\omega}_j, X_0(\vartheta_1 Z \omega_k + \vartheta_2 \tilde{Z} \omega_k + \tilde{\omega}_k)_{\delta*} + \vartheta_1 X_1 \omega_k)_{\delta*},
\]

\[
+ \vartheta_2 Y_0(\vartheta_1 Z \omega_j + \vartheta_2 \tilde{Z} \omega_j + \tilde{\omega}_j, Y_2(\vartheta_1 Z \omega_k + \vartheta_2 \tilde{Z} \omega_k + \tilde{\omega}_k)_{\delta*} + \vartheta_2 Y_1 \omega_k)_{\delta*},
\]

\[
+ \vartheta_1 \vartheta_2 (Y_2 \omega_j, Y_1(\vartheta_1 Z \omega_k + \vartheta_2 \tilde{Z} \omega_k + \tilde{\omega}_k)_{\delta*} + \vartheta_1 \vartheta_2 (Y_2 \omega_j, Y_1 \varphi_k^{(1)}), Y_2 \omega_k)_{\delta*},
\]

\[
+ \vartheta_2^2 [q|\omega_j, \vartheta_1 Z \omega_k + \vartheta_2 \tilde{Z} \omega_k + \tilde{\omega}_k)_{\delta*} + q|\vartheta_1 Z \omega_j + \vartheta_2 \tilde{Z} \omega_j + \tilde{\omega}_j, \omega_k)]_{\delta*},
\]

\[
+ \vartheta_2^2 \lambda((Q \omega_j, \vartheta_1 Z \omega_k + \vartheta_2 \tilde{Z} \omega_k + \tilde{\omega}_k)_{\delta*} + (Q \omega_j, \vartheta_1 \vartheta_2 (Y_2 \omega_j, Y_1 \varphi_k^{(1)}, Y_2 \omega_k)_{\delta*}).
\]

Observe that \( X_0 \tilde{\omega}_j = 0, Y_0 \tilde{\omega}_k = 0, \) and \( (X_0 Z + X_1) \omega_i = R \omega_i \).

We denote by \( \tilde{\Delta}_{jk}(\vartheta) \) the sum of all terms in (3.26) that involve the undefined elements \( \tilde{\omega}_j \) or \( \tilde{\omega}_k \). Then

\[
\tilde{\Delta}_{jk}(\vartheta) = \vartheta_1 (\vartheta_1 R \omega_j + \vartheta_2 X_0 \tilde{Z} \omega_j, X_1 \tilde{\omega}_k)_{\delta*} + \vartheta_1 (X_1 \tilde{\omega}_j, \vartheta_1 R \omega_k + \vartheta_2 X_0 \tilde{Z} \omega_k)_{\delta*},
\]

\[
(3.27) \quad \tilde{\Delta}_{jk}(\vartheta) = \vartheta_2 (\vartheta_1 Y_0 \omega_j + \vartheta_2 Y_0 \tilde{Z} \omega_j + \vartheta_1 Y_1 \omega_j, Y_2 \tilde{\omega}_k)_{\delta*} + \vartheta_2 \vartheta_1 (Y_1 \tilde{\omega}_j, Y_2 \omega_k)_{\delta*},
\]

\[
+ \vartheta_2 (Y_2 \tilde{\omega}_j, Y_1 \omega_k + \vartheta_2 Y_0 \tilde{Z} \omega_j + \vartheta_1 Y_1 \omega_k)_{\delta*} + \vartheta_1 \vartheta_2 (Y_2 \omega_j, Y_1 \tilde{\omega}_k)_{\delta*} + \vartheta_2^2 \lambda((Q \omega_j, \vartheta_1 \tilde{\omega}_k + \vartheta_2 \tilde{\omega}_k)_{\delta*} + \lambda(Q \omega_j, \vartheta_1 \vartheta_2 (Y_2 \omega_j, Y_1 \tilde{\omega}_k)_{\delta*}),
\]

\[
\text{Since } X_0^* R = 0 \text{ and } R^* R = S, \text{ we see that}
\]

\[
(R \omega_j, X_1 \tilde{\omega}_k)_{\delta*} = (R \omega_j, (X_0 Z + X_1) \tilde{\omega}_k)_{\delta*} = (R \omega_j, R \tilde{\omega}_k)_{\delta*} = (S \omega_j, \tilde{\omega}_k)_{\delta*}.
\]

Similarly, \( (X_1 \tilde{\omega}_j, R \tilde{\omega}_k)_{\delta*} = (\tilde{\omega}_j, S \omega_k)_{\delta*} \).

Next, by the definition of \( Z \),

\[
X_0^* X_0 Z \omega + X_0^* X_1 \omega = 0
\]
for any \( \omega \in \mathfrak{N} \), whence

\[
(X_0 \tilde{Z} \omega_j, X_1 \tilde{\omega}_k)_\delta = -(X_0 \tilde{Z} \omega_j, X_0 Z \tilde{\omega}_k)_\delta, \\
(X_1 \tilde{\omega}_j, X_0 \tilde{Z} \omega_k)_\delta = -(X_0 Z \tilde{\omega}_j, X_0 \tilde{Z} \omega_k)_\delta.
\]

Similarly, the definition of \( \tilde{Z} \) shows that

\[
X_0^* X_0 \tilde{Z} \omega + Y_0^* Y_2 \omega = 0
\]

for any \( \omega \in \mathfrak{N} \), whence

\[
(Y_0 Z \omega_j, Y_2 \tilde{\omega}_k)_\delta = -(X_0 Z \omega_j, X_0 \tilde{Z} \tilde{\omega}_k)_\delta, \\
(Y_0 \tilde{Z} \omega_j, Y_2 \tilde{\omega}_k)_\delta = -(X_0 \tilde{Z} \omega_j, X_0 Z \tilde{\omega}_k)_\delta, \\
(Y_2 \tilde{\omega}_j, Y_0 Z \omega_k)_\delta = -(X_0 \tilde{Z} \omega_j, X_0 Z \omega_k)_\delta, \\
(Y_2 \tilde{\omega}_j, Y_0 \tilde{Z} \omega_k)_\delta = -(X_0 \tilde{Z} \omega_j, X_0 \tilde{Z} \omega_k)_\delta.
\]

By using the above relations, the expression (3.27) can be rewritten as

\[
\Delta_{jk}(\vartheta) = \vartheta_1^2 (S \omega_j, \tilde{\omega}_k)_\delta - \vartheta_1 \vartheta_2 X_0 \tilde{Z} \omega_j, X_0 Z \tilde{\omega}_k)_\delta, \\
+ \vartheta_1^2 (\tilde{\omega}_j, S \omega_k)_\delta - \vartheta_1 \vartheta_2 (X_0 Z \tilde{\omega}_j, X_0 \tilde{Z} \omega_k)_\delta, \\
- \vartheta_1 \vartheta_2 (X_0 Z \omega_j, X_0 \tilde{Z} \omega_k)_\delta - \vartheta_1^2 (X_0 \tilde{Z} \omega_j, X_0 \tilde{Z} \omega_k)_\delta, \\
+ \vartheta_1 \vartheta_2 (Y_1 \omega_j, Y_2 \tilde{\omega}_k)_\delta + \vartheta_1 \vartheta_2 (Y_1 \tilde{\omega}_j, Y_2 \omega_k)_\delta - \vartheta_1 \vartheta_2 (X_0 \tilde{Z} \omega_j, X_0 Z \omega_k)_\delta, \\
- \vartheta_1^2 (X_0 \tilde{Z} \omega_j, X_0 \tilde{Z} \omega_k)_\delta + \vartheta_1 \vartheta_2 (Y_2 \tilde{\omega}_j, Y_1 \omega_k)_\delta + \vartheta_1 \vartheta_2 (Y_2 \omega_j, Y_1 \tilde{\omega}_k)_\delta, \\
+ \vartheta_2^2 (q[\omega_j, \tilde{\omega}_k] + q[\tilde{\omega}_j, \omega_k] + \lambda(Q_0 \omega_j, \tilde{\omega}_k)_\delta + \lambda(Q_0 \tilde{\omega}_j, \omega_k)_\delta).
\]

Now, recalling the expression (1.34) for the spectral germ \( S(\vartheta) \), we conclude that

\[
\Delta_{jk}(\vartheta) = (S(\vartheta) \omega_j, \tilde{\omega}_k)_\delta + (\tilde{\omega}_j, S(\vartheta) \omega_k)_\delta = \gamma_j(\omega_j, \tilde{\omega}_k)_\delta + \gamma_k(\tilde{\omega}_j, \omega_k)_\delta.
\]

(We have used (1.35).)

We summarize the results. From (3.24) and (1.28) it follows that \( \Delta_{jk}(\vartheta) = \mu_j(\vartheta) \delta_{jk} \).

Comparing this with (3.26) and (3.28), we obtain

\[
\mu_j \delta_{jk} - \gamma_j(\omega_j, \tilde{\omega}_k)_\delta - \gamma_k(\tilde{\omega}_j, \omega_k)_\delta
= \vartheta_1 \vartheta_2 (X_0 \tilde{Z} \omega_j, X_1 (\vartheta_1 Z \omega_k + \vartheta_2 \tilde{Z} \omega_k))_\delta, \\
+ \vartheta_1 (X_1 (\vartheta_1 Z \omega_j + \vartheta_2 \tilde{Z} \omega_j), \vartheta_1 R \omega_k + \vartheta_2 X_0 \tilde{Z} \omega_k)_\delta, \\
+ \vartheta_2 (Y_0 (\vartheta_1 Z \omega_j + \vartheta_2 \tilde{Z} \omega_j), \vartheta_1 Y_1 \omega_j, Y_2 (\vartheta_1 Z \omega_k + \vartheta_2 \tilde{Z} \omega_k))_\delta, \\
+ \vartheta_1 \vartheta_2 (Y_1 (\vartheta_1 Z \omega_j + \vartheta_2 \tilde{Z} \omega_j), Y_2 \omega_k)_\delta, \\
+ \vartheta_2 (Y_2 (\vartheta_1 Z \omega_j + \vartheta_2 \tilde{Z} \omega_j), Y_0 (\vartheta_1 Z \omega_k + \vartheta_2 \tilde{Z} \omega_k) + \vartheta_1 Y_1 \omega_k)_\delta, \\
+ \vartheta_1 \vartheta_2 (Y_2 \omega_j, Y_1 (\vartheta_1 Z \omega_k + \vartheta_2 \tilde{Z} \omega_k))_\delta, \\
+ \vartheta_2^2 (q[\omega_j, \tilde{\omega}_k] + q[\tilde{\omega}_j, \omega_k]) + \lambda(Q_0 \omega_j, \tilde{\omega}_k)_\delta + \lambda(Q_0 \tilde{\omega}_j, \omega_k)_\delta.
\]

By (3.23), the matrix entries \( (N(\vartheta) \omega_j, \omega_k)_\delta, j, k = 1, \ldots, n \), are given by (3.29). This yields the following representation for the operator \( N(\vartheta) \):

\[
N(\vartheta) = \vartheta_1^2 N_{11} + \vartheta_1^2 \vartheta_2 N_{12} + \vartheta_1 \vartheta_2^2 N_{21} + \vartheta_2^3 N_{22},
\]
Similarly, in (3.34), the formal expression
\[
N_{21} = (X_0 \hat{Z})^* X_1 \hat{Z} + (X_1 \hat{Z})^* X_0 \hat{Z} + (X_2 \hat{Z})^* Y_0 \hat{Z}
\]
\[
+ (Y_0 \hat{Z})^* Y_2 Z + (Y_2 \hat{Z})^* Y_0 \hat{Z} + (Y_0 \hat{Z})^* Y_2 \hat{Z} + (Y_2 \hat{Z})^* Y_1 P
\]
\[
+ (Y_1 P)^* Y_2 \hat{Z} + (Y_1 \hat{Z})^* Y_2 P + (Y_2 P)^* Y_1 \hat{Z}
\]
\[
+ Z^* QP + PQZ + \lambda (\hat{Z}^* Q_0 P + PQ_0 \hat{Z}).
\]

Note that, in (3.33), the formal expression \(Z^* QP + PQZ\) means the bounded selfadjoint operator in \(\mathfrak{H}\) generated by the form
\[
q [P \hat{u}, Z u] + q [Z u, P \hat{u}], \quad u \in \mathfrak{H}.
\]
Similarly, in (3.34), the formal expression \(\hat{Z}^* QP + PQ\hat{Z}\) is understood as the bounded selfadjoint operator in \(\mathfrak{H}\) corresponding to the form
\[
q [P \hat{u}, \hat{Z} u] + q [\hat{Z} u, P \hat{u}], \quad u \in \mathfrak{H}.
\]

### 3.5. Estimates for the operators \(K_0(\vartheta)\) and \(N(\vartheta)\)

Using (1.21), (1.25), and (1.37), we obtain the following estimate for the operator (3.21):
\[
\|K_0(\vartheta)\|_{\mathfrak{H} \to \mathfrak{H}} \leq 2 (\|Z\|_{\mathfrak{H} \to \mathfrak{H}} + \|\hat{Z}\|_{\mathfrak{H} \to \mathfrak{H}}) \|\mathcal{S}(\vartheta) P\|_{\mathfrak{H} \to \mathfrak{H}} \leq C_5,
\]
\[
C_5 = 2 \kappa^{1/2} (13 \delta)^{-1/2} (\|X_1\| + c_1 C(1)^{1/2})
\]
\[
\times (\|X_1\|^2 + 2 c_1 C(1)^{1/2} \|X_1\|^2 + c_2^2 C(1)^{1/2} + 2 C(1)^{1/2} \|Y_1\| + c_3 + |\lambda| \|Q_0\|).
\]

Relations (1.21) and (1.26) imply the following estimate for the operator (3.31):
\[
\|N_{11}\|_{\mathfrak{H} \to \mathfrak{H}} \leq 2 \|X_1\| \|Z\| \|R\| \leq 2 \kappa^{1/2} (13 \delta)^{-1/2} \|X_1\|^3 =: C_{11}.
\]

In order to estimate the operators (3.32)–(3.34), we need the next statement.

**Proposition 3.4.** We have
\[
\|Y_0 Z\|_{\mathfrak{H} \to \mathfrak{H}} \leq c_1 \|X_1\|,
\]
\[
\|Y_0 \hat{Z}\|_{\mathfrak{H} \to \mathfrak{H}} \leq c_2^2 C(1)^{1/2},
\]
\[
\|Y_2 Z\|_{\mathfrak{H} \to \mathfrak{H}} \leq (1 + \kappa (13 \delta)^{-1} C(1))^{1/2} \|X_1\|,
\]
\[
\|Y_2 \hat{Z}\|_{\mathfrak{H} \to \mathfrak{H}} \leq (1 + \kappa (13 \delta)^{-1} C(1))^{1/2} c_1 C(1)^{1/2},
\]
\[
\|Y_2 P\|_{\mathfrak{H} \to \mathfrak{H}} \leq C(1)^{1/2},
\]
\[
\|Z^* QP + PQZ\|_{\mathfrak{H} \to \mathfrak{H}} \leq 2 c_3^{1/2} \|X_1\| (c_2 + c_3 \kappa (13 \delta)^{-1})^{1/2},
\]
\[
\|\hat{Z}^* QP + PQ\hat{Z}\|_{\mathfrak{H} \to \mathfrak{H}} \leq 2 c_3^{1/2} c_1 C(1)^{1/2} (c_2 + c_3 \kappa (13 \delta)^{-1})^{1/2}.
\]

**Proof.** From (1.20) and Condition 1.2 with \(t = 0\), it follows that
\[
\|Y_0 Z\|_{\mathfrak{H} \to \mathfrak{H}} \leq c_1 \|X_0 Z\|_{\mathfrak{H} \to \mathfrak{H}} \leq c_1 \|X_1\|,
\]
which implies (3.37). Estimate (3.38) is proved similarly by using (1.24).
By Condition 1.3 with \( t = 0 \) and inequalities (1.20) and (1.21), we have
\[
\|Y_2 Z\|_{\mathcal{B}_0} \leq \|X_0 Z\|_{\mathcal{B}_0} + C(1)\|Z\|_{\mathcal{B}_0} \leq \|X_1\|^2 + C(1)\kappa(13\delta)^{-1}\|X_1\|^2,
\]
which yields (3.39). Similarly, (1.24) and (1.25) imply (3.40).

Inequality (3.41) is a direct consequence of Condition 1.3 with \( t = 0 \).

For the proof of (3.42), we use Condition 1.4(1) with \( t = 0 \), obtaining
\[
\|q[P_{u, Zv} + q[Zu, P_{v}]\| \leq c_3^{1/2}\|P_{u}\|_{\mathcal{B}_0} (c_2\|X_0 Zv\|_{\mathcal{B}_0}^2 + c_3\|Zv\|_{\mathcal{B}_0}^2)^{1/2} + (c_2\|X_0 Zu\|_{\mathcal{B}_0}^2 + c_3\|Zu\|_{\mathcal{B}_0}^2)^{1/2} c_2^{1/2}\|P_{v}\|_{\mathcal{B}_0}, \quad u, v \in \mathcal{B}_0,
\]
which implies (3.42) in view of (1.20) and (1.21). Inequality (3.43) is checked similarly by using (1.24) and (1.25).

Combining (1.21), (1.24)–(1.26), (3.37), (3.39), and (3.41), we arrive at the following estimate for the operator (3.32):
\[
\|N_{12}\|_{\mathcal{B}_0} \leq 2\|X_1\|\|(R\|\|\hat{Z}\| + \|Z\|\|X_0\|\hat{Z}\|)
+ 2\|Y_2 Z\|\|(Y_0\| Z\| + \|Y_1\|) + 2\|Y_2 P\|\|Y_1\|\|Z\| \leq C_{12},
C_{12} = 2(\kappa C(1))^{1/2}(13\delta)^{-1/2}\|X_1\| (2c_1\|X_1\| + \|Y_1\|)
+ 2(1 + \kappa(13\delta)^{-1}C(1))^{1/2}\|X_1\| (c_2\|X_1\| + \|Y_1\|).
\]

For the operator (3.33), from (1.21), (1.24), (1.25), and (3.37)–(3.42) we deduce the inequality
\[
\|N_{21}\|_{\mathcal{B}_0} \leq 2\|X_0\|\|\hat{Z}\|\|X_1\|\|\hat{Z}\| + 2\|Y_2 Z\|\|Y_0\|\|\hat{Z}\| + 2\|Y_2 \hat{Z}\|\|Y_0\|\|Z\| + \|Y_1\|)
+ 2\|Y_2 P\|\|Y_1\|\|\hat{Z}\| + \|\hat{Z}^* QP + PQ \hat{Z}\| + 2\|Z\|\|Q_0\| \leq C_{21},
C_{21} = 2c_3^{1/2}(13\delta)^{-1/2} (c_1^2 C(1)\|X_1\| + c_1 C(1)\|Y_1\| + \|\hat{Z}\|\|Q_0\|\|X_1\|)
+ 2(1 + \kappa(13\delta)^{-1}C(1))^{1/2} c_1 C(1)\|X_1\| + \|Y_1\|
+ 2c_3^{1/2} \|X_1\| (c_2 + c_3\kappa(13\delta)^{-1})^{1/2}.
\]

Finally, relations (1.25), (3.38), (3.40), and (3.43) imply the following estimate for the operator (3.34):
\[
\|N_{22}\|_{\mathcal{B}_0} \leq 2\|Y_0\|\|\hat{Z}\|\|\hat{Z}\| \|\hat{Z}^* QP + PQ \hat{Z}\| + 2\|\hat{Z}\|\|Q_0\|\|\hat{Z}\| \leq C_{22},
C_{22} = 2c_3^3 C(1)\|\hat{Z}\| (1 + \kappa(13\delta)^{-1}C(1))^{1/2} + 2c_3^{1/2} c_1 C(1)\|X_1\| + (c_2 + c_3\kappa(13\delta)^{-1})^{1/2} \|\hat{Z}\|\|Q_0\|
+ 2c_1 (\kappa C(1))^{1/2} (13\delta)^{-1/2} \|\hat{Z}\|\|Q_0\|.
\]

As a result, from (3.36), (3.44), (3.46), and (3.48) we obtain the next estimate for the operator (3.30):
\[
\|N(\vartheta)\|_{\mathcal{B}_0} \leq C_{11} + C_{12} + C_{21} + C_{22} =: C_6.
\]
Together with (3.35), this implies the following estimate for the operator (3.18):
\[
\|K(\vartheta)\|_{\mathcal{B}_0} \leq C_5 + C_6.
\]

§4. Approximation of the operator \( B_\lambda(t, \varepsilon)^{-1} \)

4.1. The principal term of approximation for the operator \( B_\lambda(\tau; \vartheta)^{-1} \). As in [Su2, Subsection 3.1], we assume that
\[
A(t) \geq c_4 t^2 I, \quad c_4 > 0, \quad |t| \leq \tau_0,
\]
for some $c_* > 0$. Then, by (1.12), we have
\begin{equation}
(4.2) \quad B_\lambda(\tau; \vartheta) \geq \tilde{c}_* \tau^2 I, \quad |\tau| \leq \tau_0, \quad \tilde{c}_* = \frac{1}{2} \min\{\kappa c_*, 2\beta\}.
\end{equation}

Recall that the number $\beta$ is defined by (1.11). From (4.2) and (1.27) it follows that
\begin{equation}
(4.3) \quad \lambda_l(\tau; \vartheta) \geq \tilde{c}_* \tau^2, \quad l = 1, \ldots, n, \quad |\tau| \leq \tau_0.
\end{equation}

By (4.3) and (1.28), we have $\gamma_l(\vartheta) \geq \tilde{c}_*, \ l = 1, \ldots, n$. Combining this with (1.36), we obtain
\begin{equation}
(4.4) \quad S(\vartheta) \geq \tilde{c}_* I_R.
\end{equation}

The principal term of approximation for the operator $B_\lambda(\tau; \vartheta)^{-1}$ for small $|\tau|$ with an error term $O(|\tau|^{-1})$ was obtained in [Su2 Subsection 3.1]. We repeat the corresponding arguments, because they are needed for the further considerations. Obviously,
\begin{equation}
(4.5) \quad B_\lambda(\tau; \vartheta)^{-1} = B_\lambda(\tau; \vartheta)^{-1} F(\tau; \vartheta)^\top + B_\lambda(\tau; \vartheta)^{-1} F(\tau; \vartheta).
\end{equation}

By (1.16),
\begin{equation}
(4.6) \quad \|B_\lambda(\tau; \vartheta)^{-1} F(\tau; \vartheta)^\top\|_{\mathcal{B}_R \rightarrow \mathcal{B}_R} \leq (3\delta)^{-1}, \quad |\tau| \leq \tau_0.
\end{equation}

Denote
\begin{equation}
(4.7) \quad \Xi(\tau; \vartheta) := (\tau^2 S(\vartheta))^{-1} P,
\end{equation}
and consider the operator
\begin{equation}
(4.8) \quad G(\tau; \vartheta) := B_\lambda(\tau; \vartheta)^{-1} F(\tau; \vartheta) - \Xi(\tau; \vartheta) = G_1(\tau; \vartheta) + G_2(\tau; \vartheta) + G_3(\tau; \vartheta),
\end{equation}
where
\begin{equation}
(4.9) \quad G_1(\tau; \vartheta) = B_\lambda(\tau; \vartheta)^{-1} F(\tau; \vartheta)(F(\tau; \vartheta) - P),
\end{equation}
\begin{equation}
(4.10) \quad G_2(\tau; \vartheta) = (F(\tau; \vartheta) - P) \Xi(\tau; \vartheta),
\end{equation}
\begin{equation}
(4.11) \quad G_3(\tau; \vartheta) = F(\tau; \vartheta) B_\lambda(\tau; \vartheta)^{-1} P - F(\tau; \vartheta) \Xi(\tau; \vartheta).
\end{equation}

By (3.3) and (4.2), the operator (4.9) satisfies the estimate
\begin{equation}
(4.12) \quad \|G_1(\tau; \vartheta)\|_{\mathcal{B}_R \rightarrow \mathcal{B}_R} \leq C_1(\tilde{c}_*)^{-1} |\tau|^{-1}, \quad |\tau| \leq \tau_0.
\end{equation}

Similarly, (3.3) and (4.4) imply that
\begin{equation}
(4.13) \quad \|G_2(\tau; \vartheta)\|_{\mathcal{B}_R \rightarrow \mathcal{B}_R} \leq C_1(\tilde{c}_*)^{-1} |\tau|^{-1}, \quad |\tau| \leq \tau_0.
\end{equation}

Writing the operator (4.11) as
\begin{equation}
(4.14) \quad G_3(\tau; \vartheta) = F(\tau; \vartheta) B_\lambda(\tau; \vartheta)^{-1} \left(\tau^2 S(\vartheta) P - B_\lambda(\tau; \vartheta) F(\tau; \vartheta)\right) \Xi(\tau; \vartheta),
\end{equation}
we estimate it with the help of (3.9), (3.10), (4.2), and (4.4):
\begin{equation}
(4.15) \quad \|G_3(\tau; \vartheta)\|_{\mathcal{B}_R \rightarrow \mathcal{B}_R} \leq C_3(\tilde{c}_*)^{-2} |\tau|^{-1}, \quad |\tau| \leq \tau_0.
\end{equation}

Now, relations (4.8), (4.12), (4.13), and (4.15) imply
\begin{equation}
(4.16) \quad \|G(\tau; \vartheta)\|_{\mathcal{B}_R \rightarrow \mathcal{B}_R} \leq \left(2C_1(\tilde{c}_*)^{-1} + C_3(\tilde{c}_*)^{-2}\right) |\tau|^{-1}, \quad |\tau| \leq \tau_0.
\end{equation}

Combining this with (4.5), (4.6), and (4.8), we arrive at the following result (which was proved in [Su2 Theorem 3.1]).
Theorem 4.1. We have
\[ \| B_\lambda(\tau; \vartheta)^{-1} - (\tau^2 S(\vartheta))^{-1} P \|_{\delta \to \delta} \leq C_0 |\tau|^{-1}, \quad |\tau| \leq \tau_0, \]
where
\[ C_0 = \tau_0 (3\delta)^{-1} + 2C_1(\tilde{c}_*)^{-1} + C_3(\tilde{c}_*)^{-2}. \]
The number \( \tau_0 \) is subject to the restriction (1.15), and the constants \( C_1, C_3, \) and \( \tilde{c}_* \) are defined as in (3.4), (3.11), and (4.2), respectively.

4.2. A sharper approximation of the operator \( B_\lambda(\tau; \vartheta)^{-1} \). Now our goal is to approximate the operator \( B_\lambda(\tau; \vartheta)^{-1} \) with an error of the form \( O(1) \).

By (4.6), the first term on the right-hand side of (4.5) still “moves to the error”. We use (3.5) to extract the principal part of the operator (4.10):
\[ G_2(\tau; \vartheta) = G^0_2(\tau; \vartheta) + \tilde{G}_2(\tau; \vartheta), \]
where
\[ G^0_2(\tau; \vartheta) = \tau F_1(\vartheta) \Xi(\tau; \vartheta), \]
and the term \( \tilde{G}_2(\tau; \vartheta) = F_2(\tau; \vartheta) \Xi(\tau; \vartheta) \) is estimated with the help of (3.6) and (4.4):
\[ \| \tilde{G}_2(\tau; \vartheta) \|_{\delta \to \delta} \leq C_2(\tilde{c}_*)^{-1}, \quad |\tau| \leq \tau_0. \]
In (4.9), first we “replace” \( B_\lambda(\tau; \vartheta)^{-1} F(\tau; \vartheta) \) by \( \Xi(\tau; \vartheta) \), using (4.8):
\[ G_1(\tau; \vartheta) = \tilde{G}_1(\tau; \vartheta) + \tilde{G}_1(\tau; \vartheta), \]
where
\[ \tilde{G}_1(\tau; \vartheta) = \Xi(\tau; \vartheta)(F(\tau; \vartheta) - P), \]
and the term \( \tilde{G}_1(\tau; \vartheta) = G(\tau; \vartheta)(F(\tau; \vartheta) - P) \) is estimated with the help of (3.3) and (4.16):
\[ \| \tilde{G}_1(\tau; \vartheta) \|_{\delta \to \delta} \leq C_1 \left( 2C_1(\tilde{c}_*)^{-1} + C_3(\tilde{c}_*)^{-2} \right), \quad |\tau| \leq \tau_0. \]
Now, we use (3.5) to extract the principal part of the operator (4.21):
\[ \tilde{G}_1(\tau; \vartheta) = G^0_1(\tau; \vartheta) + \tilde{G}_1(\tau; \vartheta), \]
where
\[ G^0_1(\tau; \vartheta) = \Xi(\tau; \vartheta) F_1(\vartheta), \]
and the operator \( \tilde{G}_1(\tau; \vartheta) = \Xi(\tau; \vartheta) F_2(\tau; \vartheta) \) is estimated with the help of (3.6) and (4.4):
\[ \| \tilde{G}_1(\tau; \vartheta) \|_{\delta \to \delta} \leq C_2(\tilde{c}_*)^{-1}, \quad |\tau| \leq \tau_0. \]
Next, in (4.14) we “replace” \( B_\lambda(\tau; \vartheta)^{-1} F(\tau; \vartheta) \) by \( \Xi(\tau; \vartheta) \), using (4.8):
\[ G_3(\tau; \vartheta) = \tilde{G}_3(\tau; \vartheta) + \tilde{G}_3(\tau; \vartheta), \]
where
\[ \tilde{G}_3(\tau; \vartheta) = \Xi(\tau; \vartheta) \left( \tau^2 S(\vartheta) P - B_\lambda(\tau; \vartheta) F(\tau; \vartheta) \right) \Xi(\tau; \vartheta), \]
and the operator \( \tilde{G}_3(\tau; \vartheta) = G(\tau; \vartheta) \left( \tau^2 S(\vartheta) P - B_\lambda(\tau; \vartheta) F(\tau; \vartheta) \right) \Xi(\tau; \vartheta) \) is estimated with the help of (3.9), (3.10), (4.4), and (4.16):
\[ \| \tilde{G}_3(\tau; \vartheta) \|_{\delta \to \delta} \leq C_3(\tilde{c}_*)^{-1} \left( 2C_1(\tilde{c}_*)^{-1} + C_3(\tilde{c}_*)^{-2} \right), \quad |\tau| \leq \tau_0. \]
We use (3.13) to extract the principal part of the operator (4.27):
\[ \tilde{G}_3(\tau; \vartheta) = G^0_3(\tau; \vartheta) + \tilde{G}_3(\tau; \vartheta), \]
where
\begin{equation}
G_0^3(\tau; \vartheta) = -\Xi(\tau; \vartheta)\tau^3K(\vartheta)\Xi(\tau; \vartheta),
\end{equation}
and, by (3.14) and (4.4), the operator \( \tilde{G}_3(\tau; \vartheta) \) satisfies
\begin{equation}
\|\tilde{G}_3(\tau; \vartheta)\|_{\mathcal{B}_0} \leq C_4(\tilde{c}_*^{-1}), \quad |\tau| \leq \tau_0.
\end{equation}

As a result, relations (4.8), (4.17), (4.20), (4.23), (4.26), and (4.29) imply that
\[ G(\tau; \vartheta) = G_0^3(\tau; \vartheta) + G_0^3(\tau; \vartheta) + G_3(\tau; \vartheta) + \tilde{G}(\tau; \vartheta), \]
where \( \tilde{G}(\tau; \vartheta) = \tilde{G}_1(\tau; \vartheta) + \tilde{G}_1(\tau; \vartheta) + \tilde{G}_2(\tau; \vartheta) + \tilde{G}_3(\tau; \vartheta) + \tilde{G}_3(\tau; \vartheta). \) By (4.19), (4.22), (4.25), (4.28), and (4.31), we have
\[ \|\tilde{G}(\tau; \vartheta)\|_{\mathcal{B}_0} \leq (C_1 + C_3(\tilde{c}_*)^{-1})(2C_1(\tilde{c}_*)^{-1} + C_3(\tilde{c}_*)^{-2}) + 2C_2(\tilde{c}_*)^{-1} + C_4(\tilde{c}_*)^{-2} \]
for \( |\tau| \leq \tau_0. \) Combining this with (4.5), (4.6), (4.8), (4.18), (4.24), and (4.30), we arrive at the following result.

**Theorem 4.2.** We have
\begin{equation}
B_\lambda(\tau; \vartheta)^{-1} = \Xi(\tau; \vartheta) + \tau(F_1(\vartheta)\Xi(\tau; \vartheta) + \Xi(\tau; \vartheta)F_1(\vartheta))
- \tau^3\Xi(\tau; \vartheta)K(\vartheta)\Xi(\tau; \vartheta) + J(\tau; \vartheta),
\end{equation}
where the operators \( \Xi(\tau; \vartheta) \) and \( F_1(\vartheta) \) are defined by (4.7) and (1.41), respectively, and \( K(\vartheta) \) is defined as in (3.18), (3.21), and (3.30)–(3.34). The reminder term \( J(\tau; \vartheta) = B_\lambda(\tau; \vartheta)^{-1}F(\tau; \vartheta)^{1/2} + \tilde{G}(\tau; \vartheta) \) satisfies
\[ \|J(\tau; \vartheta)\|_{\mathcal{B}_0} \leq C, \quad |\tau| \leq \tau_0, \]
the number \( \tau_0 \) is subject to the restriction (1.15), and the constant \( C \) is given by
\begin{equation}
C = (3\delta)^{-1} + (C_1 + C_3(\tilde{c}_*)^{-1})(2C_1(\tilde{c}_*)^{-1} + C_3(\tilde{c}_*)^{-2}) + 2C_2(\tilde{c}_*)^{-1} + C_4(\tilde{c}_*)^{-2}.
\end{equation}
The constants \( C_j, j = 1, 2, 3, 4, \) and \( \tilde{c}_* \) are defined as in (3.4), (3.7), (3.11), (3.16), and (4.2), respectively.

We transform formula (4.32). Recalling expression (1.41) for the operator \( F_1(\vartheta) \) and the relations \( Z^*P = 0 \) and \( \tilde{Z}^*P = 0, \) we obtain
\begin{equation}
F_1(\vartheta)\Xi(\tau; \vartheta) = (\vartheta_1Z + \vartheta_2\tilde{Z})\Xi(\tau; \vartheta).
\end{equation}
Next, by (3.18) and (3.22), we have \( PK(\vartheta)P = PN(\vartheta)P, \) whence
\begin{equation}
\Xi(\tau; \vartheta)K(\vartheta)\Xi(\tau; \vartheta) = \Xi(\tau; \vartheta)N(\vartheta)\Xi(\tau; \vartheta).
\end{equation}
By (4.34) and (4.35), the representation (4.32) takes the form
\begin{equation}
B_\lambda(\tau; \vartheta)^{-1} = \Xi(\tau; \vartheta) + \tau(\vartheta_1Z + \vartheta_2\tilde{Z})\Xi(\tau; \vartheta) + \tau\Xi(\tau; \vartheta)(\vartheta_1Z^* + \vartheta_2\tilde{Z}^*)
- \tau^3\Xi(\tau; \vartheta)N(\vartheta)\Xi(\tau; \vartheta) + J(\tau; \vartheta).
\end{equation}

4.3. Now we return to the initial parameters \( t \) and \( \varepsilon, \) recalling that \( t = \tau\vartheta_1, \varepsilon = \tau\vartheta_2. \) By (1.34), the operator \( \tau^2S(\vartheta) = L(t, \varepsilon) \) is represented as
\begin{equation}
L(t, \varepsilon) = t^2S + t\varepsilon(-X_0Z^*X_0\tilde{Z} - (X_0\tilde{Z})^*X_0Z + P(Y_2^*Y_1 + Y_1^*Y_2))|_{\mathfrak{R}}
+ \varepsilon^2(-X_0\tilde{Z}^*X_0\tilde{Z})|_{\mathfrak{R}} + Q_{\mathfrak{R}} + \lambda Q_{\mathfrak{R}})
\end{equation}
The operator (4.7) can be written as \( L(t, \varepsilon)^{-1}P. \) Note that, by (4.4),
\begin{equation}
L(t, \varepsilon) \geq \tilde{c}_*(t^2 + \varepsilon^2)I_{\mathfrak{R}}, \quad t \in \mathbb{R}, \quad 0 \leq \varepsilon \leq 1.
\end{equation}
By (3.30), the operator \( \tau^3N(\vartheta) = N(t, \varepsilon) \) has the following structure:
\begin{equation}
N(t, \varepsilon) = t^3N_{11} + t^2\varepsilon N_{12} + t\varepsilon^2 N_{21} + \varepsilon^3 N_{22}.
\end{equation}
From (3.50) it follows that
\begin{equation}
\|N(t,\varepsilon)\|_{\mathcal{D}\to\mathcal{D}} \leq C_0(t^2 + \varepsilon^2)^{3/2}, \quad t \in \mathbb{R}, \quad 0 \leq \varepsilon \leq 1.
\end{equation}
Writing (4.36) in terms of the initial parameters, we give an equivalent formulation of Theorem 4.2, which is convenient for further applications to differential operators. Consider the operator
\begin{equation}
K(t,\varepsilon) := (tZ + \varepsilon \bar{Z})L(t,\varepsilon)^{-1}P + L(t,\varepsilon)^{-1}P(tZ^* + \varepsilon \bar{Z}^*)
- L(t,\varepsilon)^{-1}N(t,\varepsilon)L(t,\varepsilon)^{-1}P,
\end{equation}
called the corrector. The next estimate follows from (1.21), (1.25), (4.38), and (4.40):
\begin{equation}
\|K(t,\varepsilon)\|_{\mathcal{D}\to\mathcal{D}} \leq C_7(t^2 + \varepsilon^2)^{-1/2}, \quad t \in \mathbb{R}, \quad 0 \leq \varepsilon \leq 1,
\end{equation}
\begin{equation}
C_7 = C_6(\bar{\varepsilon}_s)^{-2} + 2(\bar{\varepsilon}_s)^{-1}\kappa^{1/2}(13\delta)^{-1/2}(\|X_1\| + c_1C(1)^{1/2}),
\end{equation}
where the constant $C_6$ is defined by (3.50).

**Theorem 4.3.** Under the assumptions of Subsections 1.1–1.4, let $B_\lambda(t,\varepsilon)$ be the operator defined in Subsection 1.4. Suppose that condition (4.1) is satisfied. Then
\begin{equation}
B_\lambda(t,\varepsilon)^{-1} = L(t,\varepsilon)^{-1}P + K(t,\varepsilon) + J(t,\varepsilon).
\end{equation}
Here $P$ is the orthogonal projection onto the subspace $\mathcal{H}$; the operator $L(t,\varepsilon)$ acting in $\mathcal{H}$ is given by (4.37). The corrector $K(t,\varepsilon)$ is defined by (4.41), where $N(t,\varepsilon)$ is defined in accordance with (3.39) and (3.40); the operators $Z, \bar{Z}, R, S$ are as defined in Subsections 1.6 and 1.7. The operators $L(t,\varepsilon)$ and $K(t,\varepsilon)$ are subject to estimates (4.38) and (4.42), respectively. The reminder term $J(t,\varepsilon)$ satisfies
\begin{equation}
\|J(t,\varepsilon)\|_{\mathcal{D}\to\mathcal{D}} \leq C, \quad t^2 + \varepsilon^2 \leq \tau_0^2,
\end{equation}
the number $\tau_0$ is subject to the restriction (1.15), and the constant $C$ is given by (4.33).

**Remark 4.4.** Some cumbersome explicit expressions for the constants $C$ and $C_6$, $C_7$ can be derived from relations (2.12), (2.13), (2.15), (3.4), (3.7), (3.11), (3.16), (4.33), and relations (3.36), (3.45), (3.47), (3.49), (3.50), (4.43), respectively. The character of dependence of these constants on the initial problem data is important for us. It can be seen that the constant $C$ can be replaced by a greater constant that is a polynomial in the variables $\delta, \delta^{-1}, (\bar{\varepsilon}_s)^{-1}, \kappa^{-1/2}, \|X_1\|, \|Y_1\|, c_1, C(1), c_2, c_3, |\lambda||Q_0||$, and $\tau_0$. The constant $\bar{\varepsilon}_s$ is defined by (4.2), and the number $\tau_0$ is fixed in accordance with (1.15). Similarly, the constant $C_6$ can be replaced by a greater constant that is a polynomial in the variables $\delta^{-1/2}, \kappa^{1/2}, \|X_1\|, \|Y_1\|, c_1, C(1)^{1/2}, c_2^{1/2}, c_3^{1/2},$ and $|\lambda||Q_0||$; the constant $C_7$ can be replaced by a greater constant that is a polynomial in the same variables and $(\bar{\varepsilon}_s)^{-1}$. The coefficients of the polynomials mentioned above are some positive universal constants.

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