Алгебра и анализ
Том 26 (2014), № 1

St. Petersburg Math. J.
Vol. 26 (2015), No. 1, Pages 1–25
S 1061-0022(2014)01328-3
Article electronically published on November 21, 2014

HOCHSCHILD COHOMOLOGY RING OF THE MODULAR GROUP

A. P. ALEKHIN, YU. V. VOLKOV, AND A. I. GENERALOV

Abstract. A description in terms of generators and relations is given for the cohomology ring and the Hochschild cohomology ring of the group algebra for the even modular group over the ring of integers. The free resolution of the trivial module described by Wall is used for that. Moreover, the bimodule resolution of the group algebra in question is described.

§1. Introduction

Through the past years, the interest has grown to the investigation of the structure of the Hochschild cohomology algebra. There are many results on calculation of that algebra for finite-dimensional algebras over a field. Nevertheless, not many results are known for group algebras over the ring of integers.

In the papers [1] and [2], the Hochschild cohomology algebra was calculated for the group ring $\mathbb{Z}[D_{4m}]$ and its additive structure was described for $\mathbb{Z}[SD_{2^k}]$. In those papers, the bimodule resolution of the corresponding algebra was used as the main tool.

In his papers [3, 4, 5, 6], Hayami described the Hochschild cohomology ring for integer group rings of generalized quaternion groups, of dihedral groups, and of semidihedral groups. He used the results of [7] in his calculations.

The main result of the present paper is the description of the Hochschild cohomology algebra of the integer group ring of the even modular group M_{2^ℓ}. The key construction is the free resolution described by Wall in [8]. Using this resolution and results of [9], we immediately obtain the bimodule resolution of the group algebra in question. Nonetheless, a technique similar to that of [7] allows us to use only the free resolution of the trivial module $\mathbb{Z}$ in the calculation of the Hochschild cohomology algebra.

It should be noted that the results of calculation of the algebra $\mathrm{HH}^*(\mathbb{Z}[M_{2^\ell}])$ show that this algebra is commutative.

From calculations presented in the paper, we also derive a description of the usual cohomology algebra of the even modular group. Though the cohomology algebra for the modular groups of odd order was calculated in the works [10] by Leary and [11] by Thomas, the authors were not able to find a publication containing a description of the cohomology algebra of the modular groups of even order.

§2. Free resolution

In this paper, we consider only right modules. Moreover, we write simply $\otimes$ instead of $\otimes_{\mathbb{Z}}$. Fixing a natural number $\ell \geq 4$, we introduce the notation

$$r = 2^{\ell-2}, \quad t = r + 1.$$

2010 *Mathematics Subject Classification.* Primary 16E40.
Key words and phrases. Hochschild cohomology, group algebras, modular 2-groups.

The modular group of order $4r$ can be presented as a semidirect product of two cyclic groups:

$$G = M_{2^\ell} = C_2 \leftthreetimes C_{2r} = \langle a, b \mid a^2 = b^{2r} = 1,\ aba = b^t \rangle.$$

We consider an integer group algebra $R = \mathbb{Z}[G]$ of the group G.

Construction of the resolution is based on the results presented in [8] and employs the following notation:

$$N_b = \sum_{i=0}^{2r-1} b^i, \quad L = \sum_{i=0}^{r} b^i, \quad x_k = \frac{t^k+1}{2}, \quad y_k = \frac{t^k-1}{r}.$$

We build the following diagram:

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
A_{0,4} \arrow[u] & A_{1,4} \arrow[u] \arrow[l] & A_{2,4} \arrow[u] \arrow[l] & A_{3,4} \arrow[u] \arrow[l] & A_{4,4} \arrow[u] \arrow[l] & \cdots \arrow[l] \\
A_{0,3} \arrow[u] & A_{1,3} \arrow[u] \arrow[l] \arrow[dr, dotted] & A_{2,3} \arrow[u] \arrow[l] & A_{3,3} \arrow[u] \arrow[l] \arrow[dr, dotted] & A_{4,3} \arrow[u] \arrow[l] & \cdots \arrow[l] \\
A_{0,2} \arrow[u] & A_{1,2} \arrow[u] \arrow[l] \arrow[dr, dotted] & A_{2,2} \arrow[u] \arrow[l] \arrow[dr, dotted] & A_{3,2} \arrow[u] \arrow[l] \arrow[dr, dotted] & A_{4,2} \arrow[u] \arrow[l] \arrow[dr, dotted] & \cdots \arrow[l] \\
A_{0,1} \arrow[u] & A_{1,1} \arrow[u] \arrow[l] \arrow[dr, dotted] & A_{2,1} \arrow[u] \arrow[l] \arrow[dr, dotted] & A_{3,1} \arrow[u] \arrow[l] \arrow[dr, dotted] & A_{4,1} \arrow[u] \arrow[l] \arrow[dr, dotted] & \cdots \arrow[l] \\
A_{0,0} \arrow[u] & A_{1,0} \arrow[u] \arrow[l] & A_{2,0} \arrow[u] \arrow[l] & A_{3,0} \arrow[u] \arrow[l] & A_{4,0} \arrow[u] \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Here, any vertex $A_{i,j}$ is simply the group algebra R. The additional numbering is needed for defining the arrows, namely, each arrow represents multiplication (from the left) by the corresponding element of R:

$$A_{2m+1,n} \xrightarrow{(b-1)\cdot} A_{2m,n} \xrightarrow{N_b\cdot} A_{2m-1,n},$$

$$A_{2m,2n+1} \overset{(L^m a-1)\cdot}{\dashrightarrow} A_{2m,2n} \overset{(L^m a+1)\cdot}{\dashrightarrow} A_{2m,2n-1},$$

$$A_{2m-1,2n+1} \overset{(-L^m a+1)\cdot}{\dashrightarrow} A_{2m-1,2n} \overset{(-L^m a-1)\cdot}{\dashrightarrow} A_{2m-1,2n-1},$$

$$A_{2m-1,n} \overset{-x_m y_m\cdot}{\dotrightarrow} A_{2m,n-2}\ .$$

By a process similar to the construction of a total complex of a bicomplex, we use the above diagram to construct the following complex:

$$0 \longleftarrow \mathbb{Z} \xleftarrow{\varepsilon} Q_0 \xleftarrow{d_0} Q_1 \xleftarrow{d_1} \cdots \longleftarrow Q_i \xleftarrow{d_i} Q_{i+1} \longleftarrow \cdots, \tag{2.1}$$

where $Q_i = R^{i+1}$, ε is the augmentation map ($\varepsilon(1 \cdot g) = 1$ for any $g \in G$), and the differentials are represented by three-diagonal matrices of the following form:

$$d_0 = (a-1, b-1),$$

$$d_{2n} = \begin{pmatrix} a-1 & b-1 & & & & & \\ & -La-1 & N_b & & & \mathbb{O} & \\ & -x_1y_1 & La-1 & b-1 & & & \\ & & & \ddots & & & \\ & & & & b-1 & & \\ & \mathbb{O} & & & -L^na-1 & N_b & \\ & & & & -x_ny_n & L^na-1 & b-1 \end{pmatrix}, \quad n > 0,$$

$$d_{2n+1} = \begin{pmatrix} a+1 & b-1 & & & & & \\ & -La+1 & N_b & & & \mathbb{O} & \\ & -x_1y_1 & La+1 & b-1 & & & \\ & & & \ddots & & & \\ & & & & b-1 & & \\ & \mathbb{O} & & & -L^na+1 & N_b & \\ & & & & -x_ny_n & L^na+1 & b-1 \\ & & & & & & -L^{n+1}a+1 \quad N_b \end{pmatrix}, \ n \geq 0.$$

Proposition 1. *$Q_\bullet = (Q_n, d_n)$ is a free resolution of the trivial R-module $\mathbb{Z}$.*

Proof. This follows from [8, Theorem 1]. □

We denote by $\Lambda = R^{\mathrm{op}} \otimes R$ the envelope algebra of the algebra R. Let $\Delta : R \to \Lambda$ be an algebra homomorphism defined on the elements of G by the formula $\Delta(g) = g^{-1} \otimes g$. For brevity, we denote the image of $x \in R$ under the map Δ by $\bar{x} \in \Lambda$. Then we build the complex

$$0 \longleftarrow R \xleftarrow{\nu} T_0 \xleftarrow{\bar{d}_0} T_1 \xleftarrow{\bar{d}_1} \cdots \longleftarrow T_i \xleftarrow{\bar{d}_i} T_{i+1} \longleftarrow \cdots,$$

where $T_i = \Lambda^{i+1}$, ν is the multiplication map ($\nu(x \otimes y) = xy$ for any $x, y \in R$), and the differentials are three-diagonal matrices of the following form:

$$\bar{d}_0 = \left(\overline{a-1} \ \ \overline{b-1}\right),$$

$$\bar{d}_{2n} = \begin{pmatrix} \overline{a-1} & \overline{b-1} & & & & & \\ & \overline{-La-1} & \overline{N_b} & & & \mathbb{O} & \\ & \overline{-x_1y_1} & \overline{La-1} & \overline{b-1} & & & \\ & & & \ddots & & & \\ & & & & \overline{b-1} & & \\ & \mathbb{O} & & & \overline{-L^na-1} & \overline{N_b} & \\ & & & & \overline{-x_ny_n} & \overline{L^na-1} & \overline{b-1} \end{pmatrix},$$

$$\bar{d}_{2n+1} = \begin{pmatrix} \overline{a+1} & \begin{matrix}\overline{b-1}\\ \overline{-La+1}\end{matrix} & & & & & \\ & \begin{matrix}\overline{N_b}\\ \overline{-x_1y_1}\end{matrix} & \begin{matrix}\overline{b-1}\\ \overline{La+1}\end{matrix} & & & \mathbb{O} & \\ & & \ddots & & & & \\ & & & \begin{matrix}\overline{b-1}\\ \overline{-L^na+1}\end{matrix} & & & \\ & \mathbb{O} & & \begin{matrix}\overline{N_b}\\ \overline{-x_ny_n}\end{matrix} & \begin{matrix}\overline{b-1}\\ \overline{L^na+1}\end{matrix} & & \\ & & & & \begin{matrix}\overline{b-1}\\ \overline{-L^{n+1}a+1}\end{matrix} & \overline{N_b} \end{pmatrix}.$$

Corollary 1. *$T_\bullet = (T_n, \bar{d}_n)$ is a free resolution of the Λ-module R.*

Proof. This follows from Proposition 1 and [9, Theorem 2]. □

§3. Additive structure

We denote by e_i^n $(1 \le i \le n+1)$ the element of the module $Q_n = R^{n+1}$ that corresponds to the column with 1 at the ith position and with 0 at the other positions. In this section, we compute the groups $\mathrm{HH}^s(R) = \mathrm{Ext}^s_\Lambda(R, R)$.

Theorem 1. *For all $s \ge 0$, we have*

$$\mathrm{HH}^s(R) = \begin{cases} \mathbb{Z}^{5r/2} & \text{if } s = 0,\\ \mathbb{Z}_2^{2rk} \oplus \mathbb{Z}_r^{r/2} \oplus \mathbb{Z}_{2r}^{2r} & \text{if } s = 4k > 0,\\ \mathbb{Z}_2^{2rk} & \text{if } s = 4k+1,\\ \mathbb{Z}_2^{(4k+3)r/2} \oplus \mathbb{Z}_r^{3r/2} \oplus \mathbb{Z}_{2r}^{r} & \text{if } s = 4k+2,\\ \mathbb{Z}_2^{(4k+1)r/2} & \text{if } s = 4k+3. \end{cases}$$

We divide the proof of this theorem into several lemmas. First, we prove a statement having more general nature. Let G be an arbitrary finite group, K a commutative ring with unit, $S = KG$ the corresponding group algebra, and G^G a set of representatives of the conjugacy classes of G. For $g \in G$, let $C(g)$ be the centralizer of g in G, and $O(g)$ the conjugacy class of g. We denote by $\widetilde{S}$ an S-module that is isomorphic to S as a K-module and on which the group G acts by conjugation (i.e., $a * g = g^{-1}ag$ for any $a \in \widetilde{S}$, and $g \in G$). Let $M^{(g)}$ denote the submodule in $\widetilde{S}$ generated by the set $O(g)$.

Lemma 1. *Put $S = KG$ and fix $g \in G$. For any $s \ge 0$, we have*

$$\mathrm{H}^s(G, M^{(g)}) \simeq \mathrm{H}^s(C(g), K).$$

Proof. Put $H = C(g)$. Assume that $K \xrightarrow{\varepsilon} Q^\bullet$ is a KH-injective resolution of the trivial H-module K. Then

$$\mathrm{Hom}_{KH}(KG, K) \xrightarrow{\varepsilon_*} \mathrm{Hom}_{KH}(KG, Q^\bullet)$$

is an injective resolution of the KG-module $\mathrm{Hom}_{KH}(KG, K) \simeq M^{(g)}$. Hence,

$$\begin{aligned} \mathrm{H}^s(H, K) &= \mathrm{Ext}^s_{KH}(K, K) = \mathrm{H}^s(\mathrm{Hom}_{KH}(K \otimes_{KG} KG, Q^\bullet))\\ &\simeq \mathrm{H}^s(\mathrm{Hom}_{KG}(K, \mathrm{Hom}_{KH}(KG, Q^\bullet))) = \mathrm{H}^s(G, M^{(g)}). \end{aligned}$$ □

Corollary 2 ([12, Theorem 2.11.2])**.** *Put $S = KG$. For any $s \ge 0$, we have*

$$\mathrm{HH}^s(S) \simeq \bigoplus_{g \in G^G} \mathrm{H}^s(C(g), K). \tag{3.1}$$

Proof. It is well known (see [7], [9, Corollary 3], [13, §5], and [14, Chapter X, Theorem 5.5]) that $\mathrm{HH}^s(S) \simeq \mathrm{H}^s(G, \widetilde{S})$. It is easily seen that $\widetilde{S} \simeq \bigoplus_{g \in G^G} M^{(g)}$, and the claim follows. □

In the sequel, we put $G = M_{2^\ell}$, $R = \mathbb{Z}G$. Let

$$\tau \,:\, \mathrm{HH}^*(R) \xrightarrow{\simeq} \mathrm{H}^*(G, \widetilde{R}) \tag{3.2}$$

denote the isomorphism of Abelian groups considered in Corollary 2.

Corollary 3.

$$\begin{aligned}\mathrm{HH}^s(R) \simeq \big(\mathrm{H}^s(G,\mathbb{Z})\big)^r \oplus \big(\mathrm{H}^s(G, M^{(a)})\big)^{r/2} \\ \oplus \big(\mathrm{H}^s(G, M^{(b)})\big)^{r/2} \oplus \big(\mathrm{H}^s(G, M^{(ab)})\big)^{r/2}.\end{aligned} \tag{3.3}$$

Proof. This follows immediately from the description of conjugacy classes of the group G:

$$\begin{aligned} O(b^{2i}) &= \{b^{2i}\} && \text{for } \ 0 \le i \le r-1, \\ O(b^{2i+1}) &= \{b^{2i+1}, b^{2i+t}\} && \text{for } \ 0 \le i \le \frac{r}{2}-1, \\ O(ab^{2i}) &= \{ab^{2i}, ab^{2i+r}\} && \text{for } \ 0 \le i \le \frac{r}{2}-1, \\ O(ab^{2i+1}) &= \{ab^{2i+1}, ab^{2i+t}\} && \text{for } \ 0 \le i \le \frac{r}{2}-1, \end{aligned}$$

and, as a consequence, from the isomorphisms

$$M^{(ab^{2i})} \simeq M^{(a)}, \quad M^{(b^{2i+1})} \simeq M^{(b)}, \quad M^{(ab^{2i+1})} \simeq M^{(ab)}. \qquad \square$$

Remark 1. In the subsequent calculations it will be convenient to use the decomposition of the form (3.3) instead of (3.1). This is explained by our desire to obtain representatives of cohomology classes in terms of our resolution (2.1).

We introduce some additional notation. For the modules $M^{(a)}$, $M^{(b)}$, $M^{(ab)}$, we fix the following $\mathbb{Z}$-bases:

$$M^{(a)} = \langle a + ab^r, a\rangle, \quad M^{(b)} = \langle b + b^t, b\rangle, \quad M^{(ab)} = \langle a(b+b^t), ab\rangle. \tag{3.4}$$

Then the action of the group G on these bases is defined, respectively, by homomorphisms $\chi_a, \chi_b, \chi_{ab} \,:\, G \to \mathrm{GL}(2,\mathbb{Z})$, where

$$\chi_a(b) = \chi_b(a) = \chi_{ab}(a) = \chi_{ab}(b) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad \chi_a(a) = \chi_b(b) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We denote by χ_a, χ_b, and χ_{ab} not only the corresponding group homomorphisms, but also the induced maps from R into $\mathrm{M}(2,\mathbb{Z})$.

We denote by $\mathbb{O}^n_m$ (respectively, by $\mathbb{O}^n$ and $\mathbb{O}_m$) the zero matrix of size $n \times m$ (respectively, the zero column of height n, and the zero row of length m). We consider elements of the free module $\mathbb{Z}^n$ as columns of height n. Furthermore, if v is a row of length m, then we denote by $[v]^n_i$ $(1 \le i \le n-m+1)$ the vector $(\mathbb{O}_{i-1}, v, \mathbb{O}_{n-m-i+1})^{\mathrm{T}} \in \mathbb{Z}^n$.

If N is an R-module, we put

$$\delta^s_N = \mathrm{Hom}_R(d_s, N) \,:\, \mathrm{Hom}_R(Q_s, N) \to \mathrm{Hom}_R(Q_{s+1}, N),$$

where d_s is the differential in the resolution (2.1). Moreover, we often identify notationally a cocycle $f \in \mathrm{Ker}\,\delta^s_N$ and its cohomology class in $\mathrm{H}^s(G,N)$.

Lemma 2 (Cohomology groups)**.** *For any $s \ge 0$,*

$$\mathrm{H}^s(G,\mathbb{Z}) \simeq \begin{cases} \mathbb{Z}, & \text{if } \ s = 0, \\ \mathbb{Z}_2^k \oplus \mathbb{Z}_{2r} & \text{if } \ s = 4k > 0, \\ \mathbb{Z}_2^k & \text{if } \ s = 4k+1, \\ \mathbb{Z}_2^{k+1} \oplus \mathbb{Z}_r & \text{if } \ s = 4k+2, \\ \mathbb{Z}_2^k & \text{if } \ s = 4k+3. \end{cases}$$

Proof. It is clear that $\mathrm{Hom}_R(Q_i,\mathbb{Z})\simeq\mathbb{Z}^{i+1}$, and we may assume, modulo this isomorphism, that

$$\delta^0_{\mathbb{Z}}=\mathbb{O}^2,\quad \delta^{2n}_{\mathbb{Z}}=\begin{pmatrix}0 & & & \mathbb{O}\\ & B_1 & & \\ & & \ddots & \\ & & & B_n\\ \mathbb{O} & & & \mathbb{O}_2\end{pmatrix},\quad \delta^{2n+1}_{\mathbb{Z}}=\begin{pmatrix}2 & & & & & \mathbb{O}\\ & C_1 & & & & \\ & & \ddots & & & \\ & & & C_n & & \\ & & & & -ry_{n+1} & \\ \mathbb{O} & & & & & 2r\end{pmatrix},$$

where $B_k=\begin{pmatrix}-2x_k & -x_ky_k\\ 2r & ry_k\end{pmatrix}$ and $C_k=\begin{pmatrix}-ry_k & -x_ky_k\\ 2r & 2x_k\end{pmatrix}$ (we use the same notation for homomorphisms $\mathbb{Z}^2\to\mathbb{Z}^2$ defined by these matrices). It is easily seen that

$$\begin{aligned}\mathrm{Ker}\,B_{2k-1}&=\left\langle\begin{pmatrix}-y_{2k-1}\\ 2\end{pmatrix}\right\rangle,\quad \mathrm{Ker}\,B_{2k}=\left\langle\begin{pmatrix}-x_ky_k\\ 1\end{pmatrix}\right\rangle,\\ \mathrm{Im}\,B_{2k-1}&=\left\langle\begin{pmatrix}-x_{2k-1}\\ r\end{pmatrix}\right\rangle,\quad \mathrm{Im}\,B_{2k}=\left\langle\begin{pmatrix}-2x_{2k}\\ 2r\end{pmatrix}\right\rangle,\\ \mathrm{Ker}\,C_k&=\left\langle\begin{pmatrix}-x_k\\ r\end{pmatrix}\right\rangle,\quad \mathrm{Im}\,C_k=\left\langle\begin{pmatrix}-y_k\\ 2\end{pmatrix}\right\rangle.\end{aligned}$$

We denote

$$f_{2n+1,k}=[-x_k,r]^{2n+2}_{2k},\quad f_{2n,k}=\begin{cases}[1]^{2n+1}_1 & \text{if } k=0,\\ [-\frac{y_k}{2},1]^{2n+1}_{2k} & \text{if } k\,\vdots\,2,\ k>0,\\ [-y_k,2]^{2n+1}_{2k} & \text{if } k\not\vdots\,2.\end{cases}\tag{3.5}$$

Then

$$\begin{aligned}\mathrm{Ker}\,\delta^{2n}_{\mathbb{Z}}&=\langle\{f_{2n,k}\}_{0\le k\le n}\rangle,\ \mathrm{Im}\,\delta^{2n}_{\mathbb{Z}}=\langle\{q_kf_{2n+1,k}\}_{1\le k\le n}\rangle,\\ \mathrm{Ker}\,\delta^{2n+1}_{\mathbb{Z}}&=\langle\{f_{2n+1,k}\}_{1\le k\le n}\rangle,\\ \mathrm{Im}\,\delta^{2n+1}_{\mathbb{Z}}&=\langle\{q_kf_{2n+2,k}\}_{0\le k\le n},rq_{n+1}f_{2n+2,n+1}\rangle,\end{aligned}\tag{3.6}$$

where $q_k=2$ if $k\,\vdots\,2$, and $q_k=1$, if $k\not\vdots\,2$. Now, the claim is deduced easily. □

Lemma 3. *For any $s\ge 0$, we have*

$$\mathrm{H}^s(G,M^{(b)})\simeq\mathrm{H}^s(G,M^{(ab)})\simeq\begin{cases}\mathbb{Z} & \text{if } s=0,\\ \mathbb{Z}_{2r} & \text{if } s=2n>0,\\ 0 & \text{if } s=2n+1.\end{cases}\tag{3.7}$$

Furthermore, each of the groups $\mathrm{H}^0(G,M^{(b)})$ and $\mathrm{H}^0(G,M^{(ab)})$ is generated by the element of the form $[1]^2_1$, and for $n>0$, we have

$$\begin{aligned}\mathrm{H}^{2n}(G,M^{(b)})&=\langle[-x_ny_n,x_ny_n,1,ry_n]^{4n+2}_{4n-1}\rangle,\\ \mathrm{H}^{2n}(G,M^{(ab)})&=\langle[-y_n,1]^{4n+2}_{4n}\rangle.\end{aligned}$$

Proof. By Lemma 1, formula (3.7) follows from the fact that $C(g)\simeq\mathrm{C}_{2r}$ for $g\in\{b,ab\}$. The remaining part of the lemma will be discussed in detail only for the groups $\mathrm{H}^{2n}(G,M^{(ab)})$. For simplicity, we denote the module $M^{(ab)}$ by M.

It is easily seen that $\chi_{ab}(L^k a) = \begin{pmatrix} t^k & x_k \\ 0 & -1 \end{pmatrix}$ and $\chi_{ab}(N_b) = \begin{pmatrix} 2r & r \\ 0 & 0 \end{pmatrix}$. Clearly, we have $\mathrm{Hom}_R(Q_i, M) \simeq \mathbb{Z}^{2(i+1)}$, and, modulo this isomorphism, we may assume that

$$\delta_M^0 = \begin{pmatrix} 0 & 1 \\ 0 & -2 \\ 0 & 1 \\ 0 & -2 \end{pmatrix}, \quad \delta_M^{2n} = \begin{pmatrix} A & & & & & \mathbb{O} \\ B' & B_1 & & & & \\ & B & B_2 & & & \\ & & \ddots & \ddots & & \\ & & & B & B_n & \\ \mathbb{O} & & & & & B'' \end{pmatrix}, \quad \delta_M^{2n+1} = \begin{pmatrix} A' & & & & & \mathbb{O} \\ B' & C_1 & & & & \\ & B & C_2 & & & \\ & & \ddots & \ddots & & \\ & & & B & C_n & \\ \mathbb{O} & & & & B & C'_{n+1} \end{pmatrix},$$

where $A = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix}$, $A' = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} \mathbb{O}_2^2 & A \\ \mathbb{O}_2^2 & \mathbb{O}_2^2 \end{pmatrix}$, $B' = \begin{pmatrix} A \\ \mathbb{O}_2^2 \end{pmatrix}$, $B'' = \begin{pmatrix} \mathbb{O}_2^2 & A \end{pmatrix}$,

$$B_k = \begin{pmatrix} -2x_k & -x_k & -x_k y_k & 0 \\ 0 & 0 & 0 & -x_k y_k \\ 2r & r & r y_k & x_k \\ 0 & 0 & 0 & -2 \end{pmatrix}, \quad C_k = \begin{pmatrix} -r y_k & -x_k & -x_k y_k & 0 \\ 0 & 2 & 0 & -x_k y_k \\ 2r & r & 2x_k & x_k \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and C'_k is the matrix consisting of the first two columns of the matrix C_k.

We introduce the following notation:

$$f_{2n,k}^{(ab)} = \begin{cases} [1]_1^{4n+2} & \text{if } k = 0, \\ [-y_k, 1]_{4k}^{4n+2} & \text{if } k > 0, \end{cases} \quad (0 \le k \le n),$$
$$g_{2n,k}^{(ab)} = [1, -2]_{4k-1}^{4n+2} \qquad (1 \le k \le n).$$

It is clear that $\mathrm{Ker}\,\delta_M^0 = \langle f_{0,0}^{(ab)} \rangle$. Now we consider the case where $s = 2n > 0$. Assume that $v \in \mathrm{Ker}\,\delta_M^{2n} \subset \mathrm{Hom}_R(Q_{2n}, M) \simeq \mathbb{Z}^{4n+2}$. Considering the $(4m+2)$nd row of the matrix δ_M^{2n}, for any $0 \le m \le n$ we see that $(4m+2)$nd element of v is equal to 0. Then $v = [v_0]_1^{4n+2} + \sum_{k=1}^n [v_k]_{4k-1}^{4n+2}$, where $v_0 \in \mathbb{Z}$ and the v_k are rows with length 3 $(1 \le k \le n)$. Obviously, the condition $v \in \mathrm{Ker}\,\delta_M^{2n}$ is equivalent to $B_k(v_k, 0)^{\mathrm{T}} = 0$ for all $1 \le k \le n$. If we put $v_k = (u_{k,1}, u_{k,2}, u_{k,3})$, then, in its turn, the last identity (for k fixed) is equivalent to the relation $2u_{k,1} + u_{k,2} + y_k u_{k,3} = 0$. Consequently,

$$\mathrm{Ker}\,\delta_M^{2n} = \langle \{f_{2n,k}^{(ab)}\}_{0 \le k \le n}, \{g_{2n,k}^{(ab)}\}_{1 \le k \le n} \rangle,$$

and then

$$\mathrm{Ker}\,\delta_M^{2n} = \big\langle \{f_{2n,k}^{(ab)}\}_{0 \le k \le n-2}, \{g_{2n,k}^{(ab)}\}_{1 \le k \le n-1}, f_{2n,n-1}^{(ab)} + g_{2n,n}^{(ab)}, g_{2n,n}^{(ab)} - r f_{2n,n}^{(ab)}, f_{2n,n}^{(ab)} \big\rangle. \tag{3.8}$$

It is clear that the set in the angular brackets of the last formula is a set of free generators for $\mathrm{Ker}\,\delta_M^{2n}$. Hence, it suffices to prove that

$$\mathrm{Im}\,\delta_M^{2n-1} = \big\langle \{f_{2n,k}^{(ab)}\}_{0 \le k \le n-2}, \{g_{2n,k}^{(ab)}\}_{1 \le k \le n-1}, f_{2n,n-1}^{(ab)} + g_{2n,n}^{(ab)}, g_{2n,n}^{(ab)} - r f_{2n,n}^{(ab)}, 2r f_{2n,n}^{(ab)} \big\rangle. \tag{3.9}$$

First, we note that the columns of the matrix δ_M^{2n-1} correspond to the elements $2f_{2n,0}^{(ab)}$, $f_{2n,k}^{(ab)} + g_{2n,k+1}^{(ab)}$ $(0 \le k \le n-1)$, $2r f_{2n,k}^{(ab)} - r y_k g_{2n,k}^{(ab)}$, $r f_{2n,k}^{(ab)} - x_k g_{2n,k}^{(ab)}$ $(1 \le k \le n)$, and $2x_k f_{2n,k}^{(ab)} - x_k y_k g_{2n,k}^{(ab)}$ $(1 \le k \le n-1)$. Since $\mathrm{GCD}(x_k, r) = 1$, we see that $2f_{2n,k}^{(ab)} - y_k g_{2n,k}^{(ab)} \in \mathrm{Im}\,\delta_M^{2n-1}$ for $1 \le k \le n-1$. Hence, the element

$$g_{2n,k}^{(ab)} = \frac{r}{2}\big(2f_{2n,k}^{(ab)} - y_k g_{2n,k}^{(ab)}\big) - \big(r f_{2n,k}^{(ab)} - x_k g_{2n,k}^{(ab)}\big)$$

lies in the image of δ_M^{2n-1} for any $1 \le k \le n-1$. Consequently, for any $0 \le k \le n-2$ we have $f_{2n,k}^{(ab)} \in \operatorname{Im} \delta_M^{2n-1}$. Next, we observe that

$$g_{2n,n}^{(ab)} - r f_{2n,n}^{(ab)} = \frac{x_n+1}{2}\big(2r f_{2n,n}^{(ab)} - r y_n g_{2n,n}^{(ab)}\big) - \frac{r y_n + 2}{2}\big(r f_{2n,n}^{(ab)} - x_n g_{2n,n}^{(ab)}\big)$$

and

$$2r f_{2n,n}^{(ab)} = x_n\big(2r f_{2n,n}^{(ab)} - r y_n g_{2n,n}^{(ab)}\big) - r y_n\big(r f_{2n,n}^{(ab)} - x_n g_{2n,n}^{(ab)}\big).$$

Therefore, the set indicated on the right-hand side in (3.9) lies in the image of δ_M^{2n-1}. It remains to prove that this set generates that image. Let $T \subset \operatorname{Im} \delta_M^{2n-1}$ be a submodule generated by this set. Since

$$2f_{2n,n-1}^{(ab)} = 2\big(f_{2n,n-1}^{(ab)} + g_{2n,n}^{(ab)}\big) - 2\big(g_{2n,n}^{(ab)} - r f_{2n,n}^{(ab)}\big) - 2r f_{2n,n}^{(ab)} \in T,$$

we see that T contains the elements $2f_{2n,0}^{(ab)}$, $f_{2n,k}^{(ab)} + g_{2n,k+1}^{(ab)}$ $(0 \le k \le n-1)$, $2r f_{2n,k}^{(ab)} - r y_k g_{2n,k}^{(ab)}$, $r f_{2n,k}^{(ab)} - x_k g_{2n,k}^{(ab)}$, $2x_k f_{2n,k}^{(ab)} - x_k y_k g_{2n,k}^{(ab)}$ $(1 \le k \le n-1)$. It remains to prove that $2r f_{2n,n}^{(ab)} - r y_n g_{2n,n}^{(ab)}, r f_{2n,n}^{(ab)} - x_n g_{2n,n}^{(ab)} \in T$, but this is implied by the following facts: $2g_{2n,n}^{(ab)} = 2(g_{2n,n}^{(ab)} - r f_{2n,n}^{(ab)}) + 2f_{2n,n}^{(ab)} \in T$ and $x_n \not\vdots\, 2$.

The generators for the groups $\mathrm{H}^{2n}(G, M^{(b)})$ are obtained similarly. In this case, we use the following formulas for $\delta_{M^{(b)}}^i$:

$$\delta_{M^{(b)}}^0 = \begin{pmatrix} 0 & 1 \\ 0 & -2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \delta_{M^{(b)}}^{2n} = \begin{pmatrix} A & & & \mathbb{O} \\ & B_1 & & \\ & & \ddots & \\ & & & B_n \\ \mathbb{O} & & & \mathbb{O}_4^2 \end{pmatrix}, \quad \delta_{M^{(b)}}^{2n+1} = \begin{pmatrix} A' & & & & \mathbb{O} \\ & C_1 & & & \\ & & \ddots & & \\ & & & C_n & \\ \mathbb{O} & & & & C'_{n+1} \end{pmatrix},$$

where $A = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix}$, $A' = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$,

$$B_k = \begin{pmatrix} -2x_k & -t^k & -x_k y_k & 0 \\ 0 & r y_k & 0 & -x_k y_k \\ 2r & 0 & r y_k & t^k \\ 0 & 2r & 0 & -2x_k \end{pmatrix}, \quad C_k = \begin{pmatrix} -r y_k & -t^k & -x_k y_k & 0 \\ 0 & 2x_k & 0 & -x_k y_k \\ 2r & 0 & 2x_k & t^k \\ 0 & 2r & 0 & -r y_k \end{pmatrix},$$

and C'_k is the matrix consisting of the first two columns of the matrix C_k. □

Lemma 4. *For any $s \ge 0$, we have*

$$\mathrm{H}^s(G, M^{(a)}) \simeq \begin{cases} \mathbb{Z} & \text{if } s = 0, \\ \mathbb{Z}_2^n \oplus \mathbb{Z}_r & \text{if } s = 2n > 0, \\ \mathbb{Z}_2^n & \text{if } s = 2n+1. \end{cases} \tag{3.10}$$

Moreover, for the role of generating sets of the groups $\mathrm{H}^s(G, M^{(a)})$ we can take the following sets:

(a) $\mathrm{H}^0(G, M^{(a)})$ *is generated by the element* $[1]_1^2$;

(b) *if $s = 4\ell$ $(\ell > 0)$, then the group $\mathrm{H}^s(G, M^{(a)})$ is generated by the elements*

$$[1]_1^{8\ell+2}, \quad [1, 0-1, 1]_2^{8\ell+2}, \quad \left[\frac{y_{2i}}{2}, 0, -1\right]_{8i-1}^{8\ell+2} \qquad \text{for } 1 \le i \le \ell,$$

$$\left[\frac{-x_{2i}y_{2i}}{2}, \frac{r y_{2i}}{4}, 1, \frac{1-y_{2i+1}}{2}, -1, 1\right]_{8i}^{8\ell+2} \qquad \text{for } 1 \le i \le \ell-1,$$

where $\operatorname{ord}\big([\frac{y_{2\ell}}{2}, 0, -1]_{8\ell-1}^{8\ell+2}\big) = r$, *and the orders of the remaining elements are equal to* 2;

(c) *if* $s = 4\ell + 2$, *then the group* $\mathrm{H}^s(G, M^{(a)})$ *is generated by the elements*

$$[1]_1^{8\ell+6}, \quad [1, 0-1, 1]_2^{8\ell+6}, \quad \left[\frac{y_{2i}}{2}, 0, -1\right]_{8i-1}^{8\ell+6} \qquad \textit{for } 1 \le i \le \ell,$$

$$\left[\frac{-x_{2i}y_{2i}}{2}, \frac{ry_{2i}}{4}, 1, \frac{1-y_{2i+1}}{2}, -1, 1\right]_{8i}^{8\ell+6} \qquad \textit{for } 1 \le i \le \ell,$$

where $\mathrm{ord}\big([\frac{-x_{2\ell}y_{2\ell}}{2}, \frac{ry_{2\ell}}{4}, 1, \frac{1-y_{2\ell+1}}{2}, -1, 1]_{8\ell}^{8\ell+6}\big) = r$, *and the orders of the remaining elements are equal to* 2;

(d) *if* $s = 2n + 1$, *then the group* $\mathrm{H}^s(G, M^{(a)})$ *is generated by the*

$$\left[-x_k, \frac{r}{2}\right]_{4k}^{4n+4} \quad \textit{for } 1 \le k \le n,$$

where the orders of all elements are equal to 2.

Proof. By Lemma 1, formula (3.10) follows from the fact that $C(a) \simeq \mathrm{C}_r \times \mathrm{C}_2$. Next, it is easily seen that $\chi_a(L^k a) = \begin{pmatrix} t^k & \frac{ry_k}{2} \\ 0 & 1 \end{pmatrix}$, $\chi_a(N_b) = \begin{pmatrix} 2r & r \\ 0 & 0 \end{pmatrix}$. Clearly, we have $\mathrm{Hom}_R(Q_i, M^{(a)}) \simeq \mathbb{Z}^{2(i+1)}$; hence, modulo this isomorphism, we may assume that

$$\delta^0_{M^{(a)}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & -2 \end{pmatrix}, \quad \delta^{2n}_{M^{(a)}} = \begin{pmatrix} \mathbb{O}_2^2 & & & & & \mathbb{O} \\ B' & B_1 & & & & \\ & B & B_2 & & & \\ & & \ddots & \ddots & & \\ & & & & B & B_n \\ \mathbb{O} & & & & & B'' \end{pmatrix},$$

$$\delta^{2n+1}_{M^{(a)}} = \begin{pmatrix} A & & & & & \mathbb{O} \\ B' & C_1 & & & & \\ & B & C_2 & & & \\ & & \ddots & \ddots & & \\ & & & B & C_n & \\ \mathbb{O} & & & & B & C'_{n+1} \end{pmatrix},$$

where $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} \mathbb{O}_2^2 & X \\ \mathbb{O}_2^2 & \mathbb{O}_2^2 \end{pmatrix}$, $B' = \begin{pmatrix} X \\ \mathbb{O}_2^2 \end{pmatrix}$, $B'' = \begin{pmatrix} \mathbb{O}_2^2 & X \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix}$,

$$B_k = \begin{pmatrix} -2x_k & -\frac{ry_k}{2} & -x_ky_k & 0 \\ 0 & -2 & 0 & -x_ky_k \\ 2r & r & ry_k & \frac{ry_k}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C_k = \begin{pmatrix} -ry_k & -\frac{ry_k}{2} & -x_ky_k & 0 \\ 0 & 0 & 0 & -x_ky_k \\ 2r & r & 2x_k & \frac{ry_k}{2} \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

and C'_k is the matrix consisting of the first two columns of the matrix C_k.

It is clear that $\mathrm{H}^0(G, M^{(a)}) = \langle [1]_1^2 \rangle$. Now, we study the groups $\mathrm{H}^{2n}(G, M^{(a)})$, $n > 0$. For $0 \le k \le n$ we put

$$f^{(a)}_{2n,k} = \begin{cases} [1]_1^{4n+2} & \text{if } k = 0, \\ [\frac{y_k}{2}, 0, -1]_{4k-1}^{4n+2} & \text{if } k \,\vdots\, 2,\ k > 0, \\ [y_k, 0, -2]_{4k-1}^{4n+2} & \text{if } k \not\vdots\, 2, \end{cases}$$

and for $0 \le k \le n-1$ we put

$$g^{(a)}_{2n,k} = \begin{cases} [1, 0, -1, 1]_2^{4n+2} & \text{if } k = 0, \\ [-\frac{x_ky_k}{2}, \frac{ry_k}{4}, 1, \frac{1-y_{k+1}}{2}, -1, 1]_{4k}^{4n+2} & \text{if } k \,\vdots\, 2,\ k > 0, \\ [-x_ky_k, \frac{ry_k}{2}, 2, 1-y_{k+1}, -2, 2]_{4k}^{4n+2} & \text{if } k \not\vdots\, 2. \end{cases}$$

We prove that the set $\{f^{(a)}_{2n,k}\}_{0\le k\le n}\cup\{g^{(a)}_{2n,k}\}_{0\le k\le n-1}$ freely generates $\operatorname{Ker}\delta^{2n}_{M^{(a)}}$. It suffices to prove only that the indicated elements generate this module. It is easy to verify that the elements $f^{(a)}_{2n,k}$ for $0\le k\le n$ and $g^{(a)}_{2n,k}$ for $0\le k\le n-1$ lie in $\operatorname{Ker}\delta^{2n}_{M^{(a)}}$. Assume that $v\in\operatorname{Ker}\delta^{2n}_{M^{(a)}}$ and prove that $v=\sum_{k=0}^{n}\alpha_k f^{(a)}_{2n,k}+\sum_{k=0}^{n-1}\beta_k g^{(a)}_{2n,k}$ for some $\alpha_k,\beta_k\in\mathbb{Z}$. Obviously, we can choose α_0, β_0 such that the first two coordinates of the vector $v-\alpha_0 f^{(a)}_{2n,0}-\beta_0 g^{(a)}_{2n,0}$ are 0. We use induction on m to prove that for any $0\le m\le n-1$ we can choose numbers α_k, β_k $(0\le k\le m)$ such that the first $4m+2$ coordinates of the vector $v-\sum_{k=0}^{m}\alpha_k f^{(a)}_{2n,k}-\sum_{k=0}^{m}\beta_k g^{(a)}_{2n,k}$ are 0. For $m=0$ this statement has already been proved. Let $1\le m\le n-1$, and suppose that the statement in question is proved for $m-1$. There exist α_k, β_k $(0\le k\le m-1)$ such that the first $4m-2$ coordinates of the vector $w=v-\sum_{k=0}^{m-1}\alpha_k f^{(a)}_{2n,k}-\sum_{k=0}^{m-1}\beta_k g^{(a)}_{2n,k}$ are 0. Let w_i denote the $(4m-2+i)$th coordinate of the vector w $(1\le i\le 4)$. Then $2w_2+x_my_mw_4=0$ and $2rw_1+rw_2+ry_mw_3+\frac{ry_m}{2}w_4=0$. With the help of these identities, it is easy to show that there exist $\alpha_m,\beta_m\in\mathbb{Z}$ with

$$(w_1,w_2,w_3,w_4)=\begin{cases}\alpha_m(\frac{y_m}{2},0,-1,0)+\beta_m(0,-\frac{x_my_m}{2},\frac{ry_m}{4},1) & \text{if } m\,\vdots\,2,\\ \alpha_m(y_m,0,-2,0)+\beta_m(0,-x_my_m,\frac{ry_m}{2},2) & \text{if } m\not\vdots\,2.\end{cases}$$

Hence, the first $4m+2$ coordinates of the vector $v-\sum_{k=0}^{m}\alpha_k f^{(a)}_{2n,k}-\sum_{k=0}^{m}\beta_k g^{(a)}_{2n,k}$ are equal to 0. So, the induction step is verified. Consequently, we may assume that $v=[v_1,v_2,v_3,v_4]^{4n+2}_{4n-1}$ for some $v_1,v_2,v_3,v_4\in\mathbb{Z}$. Now, considering the rows of the matrix $\delta^{2n}_{M^{(a)}}$ with numbers $4n-2$, $4n-1$, and $4n+1$, we easily check that $v_2=v_4=0$, $2v_1+y_nv_3=0$, whence, $v=\alpha_n f^{(a)}_{2n,n}$ with $\alpha_n\in\mathbb{Z}$. Then we observe that the columns of the matrix $\delta^{2n-1}_{M^{(a)}}$ correspond to the elements $2f^{(a)}_{2n,0}$, $q_kg^{(a)}_{2n,k}+q_{k+1}f^{(a)}_{2n,k+1}$ $(0\le k\le n-1)$, $rq_kf^{(a)}_{2n,k}$, $\frac{r}{2}q_kf^{(a)}_{2n,k}$ $(1\le k\le n)$, and $x_kq_kf^{(a)}_{2n,k}$ $(1\le k\le n-1)$, where $q_k=2$ if $k\,\vdots\,2$, and $q_k=1$ if $k\not\vdots\,2$. Since $x_k\not\vdots\,2$, it follows that

$$\begin{aligned}\operatorname{Im}\delta^{2n-1}_{M^{(a)}}&=\big\langle\{q_kf^{(a)}_{2n,k}\}_{0\le k\le n-1},\{q_kg^{(a)}_{2n,k}+q_{k+1}f^{(a)}_{2n,k+1}\}_{0\le k\le n-1},\frac{r}{2}q_nf^{(a)}_{2n,n}\big\rangle\\&=\big\langle\{q_kf^{(a)}_{2n,k}\}_{0\le k\le n-1},\{q_kg^{(a)}_{2n,k}\}_{0\le k\le n-2},q_{n-1}g^{(a)}_{2n,n-1}+q_nf^{(a)}_{2n,n},\frac{r}{2}q_nf^{(a)}_{2n,n}\big\rangle\\&=\begin{cases}\big\langle\{q_kf^{(a)}_{2n,k}\}_{0\le k\le n-1},\{q_kg^{(a)}_{2n,k}\}_{0\le k\le n-2},\\ g^{(a)}_{2n,n-1}+2f^{(a)}_{2n,n},rf^{(a)}_{2n,n}\big\rangle & \text{if } n\,\vdots\,2,\\ \big\langle\{q_kf^{(a)}_{2n,k}\}_{0\le k\le n-1},\{q_kg^{(a)}_{2n,k}\}_{0\le k\le n-2},\\ 2g^{(a)}_{2n,n-1}+f^{(a)}_{2n,n},rg^{(a)}_{2n,n-1}\big\rangle & \text{if } n\not\vdots\,2.\end{cases}\end{aligned}$$

Observe that the sets

$$\{f^{(a)}_{2n,k}\}_{0\le k\le n-1}\cup\{g^{(a)}_{2n,k}\}_{0\le k\le n-2}\cup\{g^{(a)}_{2n,n-1}+2f^{(a)}_{2n,n},f^{(a)}_{2n,n}\},$$

and

$$\{f^{(a)}_{2n,k}\}_{0\le k\le n-1}\cup\{g^{(a)}_{2n,k}\}_{0\le k\le n-2}\cup\{2g^{(a)}_{2n,n-1}+f^{(a)}_{2n,n},g^{(a)}_{2n,n-1}\}$$

are free generating sets for $\operatorname{Ker}\delta^{2n}_{M^{(a)}}$. Hence, we arrive at the generating sets for the groups $\mathrm{H}^{2n}(G,M^{(a)})$ indicated in items (b) and (c).

Now we study the groups $\mathrm{H}^{2n+1}(G,M^{(a)})$. We put

$$\begin{aligned}f^{(a)}_{2n+1,k}&=\Big[-x_k,\frac{r}{2}\Big]^{4n+4}_{4k} && \text{for } 1\le k\le n,\\ g^{(a)}_{2n+1,k}&=[1,-2]^{4n+4}_{4k-1} && \text{for } 1\le k\le n+1.\end{aligned}$$

Assume that $v \in \operatorname{Ker}\delta^{2n+1}_{M^{(a)}} \subset \operatorname{Hom}_R(Q_{2n+1}, M^{(a)}) \simeq \mathbb{Z}^{4n+4}$. Considering the rows of the matrix $\delta^{2n+1}_{M^{(a)}}$ with numbers 1 and $4m+2$ for all $0 \le m \le n$, we see that the components of the vector v with there numbers are 0. Hence, $v = \sum_{k=1}^{n}[v_k]^{4n+2}_{4k-1} + [v_0]^{4n+4}_{4n+3}$, where the v_k are rows of length 3 ($1 \le k \le n$), and v_0 is a row of length 2. It is clear that the condition $v \in \operatorname{Ker}\delta^{2n+1}_{M^{(a)}}$ is equivalent to $C_{n+1}v_0^{\mathrm{T}} = 0$, $C_k(v_k, 0)^{\mathrm{T}} = 0$ for all $1 \le k \le n$. Then it is easy to check (cf. the proof of Lemma 3) that

$$\operatorname{Ker}\delta^{2n+1}_{M^{(a)}} = \big\langle \{g^{(a)}_{2n+1,k}\}_{1\le k\le n+1} \cup \{f^{(a)}_{2n+1,k}\}_{1\le k\le n}\big\rangle.$$

Clearly, the columns of the matrix $\delta^{2n}_{M^{(a)}}$ correspond to the elements

$$\left.\begin{aligned} &g^{(a)}_{2n+1,1},\ 4f^{(a)}_{2n+1,k} - 2x_k g^{(a)}_{2n+1,k},\ 2f^{(a)}_{2n+1,k} - \frac{ry_k}{2} g^{(a)}_{2n+1,k},\\ &2y_k f^{(a)}_{2n+1,k} - x_k y_k g^{(a)}_{2n+1,k},\ y_k f^{(a)}_{2n+1,k} + g^{(a)}_{2n+1,k+1},\quad \text{where}\ \ 1 \le k \le n. \end{aligned}\right\} \tag{3.11}$$

Put

$$X = \big\{g^{(a)}_{2n+1,k} + \kappa_{k-1} f^{(a)}_{2n+1,k-1})\big\}_{1\le k\le n+1} \cup \big\{2f^{(a)}_{2n+1,k}\big\}_{1\le k\le n},$$

where we assume that $f^{(a)}_{2n+1,0} = 0$, and κ_k is the remainder of division of k by 2. Since $\{g^{(a)}_{2n+1,k} + \kappa_{k-1} f^{(a)}_{2n+1,k-1}\}_{1\le k\le n+1} \cup \{f^{(a)}_{2n+1,k}\}_{1\le k\le n}$ is a free generating set for $\operatorname{Ker}\delta^{2n+1}_{M^{(a)}}$, it suffices to verify that X generates $\operatorname{Im}\delta^{2n}_{M^{(a)}}$. Put

$$\begin{aligned} X_m &= \big\{g^{(a)}_{2n+1,k} + \kappa_{k-1} f^{(a)}_{2n+1,k-1})\big\}_{1\le k\le m+1} \cup \big\{2f^{(a)}_{2n+1,k}\big\}_{1\le k\le m},\\ X'_m &= \bigcup_{k=m+1}^{n} \Big\{4f^{(a)}_{2n+1,k} - 2x_k g^{(a)}_{2n+1,k}, 2f^{(a)}_{2n+1,k} - \frac{ry_k}{2} g^{(a)}_{2n+1,k},\\ &\qquad\qquad 2y_k f^{(a)}_{2n+1,k} - x_k y_k g^{(a)}_{2n+1,k}, y_k f^{(a)}_{2n+1,k} + g^{(a)}_{2n+1,k+1}\Big\}. \end{aligned}$$

Let $T_m \subset \mathbb{Z}^{4n+4}$ denote the submodule generated by $X_m \cup X'_m$. We shall show that $T_m = \operatorname{Im}\delta^{2n}_{M^{(a)}}$ for any $0 \le m \le n$. Since $X_n = X$ and $X'_n = \varnothing$, the desired claim will then follow. We have proved that $T_0 = \operatorname{Im}\delta^{2n}_{M^{(a)}}$ (see (3.11)). We check that $T_m = T_{m-1}$ ($1 \le m \le n$). We need to show that $2f^{(a)}_{2n+1,m}, g^{(a)}_{2n+1,m+1} + \kappa_m f^{(a)}_{2n+1,m} \in T_{m-1}$, and

$$\begin{aligned} &4f^{(a)}_{2n+1,m} - 2x_m g^{(a)}_{2n+1,m}, 2f^{(a)}_{2n+1,m} - \frac{ry_m}{2} g^{(a)}_{2n+1,m},\\ &2y_m f^{(a)}_{2n+1,m} - x_m y_m g^{(a)}_{2n+1,m}, y_m f^{(a)}_{2n+1,m} + g^{(a)}_{2n+1,m+1} \in T_m, \end{aligned}$$

which is verified easily by using the fact that $2g^{(a)}_{2n+1,m} \in T_{m-1}$ and the fact that the numbers m, y_m have the same parity. □

Consequently, Corollary 3 to Lemma 1 and Lemmas 2–4 immediately imply Theorem 1.

§4. $\mathrm{HH}^*(R)$ and $\mathrm{H}^*(G, \widetilde{R})$

As above, $Q_\bullet \xrightarrow{\varepsilon} \mathbb{Z}$ denotes the free R-resolution described in §2. For R-modules N_1, N_2, we view $N_1 \otimes N_2$ as an R-module with the action of $g \in G$ determined by the rule $(n_1 \otimes n_2)g = (n_1 g) \otimes (n_2 g)$. As was mentioned above (see (3.2)), there is an isomorphism $\tau : \mathrm{HH}^*(R) \simeq \mathrm{H}^*(G, \widetilde{R})$. We can define the $\cup$-product on $\mathrm{HH}^*(R)$ and $\mathrm{H}^*(G, \widetilde{R})$. $\mathrm{HH}^*(R)$ is an algebra with respect to this product. The remaining part of the paper is devoted to the description of this algebra. For this, we need the following description of the $\cup$-product. Assume that $f \in \mathrm{H}^n(G, \widetilde{R})$ and $g \in \mathrm{H}^m(G, \widetilde{R})$. There exist homomorphisms of R-modules $\phi_i : Q_{m+i} \to Q_i \otimes \widetilde{R}$ ($i \ge 0$) such that $(\varepsilon \otimes \mathrm{id}_{\widetilde{R}})\phi_0 \in \operatorname{Ker}\delta^n_{\widetilde{R}}$, $(\varepsilon \otimes \mathrm{id}_{\widetilde{R}})\phi_0 = g$ in $\mathrm{H}^m(G, \widetilde{R})$, $(d_i \otimes \mathrm{id}_{\widetilde{R}})\phi_{i+1} = \phi_i d_{i+m}$ (here the

notation is as in (2.1)). We call ϕ_i an ith translate of g and denote it by $T^i(g)$. Since $\widetilde{R}$ coincides with R as a $\mathbb{Z}$-module, we can introduce a multiplication $\mu : \widetilde{R}\otimes\widetilde{R}\to\widetilde{R}$ on $\widetilde{R}$ that coincides with the multiplication on R. It is easily seen that μ is a homomorphism of R-modules. Then the product of elements $\tau^{-1}(f)$, $\tau^{-1}(g)$ can be calculated by the formula

$$\tau^{-1}(f)\cup\tau^{-1}(g)=\tau^{-1}\bigl(\mu\circ(f\cup g)\bigr)=\tau^{-1}\bigl(\mu\circ(f\otimes\mathrm{id}_{\widetilde{R}})\circ T^n(g)\bigr).$$

The first identity follows from [7, Proposition 3.1] and the second from [15, Chapter 5, Theorem 4.6]. We shall view $\mathrm{H}^*(G,\widetilde{R})$ as an algebra with respect to the multiplication obtained from the $\cup$-product on $\mathrm{HH}^*(R)$ via the isomorphism τ.

Denote by $\mathbb{Z}(i)$ $(0\le i\le r-1)$ the R-submodule of $\widetilde{R}$ generated by b^{2i}, and by $M^{(a)}(i)$, $M^{(b)}(i)$, and $M^{(ab)}(i)$ $\bigl(0\le i\le\frac{r}{2}-1\bigr)$ the submodules generated by ab^{2i}, b^{2i+1}, and ab^{2i+1}, respectively. Clearly, $\mathbb{Z}(i)\simeq\mathbb{Z}$, $M^{(a)}(i)\simeq M^{(a)}$, $M^{(b)}(i)\simeq M^{(b)}$, $M^{(ab)}(i)\simeq M^{(ab)}$. Furthermore, we fix the following $\mathbb{Z}$-bases for $M^{(a)}(i)$, $M^{(b)}(i)$, $M^{(ab)}(i)$:

$$M^{(a)}(i)=\langle a(b^{2i}+b^{2i+r}),ab^{2i}\rangle,\ M^{(b)}(i)=\langle b^{2i+1}+b^{2i+t},b^{2i+1}\rangle,$$
$$M^{(ab)}(i)=\langle a(b^{2i+1}+b^{2i+t}),ab^{2i+1}\rangle.$$

In this notation, the decomposition (3.3) is written in the form

$$\tilde{R}=\bigoplus_{i=0}^{r-1}\mathbb{Z}(i)\oplus\bigoplus_{i=0}^{\frac{r-2}{2}}M^{(a)}(i)\oplus\bigoplus_{i=0}^{\frac{r-2}{2}}M^{(b)}(i)\oplus\bigoplus_{i=0}^{\frac{r-2}{2}}M^{(ab)}(i). \tag{4.1}$$

If N is a summand in this decomposition, we write the elements of $\mathrm{Hom}_R(R,N)\simeq N$ with respect to the $\mathbb{Z}$-bases indicated above.

Moreover, if $f\in\mathrm{Ker}\,\delta^n_{\widetilde{R}}$ and $\mathrm{Im}\,f\subset N$, then we can construct $T^i(f)$ $(i\ge 0)$ so that $\mathrm{Im}\,T^i(f)\subset Q_i\otimes N$. In these cases, we shall construct $T^i(f)$ as maps from Q_{n+i} into $Q_i\otimes N$.

We pick the following elements of $\mathrm{H}^*(G,\widetilde{R})$:

- $\upsilon_1,\upsilon_2\in\mathrm{H}^2(G,\mathbb{Z}(0))$, $\upsilon_1=[1]^3_1$, $\upsilon_2=[-1,2]^3_2$,
- $\nu\in\mathrm{H}^4(G,\mathbb{Z}(0))$, $\nu=[-x_1,1]^5_4$,
- $\omega\in\mathrm{H}^5(G,\mathbb{Z}(0))$, $\omega=[-x_2,r]^6_4$,
- $\lambda\in\mathrm{H}^0(G,\mathbb{Z}(1))$, $\lambda=(1)$,
- $\theta_0\in\mathrm{H}^0(G,M^{(a)}(0))$, $\theta_0=[1]^2_1$,
- $\theta_1\in\mathrm{H}^2(G,M^{(a)}(0))$, $\theta_1=[1,0,-1,1]^6_2$,
- $\varphi_0\in\mathrm{H}^0(G,M^{(b)}(0))$, $\varphi_0=[1]^2_1$,
- $\varphi_1\in\mathrm{H}^2(G,M^{(b)}(0))$, $\varphi_1=[-x_1,x_1,1,r]^6_3$,
- $\rho_1\in\mathrm{H}^3(G,M^{(a)}(0))$, $\rho_1=\bigl[-x_1,\frac{r}{2}\bigr]^8_4$,
- $\rho_2\in\mathrm{H}^5(G,M^{(a)}(0))$, $\rho_2=\bigl[-x_2,\frac{r}{2}\bigr]^{12}_8$,
- $\psi_0\in\mathrm{H}^0(G,M^{(ab)}(0))$, $\psi_0=[1]^2_1$,
- $\psi_1\in\mathrm{H}^2(G,M^{(ab)}(0))$, $\psi_1=[-1,1]^6_4$.

The proofs of Lemmas 2–4 show that all these elements are indeed represented by cocycles of the corresponding degrees.

§5. Generators

In this section, we prove that the images under τ^{-1} (see (3.2)) of the elements presented in the final part of the previous section generate the algebra $\mathrm{HH}^*(R)$. To prove this, we need the following lemma.

Lemma 5. 1) *For the role of the translates of the element υ_1, we can take the maps given by the matrices*

$$T^k(\upsilon_1) = \begin{pmatrix} I_{k+1} & \mathbb{O}_2^{k+1} \end{pmatrix},$$

where I_{k+1} is the unit matrix of size $(k+1)\times(k+1)$.

2) *The translates of ν for $k \geq 1$ can be chosen so that they are given by the matrices of the form*

$$T^k(\nu) = \begin{pmatrix} & \mathbb{O}^{k-1} & \mathbb{O}^{k-1} \\ A'_k & aL^2a & 0 \\ & s_k & 1 \end{pmatrix},$$

where A'_k is a matrix of size $(k+1)\times(k+3)$, $s_k = -x_1$ if k is even, and $s_k = s$ if k is odd; here $s \in R$ is an element satisfying $(b-1)s = s(b-1) = aL^2a - 1 - x_1N_b$.

Proof. 1) It suffices to prove that $\varepsilon \circ T^0(\upsilon_1) = \upsilon_1$ and $d_i \circ T^{i+1}(\upsilon_1) = T^i(\upsilon_1) \circ d_{i+2}$ $(i \geq 0)$. This verification is not difficult, and we leave it to the reader.

2) First, we claim that there exists $s \in R$ for which we have $aL^2a - 1 - x_1N_b = (b-1)s = s(b-1)$. Indeed, this is derived from the identity $(aL^2a - 1 - x_1N_b)N_b = 0$ and the structure of the 2-periodic resolution for the group $\langle b \rangle$.

Clearly, we can take $T^0(\nu) = \begin{pmatrix} \mathbb{O}_3 & -x_1 & 1 \end{pmatrix}$. We must show that there exists a matrix with the last two columns equal to $\begin{pmatrix} aL^2a & s \end{pmatrix}^{\mathrm{T}}$ and $\begin{pmatrix} 0 & 1 \end{pmatrix}^{\mathrm{T}}$ and such that it determines a map $T^1(\nu)$ satisfying $d_0T^1(\nu) = T^0(\nu)d_4$. The last condition is equivalent to the following two equations: $\begin{pmatrix} a-1 & b-1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = b-1$ (and this is clear), and

$$\begin{pmatrix} a-1 & b-1 \end{pmatrix}\begin{pmatrix} aL^2a \\ s \end{pmatrix} = \begin{pmatrix} -x_1 & 1 \end{pmatrix}\begin{pmatrix} N_b \\ L^2a-1 \end{pmatrix},$$

which is verified easily.

Next, we prove the statement of the lemma by induction on k. Assume that $k > 1$ and

$$T^{k-1}(\nu) = \begin{pmatrix} & \mathbb{O}^{k-2} & \mathbb{O}^{k-2} \\ A'_{k-1} & aL^2a & 0 \\ & s_{k-1} & 1 \end{pmatrix}.$$

We check that we can find a matrix $T^k(\nu)$ of the desired form, i.e., a matrix with the last two columns equal to $\begin{pmatrix} \mathbb{O}_{k-1} & aL^2a & s_k \end{pmatrix}^{\mathrm{T}}$, $\begin{pmatrix} \mathbb{O}_k & 1 \end{pmatrix}^{\mathrm{T}}$, such that it determines a map satisfying $d_{k-1}T^k(\nu) = T^{k-1}(\nu)d_{k+3}$. If $k = 2n$ with $n > 0$, then this claim is equivalent to the identity

$$\begin{pmatrix} b-1 & 0 \\ 1-L^na & N_b \end{pmatrix}\begin{pmatrix} aL^2a & 0 \\ -x_1 & 1 \end{pmatrix} = \begin{pmatrix} aL^2a & 0 \\ s & 1 \end{pmatrix}\begin{pmatrix} b-1 & 0 \\ 1-L^{n+2}a & N_b \end{pmatrix}.$$

If $k = 2n+1$ with $n > 0$, then the same claim is equivalent to the identity

$$\begin{pmatrix} N_b & 0 \\ L^na-1 & b-1 \end{pmatrix}\begin{pmatrix} aL^2a & 0 \\ s & 1 \end{pmatrix} = \begin{pmatrix} aL^2a & 0 \\ -x_1 & 1 \end{pmatrix}\begin{pmatrix} N_b & 0 \\ L^{n+2}a-1 & b-1 \end{pmatrix}.$$

The verification of these two relations is easy, and we leave it to the reader. □

Proposition 2. *The set*

$$\mathcal{X} = \{\upsilon_1, \upsilon_2, \nu, \omega, \lambda, \varphi_0, \varphi_1, \theta_0, \theta_1, \rho_1, \rho_2, \psi_0, \psi_1\} \tag{5.1}$$

generates the algebra $\mathrm{H}^*(G, \widetilde{R})$. In other words, the image of this set under τ^{-1} generates $\mathrm{HH}^*(R)$ as an algebra with respect to the $\cup$-product.

Proof. Let $\mathcal{T}$ denote the subalgebra of $\mathrm{H}^*(G,\widetilde{R})$ generated by the set $\mathcal{X}$. We shall prove that $\mathrm{H}^*(G,\widetilde{R})\subset\mathcal{T}$. First, we check that multiplication by λ induces isomorphisms $\mathrm{H}^*(G,\mathbb{Z}(i))\to\mathrm{H}^*(G,\mathbb{Z}(i+1))$ $(0\le i\le r-2)$, $\mathrm{H}^*(G,M^{(a)}(i))\to\mathrm{H}^*(G,M^{(a)}(i+1))$, $\mathrm{H}^*(G,M^{(b)}(i))\to\mathrm{H}^*(G,M^{(b)}(i+1))$, and $\mathrm{H}^*(G,M^{(ab)}(i))\to\mathrm{H}^*(G,M^{(ab)}(i+1))$ $(0\le i\le\frac{r-4}{2})$. Indeed, assume that $f:Q_n\to\widetilde{R}$ takes an element e_i^n $(1\le i\le n+1)$ to $\sum_{g\in G}a_{i,g}g$ with $a_{i,g}\in\mathbb{Z}$. Then for the role of a representative of λf we can choose a map from Q_n to $\widetilde{R}$ that takes e_i^n to $\sum_{g\in G}a_{i,g}gb^2$. The isomorphisms in question are obtained with the help of considering $\mathbb{Z}$-bases for the modules $\mathbb{Z}(i)$, $M^{(a)}(i)$, $M^{(b)}(i)$, and $M^{(ab)}(i)$. It remains to show that $\mathrm{H}^*(G,\mathbb{Z}(0))$, $\mathrm{H}^*(G,M^{(a)}(0))$, $\mathrm{H}^*(G,M^{(b)}(0))$, and $\mathrm{H}^*(G,M^{(ab)}(0))$ lie in $\mathcal{T}$. Let N denote an R-module with a basis B consisting of m elements. The first part of Lemma 5 implies that if h is a row of length $m(n+1)$ such that $[h]_1^{m(n+1)}\in\mathrm{H}^n(G,N)$, then

$$\upsilon_1[h]_1^{m(n+1)}=[h]_1^{m(n+3)}. \tag{5.2}$$

Let $\mathcal{U}$ be a subalgebra of $\mathrm{H}^*(G,\mathbb{Z}(0))$ generated by the elements υ_1, υ_2, ν, and ω. We claim that $\mathrm{H}^*(G,\mathbb{Z}(0))\subset\mathcal{U}$. Clearly, the product of elements of $\mathrm{H}^*(G,\mathbb{Z}(0))$ obtained by transferring the $\cup$-product on $\mathrm{HH}^*(R)$ by using τ^{-1} coincides with the usual $\cup$-product on $\mathrm{H}^*(G,\mathbb{Z}(0))\simeq\mathrm{H}^*(G,\mathbb{Z})$. First, we prove that $\mathrm{H}^{2n}(G,\mathbb{Z}(0))\subset\mathcal{U}$ for $n\ge 0$. If $n\in\{0,1\}$, this follows from (3.6) and the definition of υ_1, υ_2. Then we use induction on n. Assume that $\mathrm{H}^{2(n-1)}(G,\mathbb{Z}(0))\subset\mathcal{U}$. From (5.2) we see that the elements $\upsilon_1 f_{2n-2,2k}=f_{2n,k}$ of $\mathrm{H}^{2n}(G,\mathbb{Z}(0))$, $0\le k\le\frac{n-1}{2}$, lie in $\mathcal{U}$ (see the notation in (3.5)). Using part 2) of Lemma 5, we easily derive by induction on k that ν^k can be presented in $\mathbb{Z}^{4k+1}$ by a vector with the last coordinate equal to 1, and that $\upsilon_2\nu^k$ can be presented in $\mathbb{Z}^{4k+3}$ by a vector with the last coordinate equal to 2. From this and (3.6), it follows that if $n\,\vdots\,2$, then $\nu^{\frac{n}{2}}=f_{2n,n}+\sum_{k=0}^{\frac{n-2}{2}}a_kf_{2n,2k}$ in $\mathrm{H}^{2n}(G,\mathbb{Z}(0))$, and if $n\not\vdots\,2$, then $\upsilon_2\nu^{\frac{n-1}{2}}=f_{2n,n}+\sum_{k=0}^{\frac{n-1}{2}}a_kf_{2n,2k}$, where all a_k are integers. This shows that $\mathrm{H}^{2n}(G,\mathbb{Z}(0))\subset\mathcal{U}$. Next, we check that $\mathrm{H}^{2n+1}(G,\mathbb{Z}(0))\subset\mathcal{U}$ for $n\ge 0$. If $n=0,1,2$, this follows from (3.6) and the definition of ω. Then we argue by induction on n. Assume that $\mathrm{H}^{2n-1}(G,\mathbb{Z}(0))\subset\mathcal{U}$. From (5.2) we see that the elements $\upsilon_1 f_{2n-1,2k}=f_{2n+1,2k}$ of $\mathrm{H}^{2n+1}(G,\mathbb{Z}(0))$ $\left(1\le k\le\frac{n-1}{2}\right)$ lie in $\mathcal{U}$. If $n\not\vdots\,2$, then (3.6) immediately implies that $\mathrm{H}^{2n+1}(G,\mathbb{Z}(0))\subset\mathcal{U}$. Suppose $n\,\vdots\,2$. Using part 2) of Lemma 5 and the fact that now we have $2f_{2n+1,n}\in\operatorname{Im}\delta_{\mathbb{Z}}^{2n}$, we prove by induction on k that $\omega\nu^k$ can be presented by a vector the last but one coordinate of which is equal to r, and the last coordinate is zero. Consequently, in $\mathrm{H}^{2n+1}(G,\mathbb{Z}(0))$ we obtain the identity $\omega\nu^{\frac{n-2}{2}}=f_{2n+1,n}+\sum_{k=1}^{\frac{n-2}{2}}a_kf_{2n+1,k}$ with $a_k\in\mathbb{Z}$. Using this formula and the above discussion, we conclude that $\mathrm{H}^{2n+1}(G,\mathbb{Z}(0))\subset\mathcal{U}$ also for even n.

Next, we claim that the element $\psi_0\nu^n$ generates $\mathrm{H}^{4n}(G,M^{(ab)}(0))$, the element $\psi_1\nu^n$ generates $\mathrm{H}^{4n+2}(G,M^{(ab)}(0))$, the element $\varphi_0\nu^n$ generates $\mathrm{H}^{4n}(G,M^{(b)}(0))$, and the element $\varphi_1\nu^n$ generates $\mathrm{H}^{4n+2}(G,M^{(b)}(0))$ for any $n\ge 0$. Since $\mathrm{H}^n(G,M^{(ab)}(0))=\mathrm{H}^n(G,M^{(b)}(0))=0$ for odd n, the inclusion

$$\mathrm{H}^*(G,M^{(ab)}(0))\cup\mathrm{H}^*(G,M^{(b)}(0))\subset\mathcal{T}.$$

will follow from the above claim. Using induction on n and part 2) of Lemma 5, it is easy to show that the elements $\varphi_0\nu^n$, $\varphi_1\nu^n$, $\psi_0\nu^n$, and $\psi_1\nu^n$ have the last but one coordinate equal to 1. Then (3.8) and (3.9) imply that $\psi_0\nu^n$ generates $\mathrm{H}^{4n}(G,M^{(ab)}(0))$, and that $\psi_1\nu^n$ generates $\mathrm{H}^{4n+2}(G,M^{(ab)}(0))$. Similarly, $\varphi_0\nu^n$ generates $\mathrm{H}^{4n}(G,M^{(b)}(0))$ and $\varphi_1\nu^n$ generates $\mathrm{H}^{4n+2}(G,M^{(b)}(0))$.

It remains to prove that $\mathrm{H}^*(G,M^{(a)}(0))\subset\mathcal{T}$. First, we prove that $\mathrm{H}^{2n}(G,M^{(a)}(0))\subset\mathcal{T}$ for $n\ge 0$. The proof of Lemma 4 shows that θ_0 generates $\mathrm{H}^0(G,M^{(a)}(0))$. Then we

use induction on n. Assume that $n > 0$ and $\mathrm{H}^{2(n-1)}(G, M^{(a)}(0)) \subset \mathcal{T}$. Then (5.2) implies that the elements $\upsilon_1 f^{(a)}_{2n-2,2k} = f^{(a)}_{2n,2k}$, $0 \le k \le \frac{n-1}{2}$, and $\upsilon_1 g^{(a)}_{2n-2,2k} = g^{(a)}_{2n,2k}$, $0 \le k \le \frac{n-2}{2}$, of $\mathrm{H}^{2n}(G, M^{(a)}(0))$ lie in $\mathcal{T}$. Using part 2) of Lemma 5, we check by induction on k that $\theta_0\nu^k$ and $\theta_1\nu^k$ can be presented in $\mathbb{Z}^{8k+2}$ and in $\mathbb{Z}^{8k+6}$, respectively, by vectors with the last but one coordinate equal to 1. The proof of Lemma 4 shows that, in $\mathrm{H}^{2n}(G, M^{(a)}(0))$, we have

$$\theta_0\nu^{\frac{n}{2}} = -f^{(a)}_{2n,n} + \sum_{k=0}^{\frac{n-2}{2}} (a_k f^{(a)}_{2n,2k} + a'_k g^{(a)}_{2n,2k})$$

if $n \vdots 2$, and

$$\theta_1\nu^{\frac{n-1}{2}} = g^{(a)}_{2n,n-1} + \sum_{k=0}^{\frac{n-3}{2}} (a_k f^{(a)}_{2n,2k} + a'_k g^{(a)}_{2n,2k}) + a_{\frac{n-1}{2}} f^{(a)}_{2n,n-1}$$

if $n \not\vdots 2$ (in the two identities, all a_k, a'_k are integers). Consequently, $\mathrm{H}^{2n}(G, M^{(a)}(0)) \subset \mathcal{T}$. Next, we check that $\mathrm{H}^{2n+1}(G, M^{(a)}(0)) \subset \mathcal{T}$ for $n \ge 0$. The proof of Lemma 4 implies that $\mathrm{H}^1(G, M^{(a)}(0)) = 0$ and that ρ_1 generates $\mathrm{H}^3(G, M^{(a)}(0))$. Then we apply induction on n. Assume that $n > 1$ and $\mathrm{H}^{2n-1}(G, M^{(a)}(0)) \subset \mathcal{T}$. Then, by (5.2), the elements $\upsilon_1 f^{(a)}_{2n-1,k} = f^{(a)}_{2n+1,k}$ of $\mathrm{H}^{2n+1}(G, M^{(a)}(0))$ $(0 \le k \le n-1)$ lie in $\mathcal{T}$. Then, using induction on k, part 2) of Lemma 5, and the fact that $2f^{(a)}_{2n+1,n} \in \operatorname{Im}\delta^*(M^{(a)})$, we see that $\rho_1\nu^k$, $\rho_2\nu^k$ can be presented in $\mathbb{Z}^{8k+8}$ and in $\mathbb{Z}^{8k+12}$, respectively, by vectors of the form $(v, \frac{r}{2}, \mathbb{O}_3)^{\mathrm{T}}$, where v is a row of suitable length. Then the proof of Lemma 4 shows that

$$\rho' = f^{(a)}_{2n+1,n} + \sum_{k=1}^{n-1} a_k f^{(a)}_{2n+1,k}$$

in $\mathrm{H}^{2n+1}(G, M^{(a)}(0))$, where $a_k \in \mathbb{Z}$ $(1 \le k \le n-1)$, and moreover, $\rho' = \rho_1\nu^{\frac{n-1}{2}}$ for n odd, and $\rho' = \rho_2\nu^{\frac{n-2}{2}}$ for n even. Thus, $f^{(a)}_{2n+1,n} \in \mathcal{T}$, which implies that $\mathrm{H}^{2n+1}(G, M^{(a)}(0)) \subset \mathcal{T}$. This completes the proof of the inclusion $\mathrm{H}^*(G, \widetilde{R}) \subset \mathcal{T}$. □

Corollary 4. *The elements υ_1, υ_2, ν, and ω generate a subalgebra of $\mathrm{H}^*(G, \widetilde{R})$ isomorphic to the cohomology algebra of the group G.*

§6. Relations

This section is devoted to deriving relations among the elements of the set $\mathcal{X}$ (see (5.1)).

Lemma 6. 1) *For υ_2, we can find translates with numbers 0, 1, 2 so that the following identities are fulfilled:*

$$T^0(\upsilon_2) = \begin{pmatrix} 0 & -1 & 2 \end{pmatrix},$$

$$T^1(\upsilon_2) = \begin{pmatrix} 0 & aLa & 2aLa & 0 \\ 0 & L\sum_{i=0}^{r-1}(r-i)b^{i(r+1)} & (b^r - L)(aLa-1) & 2 \end{pmatrix},$$

$$T^2(\upsilon_2) = \begin{pmatrix} 0 & \star & 2(La)^2 & 0 & 0 \\ 0 & \star & (b^r - L)(aLa-1) & 2aLa & 0 \\ 0 & \star & \frac{r(r+1)^2}{2} - r(r+1)a & -1 & 2 \end{pmatrix},$$

where $\big(T^2(\upsilon_2)\big)_{*2}$ denotes a suitable column.

2) *For ω, we can find translates with numbers 0, 1, 2, 3 so that the following identities are fulfilled:*

$$T^0(\omega)(e^5_1) = T^0(\omega)(e^5_2) = T^0(\omega)(e^5_3) = 0, \quad T^0(\omega)(e^5_4) = -x_2, \quad T^0(\omega)(e^5_5) = r;$$

$$T^1(\omega)(e_1^6) = T^1(\omega)(e_2^6) = T^1(\omega)(e_3^6) = 0,$$
$$T^1(\omega)(e_4^6) = x_2 \begin{pmatrix} t^2 \\ S_1 a \end{pmatrix}, \quad T^1(\omega)(e_5^6) = r \begin{pmatrix} t^2 \\ S_1 a \end{pmatrix} - x_2 \begin{pmatrix} 0 \\ S_2 \end{pmatrix};$$
$$T^2(\omega)(e_1^7) = T^2(\omega)(e_2^7) = T^2(\omega)(e_3^7) = 0, \quad T^2(\omega)(e_4^7) = -x_2 \begin{pmatrix} t^4 \\ t^2 S_1 a \\ \frac{r}{4} y_2^2 \end{pmatrix},$$
$$T^2(\omega)(e_5^7) = r \begin{pmatrix} t^4 \\ t^2 S_1 a \\ \frac{r}{4} y_2^2 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ t^2 S_2 \\ \frac{r}{2} t^2 a \end{pmatrix};$$
$$T^3(\omega)(e_1^8) = T^3(\omega)(e_2^8) = T^3(\omega)(e_3^8) = 0, \quad T^3(\omega)(e_4^8) = x_2 \begin{pmatrix} t^6 \\ t^4 S_1 a \\ \frac{r}{4} y_2^2 t^2 \\ S_3 a \end{pmatrix},$$

where S_1 and S_2 are elements in R representable as integral linear combinations of powers of b and satisfying

$$(b-1)S_1 = L^2 - t^2, \quad (b-1)S_2 = N_b - 2r \quad \text{and} \quad \varepsilon(S_1) = \varepsilon(S_2) = r;$$

moreover, $S_3 \in R$ satisfies

$$(b-1)S_3 = \frac{y_2 t^4}{2} S_1 - \frac{r y_2^2 t^2}{4} L + \frac{r y_2^2}{4} L^2 - \frac{x_2 y_2 r t^2}{2}.$$

3) *For θ_1, we can find translates with numbers 0, 1, 2 so that the following identities are fulfilled*:

$$T^0(\theta_1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix},$$
$$T^1(\theta_1) = \begin{pmatrix} 0 & \frac{r}{2} & aLa & 0 \\ 1 & 1 & 0 & 0 \\ 0 & \sum_{i=1}^{\frac{r}{2}} \sum_{j=0}^{2i-1} b^j a + 1 & -b^r \sum_{i=1}^{\frac{r}{2}} b^{2i-1} & 1 \\ 0 & -\sum_{i=1}^{\frac{r}{2}} b^{2i-1} a - 1 & \sum_{i=1}^{r} b^{2i-1} & 0 \end{pmatrix},$$
$$T^2(\theta_1) = \begin{pmatrix} 0 & \star & (La)^2 & 0 & 0 \\ 1 & \star & 0 & 0 & 0 \\ 0 & \star & \sum_{i=1}^{\frac{r}{2}} b^{2i-1} & aLa & 0 \\ 0 & \star & -\sum_{i=1}^{r} b^{2i-1} & 0 & 0 \\ 0 & \star & \frac{r^2}{4} a + 1 & -1 & 1 \\ 0 & \star & \frac{r}{2} a - 1 & 1 & 0 \end{pmatrix},$$

where $\left(T^2(\theta_1)\right)_{*2}$ denotes a suitable column.

4) *For φ_1, we can find translates such that, for $k \geq 0$, they are determined by matrices of the form*

$$T^k(\varphi_1) = \left(\mathbb{O}^{2k+2} \;\; A'_k \;\; [1, r]_{2k+1}^{2k+2}\right),$$

where A'_k is a matrix of size $(2k+2) \times (k+1)$.

5) *For ψ_1, we can find translates such that, for $k \geq 0$, they are determined by matrices of the form*

$$T^k(\psi_1) = \left(\mathbb{O}^{2k+2} \;\; A''_k \;\; [1, 0]_{2k+1}^{2k+2}\right),$$

where A''_k is a matrix of size $(2k+2) \times (k+1)$.

Remark 2. If M is one of the modules $M^{(b)}$, $M^{(ab)}$, $M^{(a)}$, and $\{m_1, m_2\}$ is the standard $\mathbb{Z}$-basis of M (see (3.4)) then, in the description of matrices for translates of $\theta_1, \varphi_1, \psi_1$, for the modules $Q_n \otimes M$, we use the sets of generators of the form $e_i^n \otimes m_j$ ($1 \leq i \leq n+1, j \in \{1, 2\}$).

Proof of Lemma 6. 1) We must prove that $\varepsilon T^0(\upsilon_2) = \upsilon_2$, $d_0T^1(\upsilon_2) = T^0(\upsilon_2)d_2$, and $d_1T^2(\upsilon_2) = T^1(\upsilon_2)d_3$. The first relation is obvious, and for verifying the last relation it suffices to consider only the elements e_1^4, e_3^4, e_4^4,e_5^4. Observe that

$$(b-1)L\sum_{i=0}^{r-1}(r-i)b^{i(r+1)} = (b^{r+1}-1)\sum_{i=0}^{r-1}(r-i)b^{i(r+1)}$$
$$= \sum_{i=1}^{r}(r-i+1)b^{i(r+1)} - \sum_{i=0}^{r-1}(r-i)b^{i(r+1)} = \sum_{i=1}^{r}b^{i(r+1)} - r = aLa-(r+1)$$

and

$$(b-1)(b^r-L)(aLa-1) = (1-b^r)(aLa-1) = (1-b^r)\sum_{i=1}^{r}b^{i(r+1)}$$
$$= \sum_{i=1}^{r}b^{i(r+1)} - \sum_{i=1}^{r}b^{(i+r)(r+1)} = 2\sum_{i=0}^{r}b^{i(r+1)} - 2 - \sum_{i=1}^{2r}b^{(i+r)(r+1)} = 2aLa-2-N_b.$$

These identities imply $d_0T^1(\upsilon_2) = T^0(\upsilon_2)d_2$. Next, it is easily seen that the maps $d_1T^2(\upsilon_2)$ and $T^1(\upsilon_2)d_3$ coincide at e_1^4, e_4^4, e_5^4. To check that they coincide at e_3^4, we need to prove the identities

$$2(a+1)(La)^2 + 2aLa - 2 - N_b = aLaN_b + 2aLa(La+1),$$
$$(1-La)(b^r-L)(aLa-1) + \left(\frac{r(r+1)^2}{2} - r(r+1)a\right)N_b$$
$$= L\sum_{i=0}^{r-1}(r-i)b^{i(r+1)}N_b + (b^r-L)(aLa-1)(La+1).$$

Since $b^iN_b = N_b$ for any $i \in \mathbb{Z}$, the first identity is equivalent to

$$(La)^2 = \frac{r+2}{2}N_b + 1, \tag{6.1}$$

and the second is equivalent to

$$((aL)^2 + (La)^2 - (b^r+1)L - (b^r+1)aLa + 2b^r)La = r(r+1)N_ba.$$

Since $(b^r+1)L = (b^r+1)aLa = N_b + b^r + 1$, the last identity can be written in the form $((aL)^2 + (La)^2 - 2)La = (r+2)(r+1)N_ba$. Because of $(aL)^2 = a(La)^2a$, it remains to verify (6.1). Observe that

$$(b-1)(La)^2 = (b-1)LaLa = (b^{r+1}-1)\sum_{i=0}^{r}b^{i(r+1)} = b^{(r+1)^2} - 1 = b-1.$$

Since $(La)^2$ is an integral linear combination of powers of b, we have $(La)^2 = \varkappa N_b + 1$ for some $\varkappa \in \mathbb{Z}$. Finally, we observe that $\varepsilon((La)^2) = (r+1)^2$, i.e., $\varkappa = \frac{(r+1)^2-1}{2r} = \frac{r+2}{2}$.

2) First, we prove the existence of elements S_1, S_2, S_3 with the required properties. Observe that the element $S_1(\gamma) = \sum_{i=0}^{r-1}(r-i)b^i(L+t) - \gamma N_b$ satisfies the equation $(b-1)S_1(\gamma) = L^2 - t^2$ for any integer γ. It remains to show that $\varepsilon\big(\sum_{i=0}^{r-1}(r-i)b^i(L+t)\big)$ has remainder r under division by $2r$: indeed, we have $\varepsilon\big(\sum_{i=0}^{r-1}(r-i)b^i(L+t)\big) = \frac{r(r+1)}{2}\cdot(2r+2) \equiv r \bmod 2r$. Similarly, the element S_2 is chosen in the form $S_2 = \sum_{i=0}^{2r-1}(2r-i)b^i - \gamma N_b$ (for suitable $\gamma \in \mathbb{Z}$). Since

$$\left(\frac{y_2t^4}{2}S_1 - \frac{ry_2^2t^2}{4}L + \frac{ry_2^2}{4}L^2 - \frac{x_2y_2rt^2}{2}\right)N_b = \left(\frac{y_2t^4r}{2} - \frac{ry_2^2t^3}{4} + \frac{ry_2^2t^2}{4} - \frac{x_2y_2rt^2}{2}\right)N_b = 0,$$

we see that an element S_3 with the required properties also exists.

Next, we prove the identities $\varepsilon T^0(\omega) = \omega$ and $d_i T^{i+1}(\omega)(e_j^{i+6}) = T^i(\omega) d_{i+5}(e_j^{i+6})$ for the pairs (i,j) such that either $i \in \{0,1\}$, $1 \le j \le 5$, or $i = 2$, $1 \le j \le 4$. The first identity is verified without difficulties. The second is obvious for $1 \le j \le 3$. It is easily seen that $d_0 T^1(\omega)(e_j^6) = T^0(\omega) d_5(e_j^6)$ for $j \in \{4,5\}$. We can establish by direct calculations that

$$d_1 T^2(\omega)(e_4^7) - T^1(\omega) d_6(e_4^7) = \begin{pmatrix} 0 \\ x_2\left(-\frac{r}{4} y_2^2 N_b + Y_1\right) \end{pmatrix},$$
$$d_1 T^2(\omega)(e_5^7) - T^1(\omega) d_6(e_5^7) = \begin{pmatrix} 0 \\ r(\frac{r}{4} y_2^2 N_b - Y_1) + x_2(\frac{r}{2}(t^2-2)N_b - Y_2)a \end{pmatrix}$$

and

$$d_2 T^3(\omega)(e_4^7) - T^2(\omega) d_7(e_4^7) = \begin{pmatrix} 0 \\ x_2 t^2 (\frac{r}{4} y_2^2 N_b - Y_1) \\ 0 \end{pmatrix},$$

where $Y_1 = S_1 a L^2 a + t^2 L a S_1 a - x_2 y_2 S_2$ and $Y_2 = -2r S_1 - S_2 L^2 + t^2 L a S_2 a$. Since $(La)^2 = \frac{r+2}{2} N_b + 1$, we can easily show that $(b-1)Y_1 = (b-1)Y_2 = 0$, $\varepsilon(Y_1) = \frac{r^2}{2} y_2^2$, and $\varepsilon(Y_2) = r^2(t^2-2)$, and the desired identities follow.

3) We must prove that $(\varepsilon \otimes \mathrm{id}_{M^{(a)}(0)}) T^0(\theta_1) = \theta_1$, $(d_0 \otimes \mathrm{id}_{M^{(a)}(0)}) T^1(\theta_1) = T^0(\theta_1) d_2$, and $(d_1 \otimes \mathrm{id}_{M^{(a)}(0)}) T^2(\theta_1) = T^1(\theta_1) d_3$. Moreover, it suffices to check the last identity only at the elements e_1^4, e_3^4, e_4^4, e_5^4. All these identities can be verified by direct, although cumbersome, calculations using formula (6.1) and the identities

$$b^j \sum_{i=1}^r b^{2i} = b^{j+1} \sum_{i=1}^r b^{2i-1} = \begin{cases} \sum_{i=1}^r b^{2i} & \text{if } j \,\vdots\, 2, \\ \sum_{i=1}^r b^{2i-1} & \text{if } j \not\vdots\, 2. \end{cases}$$

These verifications are left to the reader.

The two remaining parts of the lemma are established without difficulties. □

Proposition 3. *In the algebra* $\mathrm{H}^*(G, \widetilde{R})$*, the following relations are fulfilled*:

$$2\upsilon_1 = r\upsilon_2 = 2r\nu = 2\omega = 2r\varphi_1 = r\theta_0\nu = r\theta_1 = 2\rho_1 = 2\rho_2 = 2r\psi_1 = 0;$$
$$\upsilon_1\upsilon_2 = 0, \quad \upsilon_2^2 = 4\nu, \quad \upsilon_2\omega = 0, \quad \omega^2 = 0, \quad \lambda^r = 1;$$
$$\psi_0^2 = 2\big(\lambda + \lambda^{\frac{r+2}{2}}\big), \quad \psi_0\psi_1 = \big(\lambda + \lambda^{\frac{r+2}{2}}\big)\upsilon_2, \quad \psi_1^2 = \psi_0^2\nu;$$
$$\psi_0\upsilon_1 = r\psi_1, \quad \psi_1\upsilon_1 = r\psi_0\nu, \quad \psi_0\upsilon_2 = (2+r)\psi_1, \quad \psi_1\upsilon_2 = (2+r)\psi_0\nu,$$
$$\psi_0\omega = \psi_1\omega = 0, \quad \psi_0\lambda^{\frac{r}{2}} = \psi_0, \quad \psi_1\lambda^{\frac{r}{2}} = (1+r)\psi_1;$$
$$\theta_0^2 = 2\big(1 + \lambda^{\frac{r}{2}}\big), \quad \theta_0\theta_1 = \big(1 + \lambda^{\frac{r}{2}}\big)(\upsilon_1 + \upsilon_2), \quad \theta_1^2 = \upsilon_1^2 + \theta_0^2\nu, \quad \theta_0\rho_1 = 0,$$
$$\theta_1\rho_1 = \theta_0\rho_2 = \big(1 + \lambda^{\frac{r}{2}}\big)\omega, \quad \theta_1\rho_2 = \upsilon_1\omega, \quad \rho_1^2 = \rho_1\rho_2 = \rho_2^2 = 0;$$
$$\theta_0\upsilon_2 = 2\theta_1, \quad \theta_1\upsilon_2 = 2\theta_0\nu, \quad \theta_0\omega = \rho_1\upsilon_1, \quad \theta_1\omega = \rho_2\upsilon_1,$$
$$\rho_1\upsilon_2 = \rho_2\upsilon_2 = \rho_1\omega = \rho_2\omega = 0,$$
$$\theta_0\lambda^{\frac{r}{2}} = \theta_0, \quad \theta_1\lambda^{\frac{r}{2}} = \theta_1 + \theta_0\upsilon_1, \rho_1\lambda^{\frac{r}{2}} = \rho_1, \quad \rho_2\lambda^{\frac{r}{2}} = \rho_2 + \rho_1\upsilon_1;$$
$$\varphi_0^2 = 2\big(\lambda + \lambda^{\frac{r+2}{2}}\big), \quad \varphi_0\varphi_1 = \Big(1 + \frac{r}{2}\Big)\big(\lambda + \lambda^{\frac{r+2}{2}}\big)\upsilon_2, \quad \varphi_1^2 = \varphi_0^2\nu;$$
$$\varphi_0\upsilon_1 = \varphi_1\upsilon_1 = \varphi_0\omega = \varphi_1\omega = 0, \quad \varphi_0\upsilon_2 = 2\varphi_1, \quad \varphi_1\upsilon_2 = 2\varphi_0\nu,$$
$$\varphi_0\lambda^{\frac{r}{2}} = \varphi_0, \quad \varphi_1\lambda^{\frac{r}{2}} = (1+r)\varphi_1;$$

$$\begin{aligned}
&\psi_0\theta_0 = 2\varphi_0, \psi_0\theta_1 = \psi_1\theta_0 = 2\varphi_1, \psi_1\theta_1 = 2\varphi_0\nu,\\
&\psi_0\rho_1 = \psi_0\rho_2 = \psi_1\rho_1 = \psi_1\rho_2 = 0;\\
&\varphi_0\psi_0 = 2\theta_0\lambda, \varphi_0\psi_1 = \varphi_1\psi_0 = 2\theta_1\lambda, \varphi_1\psi_1 = 2\theta_0\nu\lambda;\\
&\varphi_0\theta_0 = 2\psi_0, \varphi_1\theta_0 = 2\psi_1, \varphi_0\theta_1 = (2+r)\psi_1, \varphi_1\theta_1 = 2\psi_0\nu,\\
&\varphi_0\rho_1 = \varphi_0\rho_2 = \varphi_1\rho_1 = \varphi_1\rho_2 = 0.
\end{aligned}$$

Proof. We have $r\theta_0\nu = 0$, because $\theta_0\nu$ is an element of the group $\mathrm{H}^4(G, M^{(a)}(0))$, and $\mathrm{H}^4(G, M^{(a)}(0)) = \mathbb{Z}_2^2 \oplus \mathbb{Z}_r$ by Lemma 4. The remaining identities in the first line follow from the definition of the elements of the set $\mathcal{X}$ (see (5.1)) and the proofs of Lemmas 2–4.

Then (5.2) and (3.6) imply that $\upsilon_1\upsilon_2 = \upsilon_1 f_{2,1} = f_{4,1} = 0$ in $\mathrm{H}^4(G, \widetilde{R})$. From part 1) of Lemma 6, it follows that $\upsilon_2^2 \in \mathbb{Z}^5$ has the first coordinate equal to 0 and the last equal to 4. Now, from (3.6) and the definition of ν it follows that $\upsilon_2^2 = 4\nu$. Since $\upsilon_2\omega \in \mathrm{H}^7(G, \mathbb{Z}(0)) \simeq \mathbb{Z}_2$ and $\upsilon_1\omega = f_{7,3}$ is a generator of this group (see the proof of Lemma 2), we have $\upsilon_2\omega = \varkappa\upsilon_1\omega$ for some $\varkappa \in \mathbb{Z}$. Multiplying this by υ_1, we obtain $\varkappa\upsilon_1^2\omega = 0$. We have $\upsilon_1^2\omega = f_{9,4} \neq 0$ in $\mathrm{H}^9(G, \mathbb{Z}(0))$, which implies that $\varkappa \vdots 2$, i.e., $\upsilon_2\omega = 0$. Furthermore, since λ^k takes 1 to b^{2k}, we see that $\lambda^r = 1$.

Since $T^0(\psi_0)$ takes 1 to $1 \otimes (ab + ab^t)$, the element ψ_0^2 takes 1 to $(ab + ab^t)^2 = 2(b^2 + b^{r+2})$, i.e., $\psi_0^2 = 2\big(\lambda + \lambda^{\frac{r+2}{2}}\big)$. Since $\mu(\psi_0 \otimes \mathrm{id})T^0(\psi_1)(e_1^2) = 0$, $\mu(\psi_0 \otimes \mathrm{id})T^0(\psi_1)(e_2^2) = -ab(ab + ab^t) = -(b^2 + b^{r+2})$, and $\mu(\psi_0 \otimes \mathrm{id})T^0(\psi_1)(e_3^2) = (ab + ab^t)^2 = 2(b^2 + b^{r+2})$, we obtain $\psi_0\psi_1 = \big(\lambda + \lambda^{\frac{r+2}{2}}\big)\upsilon_2$. Using part 3) of Lemma 6, we easily check that ψ_1^2 is represented in $\mathrm{H}^4(G, \widetilde{R})$ by an element that takes e_1^4 to 0 and e_5^4 to $2(b^2 + b^{r+2})$. Then (3.6) shows that, as an element of $\mathrm{H}^4(G, \mathbb{Z}(1)) \oplus \mathrm{H}^4\left(G, \mathbb{Z}\big(\frac{r+2}{2}\big)\right)$, ψ_1^2 is represented in the form $(2f_{4,2}, 2f_{4,2})$, whence $\psi_1^2 = 2\big(\lambda + \lambda^{\frac{r+2}{2}}\big)\nu = \varphi_0^2\nu$.

Using (3.9), we obtain the following relations in $\mathrm{H}^*(G, M^{(ab)}(0))$:

$$\begin{aligned}
\psi_0\upsilon_1 &= [1]_1^6 = f_{2,0}^{(ab)} = r f_{2,1}^{(ab)} = r\psi_1,\\
\psi_1\upsilon_1 &= f_{4,1}^{(ab)} = r f_{4,2}^{(ab)} = r\psi_0\nu
\end{aligned}$$

(earlier we have proved that $\langle\psi_0\nu\rangle = \mathrm{H}^4(G, M^{(ab)}(0))$), and

$$\psi_0 T^0(\upsilon_2) = 2f_{2,1}^{(ab)} - g_{2,1}^{(ab)} = (2+r)f_{2,1}^{(ab)} = (2+r)\psi_1.$$

Using part 1) of Lemma 6 and the fact that L acts on $M^{(ab)} = \mathbb{Z}^2$ as the matrix $\begin{pmatrix} r+1 & \frac{r}{2} \\ 0 & 1 \end{pmatrix}$, we conclude that $\psi_1 T^2(\upsilon_2)$ is presented in $\mathbb{Z}^{10}$ by a vector of the form $\big(v', \frac{r(r^2+r-1)}{2}, 0, -(r+1), -2, 2, 0\big)^{\mathrm{T}}$, where v' is a row of length 4. Then (3.8) and (3.9) imply the following identities in $\mathrm{H}^4(G, M^{(ab)})$:

$$\psi_1\upsilon_2 = \frac{r(r^2+r-1)}{2} f_{4,1}^{(ab)} - (r+1)g_{4,2}^{(ab)} + 2f_{4,2}^{(ab)} = (2+r)f_{4,2}^{(ab)} = (2+r)\psi_0\nu.$$

Next, we have $\psi_0\omega = \psi_1\omega = 0$, because $\mathrm{H}^5(G, M^{(ab)}) = \mathrm{H}^7(G, M^{(ab)}) = 0$. Since ψ_0 takes e_1^0 to $ab + ab^t$, we see that $\psi_0\lambda^{\frac{r}{2}}$ also takes e_1^0 to $ab + ab^t$, whence $\psi_0\lambda^{\frac{r}{2}} = \psi_0$. Since ψ_1 takes e_1^2, e_2^2, and e_3^2 to 0, to $-ab$, and to $ab + ab^t$, respectively, $\psi_1\lambda^{\frac{r}{2}}$ takes the same elements to 0, to $-ab^t = ab - (ab + ab^t)$, and to $ab + ab^t$, respectively, and hence, in $\mathrm{H}^2(G, M^{(ab)})$ we have $\psi_1\lambda^{\frac{r}{2}} = f_{2,1}^{(ab)} - g_{2,1}^{(ab)} = (r+1)f_{2,1}^{(ab)} = (r+1)\psi_1$.

The relations for θ_0^2 and $\theta_0\theta_1$ are established much as those for ψ_0^2 and $\psi_0\psi_1$. Using part 4) of Lemma 6, we show that θ_1^2 can be presented in $\mathrm{H}^4(G, \widetilde{R})$ by an element that takes e_1^4 to 1 and e_5^4 to $2(1 + b^r)$. From (3.6), it follows that, as an element of $\mathrm{H}^4(G, \mathbb{Z}(0)) \oplus \mathrm{H}^4\left(G, \mathbb{Z}\big(\frac{r}{2}\big)\right)$, θ_1^2 is presented in the form $(f_{4,1} + 2f_{4,2}, 2f_{4,2})$, whence $\theta_1^2 = \upsilon_1^2 + 2\big(1 + \lambda^{\frac{r}{2}}\big)\nu = \upsilon_1^2 + \theta_0^2\nu$.

We have $\theta_0\rho_1 = 0$, because $\mathrm{H}^3\left(G, \mathbb{Z}\left(\frac{r}{2}\right)\right) = \mathrm{H}^3(G, \mathbb{Z}(0)) = 0$. Next, it is easily seen that $\theta_0\rho_2 = \left(1 + \lambda^{\frac{r}{2}}\right)\omega$ in $\mathrm{H}^5(G, \mathbb{Z}(0)) \oplus \mathrm{H}^5\left(G, \mathbb{Z}\left(\frac{r}{2}\right)\right)$. Now, we observe immediately that $\lambda^{\frac{r}{2}}\theta_0 = \theta_0$,

$$\lambda^{\frac{r}{2}}\theta_1 = f_{2,0}^{(a)} - f_{2,1}^{(a)} - g_{2,0}^{(a)} = f_{2,0}^{(a)} + g_{2,0}^{(a)} = \theta_0\upsilon_1 + \theta_1,$$

$\lambda^{\frac{r}{2}}\rho_1 = \rho_1 - x_1 g_{3,1}^{(a)} = \rho_1$, and

$$\lambda^{\frac{r}{2}}\rho_2 = \rho_2 - x_2 g_{5,2}^{(a)} = \rho_2 + f_{5,1}^{(a)} = \rho_2 + \upsilon_1\rho_1.$$

The proof of Proposition 2 shows that $\theta_1\rho_1 = \left(\varkappa_1 + \varkappa_2\lambda^{\frac{r}{2}}\right)\omega$ for some $\varkappa_1, \varkappa_2 \in \mathbb{Z}$. Multiplying this by $\lambda^{\frac{r}{2}}$, we obtain $\varkappa_1 \equiv \varkappa_2 \pmod 2$, and hence, $\theta_1\rho_1 = \varkappa_1\left(1 + \lambda^{\frac{r}{2}}\right)\omega$. Since $2\omega = 0$, we may assume that $\varkappa_1$ is equal to 0 or 1. But $\theta_1^2\rho_1 = (\upsilon_1^2 + \theta_0^2\nu)\rho_1 = \upsilon_1^2\rho_1 \neq 0$, whence $\varkappa_1 \neq 0$, and, consequently, $\theta_1\rho_1 = \left(1 + \lambda^{\frac{r}{2}}\right)\omega$. Next, we have $\theta_0\upsilon_2 = -f_{2,1}^{(a)} = 2g_{2,0}^{(a)} = 2\theta_1$ and $\theta_0\omega = 2f_{5,2}^{(a)} - x_2 g_{5,2}^{(a)} = f_{5,1}^{(a)} = \upsilon_1\rho_1$. Furthermore, $\theta_1\upsilon_2 \in \mathrm{H}^4(G, M^{(a)}(0))$, so that $\theta_1\upsilon_2 = \varkappa_1\theta_0\upsilon_1^2 + \varkappa_2\theta_1\upsilon_1 + \varkappa_3\theta_0\nu$ for some $\varkappa_1, \varkappa_2, \varkappa_3 \in \mathbb{Z}$. Multiplying this by υ_1, we see that $\varkappa_1 \equiv \varkappa_2 \equiv 0 \pmod 2$, whence $\theta_1\upsilon_2 = \varkappa_3\theta_0\nu$. Multiplying the last relation by θ_0, we conclude that $\varkappa_3 \equiv 2 \pmod r$, whence we $\theta_1\upsilon_2 = 2\theta_0\nu$.

Now we check that

$$\theta_1\rho_2 = \upsilon_1\omega, \quad \theta_1\omega = \rho_2\upsilon_1. \tag{6.2}$$

Since $\theta_1\rho_2 \in \mathrm{H}^7(G, \mathbb{Z}(0)) \oplus \mathrm{H}^7\left(G, \mathbb{Z}\left(\frac{r}{2}\right)\right)$ and $\theta_1\omega \in \mathrm{H}^7(G, M^{(a)}(0))$, there exist integers $\varkappa_1, \varkappa_2, \alpha_1, \alpha_2, \alpha_3$ with

$$\theta_1\rho_2 = \left(\varkappa_1 + \varkappa_2\lambda^{\frac{r}{2}}\right)\upsilon_1\omega, \quad \theta_1\omega = \alpha_1\rho_2\upsilon_1 + \alpha_2\rho_1\upsilon_1^2 + \alpha_3\rho_1\nu$$

(see the proof of Proposition 2). Multiplying the first identity by $\lambda^{\frac{r}{2}}$ and the second by θ_0 and using identities proved above, we get $\varkappa_1 \equiv \varkappa_2 + 1 \pmod 2$ and $\alpha_1 \equiv 1 \pmod 2$. Then

$$\upsilon_1^2\omega = \theta_1^2\omega = \theta_1\rho_2\upsilon_1 + \alpha_2\theta_1\rho_1\upsilon_1^2 + \alpha_3\theta_1\rho_1\nu = \upsilon_1^2\omega + \left(1 + \lambda^{\frac{r}{2}}\right)\left((\varkappa_2 + \alpha_2)\upsilon_1^2\omega + \alpha_3\nu\omega\right).$$

Consequently, we can take $\alpha_3 = 0$ and $\alpha_2 = \varkappa_2$, obtaining $\theta_1\rho_2 = \upsilon_1\omega + \varkappa\left(1 + \lambda^{\frac{r}{2}}\right)\upsilon_1\omega$ and $\theta_1\omega = \rho_2\upsilon_1 + \varkappa\rho_1\upsilon_1^2$ for some integer $\varkappa$. From part 2) of Lemma 6, it follows that $\theta_1 T^2(\omega)(e_1^7) = \theta_1 T^2(\omega)(e_2^7) = \theta_1 T^2(\omega)(e_3^7) = 0$ and

$$\theta_1 T^2(\omega)(e_4^7) = -x_2\left(t^4 a - t^2 a * S_1 a + \frac{r}{4}y_2^2(a + ab^r)\right).$$

We shall prove that

$$a * S_1 a = 2\beta_1(a + ab^r) + \beta_2 a \quad \text{with } \beta_1, \beta_2 \in \mathbb{Z}. \tag{6.3}$$

Then the form of the elements $\rho_2\upsilon_1$, $\rho_1\upsilon_1^2$ and the proof of Lemma 4 will show immediately that $\varkappa \vdots 2$, which will prove identities (6.2). Since $S_1 = \sum_{i=1}^{2r}\gamma_i b^i$, where $\gamma_i \in \mathbb{Z}$ and $\sum_{i=1}^{2r}\gamma_i = r$, it remains to show that $\sum_{i=1}^{r}\gamma_{2i} - \sum_{i=1}^{r}\gamma_{2i-1} \vdots 4$. This follows from the fact that $S_1 = \sum_{i=0}^{r-1}(r-i)b^i(L+t) - \gamma N_b$ for some integer γ, and the sum of the coefficients of the even powers of b in the expansion of $L + t$ in powers of b differs from the sum of the coefficients of odd powers of b by the even number $t + 1$.

Now, the proof of Proposition 2 shows that, for any n, multiplication by υ_1 gives rise to an injective map from $\mathrm{H}^{2n-1}(G, M^{(a)}(0))$ into $\mathrm{H}^{2n+1}(G, M^{(a)}(0))$. Since also $\rho_1\upsilon_2 \in \mathrm{H}^5(G, M^{(a)}(0))$, $\rho_2\upsilon_2 \in \mathrm{H}^7(G, M^{(a)}(0))$, and $\upsilon_1\upsilon_2 = 0$, we obtain $\rho_1\upsilon_2 = \rho_2\upsilon_2 = 0$.

Now we claim that $\rho_1^2 = 0$. Since $\rho_1^2 \in \mathrm{H}^6(G, \mathbb{Z}(0)) \oplus \mathrm{H}^6(G, \mathbb{Z}(\frac{r}{2}))$, we have

$$\rho_1^2 = \left(\varkappa_1 + \varkappa_1'\lambda^{\frac{r}{2}}\right)\upsilon_1^3 + \left(\varkappa_2 + \varkappa_2'\lambda^{\frac{r}{2}}\right)\upsilon_1\nu + \left(\varkappa_3 + \varkappa_3'\lambda^{\frac{r}{2}}\right)\upsilon_2\nu \tag{6.4}$$

for some $\varkappa_1, \varkappa_1', \varkappa_2, \varkappa_2', \varkappa_3, \varkappa_3' \in \mathbb{Z}$. The translates $T^k(\rho_1)$, $k \geq 0$, can be constructed so that $T^k(\rho_1)(e_{k+4}^{k+3}) = 0$. Then (3.6) and the proof of Proposition 2 imply $\varkappa_3 = \varkappa_3' = 0$. It is clear that either $\rho_1^2 = 0$ or $\upsilon_1\rho_1^2 \neq 0$. But $\upsilon_1\rho_1^2 = \theta_0\omega\rho_1 = 0$, and we conclude that $\rho_1^2 = 0$.

Now we check that $\rho_1\omega = \rho_2\omega = \rho_1\rho_2 = \rho_2^2 = \omega^2 = 0$. Since $\rho_1\omega \in \mathrm{H}^8(G, M^{(a)}(0))$, we have

$$\rho_1\omega = \varkappa_1\upsilon_1^4\theta_0 + \varkappa_2\upsilon_1^2\nu\theta_0 + \varkappa_3\nu^2\theta_0 + \varkappa_4\upsilon_1^3\theta_1 + \varkappa_5\upsilon_1\nu\theta_1$$

for some integers $\varkappa_i$ $(1 \leq i \leq 5)$. Multiplying this by $\lambda^{\frac{r}{2}}$, we obtain $\varkappa_4 \equiv \varkappa_5 \equiv 0 \pmod 2$. Multiplying the resulting identity by θ_0, we conclude that $\varkappa_3 \equiv 0 \pmod r$, whence $\rho_1\omega = \varkappa_1\upsilon_1^4\theta_0 + \varkappa_2\upsilon_1^2\nu\theta_0$. Observe that $\upsilon_1^4\theta_0 = f_{8,0}^{(a)}$ and $\upsilon_1^2\nu\theta_0 = f_{8,2}^{(a)}$. Hence, by the proof of Lemma 4, it suffices to show that $\rho_1T^3(\omega)(e_1^8) = \rho_1T^3(\omega)(e_2^8) = \rho_1T^3(\omega)(e_3^8) = 0$ and $\rho_1T^3(\omega)(e_4^8) = 2\beta_1(a+ab^r)+\beta_2a$ for some integers β_1, β_2. Part 2 of Lemma 6 implies that $T^3(\omega)(e_1^8) = T^3(\omega)(e_2^8) = T^3(\omega)(e_3^8) = 0$ and

$$\rho_1T^3(\omega)(e_4^8) = x_2\left(-x_1t^4a * S_1a + \frac{r^2}{8}y_2^2t^2(a+ab^r)\right).$$

Since $a * S_1a$ is of the form (6.3), we obtain $\rho_1\omega = 0$. Multiplying this by θ_1, we see that $\big(1+\lambda^{\frac{r}{2}}\big)\omega^2 = 0$, whence $\omega^2 = 0$. Multiplying the last identity by θ_1, we obtain $\upsilon_1\rho_2\omega = 0$. Since $\rho_2\omega \in \mathrm{H}^{10}(G, M^{(a)}(0))$, we easily show that $\rho_2\omega = \varkappa\nu^2\theta_1$ for some $\varkappa \in \mathbb{Z}$. Multiplying this by θ_0, we see that $\varkappa \equiv 0 \pmod r$, whence $\rho_2\omega = 0$. Then $\upsilon_1\rho_1\rho_2 = \theta_0\omega\rho_2 = 0$, and $\upsilon_1\rho_2^2 = \theta_1\omega\rho_2 = 0$, implying that $\rho_1\rho_2 = \big(\varkappa_1 + \varkappa_1'\lambda^{\frac{r}{2}}\big)\nu^2$ and $\rho_2^2 = \big(\varkappa_2 + \varkappa_2'\lambda^{\frac{r}{2}}\big)\upsilon_2\nu^2$ for some $\varkappa_1, \varkappa_1', \varkappa_2, \varkappa_2' \in \mathbb{Z}$. To prove that $\varkappa_1 = \varkappa_1' = \varkappa_2 = \varkappa_2' = 0$, we can argue as in the proof of $\varkappa_3 = \varkappa_3' = 0$ in (6.4). Consequently, $\rho_1\rho_2 = \rho_2^2 = 0$.

The relations for φ_0^2 and $\varphi_0\varphi_1$ are proved like those for ψ_0^2 and $\psi_0\psi_1$. Moreover, using part 5) of Lemma 6, we establish that φ_1^2 is represented in $\mathrm{H}^4(G, \widetilde{R})$ by an element that takes e_1^4 to 0 and e_5^4 to $(r^2+2r+2)b^2 + (2r+2)b^{r+2}$. From (3.6), it follows that, as an element of $\mathrm{H}^4(G, \mathbb{Z}(0)) \oplus \mathrm{H}^4\big(G, \mathbb{Z}\big(\frac{r}{2}\big)\big)$, φ_1^2 is presented in the form $(2f_{4,2}, 2f_{4,2})$, whence $\varphi_1^2 = 2\big(\lambda + \lambda^{\frac{r+2}{2}}\big)\nu = \varphi_0^2\nu$.

Since $\varphi_0T^0(\upsilon_1)(e_3^2) = \varphi_1T^2(\upsilon_1)(e_5^4) = 0$ (see part 1 of Lemma 5) and $\varphi_0T^0(\upsilon_2)(e_3^2) = 2(b+b^t)$, $\varphi_1T^2(\upsilon_2)(e_5^4) = 2(b+b^t) + 2rb$ (see part 1 of Lemma 6), we obtain $\varphi_0\upsilon_1 = \varphi_1\upsilon_1 = 0$, $\varphi_0\upsilon_2 = 2\varphi_1$, $\varphi_1\upsilon_2 = 2\varphi_0\nu$. The identities $\varphi_0\omega = \varphi_1\omega = 0$ are fulfilled because $\mathrm{H}^{2n+1}(G, M^{(b)}) = 0$ for any n. Moreover, $\lambda^{\frac{r}{2}}\varphi_0 = \varphi_0$, and $\mu(\lambda^{\frac{r}{2}} \otimes \mathrm{id})T^0(\varphi_1) = (r+1)f_{2,1}^{(b)} - \frac{r+2}{2}g_{2,1}^{(b)} = (r+1)\varphi_1$.

Next, it is immediately verified that $\psi_0\theta_0 = 2\varphi_0$ and $\mu(\psi_0 \otimes \mathrm{id})T^0(\theta_1)(e_3^2) = 2(b+b^t) = \mu(\theta_0 \otimes \mathrm{id})T^0(\psi_1)(e_3^2)$. Moreover, part 4 of Lemma 6 shows that $\mu(\psi_1 \otimes \mathrm{id})T^2(\theta_1)(e_5^4) = 2(b+b^t)$. The identities obtained above (and the calculation of the groups $\mathrm{H}^{2m}(G, M^{(b)})$, see the proof of Lemma 3) show that $\psi_0\theta_1 = \psi_1\theta_0 = 2\varphi_1$ and $\psi_1\theta_1 = 2\varphi_0\nu$. The relations $\psi_0\rho_1 = \psi_0\rho_2 = \psi_1\rho_1 = \psi_1\rho_2 = 0$ are valid because $\mathrm{H}^{2n+1}(G, M^{(b)}) = 0$ for any n.

Now, direct calculations show that $\psi_0\varphi_0 = 2\lambda\theta_0$ and that in $\mathrm{H}^2(G, M^{(a)}(1))$ we have $\psi_0\varphi_1 = -\frac{r+2}{2}f_{2,1}^{(a)} = 2g_{2,0}^{(a)}$, $\varphi_0\psi_1 = -f_{2,1}^{(a)} = 2g_{2,0}^{(a)}$. Consequently, $\varphi_0\psi_1 = \varphi_1\psi_0 = 2\theta_1\lambda$. Since $\upsilon_1\varphi_1\psi_1 = 0$ and $\varphi_1\psi_1 \in \mathrm{H}^4(G, M^{(a)}(1))$, we have $\varphi_1\psi_1 = \varkappa\theta_0\nu\lambda$ for some $\varkappa \in \mathbb{Z}$. Multiplying the last identity by θ_0 and using the identities proved above, we see that $\varkappa \equiv 2 \pmod r$, whence, $\varphi_1\psi_1 = 2\theta_0\nu\lambda$.

Again, direct calculations show that $\theta_0\varphi_0 = 2\psi_0$, and that in $\mathrm{H}^2(G, M^{(ab)}(0))$ we have

$$\theta_0\varphi_1 = (r+2)f_{2,1}^{(ab)} - \frac{r+2}{2}g_{2,1}^{(ab)} = 2f_{2,1}^{(ab)} \quad \text{and} \quad \varphi_0\theta_1 = 2f_{2,1}^{(ab)} - g_{2,1}^{(ab)} = (r+2)f_{2,1}^{(ab)},$$

i.e., $\theta_0\psi_1 = 2\varphi_1$ and $\theta_1\psi_0 = (2+r)\varphi_1$. Using part 4) of Lemma 6, we check that

$$\mu(\varphi_1 \otimes \mathrm{id})T^2(\theta_1)(e_3^4) = \frac{r^2+r+2}{2}x_1(ab+ab^t),$$
$$\mu(\varphi_1 \otimes \mathrm{id})T^2(\theta_1)(e_4^4) = -\frac{r^2+5r+4}{2}(ab+ab^t) + rab,$$
$$\mu(\varphi_1 \otimes \mathrm{id})T^2(\theta_1)(e_5^4) = (r+2)(ab+ab^t).$$

From (3.8) and (3.9), it follows that in $\mathrm{H}^4(G, M^{(ab)}(0))$ we have

$$\varphi_1\theta_1 = \frac{r^2+r+2}{2}f_{4,1}^{(ab)} - \frac{r^2+5r+4}{2}g_{4,2}^{(ab)} + (r+2)f_{4,2}^{(ab)} = 2f_{4,2}^{(ab)} = 2\psi_0\nu.$$

Finally, since $\mathrm{H}^{2n+1}(G, M^{(ab)}) = 0$ for any n, we have

$$\varphi_0\rho_1 = \varphi_0\rho_2 = \varphi_1\rho_1 = \varphi_1\rho_2 = 0. \qquad \square$$

Corollary 5. *The algebra* $\mathrm{HH}^*(\mathbb{Z}[M_{2^\ell}])$ *is commutative.*

Proof. This follows from the fact that the product of any two generators of odd degree (namely, they are exactly ω, ρ_1, and ρ_2) is equal to 0. $\square$

§7. Description of the Hochschild cohomology ring

In this section, we describe the Hochschild cohomology algebra of the group G in terms of generators and defining relations. Put $\mathcal{X} = \{\lambda, \upsilon_1, \upsilon_2, \nu, \omega, \varphi_0, \varphi_1, \theta_0, \theta_1, \rho_1, \rho_2, \psi_0, \psi_1\}$; the elements of this set were defined in §4. Then we consider the new set $\widetilde{\mathcal{X}} = \{\widetilde{x} \mid x \in \mathcal{X}\}$, which is in a one-to-one correspondence with $\mathcal{X}$. On the algebra $\mathbb{Z}[\widetilde{\mathcal{X}}]$, we introduce a grading such that $\deg \widetilde{x} = \deg x$ for any $x \in \mathcal{X}$.

We consider the algebra $\mathcal{H} = \mathbb{Z}[\widetilde{\mathcal{X}}]/\mathcal{I}$, where $\mathcal{I}$ is the homogeneous ideal generated by the following elements:

$$2\widetilde{\upsilon}_1,\ r\widetilde{\upsilon}_2,\ 2r\widetilde{\nu},\ 2\widetilde{\omega},\ 2r\widetilde{\varphi}_1,\ r\widetilde{\theta}_0\widetilde{\nu},\ r\widetilde{\theta}_1,\ 2\widetilde{\rho}_1,\ 2\widetilde{\rho}_2,\ 2r\widetilde{\psi}_1,$$
$$\widetilde{\upsilon}_1\widetilde{\upsilon}_2,\ \widetilde{\upsilon}_2^2 - 4\widetilde{\nu},\ \widetilde{\upsilon}_2\widetilde{\omega},\ \widetilde{\omega}^2,\ \widetilde{\lambda}^r - 1,$$
$$\widetilde{\psi}_0^2 - 2\big(\widetilde{\lambda} + \widetilde{\lambda}^{\frac{r+2}{2}}\big),\ \widetilde{\psi}_0\widetilde{\psi}_1 - \big(\widetilde{\lambda} + \widetilde{\lambda}^{\frac{r+2}{2}}\big)\widetilde{\upsilon}_2,\ \widetilde{\psi}_1^2 - \widetilde{\psi}_0^2\widetilde{\nu},$$
$$\widetilde{\psi}_0\widetilde{\upsilon}_1 - r\widetilde{\psi}_1,\ \widetilde{\psi}_1\widetilde{\upsilon}_1 - r\widetilde{\psi}_0\widetilde{\nu},\ \widetilde{\psi}_0\widetilde{\upsilon}_2 - (2+r)\widetilde{\psi}_1,\ \widetilde{\psi}_1\widetilde{\upsilon}_2 - (2+r)\widetilde{\psi}_0\widetilde{\nu},\ \widetilde{\psi}_0\widetilde{\omega},\ \widetilde{\psi}_1\widetilde{\omega},$$
$$\widetilde{\psi}_0\widetilde{\lambda}^{\frac{r}{2}} - \widetilde{\psi}_0,\ \widetilde{\psi}_1\widetilde{\lambda}^{\frac{r}{2}} - (1+r)\widetilde{\psi}_1,$$
$$\widetilde{\theta}_0^2 - 2\big(1 + \widetilde{\lambda}^{\frac{r}{2}}\big),\ \widetilde{\theta}_0\widetilde{\theta}_1 - \big(1 + \widetilde{\lambda}^{\frac{r}{2}}\big)(\widetilde{\upsilon}_1 + \widetilde{\upsilon}_2),\ \widetilde{\theta}_1^2 - \widetilde{\upsilon}_1^2 - \widetilde{\theta}_0^2\widetilde{\nu},\ \widetilde{\theta}_0\widetilde{\rho}_1,$$
$$\widetilde{\theta}_1\widetilde{\rho}_1 - \widetilde{\theta}_0\widetilde{\rho}_2,\ \widetilde{\theta}_0\widetilde{\rho}_2 - \big(1 + \widetilde{\lambda}^{\frac{r}{2}}\big)\widetilde{\omega},\ \widetilde{\theta}_1\widetilde{\rho}_2 - \widetilde{\upsilon}_1\widetilde{\omega},\ \widetilde{\rho}_1^2,\ \widetilde{\rho}_1\widetilde{\rho}_2,\ \widetilde{\rho}_2^2,$$
$$\widetilde{\theta}_0\widetilde{\upsilon}_2 - 2\widetilde{\theta}_1,\ \widetilde{\theta}_1\widetilde{\upsilon}_2 - 2\widetilde{\theta}_0\widetilde{\nu},\ \widetilde{\theta}_0\widetilde{\omega} - \widetilde{\rho}_1\widetilde{\upsilon}_1,\ \widetilde{\theta}_1\widetilde{\omega} - \widetilde{\rho}_2\widetilde{\upsilon}_1,\ \widetilde{\rho}_1\widetilde{\upsilon}_2,\ \widetilde{\rho}_2\widetilde{\upsilon}_2,\ \widetilde{\rho}_1\widetilde{\omega},\ \widetilde{\rho}_2\widetilde{\omega},$$
$$\widetilde{\theta}_0\widetilde{\lambda}^{\frac{r}{2}} - \widetilde{\theta}_0,\ \widetilde{\theta}_1\widetilde{\lambda}^{\frac{r}{2}} - \widetilde{\theta}_1 - \widetilde{\theta}_0\widetilde{\upsilon}_1,\ \widetilde{\rho}_1\widetilde{\lambda}^{\frac{r}{2}} - \widetilde{\rho}_1,\ \widetilde{\rho}_2\widetilde{\lambda}^{\frac{r}{2}} - \widetilde{\rho}_2 - \widetilde{\rho}_1\widetilde{\upsilon}_1,$$
$$\widetilde{\varphi}_0^2 - 2\big(\widetilde{\lambda} + \widetilde{\lambda}^{\frac{r+2}{2}}\big),\ \widetilde{\varphi}_0\widetilde{\varphi}_1 - \Big(1 + \frac{r}{2}\Big)\big(\widetilde{\lambda} + \widetilde{\lambda}^{\frac{r+2}{2}}\big)\widetilde{\upsilon}_2,\ \widetilde{\varphi}_1^2 - \widetilde{\varphi}_0^2\widetilde{\nu},$$
$$\widetilde{\varphi}_0\widetilde{\upsilon}_1,\ \widetilde{\varphi}_1\widetilde{\upsilon}_1,\ \widetilde{\varphi}_0\widetilde{\omega},\ \widetilde{\varphi}_1\widetilde{\omega},\ \widetilde{\varphi}_0\widetilde{\upsilon}_2 - 2\widetilde{\varphi}_1,\ \widetilde{\varphi}_1\widetilde{\upsilon}_2 - 2\widetilde{\varphi}_0\widetilde{\nu},\ \widetilde{\varphi}_0\widetilde{\lambda}^{\frac{r}{2}} - \widetilde{\varphi}_0,\ \widetilde{\varphi}_1\widetilde{\lambda}^{\frac{r}{2}} - (1+r)\widetilde{\varphi}_1,$$
$$\widetilde{\psi}_0\widetilde{\theta}_0 - 2\widetilde{\varphi}_0,\ \widetilde{\psi}_0\widetilde{\theta}_1 - 2\widetilde{\varphi}_1,\ \widetilde{\psi}_1\widetilde{\theta}_0 - 2\widetilde{\varphi}_1,\ \widetilde{\psi}_1\widetilde{\theta}_1 - 2\widetilde{\varphi}_0\widetilde{\nu},\ \widetilde{\psi}_0\widetilde{\rho}_1,\ \widetilde{\psi}_0\widetilde{\rho}_2,\ \widetilde{\psi}_1\widetilde{\rho}_1, \widetilde{\psi}_1\widetilde{\rho}_2,$$
$$\widetilde{\varphi}_0\widetilde{\psi}_0 - 2\widetilde{\theta}_0\widetilde{\lambda},\ \widetilde{\varphi}_0\widetilde{\psi}_1 - 2\widetilde{\theta}_1\widetilde{\lambda},\ \widetilde{\varphi}_1\widetilde{\psi}_0 - 2\widetilde{\theta}_1\widetilde{\lambda},\ \widetilde{\varphi}_1\widetilde{\psi}_1 - 2\widetilde{\theta}_0\widetilde{\nu}\widetilde{\lambda},$$
$$\widetilde{\theta}_0\widetilde{\varphi}_0 - 2\widetilde{\psi}_0,\ \widetilde{\theta}_0\widetilde{\varphi}_1 - 2\widetilde{\psi}_1,\ \widetilde{\theta}_1\widetilde{\varphi}_0 - (2+r)\widetilde{\psi}_1,\ \widetilde{\theta}_1\widetilde{\varphi}_1 - 2\widetilde{\psi}_0\widetilde{\nu},\ \widetilde{\varphi}_0\widetilde{\rho}_1,\ \widetilde{\varphi}_0\widetilde{\rho}_2,\ \widetilde{\varphi}_1\widetilde{\rho}_1,\ \widetilde{\varphi}_1\widetilde{\rho}_2.$$

Theorem 2. *There is an isomorphism* $\mathrm{HH}^*(R) \simeq \mathcal{H}$ *of graded* $\mathbb{Z}$*-algebras.*

Proof. Since $\mathrm{HH}^*(R) \simeq \mathrm{H}^*(G, \widetilde{R})$, it suffices to prove that there exists an isomorphism of graded $\mathbb{Z}$-algebras $\mathcal{H} \simeq \mathrm{H}^*(G, \widetilde{R})$. From Propositions 2 and 3, it follows that there exists a surjective homomorphism $\pi : \mathcal{H} \to \mathrm{H}^*(G, \widetilde{R})$ of graded $\mathbb{Z}$-algebras, specifically, it is defined on the classes of elements of the set $\widetilde{\mathcal{X}}$ by the formula $\pi(\widetilde{x}+\mathcal{I}) = x$. For simplicity, in what follows we write $\widetilde{x}$ instead of $\widetilde{x} + \mathcal{I}$. Let $\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$ be a decomposition of $\mathcal{H}$ into homogeneous direct summands. Now it suffices to establish that $\pi_n = \pi|_{\mathcal{H}_n}$ is an isomorphism for any $n \geq 0$.

Using the proof of Proposition 2, we conclude easily that multiplication by ν induces an embedding of $\mathrm{H}^n(G, \widetilde{R})$ into $\mathrm{H}^{n+4}(G, \widetilde{R})$ for $n > 0$. By definition, we have

$$\mathcal{H}_n = {}_{\mathbb{Z}}\Big\langle \widetilde{x}_1 \dots \widetilde{x}_p \mid p \geq 0,\ x_i \in \mathcal{X}\ (1 \leq i \leq p),\ \sum_{i=1}^{p} \deg x_i = n \Big\rangle.$$

Observe that $\mathcal{X}' = \{\omega, \rho_1, \rho_2\}$ is the set of all elements of odd degree in $\mathcal{X}$, and moreover, $\widetilde{\omega}^2, \widetilde{\rho}_1^2, \widetilde{\rho}_2^2, \widetilde{\rho}_1\widetilde{\omega}, \widetilde{\rho}_2\widetilde{\omega}, \widetilde{\rho}_1\widetilde{\rho}_2 \in \mathcal{I}$. Consequently,

$$\mathcal{H}_{2k} = {}_{\mathbb{Z}}\Big\langle \widetilde{x}_1 \dots \widetilde{x}_p \mid p \geq 0,\ x_i \in \mathcal{X} \setminus \mathcal{X}'\ (1 \leq i \leq p),\ \sum_{i=1}^{p} \deg x_i = 2k \Big\rangle, \tag{7.1}$$

$$\mathcal{H}_{2k+1} = {}_{\mathbb{Z}}\Big\langle \widetilde{x}_1 \dots \widetilde{x}_p \mid p \geq 1, x_i \in \mathcal{X} \setminus \mathcal{X}' (1 \leq i \leq p-1), x_p \in \mathcal{X}', \sum_{i=1}^{p} \deg x_i = 2k+1 \Big\rangle. \tag{7.2}$$

First, we check that π_{2k} is an isomorphism. We prove this by induction on k. Analyzing the form of the generators of the ideal $\mathcal{I}$, we easily show that $\mathcal{H}_0$ consists of the $\mathbb{Z}$-linear combinations of elements of the set $\{\widetilde{\lambda}^i\}_{0\leq i\leq r-1} \cup \{\widetilde{\varphi}_0\widetilde{\lambda}^i, \widetilde{\theta}_0\widetilde{\lambda}^i, \widetilde{\psi}_0\widetilde{\lambda}^i\}_{0\leq i\leq \frac{r-2}{2}}$, $\mathcal{H}_2$ consists of the $\mathbb{Z}$-linear combinations of elements of the set $\{\widetilde{\upsilon}_1\widetilde{\lambda}^i, \widetilde{\upsilon}_2\widetilde{\lambda}^i\}_{0\leq i\leq r-1} \cup \{\widetilde{\varphi}_1\widetilde{\lambda}^i, \widetilde{\theta}_0\widetilde{\upsilon}_1\widetilde{\lambda}^i, \widetilde{\theta}_1\widetilde{\lambda}^i, \widetilde{\psi}_1\widetilde{\lambda}^i\}_{0\leq i\leq \frac{r-2}{2}}$, and $\mathcal{H}_4$ consists of the $\mathbb{Z}$-linear combinations of elements of the set

$$\{\widetilde{\upsilon}_1^2\widetilde{\lambda}^i, \widetilde{\nu}\widetilde{\lambda}^i\}_{0\leq i\leq r-1} \cup \{\widetilde{\varphi}_0\widetilde{\nu}\widetilde{\lambda}^i, \widetilde{\theta}_0\widetilde{\upsilon}_1^2\widetilde{\lambda}^i, \widetilde{\theta}_1\widetilde{\upsilon}_1\widetilde{\lambda}^i, \widetilde{\theta}_0\widetilde{\nu}\widetilde{\lambda}^i, \widetilde{\psi}_0\widetilde{\nu}\widetilde{\lambda}^i\}_{0\leq i\leq \frac{r-2}{2}}.$$

Looking at the list of the generators of the ideal $\mathcal{I}$, we see that π_0, π_2, and π_4 are isomorphisms. Let $k \geq 3$, and suppose that π_{2j} is an isomorphism for any $0 \leq j \leq k-1$. We shall prove that π_{2k} is also an isomorphism. First, we prove that $\mathcal{H}_{2k} = \widetilde{\upsilon}_1\mathcal{H}_{2k-2} + \widetilde{\nu}\mathcal{H}_{2k-4}$, i.e., any element of the form $\widetilde{x}_1 \dots \widetilde{x}_p$, where $p \geq 0$, $x_i \in \mathcal{X} \setminus \mathcal{X}'$ $(1 \leq i \leq p)$, and $\sum_{i=1}^{p} \deg x_i = 2k$, can be presented in $\mathcal{H}$ in the form $\widetilde{\upsilon}_1 X_{2k-2} + \widetilde{\nu} X_{2k-4}$ with $X_{2k-2} \in \mathcal{H}_{2k-2}$, $X_{2k-4} \in \mathcal{H}_{2k-4}$. The condition $\sum_{i=1}^{p} \deg x_i = 2k$ implies that either $x_i \in \{\upsilon_1, \nu\}$ for some $1 \leq i \leq p$, or $x_i, x_j \in \{\upsilon_2, \varphi_1, \theta_1, \psi_1\}$ for some $1 \leq i < j \leq p$. In the former case, the claim in question is evident, and in the latter case this claim follows from the fact that the product of any two elements of the set $\{\widetilde{\upsilon}_2, \widetilde{\varphi}_1, \widetilde{\theta}_1, \widetilde{\psi}_1\}$ can be presented in the required form. Now we have

$$\begin{aligned}
\mathcal{H}_{2k} &= \widetilde{\upsilon}_1^{k-1}\mathcal{H}_2 + \widetilde{\nu}\mathcal{H}_{2k-4} \\
&= \widetilde{\upsilon}_1^{k}\mathcal{H}_0 + \widetilde{\upsilon}_1^{k-1}\widetilde{\upsilon}_2\mathcal{H}_0 + \widetilde{\upsilon}_1^{k-1}\widetilde{\varphi}_1\mathcal{H}_0 + \widetilde{\upsilon}_1^{k-1}\widetilde{\theta}_1\mathcal{H}_0 + \widetilde{\upsilon}_1^{k-1}\widetilde{\psi}_1\mathcal{H}_0 + \widetilde{\nu}\mathcal{H}_{2k-4} \\
&= \widetilde{\upsilon}_1^{k}\mathcal{H}_0 + \widetilde{\upsilon}_1^{k-1}\widetilde{\theta}_1\mathcal{H}_0 + \widetilde{\nu}\mathcal{H}_{2k-4}.
\end{aligned}$$

Since

$$\begin{aligned}
\widetilde{\upsilon}_1^2\mathcal{H}_0 &= {}_{\mathbb{Z}}\Big\langle \{\widetilde{\upsilon}_1^2\widetilde{\lambda}^i\}_{0\leq i\leq r-1} \cup \{\widetilde{\upsilon}_1^2\widetilde{\varphi}_0\widetilde{\lambda}^i, \widetilde{\upsilon}_1^2\widetilde{\theta}_0\widetilde{\lambda}^i, \widetilde{\upsilon}_1^2\widetilde{\psi}_0\widetilde{\lambda}^i\}_{0\leq i\leq \frac{r-2}{2}} \Big\rangle \\
&= {}_{\mathbb{Z}}\Big\langle \{\widetilde{\upsilon}_1^2\widetilde{\lambda}^i\}_{0\leq i\leq r-1} \cup \{\widetilde{\upsilon}_1^2\widetilde{\theta}_0\widetilde{\lambda}^i\}_{0\leq i\leq \frac{r-2}{2}} \Big\rangle,
\end{aligned}$$

$$\begin{aligned}\widetilde{\upsilon}_1\widetilde{\theta}_1\mathcal{H}_0 &= {}_{\mathbb{Z}}\Big\langle \big\{\widetilde{\upsilon}_1\widetilde{\theta}_1\widetilde{\lambda}^i\big\}_{0\le i\le r-1} \cup \big\{\widetilde{\upsilon}_1\widetilde{\theta}_1\widetilde{\varphi}_0\widetilde{\lambda}^i, \widetilde{\upsilon}_1\widetilde{\theta}_1\widetilde{\theta}_0\widetilde{\lambda}^i, \widetilde{\upsilon}_1\widetilde{\theta}_1\widetilde{\psi}_0\widetilde{\lambda}^i\big\}_{0\le i\le \frac{r-2}{2}}\Big\rangle \\ &= {}_{\mathbb{Z}}\Big\langle \big\{\widetilde{\upsilon}_1\widetilde{\theta}_1\widetilde{\lambda}^i\big\}_{0\le i\le r-1} \cup \big\{2\widetilde{\upsilon}_1\widetilde{\psi}_1\widetilde{\lambda}^i, \widetilde{\upsilon}_1(1+\widetilde{\lambda}^{\frac{r}{2}})(\upsilon_1+\upsilon_2)\widetilde{\lambda}^i, (2+r)\widetilde{\upsilon}_1\widetilde{\varphi}_1\widetilde{\lambda}^i\big\}_{0\le i\le \frac{r-2}{2}}\Big\rangle \\ &= {}_{\mathbb{Z}}\Big\langle \big\{\widetilde{\upsilon}_1\widetilde{\theta}_1\widetilde{\lambda}^i\big\}_{0\le i\le r-1} \cup \big\{\widetilde{\upsilon}_1^2(\widetilde{\lambda}^i+\widetilde{\lambda}^{i+\frac{r}{2}})\big\}_{0\le i\le \frac{r-2}{2}}\Big\rangle,\end{aligned}$$

and

$$ {}_{\mathbb{Z}}\Big\langle \big\{\widetilde{\upsilon}_1\widetilde{\theta}_1\widetilde{\lambda}^i\big\}_{0\le i\le r-1}\Big\rangle = {}_{\mathbb{Z}}\Big\langle \big\{\widetilde{\upsilon}_1\widetilde{\theta}_1\widetilde{\lambda}^i, \widetilde{\upsilon}_1\widetilde{\theta}_1\widetilde{\lambda}^i + \widetilde{\upsilon}_1^2\widetilde{\theta}_0\widetilde{\lambda}^i\big\}_{0\le i\le \frac{r-2}{2}}\Big\rangle,$$

we obtain

$$\mathcal{H}_{2k} = {}_{\mathbb{Z}}\Big\langle \big\{\widetilde{\upsilon}_1^k\widetilde{\lambda}^i\big\}_{0\le i\le r-1} \cup \big\{\widetilde{\upsilon}_1^k\widetilde{\theta}_0\widetilde{\lambda}^i, \widetilde{\upsilon}_1^{k-1}\widetilde{\theta}_1\widetilde{\lambda}^i\big\}_{0\le i\le \frac{r-2}{2}}\Big\rangle + \widetilde{\nu}\mathcal{H}_{2k-4}.$$

Since $2\upsilon_1 = 0$ and $|\mathcal{H}_{2k-4}| = |\,\mathrm{H}^{2k-4}(G,\widetilde{R})|$ by the inductive hypothesis, we conclude that

$$|\mathcal{H}_{2k}| \le 2^{2r}|\mathcal{H}_{2k-4}| = 2^{2r}|\,\mathrm{H}^{2k-4}(G,\widetilde{R})| = |\,\mathrm{H}^{2k}(G,\widetilde{R})| \text{ (see Theorem 1).}$$

Since the map π_{2k} is surjective, it must be an isomorphism.

Now we prove that π_{2k+1} is an isomorphism. Using the form of the generators of the ideal $\mathcal{I}$, we can rewrite (7.2) as follows:

$$\mathcal{H}_{2k+1} = {}_{\mathbb{Z}}\Big\langle \widetilde{x}_1\dots\widetilde{x}_p \mid p\ge 1, x_i\in\{\upsilon_1,\nu,\lambda\}(1\le i\le p-1), x_p\in\mathcal{X}', \sum_{i=1}^{p}\deg x_i = n\Big\rangle. \tag{7.3}$$

We use induction on k to check that $|\mathcal{H}_{2k+1}| \le |\,\mathrm{H}^{2k+1}(G,\widetilde{R})|$. First, observe that $\mathcal{H}_1 = 0 = \mathrm{H}^1(G,\widetilde{R})$ and $|\mathcal{H}_3| = \big|\,{}_{\mathbb{Z}}\big\langle\{\widetilde{\rho}_1\widetilde{\lambda}^i\}_{0\le i\le\frac{r-2}{2}}\big\rangle\big| \le 2^{\frac{r}{2}} = |\,\mathrm{H}^3(G,\widetilde{R})|$. Next, we assume that $k\ge 2$ and $|\mathcal{H}_{2j+1}| \le |\,\mathrm{H}^{2j+1}(G,\widetilde{R})|$ for any $0\le j\le k-1$. With the help of (7.3), we verify that

$$\mathcal{H}_{2k+1} = \widetilde{\nu}\mathcal{H}_{2k-3} + {}_{\mathbb{Z}}\Big\langle \big\{\widetilde{\rho}_1\widetilde{\upsilon}_1^{k-1}\widetilde{\lambda}^i\big\}_{0\le i\le\frac{r-2}{2}} \cup \big\{\widetilde{\rho}_2\widetilde{\upsilon}_1^{k-2}\widetilde{\lambda}^i\big\}_{0\le i\le\frac{r-2}{2}} \cup \big\{\widetilde{\omega}\widetilde{\upsilon}_1^{k-2}\widetilde{\lambda}^i\big\}_{0\le i\le r-1}\Big\rangle.$$

Then $|\mathcal{H}_{2k+1}| \le 2^{2r}|\mathcal{H}_{2k-3}| \le 2^{2r}|\,\mathrm{H}^{2k-3}(G,\widetilde{R})| = |\,\mathrm{H}^{2k+1}(G,\widetilde{R})|$ (see Theorem 1). Hence, we have $|\mathcal{H}_{2k+1}| \le |\,\mathrm{H}^{2k+1}(G,\widetilde{R})|$ for any $k\ge 0$, and the surjectivity of π implies that π_{2k+1} is an isomorphism. □

Corollary 6. *The graded algebra* $\mathrm{H}^*(G,\mathbb{Z})$ *is isomorphic to* $\mathbb{Z}[\widetilde{\upsilon}_1,\widetilde{\upsilon}_2,\widetilde{\nu},\widetilde{\omega}]/I$, *where* $\deg\widetilde{\upsilon}_1 = \deg\widetilde{\upsilon}_2 = 2$, $\deg\widetilde{\nu} = 4$, $\deg\widetilde{\omega} = 5$, *and* I *is the ideal generated by the elements* $2\widetilde{\upsilon}_1$, $r\widetilde{\upsilon}_2$, $2r\widetilde{\nu}$, $2\widetilde{\omega}$, $\widetilde{\upsilon}_1\widetilde{\upsilon}_2$, $\widetilde{\upsilon}_2^2 - 4\widetilde{\nu}$, $\widetilde{\upsilon}_2\widetilde{\omega}$, *and* $\widetilde{\omega}^2$.

Proof. This follows from Corollary 4 and the proof of Theorem 2. □

References

[1] A. I. Generalov, *The Hochschild cohomology of the integer group ring of a dihedral group.* I, *The even case*, Algebra i Analiz **19** (2007), no. 5, 70–123; English transl., St. Petersburg Math. J. **19** (2008), no. 5, 723–763. MR2381942 (2009e:16015)

[2] ———, *Hochschild cohomology for the integer group ring of the semidihedral group*, Zap. Nauchn. Sem. S.-Petersburg. Otdel. Mat. Inst. Steklov. (POMI) **388** (2011), 119–151; English transl., J. Math. Sci. (N.Y.) **183** (2012), no. 5, 640–657. MR2822518 (2012e:16024)

[3] T. Hayami, *Hochschild cohomology ring of the integral group ring of the generalized quaternion group*, SUT J. Math. **38** (2002), no. 1, 83–126. MR1919950 (2003f:16014)

[4] ———, *On Hochschild cohomology ring of the integral group ring of the quaternion group*, Tsukuba J. Math. **29** (2005), no. 2, 363–387. MR2177018 (2006j:16016)

[5] ———, *Hochschild cohomology ring of the integral group ring of dihedral groups*, Tsukuba J. Math. **31** (2007), no. 1, 99–127. MR2337122 (2008d:16016)

[6] ———, *Hochschild cohomology ring of the integral group ring of the semidehidral* 2*-group*, Algebra Colloq. **18** (2011), no. 2, 241–258. MR2785780 (2012c:16031)

[7] S. F. Siegel and S. J. Witherspoon, *The Hochschild cohomology ring of a group algebra*, Proc. London Math. Soc.(3) **79** (1999), no. 1, 131–157. MR1687539 (2000b:16016)

[8] C. T. C. Wall, *Resolutions for extensions of groups*, Proc. Cambridge Philos. Soc. **57** (1961), 251–255. MR0178046 (31:2304)

[9] A. I. Generalov and S. O. Ivanov, *Bimodule resolution of a group algebra*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **365** (2009), 143–150; English transl., J. Math. Sci. (N.Y.) **161** (2009), no. 4, 537–541. MR2749138 (2012b:16016)

[10] I. J. Leary, *The cohomology of certain groups*, PhD Thesis, Cambridge, 1992.

[11] C. B. Thomas, *Chern classes and metacyclic p-groups*, Mathematika **18** (1971), 196–200. MR0299699 (45:8747)

[12] D. J. Benson, *Representations and cohomology.* II*, Cohomology of groups and modules*, Cambridge Stud. Adv. Math. vol. 30, Cambridge Univ. Press, Cambridge, 1998. MR1634407 (99f:20001b)

[13] S. Eilenberg and S. MacLane, *Cohomology theory in abstract groups.* I, Ann. of Math. (2) **48** (1947), 51–78. MR0019092 (8:367f)

[14] S. MacLane, *Homology*, Grundlehren Math. Wiss., Bd. 114, Springer-Verlag, Berlin, 1963. MR0156879 (28:122)

[15] K. Brown, *Cohomology of groups*, Grad. Texts in Math., vol. 87, Springer, Berlin, 1982. MR672956 (83k:20002)

Research Group, Era7 Bioinformatics, Plaza Campo Verde 3 Atico, Granada 18001, Spain
E-mail address: aalekhin@ohnosequences.com

Department of Mathematics and Mechanics, St. Petersburg State University, Universitetskii pr., 28, Staryi Peterhof, St. Petersburg 198504, Russia
E-mail address: wolf86_666@list.ru

Department of Mathematics and Mechanics, St. Petersburg State University, Universitetskii pr., 28, Staryi Peterhof, St. Petersburg 198504, Russia
E-mail address: ageneralov@gmail.com

Received 10/MAR/2013

Translated by A. I. GENERALOV