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SHARP ESTIMATES INVOLVING A_∞ AND $L\log L$ CONSTANTS, AND THEIR APPLICATIONS TO PDE

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Abstract. It is a well-known fact that the union $\bigcup_{p>1} RH_p$ of the Reverse Hölder classes coincides with the union $\bigcup_{p>1} A_p = A_\infty$ of the Muckenhoupt classes, but the A_∞ constant of the weight w, which is a limit of its A_p constants, is not a natural characterization for the weight in Reverse Hölder classes. In the paper, the RH_1 condition is introduced as a limiting case of the RH_p inequalities as p tends to 1, and a sharp bound is found on the RH_1 constant of the weight w in terms of its A_∞ constant. Also, the sharp version of the Gehring theorem is proved for the case of $p = 1$, completing the answer to the famous question of Bojarski in dimension one.

The results are illustrated by two straightforward applications to the Dirichlet problem for elliptic PDE's.

Despite the fact that the Bellman technique, which is employed to prove the main theorems, is not new, the authors believe that their results are useful and prove them in full detail.

§1. Definitions and main results

We say that w is a *weight* if it is a locally integrable function on the real line, positive almost everywhere (with respect to the Lebesgue measure). Let $m_J w$ be the average of a weight w over a given interval $J \subset \mathbb{R}$:

$$m_J w := \frac{1}{|J|}\int_J w\,dx.$$

A weight w belongs to the *Muckenhoupt class* A_p whenever its Muckenhoupt constant $[w]_{A_p}$ is finite:

$$[w]_{A_p} := \sup_{J\subset\mathbb{R}} m_J w\big(m_J\big(w^{-\frac{1}{p-1}}\big)\big)^{p-1} < \infty. \tag{1.1}$$

Note that, by Hölder's inequality, $[w]_{A_p} \ge 1$ for all $1 < p < \infty$, and that the following inclusion is true:

$$\text{if } 1 < p \le q < \infty \quad \text{then} \quad A_p \subseteq A_q, \quad [w]_{A_q} \le [w]_{A_p}.$$

So, for $1 < p < \infty$ the Muckenhoupt classes A_p form an increasing chain. There are two natural limits of it — as p approaches 1 and as p goes to ∞. We shall be interested in the limiting case as $p \to \infty$, $A_\infty = \bigcup_{p>1} A_p$. There are several equivalent definitions of it. We state the one that we are going to use (the natural limit of the A_p conditions, which also defines the A_∞ constant of the weight w); for other equivalent definitions, see [GaRu, Gr] or [St93]:

$$w \in A_\infty \iff [w]_{A_\infty} := \sup_{J\subset\mathbb{R}} m_J w\, e^{-m_J(\log w)} < \infty. \tag{1.2}$$

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A weight w belongs to the *Reverse Hölder class* RH_p $(1 < p < \infty)$ if

$$[w]_{RH_p} := \sup_{J \subset \mathbb{R}} \frac{(m_J w^p)^{1/p}}{m_J w} < \infty. \tag{1.3}$$

Note that, by Hölder's inequality, the Reverse Hölder classes possess the following property:

$$\text{if } 1 < p \le q < \infty, \text{ then } RH_q \subseteq RH_p \quad \text{and} \quad 1 \le [w]_{RH_p} \le [w]_{RH_q},$$

which is similar to the inclusion chain for the A_p classes, except inclusion runs in the reverse direction. Similarly, we can consider two limiting cases RH_∞ (the smallest) and RH_1 (the largest). Same as in the case of Muckenhoupt classes, we are more interested in the largest one, we call it $RH_1 := \bigcup_{p>1} RH_p$.

For A_∞ and RH_1, in 1974 Coifman and Fefferman showed that $A_\infty = \bigcup_{p>1} RH_p = RH_1$. Now it is well known (see [GaRu, Gr, St93]) that if $w \in A_p$, then $w \in RH_q$ for some $1 < q < \infty$ and *vice versa*. In [Gr], the dependencies of p and q and of A_p and RH_q constants in any dimension were traced roughly. The A_1 and RH_∞ classes were not overlooked either, a lot of information about them can be found in [CrN]. Exact dependencies are much harder to trace, but for $1 \le p \le \infty$ and $1 < q \le \infty$, in the one-dimensional case sharp relationships between A_p and RH_q were found in [Va].

The question is whether anything is missing in the precise relationships between A_p and RH_q constants.

The answer is "Yes" and now we describe the missing little piece of this puzzle.

The union of the Reverse Hölder classes is A_∞, but the A_∞ constant (the natural limit of the A_p constants) has nothing to do with the Reverse Hölder constants. The natural limit as $p \to 1^+$ of the Reverse Hölder inequalities is the following condition, which we take as a definition of the class RH_1:

$$w \in RH_1 \iff [w]_{RH_1} := \sup_{J \subset \mathbb{R}} m_J\Big(\frac{w}{m_J w} \log \frac{w}{m_J w}\Big) < \infty, \tag{1.4}$$

where log is the regular logarithm with base e, which can be negative. Nevertheless, by the Jensen inequality, the RH_1 constant defined in this way is always nonnegative.

The RH_1 constant of the weight w is the natural limit of the RH_p constants in the sense that, for every $I \subset \mathbb{R}$,

$$m_I\Big(\frac{w}{m_I w} \log \frac{w}{m_I w}\Big) = \lim_{p \to 1^+} \frac{p}{p-1} \log \frac{m_I(w^p)^{\frac{1}{p}}}{m_I w}. \tag{1.5}$$

We want to make a remark about this definition.

Remark 1. Inequality (1.4) can be rewritten in the following way:

$$m_J\left(w \log(w)\right) \le m_J w \log(m_J w) + Q m_J w.$$

Note that, since the function $x \log x$ is concave, by Jensen's inequality we also have

$$m_J w \log(m_J w) \le m_J\left(w \log(w)\right).$$

Condition (1.4) is actually much more natural whenever one deals with Reverse Hölder conditions rather than with A_p conditions, see, for example, [Fe, Cor07, HyPer].

There is no standard notation here, sometimes this class is called $RH_{L \log L}$, because (1.4) is the reverse Jensen inequality for the function $x \log x$; otherwise, it is called G_1 to emphasize the contribution of Gehring to the study of the Reverse Hölder classes. Sometimes, for the RH_1 constant one takes $\sup_{J \subset \mathbb{R}} \exp\left\{m_J\big(\frac{w}{m_J w} \log \frac{w}{m_J w}\big)\right\}$ to remove the logarithm on the right-hand side of (1.5). We keep our notation because it is shorter and it is clear that we are working with the Reverse Hölder condition.

Different ways to define the RH_1 constant of a weight w. First, observe that, trivially, the logarithm in the definition of the RH_1 constant can be replaced by $\log^+(x)$, ($\log^+(x) = \max(\log x, 0)$) or $\log(e+x)$.

Second, from the Stein lemma (see [St69]), we know that

$$3^{-n} m_I\big(M(f\chi_I)\big) \le m_I\Big(f \log\Big(e + \frac{f}{m_I f}\Big)\Big) \le 2^n m_I(f\chi_I).$$

Thus, an equivalent way to define the RH_1 constant is

$$[w]_{RH_1'} := \sup \frac{1}{m_I w} \int_I M(w\chi_I), \tag{1.6}$$

which, indeed, is one of the ways to define the class A_∞; see, for example, [Wil] or [HyPer].

The Reverse Hölder and A_∞ constants can also be defined by using Luxemburg norms. The same is true for the RH_1 constant. First, we define the Luxemburg norm of a function in the following way: for an Orlitz function $\Phi\colon [0,\infty] \mapsto [0,\infty]$, we define be

$$\|w\|_{\Phi(L),I} := \inf\Big\{\lambda > 0 \,:\, \frac{1}{|I|}\int_I \Phi\Big(\frac{|w|}{\lambda}\Big) \le 1\Big\}.$$

Iwaniec and Verde in [IV] showed that

$$\|w\|_{L\log L,I} \le \int_I \log\Big(e + \frac{w}{m_I w}\Big)\,dx \le 2\,\|w\|_{L\log L,I}$$

for every w and $I \subset \mathbb{R}^n$, so that another equivalent definition of the RH_1 constant of the weight w is

$$[w]_{RH_1''} := \sup_{I\subset\mathbb{R}} \frac{\|w\|_{L\log L,I}}{\|w\|_{L,I}}. \tag{1.7}$$

Comparability of the RH_1 and A_∞ constants. The equivalence of the RH_1 and A_∞ conditions has been known for a long time, but not the relationship between the RH_1 and A_∞ constants. In this paper we prove the following inequality.

Theorem 1.1 (Main result 1: comparability of the RH_1 and A_∞ constants)**.** *A weight w belongs to the Muckenhoupt class A_∞ if and only if $w \in RH_1$. Moreover,*

$$[w]_{RH_1} \le C[w]_{A_\infty}, \tag{1.8}$$

where the constant C can be taken to be e ($C = e$). Moreover, the constant $C = e$ is the best possible.

The Bellman function proof of this theorem can be found in Subsection 3.1. An independent proof of an analog of this theorem for the constant $[w]_{RH_1'}$ was obtained recently in [HyPer].

Moreover, using a similar Bellman function approach, one can prove the following theorem.

Theorem 1.2. *If $[w]_{RH_1} = Q$, then*

$$[w]_\infty \le C\frac{e^{e^{Q+1}}}{e^Q},$$

where C does not depend on Q. Moreover, this inequality is sharp in Q.

We give a sketch of the proof in Subsection 3.1. We also note that in the paper [HyPer] the authors got a bound similar to Theorem 1.1 (without sharpness). However, as far as we know Theorem 1.2 is new, and we find the bound very surprising.

1-Gehring Lemma. The Reverse Hölder classes have a remarkable self-improvement property, discovered by Gehring in 1973; see [Ge].

Theorem 1.3 (Gehring's theorem)**.** *Suppose $w \in RH_p$ for some $1 < p < \infty$. Then there exists $\varepsilon > 0$, depending only on p and the RH_p constant of w, such that $w \in RH_{p+\varepsilon}$.*

In 1985, Bojarski (see [Bo]) posed the question of finding the sharp dependence of ε on p and the RH_p constant of the weight w (and the dimension, in multidimensional case). The sharp asymptotics for the case of RH_p constant close to one was obtained by Bojarski [Bo] and Wik [Wik]. In 1990, Sbordone and D'Apuzzo (see [Sb] and [DaSb]) found sharp dependence for monotone functions, and in 1992 Korenovskii [Kor] showed that increasing rearrangements do not change the Reverse Hölder constant of the weight, extending the results of Sbordone and D'Apuzzo to the weights that are not monotone. In 2008, Vasyunin (see [Va2]) presented a new proof of the sharp Gehring lemma using the method of Bellman functions. All of the above was done for the case where $1 < p < \infty$ and in dimension one. Let us state the sharp version of the Gehring Lemma.

Theorem 1.4 (Sharp Gehring Lemma ($n = 1$, $1 < p < \infty$))**.** *If w is a weight, $w \in RH_p$ for some $p > 1$, then $w \in RH_{p+\delta}$ for any $\delta < \varepsilon$, where ε is the root of the equation*

$$\frac{1}{p-1}\log\frac{p+\varepsilon-1}{\varepsilon} - \log\frac{p+\varepsilon}{p+\varepsilon-1} = \frac{p}{p-1}\log[w]_{RH_p}. \tag{1.9}$$

In the following theorem we show that, with the RH_1 constant defined as above, the Gehring Lemma works for $p = 1$, and obtain the sharp dependence of ε on the RH_1 constant of the weight in dimension one.

Theorem 1.5 (Main result 2: Sharp Gehring Lemma ($n = 1$, $p = 1$))**.** *If $w \in RH_1$, then $w \in RH_{1+\varepsilon}$, $0 < \varepsilon < \varepsilon_-$, where ε_- is the smallest solution of the equation*

$$\frac{1}{t} - \log\left(\frac{1}{t}+1\right) = [w]_{RH_1}. \tag{1.10}$$

This result is sharp in the sense that for any constant C there exists a weight $w \in RH_1$ with $[w]_{RH_1} = C$ and such that w does not belong to $RH_{1+\varepsilon_-}$ with ε_- defined by (1.10)*.*

The proof of this theorem can be found in Subsection 3.1.

We note that this result together with Theorem 1.1 turns out to be useful, see [HyPer]. Using these results (in any dimension), the authors improved the famous A_2 conjecture and somewhat disproved the so-called "reverse A_2 conjecture".

We notice that the result of Theorem 1.2 is absolutely new and essentially improves the best known result from [HyPer].

No extensions of the above sharp results to higher dimension are known. The nonsharp dependence of ε on p and the RH_p constant of the weight w is not hard to trace, even in the more general case of $\mathbb{R}^n$. Following [Gr, St93] or [GaRu], one can easily show that

$$w \in RH_1 \;\Rightarrow\; w \in RH_{1+\varepsilon} \quad \text{with} \quad \varepsilon = \frac{\log 4}{n\log 2 + 8[w]_{RH_1}}, \tag{1.11}$$

but this result is far from being sharp. We include the proof of (1.11) in Subsection 3.1 for completeness.

1-Gehring v.s. p-Gehring. In the end of this section we will show that p-Gehring (unfortunately not a sharp one) for any $p > 1$ follows from the 1-Gehring in dimension n. Except for one step where we use the 1-Gehring Lemma, we follow Iwaniec; see [IV].

We start with $p > 1$ and $w \in RH_p$, i.e., for any interval $I \subset \mathbb{R}$,

$$(m_I(w^p))^{\frac{1}{p}} \leq [w]_{RH_p} m_I w.$$

We would like to show that $w \in RH_{p+\delta}$ for some $\delta > 0$. Trivially, we have a pointwise inequality for the Hardy–Littlewood maximal function M:

$$(M(w^p))^{\frac{1}{p}} \leq [w]_{RH_p} M(w).$$

Since $w \in L_p(I)$, by our assumption $M(w)$ is also in $L_p(I)$, so by the above inequality $M(w^p) \in L_1$. By the famous result of Stein [St69], this implies that $w^p \in L\log L(I)$ and

$$m_I\Big(w^p \log\Big(e + \frac{w^p}{m_I(w^p)}\Big)\Big) \leq 2^n m_I(M(w^p)),$$

which, by Weiner, is bounded from above by

$$2^n [w]_{RH_p}^p 3^n \frac{p}{p-1} 2^p m_I(w^p).$$

Thus, by the above, $w^p \in RH_1$ with

$$[w]_{RH_1} = \sup_{I\subset\mathbb{R}} m_I\Big(\frac{w^p}{m_I(w^p)} \log\Big(e + \frac{w^p}{m_I(w^p)}\Big)\Big) \leq 6^n [w]_{RH_p}^p 2^p \frac{p}{p-1}.$$

Now all we need is to apply the 1-Gehring Lemma: there exists $\varepsilon > 0$ such that $w^p \in RH_{1+\varepsilon}$, which trivially implies that $w \in RH_{p+\delta}$ with $\delta = p\varepsilon$.

Some useful technical lemmas. We also prove two technical lemmas; we think that they can be of interest on their own. The first lemma is the RH_1 case missing in [RVV], where certain analogs were shown for RH_p and A_q for $1 < p < \infty$ and $1 < q \leq \infty$.

Lemma 1.6. *Take a function $w \in RH_1$ and define*

$$w_n(t) = \begin{cases} \frac{1}{n} & \text{if } w(t) \leq \frac{1}{n}, \\ w(t) & \text{if } \frac{1}{n} \leq w(t) \leq n, \\ n & \text{if } w(t) \geq n. \end{cases}$$

Then

$$[w_n]_{RH_1} \leq [w]_{RH_1}.$$

Moreover, the same is true for any function $w \in A_\infty$ with replacing $[.]_{RH_1}$ by $[.]_\infty$.

The next lemma is the RH_1 analog of Vasyunin's lemmas from [Va] and [Va2]. Lemma 1.8 is taken from these papers.

Lemma 1.7. *Fix $Q_1 > Q > 0$ and denote $\Omega_{Q_1} = \{(x, y) : x\log(x) \leq y \leq x\log(x)+Q_1 x\}$. Then for every $w \in RH_1$ with $[w]_{RH_1} < Q$, there are two intervals I^+ and I^- such that $I = I^- \cup I^+$ and if $x^\pm = (m_{I^\pm} w, m_{I^\pm}(w\log(w)))$, then $[x^-, x^+] \subset \Omega_{Q_1}$. Also, the parameters $\alpha^\pm = \frac{|I^\pm|}{|I|}$ can be taken separated from 0 and 1 uniformly with respect to w.*

Lemma 1.8. *Fix $Q_1 > Q > 0$ and denote $\Omega_{Q_1} = \{(x, y) : 1 \leq xe^{-y} \leq Q_1\}$. Then for every $w \in A_\infty$ with $[w]_\infty < Q$, there are two intervals I^+ and I^- such that $I = I^- \cup I^+$ and if $x^\pm = \big(m_{I^\pm} w, m_{I^\pm}(\log(w))\big)$ then $[x^-, x^+] \subset \Omega_{Q_1}$. Also, the parameters $\alpha^\pm = \frac{|I^\pm|}{|I|}$ can be taken separated from 0 and 1 uniformly with respect to w.*

The proofs of Lemma 1.6 and Lemma 1.7 are quite similar to those of their RH_p analogs from [RVV] and [Va2].

We would like, however, to give a heuristic idea why these lemmas are true. Fix $Q_1 > Q$ and take a weight w such that $[w]_\infty \leq Q$. First, we take intervals $I_\pm$, such that $|I_\pm| = \frac{1}{2}|I|$. If the line segment described above is in Ω_{Q_1}, then we stop. If no, we start enlarging I_+. The line segment that connects (x_-, y_-) and (x_+, y_+) starts turning and finally gets into Ω_{Q_1}.

The only detail is that the parameters $\frac{|I_\pm|}{|I|}$ can be chosen bounded away from 0 and 1, independently on w. This is a technical calculation, and we refer the curious reader to the paper [Va2].

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§2. Applications

2.1. Dirichlet problem for elliptic PDE's. In this section, we implicitly follow [HoMa]. This is the reason we shall work with $\mathbb{R}^{n+1}_+$.

We start with real symmetric second order elliptic operator

$$Lf(X) := -\operatorname{div} A(X)\nabla f(X), \quad X \in \mathbb{R}^{n+1}_+, \tag{2.1}$$

where $A(X) = (a_{i,j}(X))_{1\le i,j\le n+1}$ is a real, symmetric matrix of size $(n+1)\times(n+1)$ such that $a_{i,j} \in L_\infty(\mathbb{R}^{n+1}_+)$ for $1 \le i, j \le n+1$, and A is uniformly elliptic, that is, there exists $0 < \lambda \le 1$ such that

$$\lambda\,|\xi|^2 \le A(X)\xi\cdot\xi \le \lambda^{-1}\,|\xi|^2$$

for all $\xi \in \mathbb{R}^{n+1}$ and almost every $X \in \mathbb{R}^{n+1}_+$.

If f is a continuous function on $\mathbb{R}^n$, then there exists a unique function u, continuous on $\mathbb{R}^{n+1}_+$, so that $Lu = 0$ in $\mathbb{R}^{n+1}_+$ and $u = f$ on $\mathbb{R}^n$. Then for a point $X_0 \in \mathbb{R}^{n+1}_+$, the mapping $f \in \mathcal{C}(\mathbb{R}^n) \to u(X_0)$ is a positive linear functional, so that there exists a unique nonnegative measure ω^{X_0} on $\mathbb{R}^n$ such that for every $f \in \mathcal{C}$ we have $(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} f\,d\omega^{X_0} = u(X_0).$$

This measure ω^{X_0} is called the harmonic measure associated with L. We fix the point X_0 and drop the index $\omega = \omega^{X_0}$. It is often important for applications to know whether or not ω is absolutely continuous with respect to the Lebesgue (surface) measure dx on $\mathbb{R}^n$. If this is the case, it is also of interest to know how nice the Radon–Nikodym derivative $\kappa = \frac{d\omega}{dx}$ (the Poisson kernel) is. It is a well-known fact that the Dirichlet problem for L is solvable in $L_{p'}, \frac{1}{p} + \frac{1}{p'} = 1$, if and only if $\kappa \in RH_p$. (For the precise statement of this theorem, see [HoMa, FeKPi], or [Ke]).

Caffarelli, Fabbes, and Kenig [CFK] showed that there exist elliptic operators L of the form (2.1) such that the measure ω associated with L is not absolutely continuous with respect to the Lebesgue measure dx. Later, Fabes, Jerison, and Kenig showed in [FJK] that if the matrix $A(X) = (a_{i,j}(X))_{1\le i,j\le n+1}$ of our operator L has continuous entries on $\mathbb{R}^{n+1}_+$ and the modulus of continuity is sufficiently good, then ω is absolutely continuous with respect to the Lebesgue (surface) measure dx, and moreover, its Radon–Nikodym derivative κ belongs to the Reverse Hölder class RH_2. Then in [Da] Dahlberg extended this to the following result for the solvability of L in $L_{p'}$ in the case where L is a small perturbation of a solvable operator L_0. Given two elliptic operators L_0 and L as above with associated matrices A_0 and A, we define their disagreement as

$$a(X) := \sup_{|X-Y|_\infty < \rho(X)/2} |A(Y) - A_0(Y)|.$$

Theorem 2.1 (Dahlberg'86). *Let L_0 and L be two operators as above, let a be their disagreement, and let ω_0, ω denote their respective harmonic measures. Assume that the measure $\frac{a(X)^2}{\rho(X)} dX$ is a Carleson measure*:

$$\sup_{Q \in \mathbb{R}^n} \frac{1}{|Q|} \int_{R_Q} \frac{a(X)^2}{\rho(X)} \, dX < \infty, \tag{2.2}$$

where R_Q is a Carleson box associated with Q.

Suppose also that the Carleson measure $\frac{a(X)^2}{\rho(X)} dX$ has vanishing trace:

$$\lim_{r \to 0^+} \sup_{Q \in \mathbb{R}^n, \ell(Q) \le r} \frac{1}{|Q|} \int_{R_Q} \frac{a(X)^2}{\rho(X)} \, dX = 0. \tag{2.3}$$

Then the condition $\kappa_0 \in RH_p$ for some $1 < p < \infty$ implies $\kappa \in RH_p$, i.e., if L_0 is solvable in $L_{p'}$, then L is also solvable in $L_{p'}$.

In [Fe], Robert Fefferman showed that in the limiting case $p = 1$ condition (2.3) can be relaxed significantly.

Theorem 2.2 (Fefferman). *Let L_0 and L be two operators as above, let a be their disagreement, and let ω_0, ω denote their respective harmonic measures. Assume that the measure $\frac{a(X)^2}{\rho(X)} dX$ is a Carleson measure (i.e., it satisfies* (2.2)*). Suppose also that we have*

$$\|\mathcal{A}(x)\|_{L_\infty(\mathbb{R}^n)} < \infty, \quad \textit{where} \quad \mathcal{A}(x) := \left(\int_{\Gamma(x)} \frac{a(X)^2}{\rho(X)^n} \, dX \right)^{\frac{1}{2}}. \tag{2.4}$$

Then $\kappa_0 \in A_\infty (= RH_1)$ implies $\kappa \in A_\infty (= RH_1)$, i.e., if L_0 is solvable in $L_{p'}$, $1 < p' < \infty$, then L is solvable in $L_{q'}$ for the some $1 < q' < \infty$.

Moreover, $[\kappa]_{RH_1} \le C [\kappa_0]_{RH_1}$ with a constant C depending on the L_∞ norm of $\mathcal{A}(x)$, the ellipticity constants of the operators L_0 and L, and the dimension n.

Robert Fefferman did not state the dependence of the RH_1 constants, but it follows from his proof.

In 1991, Fefferman, Kenig, and Pipher came up with a different method and show that even if condition (2.4) is omitted, having that the measure $\frac{a(X)^2}{\rho(X)} dX$ is Carleson is enough to keep Radon–Nikodym derivatives in A_∞.

Theorem 2.3 (Fefferman–Kenig–Pipher). *Let L_0 and L be two operators as above, let a be their disagreement, and let ω_0, ω denote their respective harmonic measures. Assume that $\frac{a(X)^2}{\rho(X)} dX$ is a Carleson measure (i.e., it satisfies* (2.2)*).*

Then $\kappa_0 \in A_\infty (= RH_1)$ implies $\kappa \in A_\infty (= RH_1)$. More precisely, if L_0 is solvable in some $L_{p'}$, $1 < p' < \infty$, then there exists $1 < q' < \infty$ such that L is solvable in $L_{q'}$.

This theorem looks like a clear generalization of Fefferman's result, but notice that the relationship between the RH_1 constants of κ and κ_0 is not traced anymore. In this area, people normally do not need estimates on the Reverse Hölder constants, what matters is the value of p for which $\kappa \in RH_p$. Examples (see [FeKPi]) suggest that under conditions of Theorem 2.3, weaker than the vanishing trace conditions in Dahlberg's theorem, p will not be preserved (i.e., $\kappa_0 \in RH_p$ will not imply that $\kappa \in RH_p$), we can only claim that for a given p such that $\kappa_0 \in RH_p$ there exists q such that $\kappa \in RH_q$. It is natural to ask here whether there is anything we can say about q.

This is where Fefferman's estimates on the RH_1 constant of κ turn out to be very handy. When we know $[\kappa]_{RH_1}$, we can use the limiting case of Gehring's theorem for $p = 1$, see (1.11), in the following way.

Theorem 2.4. *Let L_0 and L be two operators as above, let a be their disagreement, and let ω_0, ω denote their respective harmonic measures. Assume that $\frac{a(X)^2}{\rho(X)} dX$ is a Carleson measure (i.e., it satisfies (2.2)).*

(1) (Fefferman–Kenig–Pipher) *The condition $\omega_0 \in A_\infty (= RH_1)$ implies $\omega \in A_\infty (= RH_1)$. More precisely, if L_0 is solvable in some $L_{p'}$, $1 < p' < \infty$, there exists $1 < q' < \infty$ such that L is solvable in $L_{q'}$.*

(2) (R. Fefferman) *In addition to (2.2), suppose that Fefferman's condition (2.4) is satisfied. Then $\kappa_0 \in A_\infty (= RH_1)$ implies $\kappa \in A_\infty (= RH_1)$, and moreover, $[\kappa]_{RH_1} \leq C[\kappa_0]_{RH_1}$ with $C = C(\|\mathcal{A}(x)\|_{L_\infty(\mathbb{R}^n)}, \lambda, n)$, which means that*

$$\kappa \in RH_{1+\varepsilon} \quad \textit{with} \quad \varepsilon = \frac{\log 4}{n \log 2 + 8C[\kappa_0]_{RH_1}}.$$

i.e., if L_0 is solvable in $L_{p'}$ ($\kappa_0 \in RH_p$), $1 < p' < \infty$, then L is solvable in $L_{q'}$ for $q = 1 + \frac{\log 4}{n \log 2 + 8C[\kappa_0]_{RH_1}}$. Note also that for any $1 < p < \infty$ we have $[\kappa_0]_{RH_1} \leq \frac{p}{p-1} \log[\kappa_0]_{RH_p}$.

(3) (Dahlberg) *Suppose also that the measure $\frac{a(X)^2}{\rho(X)} dX$ has vanishing trace, i.e., satisfies (2.3). Then if $\kappa_0 \in RH_p$ for some $1 < p < \infty$ implies $\kappa \in RH_p$, i.e., if L_0 is solvable in $L_{p'}$, $1 < p' < \infty$, then L is solvable in $L_{p'}$ for the same p.*

This theorem (the ε part of (2)) is not sharp. Sharp 1-Gehring would help part (2), and we shall try to get it, but it would not help to trace the dependence of q on p in part (1). In fact, it is not clear here if Fefferman's assumption can be relaxed.

§3. Proofs

3.1. Proof of Theorem 1.1 (Bellman function proof)**.** We shall prove that if w belongs to the Muckenhoupt class A_∞ on the interval J, $w \in A_\infty(J)$, i.e.,

$$\sup_{I \subset J} m_I w \, e^{-m_I(\log w)} = [w]_{A_\infty, J} < \infty, \tag{3.1}$$

then

$$m_J(w \log w) \leq m_J w \, m_J(\log w) + e[w]_{A_\infty, J} m_J w. \tag{3.2}$$

We start with the following lemma.

Lemma 3.1. *In order to prove inequality (3.2), it suffices to show that for every small $\varepsilon > 0$ there exists a Bellman function $B_{Q,\varepsilon} = B_{Q,\varepsilon}(x, y) = B(x, y)$ (we shall drop the index Q for simplicity) defined on the domain*

$$\Omega_{Q+\varepsilon} = \left\{ \vec{x} = (x, y) \in \mathbb{R}^2 \, : \, x \geq 0, \ 1 \leq x e^{-y} \leq Q + \varepsilon \right\}$$

and satisfying the following conditions:

(1) *B is continuous on $\Omega_{Q+\varepsilon}$;*

(2) *$B(x, y)$ is bounded from above by $x \log x + eQx$:*

$$B(x, y) \leq x \log x + eQx \quad \textit{for all} \quad (x, y) \in \Omega_{Q+\varepsilon}, \tag{3.3}$$

and

$$B(x, y) \geq x \log(x); \tag{3.4}$$

(3) *$B(x,y)$ is locally concave on $\Omega_{Q+\varepsilon}$:*

$$B''_{yy}(x,y)\le 0 \quad \textit{and} \quad \det\begin{pmatrix} B''_{xx} & B''_{xy} \\ B''_{xy} & B''_{yy}\end{pmatrix}=0 \ \textit{for all} \quad (x,y)\in\Omega_{Q+\varepsilon}. \tag{3.5}$$

First, we prove Lemma 3.1 and then present a function B having the above properties.

Proof of Lemma 3.1. Let w be an A_∞-weight on the interval J. We truncate it by $\frac{1}{n}$ from below and by n from above,

$$w_n(t):=\begin{cases} n & \text{if } \ w(t)\ge n,\\ w(t) & \text{if } \ \frac{1}{n}\le w(t)\le n,\\ \frac{1}{n} & \text{if } \ w(t)\le\frac{1}{n},\end{cases} \tag{3.6}$$

and show that Lemma 3.1 holds true for the weight w_n with all constants independent of n. Then by sending n to infinity and applying the Lebesgue dominated convergence theorem, we obtain inequality (3.2) for any $w\in L_1^{\mathrm{loc}}(\mathbb{R})$.

Thus, we consider the truncated weight $w_n(t)$ on the interval $J\subset\mathbb{R}$. By Lemma 1.6, we know that the A_∞ constant of the truncated weight $w_n(t)$ does not exceed the A_∞ constant of the original weight w.

Now, for every interval $I\subset J\subset\mathbb{R}$, let

$$\vec{x}_I=(x_I,y_I):=(m_I(w_n),m_I(\log w_n))\,.$$

Then, for every such I, by the Reznikov–Vasyunin–Volberg theorem, $\vec{x}_I\in\Omega_{[w]_{A_\infty}}$ and, moreover, $\frac{1}{n}\le x_I\le n$.

Next, we use Lemma 1.8 to construct a sequence $\{I_k^j\}_{1\le j\le 2^k,\ k,j\in\mathbb{N}}$ of subintervals of I with the properties that for all $k\in\mathbb{N}$ the set $\mathcal{J}_k:=\{I_k^j\}_{1\le j\le 2^k}$ forms a partition of J, the lengths of I_k^j approach 0 as $k\to\infty$, and for every $k,j\in\mathbb{N}$ with $1\le j\le 2^k-1$, the line segment connecting the points $\vec{x}_{I_k^j}$ and $\vec{x}_{I_k^{j+1}}$ belongs to the extended domain $\Omega_{[w]_{A_\infty}+\varepsilon}$, while the points $\{\vec{x}_{I_k^j}\}$ lie in $\Omega_{[w]_{A_\infty}}$.

We apply Lemma 1.8 to the interval $J=:I_0^1$ with $\varepsilon>0$ from conditions of Lemma 3.1 to split it into $J=I_0^1=I_1^1\cup I_1^2$. We repeat this procedure with the same ε for I_1^1 and I_1^2, obtaining I_2^1, I_2^2, I_2^3, and I_2^4. This way we build $\{I_k^j\}_{k,j\in\mathbb{N},\,1\le j\le 2^k}$.

Since both δ^k and $(1-\delta)^k$ tend to 0 as $k\to\infty$, we have $\lim_{k\to\infty}\max_j|I_k^j|=0$. By construction, $J=\bigcup_j I_k^j$ for all $k\in\mathbb{N}$, and finally, for every $k,j\in\mathbb{N}$ with $1\le j\le 2^k$, we have $\vec{x}_{I_k^j}\in\Omega_{[w]_{A_\infty}}$ and the closed interval $[\vec{x}_{I_k^j};\vec{x}_{I_k^{j+1}}]$ is included in $\Omega_{[w]_{A_\infty}+\varepsilon}$ whenever I_k^j and I_k^{j+1} come from the same parent I_{k-1}^i.

Denote

$$\begin{aligned} x_{k,n}(s)&:=m_{I_k^j}(w_n), \quad s\in I_k^j,\\ y_{k,n}(s)&:=m_{I_k^j}(\log w_n), \quad s\in I_k^j.\end{aligned}$$

Both $x_{k,n}$ and $y_{k,n}$ are step functions, and for almost every s we have $\big(x_{k,n}(s),y_{k,n}(s)\big)\to(w_n(s),\log w_n(s))$ as $k\to\infty$.

To finish the proof of Lemma 3.1, we observe that, by the concavity of the function B,

$$\begin{aligned} B(x_I,y_I) &\geq \frac{|I_+|}{|I|}B(x_{I_+},y_{I_+}) + \frac{|I_-|}{|I|}B(x_{I_-},y_{I_-}) \\ &\geq \frac{|I_+|}{|I|}\frac{|I_{++}|}{|I_+|}B(x_{I_{++}},y_{I_{++}}) + \frac{|I_+|}{|I|}\frac{|I_{+-}|}{|I_+|}B(x_{I_{+-}},y_{I_{+-}}) \\ &\qquad + \frac{|I_-|}{|I|}\frac{|I_{-+}|}{|I_-|}B(x_{I_{-+}},y_{I_{-+}}) + \frac{|I_-|}{|I|}\frac{|I_{--}|}{|I_-|}B(x_{I_{--}},y_{I_{--}}) \\ &\geq \dots \\ &\geq \sum_{\substack{k,j\in\mathbb{N},\\ 1\leq j\leq 2^k}} \frac{|I^j_k|}{|I|}B\big(x_{I^j_k},y_{I^j_k}\big). \end{aligned}$$

Therefore, $B(x_J,y_J) \geq \frac{1}{|J|}\int_J B(x_{k,n}(s),y_{k,n}(s))\,ds$.

Since w_n is bounded from above and from below, $\frac{1}{n} \leq w_n(t) \leq n$, the points $(x_{k,n},y_{k,n})$ belong to a compact set $K_{w,n} \subset \mathbb{R}^2$. Since B is continuous, it is bounded on K_w, and the Lebesgue dominated convergence theorem and the boundedness property (3.4) of B yield

$$\begin{aligned} B(x_J,y_J) &\geq \lim_{k\to\infty}\frac{1}{|J|}\int_J B(x_{n,k}(s),y_{n,k}(s))\,ds \\ &\geq \lim_{k\to\infty}\frac{1}{|J|}\int_J x_{n,k}(s)\ \log x_{n,k}(s)\,ds \\ &= \frac{1}{|J|}\int_J w_n(s)\ \log w_n(s)\,ds = m_J(w_n\log w_n), \end{aligned}$$

which, in its turn, implies the relations

$$m_J(w_n\log w_n) \leq B(x_J,y_J) \leq x_J\log x_J + eQx_J = m_Jw_n\ \log m_Jw_n + eQ\ m_Jw_n.$$

Since this bound does not depend on n, we send n to ∞, obtaining the desired inequality for all A_∞-weights w.

The proof of Lemma 3.1 is complete. □

Now we need to show that B with the above properties exists. The following lemma will help us to define such a function $B(x,y)$.

In fact, for any $Q>1$ we will construct the *exact* Bellman function:

$$\mathcal{B}_Q(x,y) = \sup\{m_I(w\log(w))\ :\ m_Iw = x,\ m_I(\log(w)) = y,\ [w]_\infty \leq Q\}.$$

We need some preparation. First, let γ be the root of the equation

$$t - \log(t) = 1 + \log(Q)$$

such that $\gamma < 1$. Next, fix a point $(x,y) \in \Omega_Q = \{(x,y)\ :\ 1 \leq xe^{-y} \leq Q\}$, and let $v = v(x,y)$ be a root of the equation

$$y = \frac{\gamma\cdot x}{v} + \log(v) - \gamma$$

such that $v \leq x$.

In fact, the last equation is an equation of a line ℓ such that $(v,\log(v)) \in \ell$ and ℓ is tangent to the curve $xe^{-y} = Q$. So basically we take a point (x,y) and a tangent line that passes through this point and goes to the right. This line "hits" the curve $xe^{-y} = 1$ exactly at the point $(v,\log(v))$.

We are ready to state the following lemma.

Lemma 3.2. *Let γ be as above, and let $v = v(x,y)$ be a function defined implicitly (on the domain Ω_Q) by the equation*

$$y = \frac{\gamma \cdot x}{v} + \log(v) - \gamma$$

and such that $v \le x$.

Denote

$$B(x,y) = x\log(v) + \frac{x-v}{\gamma}.$$

Then $B(x,y)$ satisfies all properties listed in Lemma 3.1.

Remark 1. We remark that instead of denoting $Q+\varepsilon$ by Q_1 we write simply Q. It is fine because we do it for every $Q > 1$.

Proof. We leave the differentiation of the function B to the reader. However, we state the answer for several derivatives. First, we have

$$v'_x = \frac{\gamma v}{\gamma x - v}, \quad v'_y = \frac{v^2}{v - \gamma x}.$$

Next,

$$B''_{xx} = \frac{\gamma}{\gamma x - v}, \quad B''_{yy} = -\frac{1}{\gamma}\frac{v^2}{v-\gamma x}, \quad B''_{xy} = \frac{v}{v-\gamma x}.$$

Finally, by the definition of v, we have $\gamma x \le v$, whence $B''_{xx} \le 0$. We also observe that $B(v, \log(v)) = v\log(v)$. Thus, we need to prove that $x\log(x) \le B(x,y) \le x\log(x) + eQx$. We observe that

$$\frac{B(x,y) - x\log(x)}{x} = \log\frac{v}{x} + \frac{1-\frac{v}{x}}{\gamma}.$$

Denote $s = \frac{v}{x}$ and notice that $s \in [\gamma, 1]$. Then

$$\frac{B(x,y) - x\log(x)}{x} = \log(s) + \frac{1-s}{\gamma} = \varphi(s).$$

Since $\varphi'(s) = \frac{1}{s} - \frac{1}{\gamma} \le 0$, we get

$$\frac{B(x,y) - x\log(x)}{x} \le \varphi(\gamma) = \log(\gamma) + \frac{1}{\gamma} - 1.$$

It is not hard to check that the last expression is not larger than eQ. Moreover,

$$\lim_{Q\to\infty} \frac{\log(\gamma) + \frac{1}{\gamma} - 1}{Q} = e,$$

so that the constant e is sharp. Finally,

$$\frac{B(x,y) - x\log(x)}{x} \ge \varphi(1) = 0,$$

whence $B(x,y) \ge x\log(x)$, which finishes the proof. □

We have proved that our function $B(x,y)$ is larger than or equal to the exact Bellman function $\mathcal{B}(x,y)$. This proves inequality (1.8) with constant $C = e$. In order to prove that this is the best possible constant, we need to do the following: for every point $(x,y) \in \Omega_Q$, present a weight w such that $[w]_\infty \le Q$, $m_I w = x$, $m_I(\log(w)) = y$, and $B(x,y) = m_I(w\log(w))$. The following lemma takes care of this issue.

Lemma 3.3. *For a point $(x, y) \in \Omega_Q$, consider a function*

$$w(t) = \begin{cases} v\left(\frac{t}{a}\right)^{\gamma-1} & \text{if } t \in [0, a], \\ v & \text{if } t \in [a, 1], \end{cases}$$

where a is taken so that $(x, y) = (m_I w, m_I(\log(w)))$. Then $[w]_\infty \leq Q$ and $B(x, y) = m_I(w \log(w))$.

This lemma is technical and we skip the proof. Later we shall prove a similar Lemma 3.9.

3.2. *Proof of Theorem* 1.5. The proof of this theorem is basically the same as the proof of Theorem 1.1. However, we give a detailed argument. Fix an interval $J \subset \mathbb{R}$. We define the RH_1 constant of the weight w on the interval J to be

$$[w]_{RH_{1,J}} := \sup_{I \subset J} m_I \Big(\frac{w}{m_I w} \log \frac{w}{m_I w} \Big) < \infty, \tag{3.7}$$

or equivalently,

$$\text{for any} \quad I \subset J,\ m_I(w \log w) \leq m_I w \log m_I w + [w]_{RH_{1,J}} m_I w. \tag{3.8}$$

Given $Q > 0$, we shall show that if $[w]_{RH_{1,J}} \leq Q$, then for every $0 < \varepsilon < \frac{1}{\gamma_+ - 1}$, where γ_+ is the larger solution of the equation

$$\gamma - \log(\gamma) = Q + 1,$$

the weight w satisfies the Reverse Hölder inequality with exponent $1+\varepsilon$ on the interval J:

$$\left(m_J w^{1+\varepsilon}\right)^{\frac{1}{1+\varepsilon}} \leq C m_J w.$$

Remark 2. We remark that the number ε_- defined in (1.10) is equal to $\frac{1}{\gamma_+ - 1}$. This is because

$$\left(1 + \frac{1}{\varepsilon_-}\right) - \log\left(1 + \frac{1}{\varepsilon_-}\right) = Q + 1,$$

and now it is clear that $\gamma_+ = 1 + \frac{1}{\varepsilon_-}$.

Given $J \subset \mathbb{R}$ and $Q > 0$, we introduce the following function $\mathcal{B}(x, y)$:

$$\mathcal{B}(x, y) = \sup\{m_J w^{1+\varepsilon} \,:\, m_J w = x,\ m_J w \log(w) = y,\ [w]_{RH_1} \leq Q\}.$$

Note that for every subinterval $I \subset J$ and any weight w satisfying $[w]_{RH_{1,J}} \leq Q$, the pair of points $(x_I, y_I) := (m_I w, m_I w \log w)$ should lie in the domain

$$\Omega = \Omega_Q = \{(x, y) \,:\, x \log x \leq y \leq x \log x + Qx\}.$$

The boundary curves of Ω will be denoted by Γ and Γ_Q:

$$\Gamma = \{(x, y) \,:\, x \log x = y\},$$
$$\Gamma_Q = \{(x, y) \,:\, y = x \log x + Qx\}.$$

So, for every weight $w \in RH_{1,J}$ and every subinterval $I \subset J$, the point (x_I, y_I) lies in the domain Ω. It is not hard to show that the opposite is also true, for every point $(x, y) \in \Omega$ there is a function w satisfying all properties from the definition of $\mathcal{B}(x, y)$ and such that $(x, y) = (m_J w, m_J w \log w)$. In fact, if $(x, y) \in \Omega$, then there are two points $V = (v, v \log v)$ and $U = (u, u \log u)$ on Γ such that the point (x, y) belongs to the line segment connecting V and U, $x = sv + (1-s)u$, $y = sv \log v + (1-s)u \log u$ with $s \in [0, 1]$, and the entire interval $[U,V]$ lies inside the domain Ω. To see the existence of w, simply observe that for $J=[0,1]$ the weight

$$w(t)=\begin{cases} v & \text{if } t\in[0,s],\\ u & \text{if } t\in[s,1]\end{cases}$$

has the above properties. Indeed,

$$m_J w = sv+(1-s)u = x,$$
$$m_J w\log(w) = s\; v\log v + (1-s)\; u\log u = y,$$

and for every interval $I\subset J=[0,1]$ we see that the point $(m_I w, m_I w\log(w))$ is a convex combination of V and U, and moreover $[w]_{RH_1}\le Q$ because the line segment $[U,V]$ is inside Ω. A simple rescaling argument proves this for a general J. Therefore, Ω is indeed the domain of $\mathcal{B}$. □

3.2.1. Geometry of Ω. We need some basic facts about the geometry of Ω. Namely, we want to investigate the following: if $(x,y)\in\Omega$, then what are the equations of the tangents to Γ_Q that pass through (x,y)? In particular, what happens if $y=x\log x$?

Lemma 3.4. *Let $V=(v,v\log v)\in\Gamma$ and $a_v=\gamma_+ v$. Then the line*

$$\ell_v\;:\; y=(\log v+\gamma_+)x - v\gamma_+$$

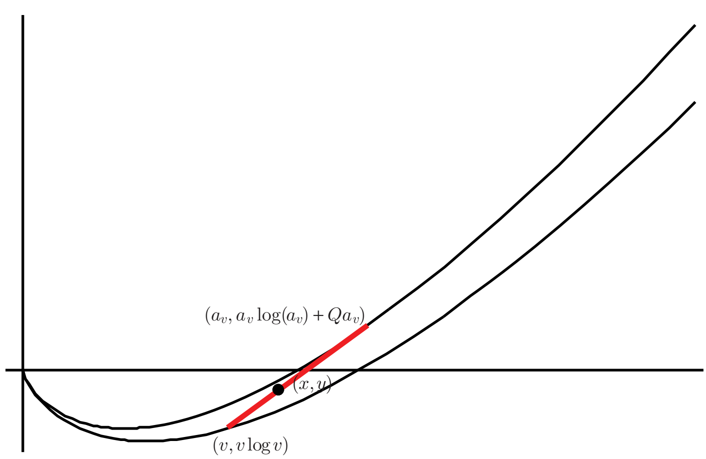
is tangent to Γ_Q. Moreover, $\ell_v\cap\Gamma_Q=\{(a_v, a_v\log(a_v)+a_vQ)\}$.

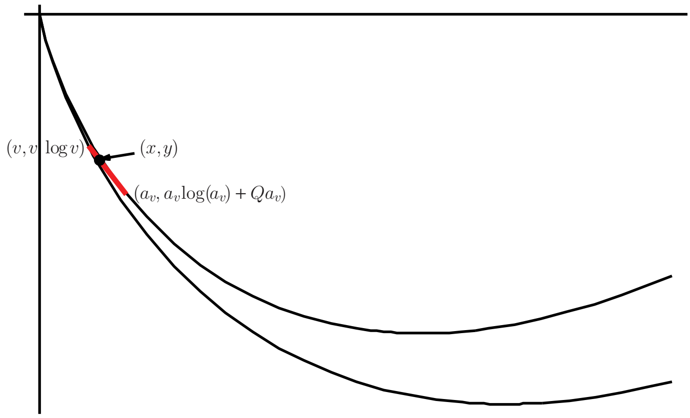
The proof of this lemma is a simple exercise in calculus, so we leave it to the reader. For any point $(x,y)\in\Omega$, now we have a line $\ell(x,y)$ tangent to Γ_Q that passes through (x,y) and has an equation

$$y=(\log v+\gamma_+)x - v\gamma_+,$$

where $v\le x$.

Take $a_v=\gamma_+ v$, so that $v\le x\le a_v$.





Now we are ready to formulate the following theorem.

Theorem 3.5. *Assume that* $0<\varepsilon<\frac{1}{\gamma_+-1}$. *Then*

$$\mathcal{B}(x,y)=\frac{v(x,y)^\varepsilon}{1+\varepsilon-\gamma\varepsilon}\big(x(1+\varepsilon)-\varepsilon\gamma v(x,y)\big),$$

where $v(x,y)$ *satisfies an implicit formula*

$$y=(\log v+\gamma_+)x-v\gamma_+,\quad v(x,y)\le x\le\gamma_+v(x,y).$$

Moreover, if $(x,y)\in\Gamma_Q$, *then the supremum is attained at the function* $w_{ex}(t)=\frac{x}{\gamma_+}t^{\frac{1-\gamma_+}{\gamma_+}}$, *while for an arbitrary* $(x,y)\in\Omega$ *the supremum is attained at a function of the form*

$$w_{ex}(t)=\begin{cases}Ct^{\frac{1-\gamma_+}{\gamma_+}} & \text{if } t\in(0,a],\\ Ca^{\frac{1-\gamma_+}{\gamma_+}} & \text{if } t\in[a,1].\end{cases}$$

Proof. We denote

$$B(x,y)=\frac{v(x,y)^\varepsilon}{1+\varepsilon-\gamma\varepsilon}(x(1+\varepsilon)-\varepsilon\gamma v(x,y)).$$

The goal is, therefore, to show that $\mathcal{B}=B$.

We break the proof into several lemmas.

Lemma 3.6. *The function* $B(x,y)$ *is locally concave in* Ω. *That is, the Hessian*

$$\begin{pmatrix}B''_{xx} & B''_{xy}\\ B''_{xy} & B''_{yy}\end{pmatrix}$$

of B *is a negative semidefinite matrix.*

Checking this condition requires nothing but careful differentiation. However, we want to point out that this lemma is true because $\gamma v(x,y)-x\ge 0$ for every $(x,y)\in\Omega$. It shows that we could not consider another tangent line from (x,y) to Γ_Q.

The local concavity of the function B implies the following lemma.

Lemma 3.7. *The following inequality is true*:

$$B(x,y)\ge\mathcal{B}(x,y).$$

Proof. First, observe that on the boundary curve Γ we have $\mathcal{B}(v,v\log v)=v^{1+\varepsilon}=B(v,v\log v)$, because the only admissible function w for the point $(v,v\log v)$ is the constant function $w(t)\equiv v$.

We consider a function B_{Q_1} that is defined like B, but with Q_1 instead of Q. Take a point (x, y) and an arbitrary w, $[w]_{RH_1} \le Q$, such that $(x, y) = (m_J w, m_J w \log(w))$. Assume that $\frac{1}{n} \le w(t) \le n$ for every t. Then, in particular,

$$m_J w \in \left[\frac{1}{n}, n\right].$$

Therefore, the set $\Upsilon = \{(m_I w, m_I w \log w) : I \subset J = [0,1]\}$ is compact, implying that B_{Q_1} is bounded on Υ. Now we take $I^\pm$ from Lemma 1.7.

Let D_n denote the set of intervals of nth generation. For example, $D_0 = \{I\}$ and $D_1 = \{I^-, I^+\}$. For every interval $J \in D_n$, we denote

$$x^J = (m_J w, m_J w \log(w)).$$

Since B_{Q_1} is locally concave, we can write

$$B_{Q_1}(x, y) \ge |I^+| B_{Q_1}(x^+) + |I^-| B_{Q_1}(x^-).$$

Repeating this procedure, we get

$$B_{Q_1}(x, y) \ge \sum_{J \in D_n} |J| B_{Q_1}(x^J) = \int_0^1 B_{Q_1}(x^n(t))\, dt,$$

where $x^n(t)$ is a step function defined in the following way: take $J \in D_n$ and denote $x^n(t) = x^J$, $t \in J$. By the Lebesgue differentiation theorem, $x^n(t) \to (w(t), w(t)\log w(t))$ for a.e. t. Moreover, since B is bounded on the set $\{x^J\}$, we can pass to the limit under the integral sign. We get

$$B_{Q_1}(x, y) \ge \int_0^1 B_{Q_1}(w(t), w(t)\log(w(t)))\, dt = \int_0^1 w^{1+\varepsilon}(t) = m_J w^{1+\varepsilon}.$$

If w is unbounded, we consider

$$w_n(t) = \begin{cases} \frac{1}{n} & \text{if } \ w(t) \le \frac{1}{n}, \\ w(t) & \text{if } \ w(t) \in [\frac{1}{n}, n], \\ n & \text{if } \ w(t) \ge n. \end{cases}$$

Then, $[w_n]_{RH_1} \le [w]_{RH_1} \le Q$ by Lemma 1.6 and

$$B_{Q_1}(x, y) \ge m_J w_n^{1+\varepsilon}.$$

Using the Lebesgue monotonic convergence theorem, we get

$$B_{Q_1}(x, y) \ge m_J w^{1+\varepsilon}$$

for every admissible function w. After taking the supremum over w, we have

$$B_{Q_1}(x, y) \ge \mathcal{B}(x, y)$$

for every $Q_1 > Q$. Since B_{Q_1} is continuous in Q, we can write

$$B(x, y) \ge \mathcal{B}(x, y). \qquad \square$$

We have shown that $B(x, y) \ge \mathcal{B}(x, y)$. In order to complete the proof of the theorem, we need to show the reverse inequality,

$$B(x, y) \le \mathcal{B}(x, y).$$

Lemma 3.8. *For every point $(x, y) \in \Omega$, there exists a function w_{ex} such that*

$$\begin{aligned}&m_I w_{ex} = x,\\&m_I(w_{ex}\log(w_{ex})) = y,\\&[w_{ex}]_{RH_1} \le Q,\\&B(x,y) = m_I(w_{ex}^{1+\varepsilon}).\end{aligned}$$

Consequently, $B(x, y) \le \mathcal{B}(x, y)$.

First, we consider a point $(x, y) \in \Gamma_Q$ and

$$w_{ex}(t) = \frac{x}{\gamma_+} t^{\frac{1-\gamma_+}{\gamma_+}}.$$

Lemma 3.9. *The function w_{ex} satisfies $[w_{ex}]_{RH_1} \le Q$. Moreover, for every $\varepsilon < \frac{1}{\gamma_+ - 1}$ we have*

$$B(m_I w_{ex}, m_I(w_{ex}\log(w_{ex}))) = m_I(w_{ex}^{1+\varepsilon}).$$

Finally,

$$m_I\left(w_{ex}^{1+\frac{1}{\gamma_+-1}}\right) = \infty.$$

Remark 3. Notice that a big part of this lemma repeats conditions from Lemma 3.8. However, the last formula shows the sharpness declared in Theorem 1.5.

Proof of Lemma 3.9. To prove that $[w_{ex}]_{RH_1} \le Q$, we take an interval $J = [a, b]$ and write

$$\begin{aligned}m_{[a,b]}w &= x\frac{1}{b-a}\big(b^{\frac{1}{\gamma_+}} - a^{\frac{1}{\gamma_+}}\big),\\ m_{[a,b]}w\log(w) &= x\log\Big(\frac{x}{\gamma_+}\Big)\big(b^{\frac{1}{\gamma_+}} - a^{\frac{1}{\gamma_+}}\big) + x\frac{1-\gamma_+}{\gamma_+}\big(b^{\frac{1}{\gamma_+}}\log b - a^{\frac{1}{\gamma_+}}\log a\big)\\ &\qquad - x(1-\gamma_+)\big(b^{\frac{1}{\gamma_+}} - a^{\frac{1}{\gamma_+}}\big).\end{aligned}$$

We substitute

$$\begin{aligned}&\alpha = a^{\frac{1}{\gamma_+}}, \quad \beta = b^{\frac{1}{\gamma_+}},\\&\alpha = s\beta,\end{aligned}$$

Then, after some technical calculations involving the definition of γ_+, we see that

$$m_J w\log(w) - m_J w\log(m_J w) - Q m_J w$$

has the same sign as

$$(\gamma_+ - 1)s\log s - (1-s)\log\frac{1-s}{1-s^{\gamma_+}}.$$

Now we use the following trick. Fix $s \in (0, 1)$ and denote

$$\varphi(\gamma) = (\gamma - 1)s\log s - (1-s)\log\frac{1-s}{1-s^{\gamma}}.$$

Obviously, $\varphi(1) = 0$. Simple calculation shows that $\varphi'(\gamma) \le 0$ if $\gamma \ge 1$, which yields

$$\varphi(\gamma_+) \le 0,$$

because $\gamma_+ > 1$. Therefore, if $s \in (0, 1)$, then

$$(\gamma_+ - 1)s\log s - (1-s)\log\frac{1-s}{1-s^{\gamma_+}} \le 0.$$

It is easy to check that the same inequality is valid for $s = 0$ and $s = 1$. Therefore,

$$m_J w\log(w) - m_J w\log(m_J w) - Q m_J w \le 0,$$

whence $[w_{ex}]_{RH_1} \le Q$. Moreover,

$$m_J w_{ex}^{1+\varepsilon} = \frac{x^{1+\varepsilon}}{\gamma_+^\varepsilon}\frac{1}{1+\varepsilon-\gamma_+\varepsilon} = B(x,y)$$

because $(x,y)\in\Gamma_Q$, so that $x=\gamma_+ v$. Finally, it is clear that $m_I\big(w_{ex}^{1+\frac{1}{\gamma_+-1}}\big)=\infty$, which finishes the proof of Lemma 3.9. □

We now proceed to an arbitrary $(x,y)\in\Omega_Q$. Take the tangent $\ell(x,y)$ and let $v = v(x,y)$, $a_v=\gamma_+ v(x,y)$ be as defined before. Define

$$w_{ex}(t) = \begin{cases} v\big(\frac{t}{u}\big)^{\frac{1-\gamma_+}{\gamma_+}} & \text{if } t\in[0,u],\\ v & \text{if } t\in[u,1].\end{cases}$$

Note that we "glue" two functions: the extremal function for the point $(v, v\log v)$ and the extremal function for $(a, a\log a + Qa)$. We should glue them so $x = m_I w$. Since

$$x = v\frac{a-x}{a-v} + a\frac{x-v}{a-v},$$

we take $u=\frac{x-v}{a-v}$. The inequality $[w]_{RH_1}\le Q$ is left to the reader. However, it is a big pleasure to point out that no calculations are needed because of the proof of such facts (a "maximizer" for Bellman function has the desired constant) given by P. Ivanishvili, N. Osipov, D. Stolyarov, V. Vasyunin, and P. Zatickiy; see [IOSVZ].

It remains to show that $m_I(w_{ex}^{1+\varepsilon}) = B(x,y)$, but this follows from the fact that B is linear on lines tangent to Γ_Q. We have proved that

$$B(x,y) = m_J w_{ex}^{1+\varepsilon} \le \mathcal{B}(x,y),$$

which finishes the proof of Theorem 3.5 and the proof of the identity $B(x,y)=\mathcal{B}(x,y)$. □

3.3. Proof of the Gehring theorem for the case of $p=1$ in dimension n.

Theorem 3.10. *If $w\in RH_1$, then*

$$w\in RH_{1+\varepsilon} \ \textit{for all}\ \varepsilon \le \frac{\log 4}{n\log 2 + 8[w]_{RH_1''}},$$

where n is the dimension of the underlying space (or is related to the doubling constant of the underlying measure).

Proof. Let $w\in RH_1$, then, by (1.7),

$$\|w\|_{L\log L, I} \le [w]_{RH_1''}\,\|w\|_{L,I} \quad \text{for all} \quad I\subset\mathbb{R}^n, \tag{3.9}$$

where $\|w\|_{\Phi(L)}$ is the Orlitz norm of w,

$$\text{if } \Phi = L\log L, \text{ then } \bar\Phi_n(t)\cong e^L-1 \tag{3.10}$$

and one can write the generalized Hölder inequality:

$$\text{for all} \quad I, \quad \int_I |f(x)\,g(x)|\,dx \le \|f\|_{L\log L,I}\,\|g\|_{e^L-1,I}\,.$$

Applying this to $\int_I |f|\,w\,dx$, we see that for every f, every $w\in RH_1$, and every $I\in D$ we have

$$\int_I |f|w \le 2\,\|w\|_{L\log L,I}\,\|f\|_{e^L-1} \le [\text{by } (3.9)] \le 2[w]_{RH_1''}\,\|w\|_{L,I}\,\|f\|_{e^L-1,I}\,. \tag{3.11}$$

First, note that $\|w\|_{L,I} = \frac{1}{|I|}\int_I w$:

$$\|w\|_{L,I} = \inf\left\{\lambda > 0 \,:\, \frac{1}{|I|}\int_I \frac{w}{\lambda} \leq 1\right\} = \frac{1}{|I|}\int_I w.$$

So, (3.11) becomes

$$\frac{1}{|I|}\int_I |f|w \leq 2[w]_{RH_1''}\frac{1}{|I|}\int_I w\, \|f\|_{e^L-1,I}\,. \tag{3.12}$$

In order to apply inequality (3.12) to $f = \chi_E$ for $E \subset I$, we note that

$$\begin{aligned}\|\chi_E\|_{e^L-1,I} &= \inf\left\{\lambda > 0 \,:\, \frac{1}{|I|}\int_I e^{\frac{\chi_E}{\lambda}} - 1 \leq 1\right\}\\ &= \inf\left\{\lambda > 0 \,:\, \frac{1}{|I|}\int_E e^{\frac{1}{\lambda}} - 1 \leq 1\right\}\\ &= \inf\left\{\lambda > 0 \,:\, \frac{|E|}{|I|}\left(e^{\frac{1}{\lambda}} - 1\right) \leq 1\right\} = \frac{1}{\log\left(1 + \frac{|I|}{|E|}\right)},\end{aligned}$$

and then (3.12) applied to $f = \chi_E$ implies

$$\frac{w(E)}{w(I)} \leq 2[w]_{RH_1''}\frac{1}{\log\left(1 + \frac{|I|}{|E|}\right)} \tag{3.13}$$

or

$$\frac{w(E)}{w(I)}\log\left(1 + \frac{|I|}{|E|}\right) \leq 2[w]_{RH_1''}.$$

Note also that, since $\frac{|I|}{|E|} \geq 1$, we have $\log\left(1 + \frac{|I|}{|E|}\right) \leq \log\left(2\frac{|I|}{|E|}\right)$. Take $\alpha = \frac{1}{e^{8[w]_{RH_1''}} - 1}$; then, whenever $\frac{|E|}{|I|} \leq \alpha$, since $\log\left(1 + \frac{1}{\alpha}\right)$ is a monotone decreasing function of α, we have

$$\log\left(1 + \frac{|I|}{|E|}\right) = \log\left(1 + \frac{1}{\frac{|E|}{|I|}}\right) \geq \log\left(1 + \frac{1}{\alpha}\right) = \log\left(e^{8[w]_{RH_1''}}\right) = 8[w]_{RH_1''},$$

i.e., whenever $\frac{|E|}{|I|} \leq \alpha = \frac{1}{e^{8[w]_{RH_1''}} - 1}$, we get $\log\left(1 + \frac{|I|}{|E|}\right) \geq 8[w]_{RH_1''}$. So, we can write (3.13) as

$$\frac{w(E)}{w(I)} \leq 2[w]_{RH_1''}\frac{1}{\log\left(1 + \frac{|I|}{|E|}\right)} \leq \frac{2[w]_{RH_1''}}{8[w]_{RH_1''}} = \frac{1}{4},$$

or, for simplicity,

$$\frac{|E|}{|I|} \leq \frac{1}{e^{8[w]_{RH_1''}} - 1} = \alpha \Rightarrow \frac{w(E)}{w(I)} \leq \frac{1}{4} =: \beta. \tag{3.14}$$

By page 398 in the book by Rubio de Francia–García-Cuerva, in order to know that w belongs to $RH_{1+\varepsilon}$, it suffices to pick an ε such that $(2^n\alpha^{-1})^{\varepsilon}\beta < 1$. For our choice of α and β in (3.14), we need to solve the following inequality for ε:

$$\left(2^n\left(e^{8[w]_{RH_1''}} - 1\right)\right)^{\varepsilon}\frac{1}{4} < 1.$$

To satisfy this inequality, it suffices to choose ε so that $\left(2^n e^{8[w]_{RH_1''}}\right)^{\varepsilon} = 4$, which yields $\varepsilon = \frac{\log(4)}{n\log(2) + 8[w]_{RH_1''}}$. Thus, if $w \in RH_1''$, then $w \in RH_{1+\varepsilon}$ with the above choice of ε. $\square$

3.4. Proof of Theorem 1.2. We give a sketch of the proof in spirit of the proof of the 1-Gehring lemma. Given a function $w \in RH_1$, we want to estimate $m_I w \exp(-m_I(\log(w)))$ from above and, therefore, we want to estimate $m_I(\log(w))$ from below. Therefore, we denote

$$B(x) = \inf\{m_I(\log(w)) \,:\, m_I w = x,\;\; m_I(w\log(w)) = y,\;\; [w]_{RH_1} \le Q\}.$$

The function B is locally convex (we remind the reader that the two functions B that occurred before were locally concave, because we considered a *sup* of something). We now denote by γ_- the smaller root of the equation $t - \log(t) = Q + 1$. We note that, for Q large, our γ_- has the following asymptotics:

$$\gamma_- = \frac{1}{e^{Q+1}} + \frac{1}{e^{2(Q+1)}} + O(e^{-3(Q+1)}).$$

Now we define a function $v(x, y)$ by the equation $y = (\log v + \gamma_-)x - v\gamma_-$, $v \ge x$.

The picture is the following: we take a point (x, y) and a tangent line to $\Gamma_Q = \{(x, y) \,:\, y = x\log(x) + Qx\}$ such that it "kisses" Γ_Q on the left-hand side of (x, y). Then $(v, v\log(v))$ is the point on the right-hand side of (x, y) where this tangent hits $\Gamma = \{(x, y) \,:\, y = x\log(x)\}$.

Then the Bellman function B is equal to $B(x, y) = \log(v) + \frac{x-v}{\gamma_- v}$.

We skip details of the proof, because they are identical to both previous proofs.

We now notice that we are interested in the quantity $x\exp(-B(x, y))$. This is because $B(x, y)$ is $m_I(\log(w))$, and $x = m_I w$. We write

$$x\exp(-B(x, y)) = \frac{x}{v}\exp\Big(\frac{1-\frac{x}{v}}{\gamma_-}\Big) = s\exp\Big(\frac{1-s}{\gamma_-}\Big),$$

where $s = \frac{x}{v} \in [\gamma_-, 1]$. We set $f(s) = s\exp\big(\frac{1-s}{\gamma_-}\big)$. Then

$$f'(s) = \exp(\dots)\Big[1 - \frac{s}{\gamma_-}\Big] \le 0,$$

whence $f(s) \le f(\gamma_-) = \gamma_- \exp\big(\frac{1-\gamma_-}{\gamma_-}\big)$.

Since $\gamma_- \sim e^{-(Q+1)} + e^{-2(Q+1)}$, we have $\frac{1}{\gamma_-} \sim e^{Q+1} - 1$ and

$$f(\gamma_-) \sim e^{-(Q+1)} e^{e^{Q+1}-2} = e^{e^{Q+1}-Q-3},$$

which finishes our proof.

To illustrate that this result is sharp, we state the following proposition.

Lemma 3.11. *Consider*

$$w(t) = \frac{1}{\gamma_-} t^{\frac{1-\gamma_-}{\gamma_-}}.$$

Then $[w]_{RH_1} = Q$, *and* $m_I w e^{-m_I(\log(w))} = \gamma_- \exp\big(\frac{1-\gamma_-}{\gamma_-}\big)$.

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