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PRETREES AND THE SHADOW TOPOLOGY

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ABSTRACT. A further development of the theory of pretrees, started by the work of L. Ward, P. Duchet, B. Bowditch, S. Adeleke and P. Neumann, and others, is presented. In particular, a relationship between this theory and the theory of convex structures is established. The shadow topology is investigated in detail. This remarkable topology emerges on tree-like objects of various types and has broad application.

INTRODUCTION

In this paper, we present a further development of the theory of pretrees. Pretrees are a broad generalization of ordinary trees[1], which covers the Λ-trees (in particular, $\mathbb{R}$-trees), the dendritic spaces (including dendrons, dendrites, etc.), the pseudotrees, and several other classes of tree-like structures. A pretree can be characterized as a set with a ternary relation, interpreted as a betweenness relation, in which every finite subspace embeds in a tree[2]. It turns out that in this case the embeddability of all finite subspaces is guaranteed by the embeddability of at most 4-point ones[3], whereby the class of pretrees can be defined by a simple finite system of axioms. Apparently, for the first time the class of pretrees were considered (under the name "*cutpoint structures*") by L. Ward in [40]. The same class of objects was introduced (in different terms, in the framework of the theory of convex structures) in the paper [14] by P. Duchet, as the "*variety of arborescent convexities*". The theory of pretrees has been developed considerably in the monographs by B. Bowditch [5], S. Adeleke and P. Neumann [1] (in [1], the pretrees appear under the term "*B-sets*"). The term "*pretree*", which gained popularity, was introduced by B. Bowditch.

There are several approaches to defining the pretrees: the ternary structures, partitions, interval spaces, the theory of convex structures, etc. (Below, several approaches are discussed in more detail.) At present, the approach with ternary structures mentioned above has become the most widespread. Within this approach, a pretree is defined as a set (say $\mathcal{T}$) with a ternary relation $\mathcal{S} \subset \mathcal{T}^3$ satisfying the following axioms:

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[1]In the mathematical literature, the term "(ordinary) trees" usually refers to one of the following classes of objects:

(i) the connected simply connected CW-complexes of dimension at most 1,

(ii) the so-called $\mathbb{Z}$-trees (edges are pairs of vertices),

(iii) the complete simplicial $\mathbb{R}$-trees (often, with edges of unit length).

Here (as well as in most of the cases where the term "(ordinary) tree" is used in this paper), the statement holds true for all these classes.

[2]Here, the remark in footnote 1 is also relevant.

[3]Cf. the characterization of classes of metric spaces by their four-point subspaces [16, no. 1.19].

(T0) if $(x,y,z) \in \mathcal{S}$, then $x \neq z$;
(T1) if $(x,y,z) \in \mathcal{S}$, then $(z,y,x) \in \mathcal{S}$;
(T2) if $(x,y,z) \in \mathcal{S}$, then $(x,z,y) \notin \mathcal{S}$;
(T3) if $(x,y,z) \in \mathcal{S}$ and $w \neq y$, then $(x,y,w) \in \mathcal{S}$ or $(w,y,z) \in \mathcal{S}$.

In a pretree, $\mathcal{S}$ is understood as a strict betweenness relation, i.e., the condition $(x,y,z) \in \mathcal{S}$ should be interpreted as "y lies strictly between x and z". For a pair x, y of points in the pretree $(\mathcal{T}, \mathcal{S})$, we set

$$[x,y] := \{t \in \mathcal{T} : (x,t,y) \in \mathcal{S}\} \cup \{x,y\}.$$

Using this notation, we rewrite the system of axioms (T0)–(T3) in the form

(A0) $[x,y] \supset \{x,y\}$;
(A1) $[x,y] = [y,x]$;
(A2) if $z \in [x,y]$ and $y \in [x,z]$, then $y = z$;
(A3) $[x,y] \subset [x,z] \cup [z,y]$,

i.e., in terms of interval spaces. (We recall that an *interval space* is a set X with a map $X \times X \to 2^X$, $(x,y) \mapsto [x,y]$, having properties (A0) and (A1).)

An alternative approach to the theory of pretrees is given by the theory of convex structures (this theory is presented, for example, in [35, 37]). In the framework of the convex structures theory, the aforementioned technique with the transition from a class of spaces to a wider class of spaces with the same finite subspaces was used by R. E. Jamison-Waldner for defining the *varieties* of convex structures in [21]. The variety of convex structures generated by trees[4] was considered in the paper [14] by P. Duchet. In [23] it was shown that this variety is equivalent to the class of pretrees: there is a natural one-to-one correspondence between the elements of this variety and the pretrees. We remark that in the class of ternary structures as well as in the class of intervals, the subclass of pretrees is *finitely based* (it can be defined by a finite list of forbidden subspaces), while the variety of convex structures that is equivalent to the class of pretrees is no longer finitely based in the class of all convex structures. In this paper, we address the relationship between the pretrees and convexity theory by giving a series of axiomatic characterizations of pretrees in terms of convex structures. For example, it turns out that the pretrees is precisely the same thing as the *coherent antimatroids having the first separation property* (the separation property in the sense of convex structures is implied).

In a broader context that goes beyond the theory of tree-like structures, the relationship between systems of subsets and betweenness relations was studied in a recent paper [3]. In [3], the pretrees appear under the name "*SAR-relations*".

A significant part of this paper is devoted to the development of the research area related to a remarkable topology, which has been described many times for different classes of tree-like objects. In a pretree, this topology is generated by the sets of the form

$$\{x \in \mathcal{T} : r \notin [x,t]\}, \quad \text{xxx } r, t \in \mathcal{T},\ r \neq t,$$

which will be called *branches* (while their complements will be called *shadows*). In the case of dendritic spaces (see the definition in Subsection 1.2(3)), in particular, in the cases of ordinary CW-trees (i.e., connected and simply connected CW-complexes of dimension at most 1) and $\mathbb{R}$-trees, the branches are the connected components of the subsets of the form $\mathcal{T} \setminus \{r\}$, $r \in \mathcal{T}$, so that, for the dendritic spaces, the connected components of the spaces of the form $\mathcal{T} \setminus F$, where $F \subset \mathcal{T}$ is finite, give a base of our topology.

The topology described above has no established name. When defining this topology for the general case of pretrees, L. Ward [40] used the terms "*augmented cutpoint topology*" and "*augmented nodal topology*". In the preceding paper [39], Ward denoted by σ

[4]See footnote 1.

the same topology in the case of dendritic spaces. In the monograph [5] by B. Bowditch, this topology was called the *order topology*. P. de la Harpe and J.-P. Préaux, following N. Monod and Y. Shalom, described a version of the same topology for the case of the space that is the union of a tree and its ends, and referred to it as to "the *shadow topology*". C. Favre and M. Jonsson used the term "*weak topology*". T. Coulbois, A. Hilion, and M. Lustig [12] described this topology for the case of $\mathbb{R}$-trees, calling it "the *observers' topology*" and attributing the invention of this term to V. Guirardel. In some cases (see, e.g., P. Papasoglu and E. L. Swenson [36, 32]), this topology has been employed without giving it a name. In the monograph [31] by J. Nikiel, a related topology on pseudotrees played a central role, but also was not named and was denoted by $T'_{\leq}$. On the pretrees that represent dendritic spaces, the topology we are interested in coincides with the *Lawson topology*. In this paper, we use the term "*shadow topology*".

It is not difficult to show that, on a usual tree[5], the shadow topology is contained in the standard one and coincides with it if and only if the tree is locally finite. On an $\mathbb{R}$-tree, the shadow topology is contained in the topology of the metric and coincides with it if and only if the metric completion of the given $\mathbb{R}$-tree is locally compact in the topology of the metric. Generally, on any dendritic space, the shadow topology is contained in the original one, and on a dendron these topologies coincide (see Subsection 6.2(3)). Furthermore, as was shown in [39, Theorems 20, 21], the transition to the shadow topology turns any dendritic space into a regular locally connected and arcwise connected dendritic space[6].

The shadow topology has the following property, which is important for applications: by completing an arbitrary pretree in a certain way, we obtain a pretree with compact shadow topology. This property is particularly useful in the study of group actions on tree-like structures. For example, in the case of an ordinary tree T, the role of this "compact completion" is played by the union $T \cup \operatorname{Ends}(T)$, where $\operatorname{Ends}(T)$ is the space of ends. On $T \cup \operatorname{Ends}(T)$, there is a natural pretree structure, and the shadow topology of this pretree is compact for any tree T. (The standard topology usually considered on $T \cup \operatorname{Ends}(T)$ is compact if and only if T is locally finite. The shadow topology on $T \cup \operatorname{Ends}(T)$ is contained in the standard one and coincides with it if T is locally compact.)

One of our purposes in this paper is to prove a series of theorems concerning the basic properties of the shadow topology (separation properties, compactness, etc.) in the general case of an arbitrary pretree. These theorems are close to the results of [31]. In several important special cases, results related to the basic properties of the shadow topology were also proved in [5, 12, 15, 28, 29, 32, 33, 39, 40].

Structure of the paper. In §1, several standard axiomatics of the theory of pretrees are presented, their equivalence is proved, and examples and some properties of the pretrees are given.

In §2, we study the properties of the convex subsets in the pretrees and investigate the relationship between the theory of pretrees and the theory of convex structures.

In §3, we study the properties of the linear subsets in the pretrees; the pretrees are considered in the context of more general classes of interval spaces.

In §4, some classes of pretrees are defined.

In §5, we introduce the notions of branches and shadows of a pretree and study their properties (which are used in the proofs of properties of the shadow topology). Also, in §5, we give an alternative definition of pretrees via systems of partitions.

In §6, the definition of the shadow topology is presented and some properties of this topology are proved.

[5]See footnote 1.

[6]In the paper [39], there is a gap in the proof of the key Lemma 8.2. This gap was filled in [29].

§7 is devoted to the separation properties of the shadow topology.

In §8, we prove a criterion for the shadow topology compactness.

In §9, it is proved that the shadow topology is sequentially compact if and only if it is countably compact.

In §10, we prove a series of assertions about the metrizability of the shadow topology.

In §11, we define the space of ends of a pretree, describe the extension of the pretree structure to the set of ends, and prove a series of assertions concerning properties of the shadow topology on the pretree "completed" by the ends.

In §12, we discuss the properties of the shadow topology on an ordinary $\mathbb{Z}$-tree and on the union of a $\mathbb{Z}$-tree with the set of its ends.

§1. Pretrees

In the present section, we give definitions and examples, and describe some properties of the pretrees. As was noted in the Introduction, there are several approaches to defining the pretrees. In this section, the standard axiomatic ways of defining the pretrees are presented: in terms of ternary relations and intervals. As an initial one, we take the definition via the strict betweenness relation, this variant of axiomatic currently dominates in the literature. Next, a system of axioms for the nonstrict betweenness relation is given. Then we turn to a description via the systems of "closed intervals". The terminology of intervals is used as the basic one throughout the paper. In §2 and 5, new (to my knowledge) systems of axioms for the theory of pretrees are presented, they are based on convex sets and on partitions, respectively.

1.1. Definition. The first system of axioms**.** A *ternary relation* (or *ternary structure*) on a set X is a subset

$$R \subset X^3 = X \times X \times X.$$

Following [40], we will usually write $Rxyz$ instead of $(x, y, z) \in R$. A set $\mathcal{T}$ with a ternary relation $\mathcal{S} \subset \mathcal{T}^3$ is called a *pretree* if the following axioms are satisfied:

(T0) if $\mathcal{S}xyz$, then $x \neq z$;
(T1) if $\mathcal{S}xyz$, then $\mathcal{S}zyx$;
(T2) if $\mathcal{S}xyz$, then $(x, z, y) \notin \mathcal{S}$;
(T3) if $\mathcal{S}xyz$ and $w \neq y$, then $\mathcal{S}xyw$ or $\mathcal{S}wyz$.

In a pretree, $\mathcal{S}$ is understood as a strict betweenness relation, i.e., the condition $\mathcal{S}xyz$ should be interpreted as "y lies strictly between x and z".

1.2. Examples. 1. The ordinary trees (see footnote 1) with the ordinary betweenness relation.

2. $\mathbb{R}$-trees and, more generally, Λ-trees (see the definitions, e.g., in [9]).

3. Dendritic spaces (including dendrons etc.).

A *dendritic space* is a connected topological space in which every two distinct points can be separated by a third one (a point $c \in D$ in a topological space D *separates* two points $a, b \in D$ if the space $D \setminus \{c\}$ can be presented as the disjoint union of two open sets A and B such that $a \in A$ and $b \in B$)[7]. It is easy to verify that all dendritic spaces are Hausdorff.

[7]A point $c \in D$ in a topological space D separates two points $a, b \in D$ if and only if a and b lie in distinct quasicomponents of the space $D \setminus \{c\}$. Recall that a *quasicomponent* of a point in a topological space is the intersection of all the clopen subsets containing this point. Quasicomponents are closed and provide a partition of the space. The (connected) component of a point is contained in its quasicomponent. If D is a dendritic space and $c \in D$, then the components of $D \setminus \{c\}$ coincide with its quasicomponents (this follows, e.g., from [33, Theorem 4] or [39, Theorem 19]).

Problem. Is it true that a space E is dendritic whenever for every two distinct points $a, b \in E$ there exists a point $c \in E$ such that a and b lie in distinct components of the space $D \setminus \{c\}$?

Compact dendritic spaces are called *dendrons*.

Metrizable dendrons are called *dendrites*.

4. Any subset of a pretree is a pretree.

As a matter of fact, the above examples exhaust the list, because each pretree embeds in a Λ-tree (see [5] and [10]). We note also that each pretree embeds in a dendron (see Theorem 7.3).

5. Any linearly ordered set with the order-betweenness relation is a pretree.

6. Any set with the empty ternary relation is a pretree.

7. A *pseudotree* is a partially ordered set $(P, \leq)$ where each subset of the form $\downarrow p := \{t \in P : t \leq p\}$, $p \in P$, is linearly ordered. The *infimum* $\inf\{x, y\}$ of elements $x, y \in P$ is the largest element in the set $(\downarrow x) \cap (\downarrow y)$ (if such an element exists). For two points x, y in a pseudotree $(P, \leq)$, we denote by $V'(x, y)$ the set of all points $z \in P$ that satisfy exactly one of the relations $z < x$ and $z < y$, and put $V(x, y) := V'(x, y) \cup \inf\{x, y\}$. Then the ternary relation $\mathcal{S}$ defined by the rule

$$\mathcal{S}abc \Leftrightarrow b \in V(a, c),$$

determines a pretree structure on P. (See also a discussion in [5, p. 25].)

8. An important example: the union of any tree (or, more generally, of a pretree) with the space of its ends is naturally endowed with the structure of a pretree (see §11 and Theorem 11.3).

9. The following example is taken from [40] (see also [3, § 6.2]).

1.3. Proposition. *Let T be a connected topological space* (*no separation axioms are required*). *Define a ternary relation $\mathcal{S} \subset T^3$ on T by setting $\mathcal{S}xyz$ if x and z are separated by y* (*i.e., lie in distinct quasicomponents of the space $T \setminus \{y\}$*). *Then $\mathcal{S}$ satisfies Axioms* (T0)–(T3), *i.e., $\mathcal{S}$ is a pretree structure on T.*

Proof. The fact that Axioms (T0) and (T1) are satisfied is obvious by the definition of $\mathcal{S}$. Axiom (T3) is satisfied because, as is well known, the quasicomponents give a partition of the space. The fact that Axiom (T2) is satisfied is a consequence of the following statement (see [30, Chapter IV, Theorem 3.4] or [3, Lemma 6.11]).

Let A be a connected subset of a connected topological space X, and let B be a clopen subset in the space $X \setminus A$. Then the union $A \cup B$ is connected. □

1.4. *Remark* (see [3, §6.1]). Proposition 1.3 remains true if we consider connected components instead of quasicomponents. In this case, the fact that Axioms (T0) and (T1) are satisfied is also obvious, Axiom (T3) is satisfied because the connected components give a partition of the space, while the fact that Axiom (T2) is satisfied is a consequence of the following statement (see [30, Chapter IV, Theorem 3.3] or [3, Lemma 6.11]).

Let A be a connected subset of a connected topological space X, and let B be a connected component of the space $X \setminus A$. Then the union $A \cup B$ is connected.

Another source of examples of pretrees stems from the following notion.

1.5. Tree-like collections of partitions. A *partition* of a set is a family of its nonempty pairwise disjoint subsets (*elements* of the partition) that cover the set. A partition is *trivial* if it consists of one element. A collection Z of nontrivial partitions of a set X is said to be *tree-like* if every pair ζ, ξ of distinct partitions in Z has elements $\zeta_\xi \in \zeta$ and $\xi_\zeta \in \xi$ such that $\zeta_\xi \cup \xi_\zeta = X$. Clearly, the condition $\zeta_\xi \cup \xi_\zeta = X$ determines the elements $\zeta_\xi \in \zeta$ and $\xi_\zeta \in \xi$ uniquely (for nontrivial distinct partitions ζ, ξ).

1.6. Proposition. *Let Z be a tree-like collection of nontrivial partitions of a set X. Define a ternary relation $\mathcal{S} \subset Z^3$ on Z by setting $\mathcal{S}\alpha\beta\gamma$ if $\alpha \neq \beta \neq \gamma$ and $\beta_\alpha \neq \beta_\gamma$. Then $\mathcal{S}$ satisfies Axioms* (T0)–(T3), *i.e., $\mathcal{S}$ is a pretree structure on Z.*

Proof. Axioms (T0) and (T1) are satisfied because the condition $\beta_\alpha \neq \beta_\gamma$ implies that $\alpha \neq \gamma$ and $\beta_\gamma \neq \beta_\alpha$. In order to check Axiom (T2), suppose that $\mathcal{S}\alpha\beta\gamma$ and $\mathcal{S}\alpha\gamma\beta$ for some $\alpha, \beta, \gamma \in Z$. This means by definition that $\alpha \neq \beta \neq \gamma \neq \alpha$, $\beta_\alpha \neq \beta_\gamma$, and $\gamma_\alpha \neq \gamma_\beta$, i.e., $\beta_\alpha \cap \beta_\gamma = \varnothing$ and $\gamma_\alpha \cap \gamma_\beta = \varnothing$. However, since Z is assumed to be tree-like, the condition $\alpha \neq \beta \neq \gamma \neq \alpha$ shows that

$$(1) \qquad \alpha_\beta \cup \beta_\alpha = \beta_\gamma \cup \gamma_\beta = \alpha_\gamma \cup \gamma_\alpha = X,$$

whereby the condition $\beta_\alpha \cap \beta_\gamma = \varnothing$ implies

$$(2) \qquad \beta_\alpha \subset \gamma_\beta \quad \text{and} \quad \beta_\gamma \subset \alpha_\beta,$$

while the condition $\gamma_\alpha \cap \gamma_\beta = \varnothing$ implies

$$(3) \qquad \gamma_\alpha \subset \beta_\gamma \quad \text{and} \quad \gamma_\beta \subset \alpha_\gamma.$$

The conditions $\beta_\gamma \subset \alpha_\beta$, $\gamma_\beta \subset \alpha_\gamma$, and $\beta_\gamma \cup \gamma_\beta = X$ imply by the nontriviality of α that $\alpha_\beta \neq \alpha_\gamma$ and therefore $\alpha_\beta \cap \alpha_\gamma = \varnothing$, showing by (1) that

$$(4) \qquad \alpha_\beta \subset \gamma_\alpha \quad \text{and} \quad \alpha_\gamma \subset \beta_\alpha.$$

From (2)–(4) it follows that

$$(5) \qquad \alpha_\beta = \gamma_\alpha = \beta_\gamma \quad \text{and} \quad \alpha_\gamma = \beta_\alpha = \gamma_\beta,$$

whence, by (1), we have $\alpha = \beta = \gamma$, a contradiction.

Axiom (T3) is satisfied because from $\beta_\alpha \neq \beta_\gamma$ it follows that either $\beta_\alpha \neq \beta_\delta$ or $\beta_\delta \neq \beta_\gamma$. □

1.7. *Remark.* Let X be a set, and let Z be a collection of its partitions. Let X/Z be the partition of X generated by Z (two points $x, y \in X$ are in one and the same element of X/Z if and only if each partition in Z contains x and y in one and the same element). If Z is tree-like, then the set $Z \cup (X/Z)$ has a natural pretree structure that extends the structure described in Proposition 1.6.

1.8. *Remark.* Syzygetic collections. A collection $\mathcal{P}$ of subsets in a set X is said to be *syzygetic* if for each pair of subsets $A, B \in \mathcal{P}$ one of the following relations hold true:

$$A \cap B = \varnothing, \quad A \cup B = X, \quad A \subset B, \quad A \supset B.$$

Let $\mathcal{P}$ be a syzygetic collection of subsets in a set X such that at least one proper subset of X is in $\mathcal{P}$, and let $Z_\mathcal{P}$ be the collection of all two-element partitions of X each of which has at least one element in $\mathcal{P}$. Then, obviously, $Z_\mathcal{P}$ is tree-like.

Note that the family of partition elements of a tree-like collection of partitions is syzygetic.

1.9. *Remark.* With any nontrivial pretree, we can associate some tree-like collections of partitions. In §5, the system of *nodal partitions* associated with a pretree is described (the pretree structure of this system is isomorphic to the initial pretree). Next, a pretree has an associated tree-like collection of two-element partitions formed by the partitions consisting of *branches* and their complementary *shadows* (see §5).

1.10. The second system of axioms. In the paper [1], a *B-set* was defined as a set $\mathcal{T}$ with a ternary relation $\bar{\mathcal{S}} \subset \mathcal{T}^3$ satisfying the following axioms:

(B1) $(\bar{\mathcal{S}}abc) \Rightarrow (\bar{\mathcal{S}}cba)$;
(B2) $(\bar{\mathcal{S}}abc) \wedge (\bar{\mathcal{S}}acb) \Leftrightarrow b = c$;
(B3) $(\bar{\mathcal{S}}abc) \Rightarrow (\bar{\mathcal{S}}abd) \vee (\bar{\mathcal{S}}dbc)$.

The relation $\bar{S}abc$ is interpreted as "b lies (nonstrictly) between a and c".

For an arbitrary set X, we set

$$D_X := \{(x, y, z) \in X^3 \,:\, (x = y) \vee (y = z)\}. \tag{6}$$

We remark that each structure of a B-set on X contains D_X (by Axiom (B2)), while no structure of a pretree on X intersects D_X (the structure of a pretree contains no triples of the form (x, y, y) due to (T2), therefore it contains no triples of the form (x, x, y) either, due to (T1)).

1.11. Theorem. *For any set X, the endomorphism of ternary structures on X that sends a structure $S \subset X^3$ to the structure $S \cup D_X$ establishes a one-to-one correspondence between the structures of pretrees and the structures of B-sets on X.*

Proof. In order to prove this theorem, it suffices to check the following:

1) if a structure $\mathcal{S} \subset X^3$ satisfies Axioms (T0)–(T3), then the structure $\mathcal{S} \cup D_X$ satisfies Axioms (B1)–(B3);

2) if a structure $\bar{\mathcal{S}} \subset X^3$ satisfies Axioms (B1)–(B3), then there exists a structure $\mathcal{S} \subset X^3$ satisfying Axioms (T0)–(T3) and with $\bar{\mathcal{S}} = \mathcal{S} \cup D_X$;

3) if two structures $\mathcal{S}_1, \mathcal{S}_2 \subset X^3$ satisfying Axioms (T0)–(T3) are distinct, then $\mathcal{S}_1 \cup D_X \neq \mathcal{S}_2 \cup D_X$.

In assertion 1, the fact that $\mathcal{S}' := \mathcal{S} \cup D_X$ satisfies (B1), (B2), and (B3) follows by a simple direct verification. For example, Axiom (B3) is verified as follows. Suppose $(a, b, c) \in \mathcal{S}'$; if $\mathcal{S}abc$, then the condition $(\mathcal{S}'abd) \vee (\mathcal{S}'dbc)$ is fulfilled by (T2); if $D_X abc$, then the condition $(\mathcal{S}'abd) \vee (\mathcal{S}'dbc)$ is fulfilled because D_X, by its definition, is a B-set structure.

To prove assertion 2, it suffices to observe that if $\bar{\mathcal{S}}$ satisfies (B1)–(B3), then $\bar{\mathcal{S}} \setminus D_X$ satisfies (T0)–(T3).

Assertion 3 follows from the fact that (as explained before Theorem 1.11) neither $\mathcal{S}_1$ nor $\mathcal{S}_2$ intersect D_X, and in view of this, obviously, the relation $\mathcal{S}_1 \cup D_X \neq \mathcal{S}_2 \cup D_X$ is equivalent to the relation $\mathcal{S}_1 \neq \mathcal{S}_2$. □

1.12. Intervals. The third system of axioms. We introduce some notation. For points x, y in a pretree $(\mathcal{T}, \mathcal{S})$, we set

$$\begin{aligned} \langle x, y\rangle &:= \{t \in \mathcal{T} \,:\, \mathcal{S}xty\}, \\ [x, y] &:= \langle x, y\rangle \cup \{x, y\}, \\ [x, y\rangle &:= \langle y, x] := [x, y] \setminus \{y\}. \end{aligned} \tag{7}$$

We note that $[x, x\rangle = \varnothing$. The sets of the form $[x, y]$, $[x, y\rangle$, $\langle y, x]$, and $\langle x, y\rangle$ will be called *intervals*. Intervals of the form $[x, y]$ are *closed intervals*.

1.13. Lemma. *For any points a, b, c in a pretree $\mathcal{T}$, the following relations hold true*:

(A0) $[a, b] \supset \{a, b\}$;
(A1) $[a, b] = [b, a]$;
(A2) *if $c \in [a, b]$ and $b \in [a, c]$, then $b = c$*;
(A3) $[a, b] \subset [a, c] \cup [c, b]$.

Lemma 1.13 is implied by the following Theorem 1.14, which states, in essence, that the family of properties (A0)–(A3) can serve as a system of axioms for the theory of pretrees. In Theorem 1.14, the notion of a *system* of subsets is employed: we interpret an indexed system $\{X_i\}_{i \in I}$ of subsets X_i of a set X as a map $I \to 2^X$.

1.14. Theorem. *The map assigning to a pretree the system of its closed intervals establishes a one-to-one correspondence between the pretrees and the systems of the form $\{[a, b]\}_{a, b \in X}$ consisting of subsets of a set X that satisfy* (A0)–(A3).

Proof. Let $\mathcal{S}$ be an arbitrary ternary relation on a set X. We set

$$\bar{\mathcal{S}} := \mathcal{S} \cup D_X,$$

where D_X is defined in (6). Also, we set

$$[x, y] := \{t \in X \,:\, \mathcal{S}xty\} \cup \{x, y\},$$

as in (7). With these definitions, we see that

$$[x, y] := \{t \in X \,:\, \bar{\mathcal{S}}xty\},$$

which implies that the relation $\bar{\mathcal{S}}abc$ is equivalent to the relation $b \in [a, c]$, i.e.,

$$\bar{\mathcal{S}} = \{(a, b, c) \in X \,:\, b \in [a, c]\}.$$

It is easily seen that the system of axioms (A0)–(A3) is none other than the result of translation of the above system of axioms (B1)–(B3) from the language of the notation $\bar{\mathcal{S}}abc$ into the language of the notation $b \in [a, c]$ (the forward and backward implications of Axiom (B2) are presented by Axioms (A2) and (A1), respectively), while Theorem 1.14, which we are proving, is a straightforward reformulation of Theorem 1.11. □

1.15. *Remark.* In the literature, various modifications of the system of axioms (A0)–(A3) can be found. For example (see [7]), an equivalent system of axioms is obtained by replacing Axiom (A0) with the relation

(A$\frac{1}{2}$) $[a, a] = \{a\}$.

Relation (A$\frac{1}{2}$) follows from Axiom (A0), which gives the inclusion $a \in [a, a]$, and Axiom (A2), by which the relation $b \in [a, a]$ implies the identity $b = a$ (because $a \in [a, b]$ by (A0)). Axiom (A0) follows from (A$\frac{1}{2}$), (A3), and (A1), because $[a, a] \subset [a, b] \cup [b, a]$ by (A3).

1.16. Lemma. *For any points a, b, c, d, x in a pretree, the following properties hold true*:

- (A4) *if $b \in [a, c]$, then $[a, b] \subset [a, c]$;*
- (A5) *if $b \in [a, c]$ and $c \in [a, d]$, then $c \in [b, d]$;*
- (A6) *if $b \in [a, c]$, then $[a, b] \cap [b, c] = \{b\}$;*
- (A7) *if $b \in [a, c]$, then $[a, b] \cup [b, c] = [a, c]$;*
- (A8) *if $b \in [a, c]$, $c \in [b, d]$, and $b \neq c$, then $\{b, c\} \subset [a, d]$;*
- (A9) *if $b \in [a, c]$, then $[x, a] \cap [x, c] \subset [x, b]$.*

Proof. (A8) Since $b \in [a, c]$, while $[a, c] \subset [a, d] \cup [c, d]$ by (A3), it follows that at least one of the relations $b \in [a, d]$ and $b \in [c, d]$ holds true. Since $c \in [b, d]$ and $b \neq c$, we have $b \notin [c, d]$ by (A2). Consequently, $b \in [a, d]$. The fact that $c \in [a, d]$ follows by (A1).

(A5) In the case where $b = c$, the claim follows by (A0). Suppose that $b \neq c$. Then the condition $b \in [a, c]$ implies by (A2) that $c \notin [a, b]$. The conditions $c \in [a, d]$ and $c \notin [a, b]$ imply by (A3) that $c \in [b, d]$, as required.

(A4) Assume that $x \in [a, b]$. The conditions $b \in [a, c]$ and $x \in [a, b]$ imply by (A5) that $b \in [x, c]$. If $x \neq b$, then the conditions $x \in [a, b]$ and $b \in [x, c]$ imply that $x \in [a, c]$ by (A8). If $x = b$, then the fact that $x \in [a, c]$ follows from the condition $b \in [a, c]$. Therefore, we have $x \in [a, c]$ for all $x \in [a, b]$.

(A7) We have $[a, c] \subset [a, b] \cup [b, c]$ by (A3). The inclusion $[a, c] \supset [a, b] \cup [b, c]$ follows from (A4) via (A1).

(A9) Suppose that $p \in [x, a] \cap [x, c]$. Since $b \in [a, c]$, we have either $b \in [p, a]$ or $b \in [p, c]$ by (A3). If $b \in [p, a]$, then $p \in [x, b]$ by (A5) because $p \in [x, a]$. If $b \in [p, c]$, then $p \in [x, b]$ by (A5) because $p \in [x, c]$.

(A6) This follows from (A9) with $x := b$ by (A$\frac{1}{2}$), (A0), and (A1). □

1.17. *Remark.* Since applications of Axioms (A0) and (A1) are rather obvious, we usually omit their mentioning below (in this, we follow [1, 5]).

1.18. Lemma. *For any points a, b, x, y, z in a pretree, the following properties hold true*:

(1) *if* $\{x, y\} \subset [a, b]$, *then* $[x, y] \subset [a, b]$;

(2) *if* $\{x, y\} \subset [a, b]$, *then* $(x \in [a, y]) \vee (y \in [a, x])$.

Proof. (1) The condition $\{x, y\} \subset [a, b]$ implies by (A7) that either $y \in [a, x]$ or $y \in [x, b]$. If $y \in [a, x]$, then by (A4) we get $[x, y] \subset [a, x] \subset [a, b]$; if $y \in [x, b]$, then the same (A4) yields $[x, y] \subset [x, b] \subset [a, b]$.

(2) The condition $\{x, y\} \subset [a, b]$ implies by (A7) that either $y \in [a, x]$ or $y \in [x, b]$, and that either $x \in [a, y]$ or $x \in [y, b]$. Consequently, should none of the relations $x \in [a, y]$ and $y \in [a, x]$ be true, this would imply that $y \in [x, b]$ and $x \in [y, b]$, whence $x = y$ by (A2) and $(x \in [a, y]) \wedge (y \in [a, x])$ by (A0). ☐

1.19. *Remark.* If a map $\mathcal{I}\colon X \times X \to 2^X$ on a set X satisfies (A0) and (A1) (when we set $[x, y] := \mathcal{I}(x, y)$), then the pair $(X, \mathcal{I})$ is called an *interval space* (see, e.g., [37]).

The interval spaces satisfying (A$\frac{1}{2}$), (A4), and (A5) were studied independently (see, e.g., [37, I, §4] and [19, 24, 11]). As in [37], such interval spaces will be called *geometric*. (It is easily seen that all geometric interval spaces possess properties (A2) and (A6).) Therefore, the pretrees are geometric interval spaces.

Geometric interval spaces are of independent interest; in a certain sense, they "generalize and well approximate" the betweenness relations arising on metric spaces. This is explained as follows. With an arbitrary metric space (M, ρ), we associate an interval space $(M, \mathcal{I}_\rho)$ by setting

$$\mathcal{I}_\rho(a, b) = \{x \in X \,:\, \rho(a, x) + \rho(x, b) = \rho(a, b)\}. \tag{8}$$

We say that an interval space $(X, \mathcal{I})$ is *metrizable* if there exists a metric ρ on X such that $\mathcal{I} = \mathcal{I}_\rho$. Let $\mathscr{E}_F(\mathcal{M})$ be the class formed by the interval spaces all of whose finite subspaces are metrizable, and let $\mathscr{E}_k(\mathcal{M})$ be the class of interval spaces all of whose subspaces of cardinality at most k are metrizable. The class $\mathscr{E}_F(\mathcal{M})$ (in contrast to the pretrees) is not finitely based (it cannot be defined by a finite list of forbidden subspaces) and does not coincide with $\mathscr{E}_k(\mathcal{M})$ for any $k \in \mathbb{N}$ (see [24], the proof[8] of assertion (v) of the main theorem). The class $\mathscr{E}_2(\mathcal{M})$ does coincide with the class of all interval spaces. The class $\mathscr{E}_3(\mathcal{M})$ coincides with the class of interval spaces with property (A2). The class $\mathscr{E}_4(\mathcal{M})$ coincides with the class of geometric interval spaces. We remark that $\mathscr{E}_4(\mathcal{M}) = \mathscr{E}_5(\mathcal{M}) \neq \mathscr{E}_6(\mathcal{M})$ (see [11]).

As was noted above, in any metric space, the structure of intervals (8), alongside with Axioms (A0) and (A1), satisfies Axiom (A2) and properties (A$\frac{1}{2}$), (A4), (A5), (A6). It is easy to check that Property (A7) is also fulfilled in the uniquely geodesic metric spaces, while the geodesic metric spaces of nonpositive curvature satisfy (A8) as well. Property (A9) (which, unlike the rest of the above properties, concerns quintuples of points) is not satisfied in some CAT(0) spaces (a counterexample: a triple of half-planes with edges glued together).

§2. Convex subsets

In this section, we discuss the relationship between the theory of pretrees and the *theory of convex structures* (see, e.g., [21, 35, 37]), introduce the notion of a *convex subset* in a pretree, and prove several assertions related to this notion.

[8]Formulas in that proof contain misprints; however, the appropriate values can be found from the context.

In the theory of convexity, a collection $\mathcal{C}$ of subsets in a set X is called a *convex structure* or *convexity* if it has the following properties (see, e.g., [37]):

(C0) the empty set and the set X are in $\mathcal{C}$;
(C1) the intersection of any collection of sets in $\mathcal{C}$ is in $\mathcal{C}$;
(C2) the union of any collection of sets in $\mathcal{C}$ that are linearly ordered by inclusion is in $\mathcal{C}$.

If $\mathcal{C}$ is a convexity on a set X and $A \subset X$, then the intersection of all the sets in $\mathcal{C}$ containing A is called the *convex hull* of A and is denoted by $\operatorname{hull}(A) := \operatorname{hull}_{\mathcal{C}}(A)$. By (C1) it follows that $\operatorname{hull}(A) \in \mathcal{C}$.

There is a natural relationship between convex structures and interval spaces (see the definition in Subsection 1.19). It is easily seen that in any interval space $(X, \mathcal{I})$, the collection $\mathcal{C}_{\mathcal{I}} \subset 2^X$ formed by all subsets $D \in 2^X$ that, together with each pair of points $x, y \in D$, contain the entire interval $\mathcal{I}(x, y)$, is a convex structure on X. On the other hand, in any convexity $(X, \mathcal{C})$, the map $\mathcal{I}_{\mathcal{C}} \colon X \times X \to 2^X$ defined by the rule $\mathcal{I}_{\mathcal{C}}(a, b) := \operatorname{hull}_{\mathcal{C}}\{a, b\}$ establishes an interval space structure on X.

It is easy to check that the above maps $\mathcal{I} \mapsto \mathcal{C}_{\mathcal{I}}$ and $\mathcal{C} \mapsto \mathcal{I}_{\mathcal{C}}$ between the class of interval spaces and the class of convexities are neither injective nor surjective. However, it can be shown (see [23]) that the restrictions of these maps to the images of the "duals" are mutually inverse bijections; furthermore, by restricting these maps further, to the class of the convex structures with Carathéodory number at most 2 and the corresponding class of interval spaces, we obtain mutually inverse functors (with respect to the corresponding categories, in which inclusions act as morphisms).

In particular, the restriction of the map $\mathcal{I} \mapsto \mathcal{C}_{\mathcal{I}}$ to the pretrees is injective and its composition with $\mathcal{C} \mapsto \mathcal{I}_{\mathcal{C}}$ yields the identity map (Theorem 2.2), which means that the pretrees can be viewed as a certain class of convexities. In [23] it was proved that the class of convexities corresponding to the pretrees coincides with the class of *arborescent convexities* introduced in [14]. In the following Theorem 2.2, an axiomatic characterization of the class of convexities corresponding to the pretrees is given (see also Theorem 2.6).

In what follows, we will mostly treat pretrees as interval spaces and rest upon Properties (A0)–(A3), which will be called "axioms". The systems of axioms (T0)–(T3) and (B1)–(B3) will be used only occasionally.

2.1. Definition. A subset C of a pretree $\mathcal{T}$ is said to be *convex* if[9] $[x, y] \subset C$ for all $x, y \in C$.

2.2. Theorem. *The map assigning to a pretree the collection of all of its convex subsets establishes a one-to-one correspondence between the pretrees and the collections $\Delta \subset 2^X$ of subsets of a set X that have the following properties*:

(Δ0) *the empty set and the set X are in Δ*;
(Δ1) *the intersection of any collection of sets in Δ is in Δ*;
(Δ2) *the union of any collection of sets in Δ that have a common point is in Δ*;
(Δ3) *if $a, b, c \in X$, then there exists an element in Δ that contains the point a and precisely one point of the set $\{b, c\}$.*

In other words, the following is true.

1) *The collection Δ of convex sets of an arbitrary pretree satisfies* (Δ0)–(Δ3).
2) *Any collection $\Delta \subset 2^X$ of subsets of an arbitrary set X satisfying* (Δ0)–(Δ3) *is the collection of convex subsets of a pretree structure on X.*
3) *Any pretree structure is uniquely determined by the totality of its convex subsets.*

[9]For the notation $[x, y]$, see Subsection 1.12.

Proof. In order to prove this theorem, it suffices to prove assertions 1)–3).

1) Let Δ be the collection of all convex sets of a pretree $\mathcal{T}$. The fact that Δ satisfies (Δ0) and (Δ1) follows directly from the definition of a convex subset (Definition 2.1). We check that property (Δ2) is satisfied. If U is the union of a collection of sets in Δ with a common point $x \in \mathcal{T}$, then for any $y, z \in U$ the inclusions $[x, y] \subset U$ and $[x, z] \subset U$ hold true, whence by (A3) it follows that $[y, z] \subset U$. Consequently, $U \in \Delta$. We proceed with the verification of (Δ3). Let $a, b, c \in \mathcal{T}$. In the case where $b = c$, the set $\{b, c\}$ consists of a unique element, so that the requirement of property (Δ3) is satisfied by the set $\mathcal{T} \in \Delta$. If $b \neq c$, then, by Axioms (A2) and (A0), at least one of the intervals $[a, c]$ and $[a, b]$ "contains the point a and precisely one point of the set $\{b, c\}$". Thereby, Δ has Property (Δ3), because the (closed) intervals of a pretree are convex (Lemma 1.18).

2) Assume that a collection $\Delta \subset 2^X$ of subsets of a set X has properties (Δ0)–(Δ3). We describe a pretree structure on X whose collection of convex subsets coincides with Δ. For two points $a, b \in X$, we denote by $[a, b]_\Delta$ the intersection of all sets in Δ that contain the set $\{a, b\}$. We show that the system $\{[a, b]_\Delta\}_{a,b\in X}$ satisfies Axioms (A0)–(A3).

Axioms (A0) and (A1) are satisfied by construction.

(A2) Suppose $a, b, c \in X$. If $b \neq c$, then, by (Δ3), Δ contains an element D that satisfies either the condition $\{a, b\} \subset D \not\ni c$ (in this case, we have $c \notin [a, b]_\Delta$) or the condition $\{a, c\} \subset D \not\ni b$ (in this case, we have $b \notin [a, c]_\Delta$). This shows that (A2) is satisfied.

(A3) Suppose $a, b, c \in X$. We note that $[a, c]_\Delta \supset \{a, c\}$ and $[c, b]_\Delta \supset \{c, b\}$ by definition (in other words, as has already been noted, Axiom (A1) is satisfied). It follows from (Δ1) by construction that $[a, c]_\Delta$ and $[c, b]_\Delta$ are in Δ. Since $c \in [a, c]_\Delta \cap [c, b]_\Delta$, property ($\Delta$2) shows that $[a, c]_\Delta \cup [c, b]_\Delta \in \Delta$. Hence, since $[a, c]_\Delta \cup [c, b]_\Delta \supset \{a, b\}$, we have $[a, b]_\Delta \subset [a, c]_\Delta \cup [c, b]_\Delta$.

Thus, the system $\{[a, b]_\Delta\}_{a,b\in X}$ satisfies Axioms (A0)–(A3), which means by Theorem 1.14 that $\{[a, b]_\Delta\}_{a,b\in X}$ is the system of intervals of some pretree on X. We denote this pretree by X_Δ and show that the collection of its convex subsets coincides with Δ. Indeed, if $D \in \Delta$, then for all $x, y \in D$ the set $[x, y]_\Delta$ lies in D by construction, whence D is convex in X_Δ. Conversely, let C be a convex subset in a pretree X_Δ. If $C = \varnothing$, then $C \in \Delta$ by Axiom (Δ0). In the case where $C \neq \varnothing$, for each point $x \in C$ the identity $C = \bigcup_{y\in C} [x, y]_\Delta$ is fulfilled by (A0) and by the definition of a convex subset (Definition 2.1). We observe that all the intervals of the collection $\{[x, y]_\Delta : y \in C\}$ lie in Δ (by Axiom (Δ1)) and have a common point x (by Axiom (A0)), whence by Axiom (Δ2) we see that $C = \bigcup_{y\in C} [x, y]_\Delta$ belongs to Δ. Assertion 2) is proved.

In order to prove assertion 3), it suffices to observe that for any pair a, b of points in a pretree $\mathcal{T}$, the intersection of all the convex subsets in $\mathcal{T}$ containing the set $\{a, b\}$ coincides with the interval $[a, b]$ (because each convex subset containing a and b contains $[a, b]$ as well, while $[a, b]$ is convex by Lemma 1.18). Therefore, the collection of all convex subsets in a pretree uniquely determines the system of its closed intervals, and this system uniquely determines the pretree structure by Theorem 1.14.

Assertion 3), and with it the theorem, is proved. □

2.3. *Remark.* As a matter of fact, Axiom (Δ0) follows from Axioms (Δ1) and (Δ2).

2.4. *Remark.* We remark that when enlarging a collection having property (Δ3), this property is preserved; therefore, given any collection with property (Δ3) we can obtain a collection satisfying Axioms (Δ0)–(Δ3) by completing this initial collection with the intersections and unions of prescribed form (see Axioms (Δ1) and (Δ2)).

2.5. *Remark.* Let $\mathcal{C}$ be a convexity on a set X (see the definition at the beginning of this section).

If the union of any two intersecting sets in $\mathcal{C}$ lies in $\mathcal{C}$, then $\mathcal{C}$ is said to be *coherent* (see [14]). (As is easily verified, the axiom of choice and Axiom (C2) imply that, in a coherent convexity $\mathcal{C}$, the union of any collection of sets in Δ that have a common point is in Δ. See property $\Delta 2$.)

If for each subset A in X and each pair of distinct points p, q in $X \setminus \operatorname{hull}_{\mathcal{C}}(A)$ we have either $p \notin \operatorname{hull}_{\mathcal{C}}(\{q\} \cup A)$ or $q \notin \operatorname{hull}_{\mathcal{C}}(\{p\} \cup A)$, then $\mathcal{C}$ is called a *convex geometry* (*antimatroid*).

If all of the one-point subsets of X lie in $\mathcal{C}$, then the convexity $\mathcal{C}$ is said to have the *first separation property*.

If for each pair a, b of distinct points in X there are $A, B \in \mathcal{C}$ such that $a \in A$, $b \in B$, and $A = X \setminus B$, then $\mathcal{C}$ is said to have the *second separation property*.

2.6. Theorem. *Let $\mathcal{C}$ be a convexity. Then the following conditions are equivalent.*

(1) *$\mathcal{C}$ is the collection of convex sets of a pretree.*
(2) *$\mathcal{C}$ is a coherent antimatroid with the first separation property.*
(3) *$\mathcal{C}$ is coherent and has the second separation property.*

We do not give here the proof of Theorem 2.6. It can be deduced from Theorem 2.2.

2.7. Definition. The *convex closure* (or *convex hull*) of a subset S in a pretree $\mathcal{T}$ is the intersection of all the convex sets containing S. The convex hull of a set S is denoted by $\operatorname{hull}(S)$. (Obviously, this definition implies that $\operatorname{hull}(S)$ is convex and contains S.)

2.8. Lemma. *For any subset S in a pretree, the following identity holds true*:

$$\operatorname{hull}(S) = \bigcup_{r,s \in S} [r, s].$$

Proof. In [1], the *convex closure* of a subset S in a pretree is defined as the union $\operatorname{Cl}(S) := \bigcup_{r,s\in S}[r, s]$. In [1, Lemma 15.5], it was proved that $\operatorname{Cl}(S)$ is convex in the sense of Definition 2.1. This yields the inclusion $\operatorname{Cl}(S) \supset \operatorname{hull}(S)$ (because $\operatorname{Cl}(S) \supset S$). The inclusion $\operatorname{Cl}(S) \subset \operatorname{hull}(S)$ is valid by definitions. Therefore, $\operatorname{hull}(S) = \operatorname{Cl}(S)$, as required. □

2.9. *Remark.* In the language of the convexity theory, Lemma 2.8 is formulated as follows: the Carathéodory number of the system of convex sets in a pretree does not exceed two (for the definition of the Carathéodory number and its properties, see, e.g., [35, 37]).

2.10. Assertion. *Let S be a subset in a pretree, and let $s \in S$. Then the following conditions are equivalent*:

(a) *the set* $\operatorname{hull}(S) \setminus \{s\}$ *is convex*;
(b) *the set* $\operatorname{hull}(S \setminus \{s\})$ *does not contain* s;
(c) *there are no points $x, y \in S \setminus \{s\}$ with $s \in [x, y]$.*

Proof. (a) ⇒ (b). If $\operatorname{hull}(S) \setminus \{s\}$ is convex, then together with $S \setminus \{s\}$ it contains $\operatorname{hull}(S \setminus \{s\})$, which implies that $s \notin \operatorname{hull}(S \setminus \{s\})$ (because $s \notin \operatorname{hull}(S) \setminus \{s\}$).

¬(a) ⇒ ¬(b). If $\operatorname{hull}(S) \setminus \{s\}$ is not convex, then (since $\operatorname{hull}(S)$ is convex) there exist $x, y \in \operatorname{hull}(S)$ such that $s \in \langle x, y\rangle$. By Lemma 2.8, there exist $p, q \in S$ such that $y \in [p, q]$. At the same time, $y \in [x, p] \cup [x, q]$ by (A3). Without loss of generality we may assume that $y \in [x, p]$. The conditions $y \in [x, p]$ and $s \in \langle x, y\rangle$ imply by (A4) and (A2) that $s \in \langle x, p\rangle$. Applying the same argument to x, we see that $s \in \langle r, p\rangle$ for some $r \in S$. This means that $s \in \operatorname{hull}(S \setminus \{s\})$.

(b) ⇔ (c). This follows from Lemma 2.8. □

2.11. Definition. A point s in a subset S of a pretree is *extreme* (in S) if the conditions of Assertion 2.10 are satisfied. A point t in a pretree $\mathcal{T}$ is said to be *terminal* if it is extreme in $\mathcal{T}$, i.e., if there are no points $x, y \in \mathcal{T}$ such that $t \in \langle x, y\rangle$.

2.12. *Remark.* The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) of Assertion 2.10 are valid in any convexity (if we understand there $[a, b]$ as $\mathrm{hull}\{a, b\}$ and $\mathrm{hull}(S)$ as the smallest convex set containing S). In the general case, (c) does not imply (b), while (b) does not imply (a).

2.13. Lemma. *For any subset S of a pretree $\mathcal{T}$, the set $\mathrm{ex}(S)$ of its extreme points coincides with the set $\mathrm{ex}(\mathrm{hull}(S))$ of the extreme points of its convex closure.*

Proof. We show that $\mathrm{ex}(S) \subset \mathrm{ex}(\mathrm{hull}(S))$. Indeed, if $s \in \mathrm{ex}(S)$, then $\mathrm{hull}(S) \setminus \{s\}$ is convex (property (a) in Assertion 2.10). Therefore, since $\mathrm{hull}(\mathrm{hull}(S)) = \mathrm{hull}(S)$, the set $\mathrm{hull}(\mathrm{hull}(S)) \setminus \{s\}$ is convex. This means by definition that $s \in \mathrm{ex}(\mathrm{hull}(S))$.

We show that $\mathrm{ex}(\mathrm{hull}(S)) \subset \mathrm{ex}(S)$. Let $x \in \mathrm{ex}(\mathrm{hull}(S))$. Then $x \in S$ because the set $\mathrm{hull}(S) \setminus \{y\}$ contains S for any $y \in \mathrm{hull}(S) \setminus S$. Hence, $\mathrm{hull}(\mathrm{hull}(S) \setminus \{y\})$ coincides with $\mathrm{hull}(S)$ and contains y, which means by property (a) of Assertion 2.10 that $y \notin \mathrm{ex}(\mathrm{hull}(S))$. Thus, $x \in S$, while the set

$$\mathrm{hull}(S) \setminus \{x\} = \mathrm{hull}(\mathrm{hull}(S)) \setminus \{x\}$$

is convex (because $x \in \mathrm{ex}(\mathrm{hull}(S))$). This means that $x \in \mathrm{ex}(S)$. □

2.14. *Remark.* The proof of Lemma 2.13 shows that this lemma holds true for any convexity if we understand the extremity of a point $s \in S$ in a subset S in accordance with condition (a) in Assertion 2.10. If we understand the extremity in the sense of condition (b), then we have the inclusion $\mathrm{ex}(\mathrm{hull}(S)) \subset \mathrm{ex}(S)$ also in the general case, but there exist convexities in which $\mathrm{ex}(S) \not\subset \mathrm{ex}(\mathrm{hull}(S))$.

§3. Linear subsets

This section is devoted to linear subsets in a pretree. We remark that the constructions and statements of this section are valid for classes of interval spaces that are significantly broader than the pretrees. For example, they are valid for the class $\mathscr{E}_4(\mathrm{CAT}(0))$ formed by the interval spaces all of whose subsets of cardinality not greater than 4 embed in geodesic metric spaces of nonpositive curvature[10]. The class $\mathscr{E}_4(\mathrm{CAT}(0))$ includes the pretrees, linear vector spaces, Hadamard spaces, subspaces thereof, etc. Up to isomorphism, there are only four four-point pretrees (they uniquely determine the class of pretrees) and five four-point spaces of the class $\mathscr{E}_4(\mathrm{CAT}(0))$ (to the four pretrees, add the space $\{x, y, z, w\}$ in which $[x, z] = \{x, y, z\}$ while each of the other intervals consists of its own ends only). The class $\mathscr{E}_4(\mathrm{CAT}(0))$ can be characterized axiomatically as the class of spaces that have properties (A0)–(A2), (A4), (A7), (A8) (which imply properties (A5) and (A6) also but not (A3)). In some of the proofs of this section, we deliberately avoid using Axiom (A3) in order to demonstrate their validity also for the spaces of the class $\mathscr{E}_4(\mathrm{CAT}(0))$.

3.1. Definitions. A subset L in a pretree is said to be *linear* if for any points $x, y, z \in L$, at least one of the relations $x \in [y, z]$, $y \in [z, x]$, and $z \in [x, y]$ holds true. A nonempty convex linear set is called an *arc*. By a *direction* on a linear subset L in a pretree $\mathcal{T}$, we mean a linear order $<$ on L such that, for each triple $x, y, z \in L$, we have

$$y \in \langle x, z\rangle \Leftrightarrow (x < y < z) \vee (z < y < x).$$

In this case, the ordered set $(L, <)$ is called a *directed* linear set.

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[10] It is assumed that a metric space (M, d) is endowed with the structure of interval space: for $x, y, z \in M$, we have $y \in [x, z]$ if and only if $d(x, z) = d(x, y) + d(y, z)$.

Following [5], we say that a point b in a pretree $\mathcal{T}$ is a *supremum* of a directed arc $(A,<)$ if either b is the largest element in $(A,<)$ or if $(A,<)$ has no largest element while the interval[11] $[x,b\rangle$ is an upper section[12] in $(A,<)$ for some (hence any) point $x\in A$. We remark that a directed arc can have several suprema.

3.2. Lemma ([1, Lemma 15.5]**).** *In a pretree, the convex hull of a linear set is linear* (*and is therefore an arc if nonempty*)*.*

3.3. Lemma. (1) *In a pretree, each linear set admits a direction. A linear set containing at least two points admits precisely two directions* (*these directions are mutually inverse*)*.*

(2) *A subset in a directed arc is convex in a pretree if and only if it is order-convex with respect to the direction.*

(3) *In a pretree, all the nonempty intervals are arcs.*

(4) *Let x and y be points in a pretree. Then the interval $[x,y]$ admits a unique direction in which x is the smallest element.*

(5) *Let $([a,b],<)$ be a directed interval in a pretree $\mathcal{T}$. Assume that a is the smallest element of $([a,b],<)$. Let S be a convex subset in $\mathcal{T}$ containing a. Then the intersection $[a,b]\cap S$ is a lower section in the ordered set $([a,b],<)$.*

Proof. (1) We consider a linear subset L in a pretree as an independent pretree (with the induced structure). The pretree L satisfies Axioms (A0)–(A3) and, by the definition of a linear subset, the additional "axiom"

$$x\in[y,z]\quad\text{or}\quad y\in[z,x]\quad\text{or}\quad z\in[x,y]\quad\text{for any}\quad x,y,z\in L.$$

This system of axioms is equivalent to the system of axioms by Pasch and Hilbert for a set with a linear betweenness relation (see [1, p. 6]), which turns assertion (1) into a well-known classical fact. See also [5, Lemma 2.7].

(2) This follows directly from the definition of the directions.

(3) It is required to prove that for any pair x,y of points in a pretree, the intervals $[x,y]$, $[x,y\rangle$, and $\langle x,y\rangle$ are linear and convex. In [1, Lemma 15.3] it was proved that the closed intervals (for example, the interval $[x,y]$) are linear and convex (see also Lemma 1.18). The linearity of $[x,y]$ implies that $[x,y\rangle$ and $\langle x,y\rangle$ are linear, because a subset of a linear set is linear by definition. Thus, it only remains to show that $[x,y\rangle$ and $\langle x,y\rangle$ are convex. Since $[x,y]$ is linear, assertion (1) implies that it admits a direction $<$. By the definition of a direction, for any point $t\in[x,y]$ we have either $x\le t\le y$ or $y\le t\le x$. Obviously, this means that x and y are the largest and smallest elements of the linearly ordered set $([x,y],<)$. From the definition of directions it follows directly that a subset of a directed arc in a pretree is convex if and only if it is order-convex with respect to the direction (assertion (2)). Then, obviously, $[x,y\rangle$ and $\langle x,y\rangle$ are convex.

(4) Since the case of a one-point interval is obvious, we assume that $[x,y]$ contains at least two points (i.e., $x\ne y$). Then $[x,y]$ is linear by assertion (3) and admits exactly two directions by assertion (1). By repeating the argument from the proof of assertion (3), we see that x and y are the largest and smallest elements of these two directions. Since these directions on $[x,y]$ are mutually inverse, it follows that x is the largest element with respect to one of these directions and the smallest element with respect to another direction. In particular, $[x,y]$ admits a unique direction in which x is the smallest element.

(5) We observe that the intersection $Q:=[a,b]\cap S$ is convex (being the intersection of convex sets) and contains a. Thus, Q is an order-convex subset in $([a,b],<)$ (by

[11] See the notation in Subsection 1.12.

[12] A subset E of a linearly ordered set $(O,<)$ is called a *lower section* if O has no elements x,y such that $x<y$, $y\in E$, and $x\notin E$. The *upper sections* of an ordered set are the lower sections of the same set with the reverse order.

assertion (2)) and contains the smallest element a. Consequently, Q is a lower section in $([a,b],<)$. □

3.4. Definition. Suppose that a linear set L consists of at least two points and is contained in a linear set L', and let $<$ be a direction on L. By Lemma 3.3(1), each of the sets L and L' admits precisely two directions. The definition of directions shows that the directions of L are the restrictions of the directions of L'. Consequently, there exists a unique direction $<'$ on L' such that $<$ is its restriction. We say that $<'$ is *induced* by $<$.

3.5. Lemma. *Let $(I,<_I)$ and $(J,<_J)$ be directed linear sets in a pretree $\mathcal{T}$. Assume that the intersection $I\cap J$ contains at least two points, being at the same time an upper section in $(I,<_I)$ and a lower one in $(J,<_J)$, and that the restrictions of the orders $<_I$ and $<_J$ to $I\cap J$ coincide. Then the union $I\cup J$ is linear, and I is its lower section with respect to the direction on $I\cup J$ induced by $<_I$.*

Proof. In order to check that $I\cup J$ is linear, it suffices to show that each three-point subset $K\subset I\cup J$ is linear. Due to the finiteness of K, there exist $p<_I q\in I$ and $r<_J s\in J$ such that

$$K\subset [p,q]\cup[r,s]. \tag{9}$$

Since by "increasing" q in $(I,<_I)$ and "decreasing" r in $(J,<_J)$ we preserve property (9), it may be assumed that q and r are in $I\cap J$. Moreover, since $|I\cap J|\geq 2$, we may assume that $r<_I q$. Furthermore, replacing if necessary p by a smaller element in $(I,<_I)$, and s by a larger element in $(J,<_J)$, we arrive at the situation where

$$p\leq_I r<_I q \quad\text{and}\quad r<_J q\leq_J s.$$

Then, by (A8), we have $r,q\in[p,s]$, whence by (9) we see that K is contained in the interval $[p,s]$ and therefore is linear (because intervals are linear, see Lemma 3.3(3)). Thus, $I\cup J$ is linear.

Furthermore, the directions $<_I$ and $<_J$, being equal on $I\cap J$, induce one and the same direction $<$ on $I\cup J$ (because $|I\cap J|\geq 2$). Hence, using the transitivity of the order relations, we find that $a<b$ whenever $a\in I$ and $b\in J\setminus I=J\setminus(I\cap J)$ (because $I\cap J$ is a lower section in $(J,<_J)$ by assumption). Therefore, I is a lower section in $(I\cup J,<)$. □

3.6. Definition. We say that a linear subset in a pretree is *bounded* if it is contained in a (closed) interval. We say that a pretree is *bounded* if all its linear subsets are bounded.

3.7. Definitions. We say that an arc in a pretree is a *ray* if it is unbounded and has an extreme point. (We say that a ray *emanates* from its extreme point.)

An arc in a pretree will be called a *half-line*[13] if it is unbounded and contained in a ray. (Thus, all rays are half-lines.)

Finally, we say that an arc in a pretree is a *line* if it is unbounded and is not a half-line.

3.8. Lemma. *In an arbitrary pretree, the following properties hold true.*

(0) *Every unbounded arc is either a line or a half-line.*

(1) *A ray has a unique extreme point and a unique direction in which this extreme point is minimal.*

(2) *Every linear set that can be presented as the union of two bounded sets is bounded.*

(3) *Every arc that can be presented as the union of a half-line with a bounded set is itself a half-line.*

(4) *No line is a proper part of an arc in the same pretree.*

(5) *All proper upper and lower sections of a directed line are half-lines.*

[13]We borrow this term from [1], but unlike [1] we do not require a half-line to be contained in a line.

(6) *Every half-line admits a unique direction in which all of its nonempty upper sections are half-lines and all of its proper lower sections are bounded.*

(7) *The set of all the half-lines that are contained in a given half-line is linearly ordered by inclusion.*

(8) *Let R be a ray, let x be its extreme point, and let $y \in R$. Then the set $R_y := R\setminus[x, y\rangle$ is a ray emanating from y.*

(9) *An arc is a line if and only if it contains two disjoint half-lines.*

(10) *Let H_1, H_2, and H_3 be half-lines; if the intersections $H_1 \cap H_2$ and $H_1 \cap H_3$ are half-lines, then $H_2 \cap H_3$ is also a half-line.*

Proof. (0) This follows immediately from the definitions.

(1) We use the fact that each linear set admits a direction (Lemma 3.3(1)). The definition of the directions shows that the extreme points of a linear set should also be order-extreme (maximal or minimal) with respect to the directions of the set (so that a linear set has at most two extreme points) and that a linear set with two extreme points is contained in the closed interval whose ends are these two points. Therefore, since a ray is unbounded, its extreme point is unique. It remains to note that the definition of directions implies also by Lemma 3.3 that an arbitrary linear set with an extreme point admits precisely one direction where this point is minimal.

(2) This assertion was proved in [1, Lemma 16.4]. It can also be proved along the lines of the proof of assertion (3) below.

(3) Let an arc A be the union of a half-line H with a bounded set B. Let $<$ be an arbitrary direction on A (the existence of $<$ follows by Lemma 3.3(1)). Since H is convex and unbounded, it follows that H is either a lower or an upper section of $(A, <)$. Since any direction has an inverse one, we may assume without loss of generality that H is an upper section. Let $B' := A \setminus H$ be the complementary lower section. If $B' = \varnothing$, then $A = H$ is a half-line. It remains to consider the case where $B' \neq \varnothing$ while $(A, <)$ has no minimal element. In this case, the set B' is infinite. Observe that B' is bounded because $B' \subset B$; therefore, there exists a closed interval $[x, y]$ containing B'. Let $\prec$ be the direction on $[x, y]$ induced by the direction $<$. We may assume without loss of generality that x is the smallest element in $([x, y], \prec)$. Let I be the smallest lower section in $([x, y], \prec)$ containing B'. Then, applying Lemma 3.5 to the directed arcs $(I, \prec_I)$ and $(A, <)$ (where $\prec_I$ is the restriction of $\prec$ to I), we see that the union $I \cup A$ is linear, and x, which is extreme in I, is extreme also in $I \cup A$. Therefore, $I \cup A$ is a ray and A is a half-line.

(4) Assume that some pretree contains a line L that is a proper part of an arc A. Let $<$ be an arbitrary direction on A (see Lemma 3.3(1)). Since the lines are convex and unbounded by definition, it follows that L is either a lower or an upper section in $(A, <)$. We may assume without loss of generality that L is an upper section. Since we assume that $L \neq A$, there exists a point $x \in A \setminus L$. Then L is contained in the upper section $A_x := \{t \in A : x \leq t\}$. The definition of directions implies that A_x is an arc and x is its extreme point. At the same time, A_x is unbounded because it contains L. Any unbounded arc having an extreme point is a ray. Therefore, L is contained in a ray, A_x, and hence is either bounded or a half-line. This contradiction completes the proof.

(5) Let Y be a proper upper or lower section in a directed line $(L, <)$. We show that Y is a half-line. We set $X := L \setminus Y$. The definition of directions implies that X and Y are both convex, whence it follows that X and Y are arcs. By assertion (4), neither X nor Y is a line. In particular, X is either bounded or a half-line. It follows that Y is unbounded because otherwise assertions (2) and (3) would show that the line $L = X \cup Y$ is either bounded or a half-line. Thus, the arc Y is unbounded and is not a line. This means that Y is a half-line.

(6), (7) Let H be a half-line, let R be a ray containing H, let p be the extreme point of R, and let $<$ be that direction on R in which p is the minimal element (see assertion (1)). We denote by $<_H$ the restriction of $<$ to H and show that $<_H$ satisfies the requirements of assertion (6). Every proper lower section X of $(H, <_H)$ is bounded because $X \subset [p, y]$ for all $y \in H \setminus X$. Suppose Y is a nonempty upper section of $(H, <_H)$ and put $X := H \setminus Y$. (We note that Y is an arc by the definition of directions.) Then X is bounded because $X \subset [p, y]$ for all $y \in Y$. Consequently, the arc Y is unbounded because otherwise the half-line H would be bounded, by assertion (2), being the union of the two bounded linear sets X and Y. Thus, Y is an unbounded arc contained in a half-line, which implies that Y is a half-line itself. Therefore, the direction $<_H$ satisfies the requirements of assertion (6). By Lemma 3.3(1), the inverse $<'_H$ of $<_H$ is the only direction on H distinct from $<_H$. Consequently, $<_H$ is the only direction on H that satisfies the requirements of assertion (6) (because all of the proper upper sections of $(H, <'_H)$ are proper lower sections of $(H, <_H)$, which are bounded).

Now, we observe that each half-line contained in H is a nonempty upper section of $(H, <_H)$. (Indeed, by the definition of directions, every unbounded arc contained in a directed arc $(I, <)$ should be a lower or an upper section of $(I, <)$, while all of the proper lower sections of $(H, <_H)$ are bounded.) This implies (7), because the set of all of the upper sections of a linearly ordered set is linearly ordered by inclusion.

(8) Let $<$ be the direction on R in which the point x is the minimal element (see assertion (1)). Then the definition of directions shows that $[x, y\rangle = \{t \in R : t < y\}$ and $R_y = \{t \in R : y \leq t\}$. The same definition implies also that R_y is an arc and y is the only extreme point of R_y (because x is the only extreme point of R), while assertion (2) implies that R_y is unbounded. This means that R_y is a ray emanating from y.

(9) If an arc A contains two disjoint half-lines, then it is unbounded and, by assertion (7), is not a half-line. Consequently, A is a line by assertion (0). The reverse implication readily follows from (5).

(10) We note that the half-lines $H_1 \cap H_2$ and $H_1 \cap H_3$ are contained in the half-line H_1. It then follows by assertion (7) that $H_1 \cap H_2 \cap H_3$ coincides with one of $H_1 \cap H_2$ and $H_1 \cap H_3$. In particular, $H_1 \cap H_2 \cap H_3$ is unbounded. We recall that the intersection of any number of linear convex sets is linear and convex, which implies that the intersection of arcs is an arc if nonempty. It follows that $H_2 \cap H_3$ is an arc. The arc $H_2 \cap H_3$ is unbounded because it contains the half-line $H_1 \cap H_2 \cap H_3$. Therefore, $H_2 \cap H_3$ is a half-line because it is contained in the half-line H_2. □

§4. Classes of pretrees

4.1. Definition. A point m in a pretree $\mathcal{T}$ is called a *median* of a triple a, b, c of points in $\mathcal{T}$ if $m \in [a, b] \cap [a, c] \cap [b, c]$. A triple of points has at most one median (see, e.g., [1, Lemma 15.2]). A pretree is said to be *median* if each triple of its points has a median.

4.2. *Remark.* The pretrees in Examples 1.2(1)–(3) are median.

4.3. Definition. If a triple a, b, c of points in a pretree $\mathcal{T}$ has no median, we say that the polytope[14] $\operatorname{hull}\{a, b, c\} = [a, b] \cup [b, c] \cup [c, a]$ is a *singularity* in $\mathcal{T}$. Let $<_a$, $<_b$, and $<_c$ be the directions on the arcs $I_a := [a, b] \cap [a, c]$, $I_b := [b, a] \cap [b, c]$, and $I_c := [c, a] \cap [c, b]$ respectively, where the points a, b, c are the minimal elements. We say that $\operatorname{hull}\{a, b, c\}$ is a singularity *of the type* Y_0, Y_1, Y_2, or Y_3 in accordance with the number (0, 1, 2, or 3) of those directed arcs in the triple $(I_a, <_a)$, $(I_b, <_b)$, $(I_c, <_c)$ that have largest elements.

[14]We adopt this term from the theory of convex structures, where the convex hull of a finite set of points is called a *polytope*. See also Lemma 2.8.

4.4. Example. We take three copies $[0,1]_i$, $i = 1,2,3$, of the unit interval. By gluing together the endpoints 0_1, 0_2, and 0_3, we obtain the *tripod* Y, which has a natural pretree structure. By removing the point 0 from Y, we obtain a singularity of type Y_0. By removing from Y a set of $k(\in \{1,2,3\})$ intervals of the form $[0_i, 1_i\rangle$, we obtain a singularity of type Y_k.

4.5. Definition. A subset S in a pretree is called a *star* if $[x,y] = \{x,y\}$ for all $x,y \in S$. (Every star is convex.)

4.6. *Remark.* A three-point star is a singularity of type Y_3. Each singularity of type Y_3 contains a three-point star.

4.7. *Remark.* A pretree $\mathcal{T}$ has a singularity of type Y_3 if and only if $\mathcal{T}$ contains a directed arc having more than one supremum.

4.8. Definition. A point t in a pretree $\mathcal{T}$ is said to be *regular* if $[x,t] \cap [t,y] \neq \{t\}$ whenever $t \notin [x,y]$. A pretree is *regular* if all its points are regular.

4.9. *Remark.* The regular pretrees were called *saturated* ones in [5]. In terms of [1], a regular pretree is a B-set with a *true* betweenness relation.

4.10. Assertion. *Every median pretree is regular.*

Proof. If a pretree $\mathcal{T}$ is not regular, then there are points x,y,z in $\mathcal{T}$ such that $z \notin [x,y]$ and $[x,z] \cap [z,y] = \{z\}$, which implies that the intersection $[x,y] \cap [x,z] \cap [z,y]$ is empty, i.e., the triple x,y,z has no median. □

4.11. *Remark.* It is easily seen that a nonmedian pretree is regular if and only if all of its singularities are of type Y_0.

4.12. Definition. We say that a pretree is *Dedekind complete* if all of its directed closed intervals (hence all of its arcs) are *order complete*, i.e., complete as linearly ordered sets. (A linearly ordered set is *complete* if each of its subsets bounded from above has a least upper bound.)

We recall that a linear subset in a pretree is *bounded* if it is contained in a (closed) interval (Definition 3.6). We say that a pretree is *weakly complete* if all its bounded arcs are intervals.

4.13. Lemma. *A Dedekind complete pretree is weakly complete.*

Proof. Suppose A is a bounded arc in a Dedekind complete pretree $\mathcal{T}$. Then, since A is bounded, there exists a closed interval (say I) in $\mathcal{T}$ that contains A. Since $\mathcal{T}$ is Dedekind complete, it follows that I is order complete (with any of its directions). From the definition of directions, it follows immediately that a subset of a directed arc in a pretree is convex if and only if it is order-convex with respect to the direction. Therefore, the arc A is an order-convex subset in an order complete closed interval. Obviously, this implies that A is a bounded order interval in I, and hence A is an interval in the pretree. Thus, every bounded arc in a Dedekind complete pretree is an interval. □

4.14. *Remark.* It can be shown that a pretree is Dedekind complete if and only if it is weakly complete and has no singularities of type Y_1. In particular, for the median and, more generally, regular pretrees, the weak completeness and Dedekind completeness are equivalent.

We note that the Dedekind complete pretrees have neither singularities of type Y_0 nor singularities of type Y_1, and that a weakly complete pretree with singularities of type Y_0 has also singularities of type Y_1. It then follows by Remark 4.11 that every regular weakly complete pretree is Dedekind complete and median.

4.15. *Remark.* In [5, 6], the term "complete pretree" meant a pretree all of whose arcs are intervals; in our terminology, this is a bounded weakly complete pretree (see Theorem 8.2). In the papers [40, 32], the Dedekind complete pretrees were called complete ones.

4.16. *Remark.* The ordinary trees (see footnote 1), $\mathbb{R}$-trees, dendritic spaces are Dedekind complete (hence, weakly complete). Not all of the Λ-trees are weakly complete.

4.17. Definition. A metric space (T, ρ) is called a *metric pretree* if the map $T^2 \to 2^T$, $(a, b) \mapsto [a, b]_\rho$, where

$$[a, b]_\rho := \{t \in T \,:\, \rho(a, t) + \rho(t, b) = \rho(a, b)\},$$

satisfies the axioms of pretrees. We say that a pretree is *metrizable* if it is isomorphic to a metric pretree.

4.18. *Remark.* A median pretree is metrizable if and only if it embeds in an $\mathbb{R}$-tree. There are metric pretrees admitting no isometric embeddings in $\mathbb{R}$-trees. Example: the metric pretree composed of four points a, b, c, d with $\rho(a, b) = 3$ and all of the other nonzero distances equal to 2.

4.19. *Remark.* The question about the metrizability of pretrees is related to fundamental questions of the set theory. There exist nonmetrizable pretrees each of whose arcs is metrizable (i.e., embeds in $\mathbb{R}$). Known examples that give such pretrees are the *order trees* $\sigma\mathbb{R}$ and $w\mathbb{R}$ introduced by D. Kurepa [20] ($w\mathbb{R}$ is the partially ordered set whose elements are all the subsets of $\mathbb{R}$ that are well-ordered sets with respect to the usual order on $\mathbb{R}$, equipped with the order $\prec$ in which $A \prec B$ if A is a lower section for B). The nonmetrizability of the pretrees $\sigma\mathbb{R}$ and $w\mathbb{R}$ follows from the main result of [22].

The *Souslin trees*[15] and *Souslin dendrons*[16] are also nonmetrizable (in the above sense) pretrees each of whose arcs is metrizable. The nonmetrizability of the pretrees corresponding to the Souslin trees follows, for example, from [13, Theorem 5]. The nonmetrizability of the pretrees corresponding to the Souslin dendrons can easily be derived from the definitions.

§5. Partitions, branches, and shadows

In this section, we introduce the notions of branches and shadows for a pretree and prove a series of their properties. These properties are used in the subsequent sections in the study of the shadow topology.

5.1. Definitions. We recall that a *partition* of a set is a family of its nonempty pairwise disjoint subsets (*elements* of the partition) that cover the set. If ζ is a partition of a set X and $x \in X$, we denote by $\zeta(x)$ the element in ζ containing x.

Let r be a point in a pretree $\mathcal{T}$. We define a binary relation R_r on the set $\mathcal{T} \setminus \{r\}$ by setting

$$xR_ry \Leftrightarrow r \notin [x, y]. \tag{10}$$

Observe that R_r is an equivalence relation (it is reflexive by (A$\frac{1}{2}$), symmetric by (A1), and transitive by (A3)).

We denote by ζ_r the partition of $\mathcal{T}$ whose elements are the set $\{r\}$ and all the equivalence classes of R_r, and say that ζ_r is the *nodal partition of the pretree $\mathcal{T}$ at the point r.*

[15] See the definition, e.g., in [13].

[16] See [25]: the existence of a Souslin dendron is equivalent to the existence of a Souslin tree, and also to the existence of a Souslin line, and is independent of ZFC.

In $\mathcal{T}$, the subsets of the form $\zeta_r(t)$, $r \neq t$, are called *branches*[17] of $\mathcal{T}$. Thus, if $r \neq t$, we have by definition

$$\zeta_r(t) = \{x \in \mathcal{T} : r \notin [x,t]\}. \tag{11}$$

In this case, the point r will be called a *pin* of the branch $\zeta_r(t)$.

The complements of branches are called *shadows*. The shadow $\mathcal{T} \setminus \zeta_r(t)$ is cast by r if t is the light source:

$$\mathcal{T} \setminus \zeta_r(t) = \{x \in \mathcal{T} : r \in [x,t]\}. \tag{12}$$

5.2. *Remark.* If we consider the standard pretree structure on a connected topological space X (see Proposition 1.3), then the branches of the nodal partition ζ_x, where $x \in X$, are quasicomponents of the space $X \setminus \{x\}$. If X is a dendritic space (e.g., an $\mathbb{R}$-tree or dendron), then the branches of ζ_x are the connected components of $X \setminus \{x\}$.

5.3. *Remark.* In a $\mathbb{Z}$-tree, the sets of branches and of shadows coincide (see Lemma 5.14).

5.4. *Remark.* For a pretree $\mathcal{T}$ and its point $a \in \mathcal{T}$, we define a binary relation R'_a on $\mathcal{T} \setminus \{a\}$ as follows:

$$xR'_a y \Leftrightarrow [a,x] \cap [a,y] \neq \{a\}.$$

In [1, Theorem 15.6] it was proved that R'_a is an equivalence relation on $\mathcal{T} \setminus \{a\}$ and that $R'_a \subset R_a$. Moreover, $R'_a = R_a$ if and only if a is a regular point (see Definition 4.8).

5.5. *Remark.* There is a uniform formula describing the elements of $\zeta_r(t)$ in the two cases where $r \neq t$ and $r = t$. Indeed, it is easy to verify that

$$\zeta_r(t) = \{x \in \mathcal{T} : r \notin [x,t\rangle\} \tag{13}$$

for any $r, t \in \mathcal{T}$.

5.6. *Remark.* In a pretree $(\mathcal{T}, \mathcal{S})$, where $\mathcal{S}$ denotes the pretree ternary structure (see Definition 1.1), for the nodal partitions we have, by definition,

$$\mathcal{S}prq \Leftrightarrow (p \neq r \neq q) \wedge (\zeta_r(p) \neq \zeta_r(q)). \tag{14}$$

5.7. *Remark.* The collection of all nodal partitions in a pretree is tree-like in the sense of Definition 1.5. Furthermore, the pretree structure of this collection, which is described in Proposition 1.6, is isomorphic to the initial pretree.

5.8. Lemma. *In every pretree $\mathcal{T}$, the following properties hold true*:

(Z0) $\zeta_x(x) = \{x\}$ *for all* $x \in \mathcal{T}$;
(Z1) $\zeta_x(y) \cup \zeta_y(x) = \mathcal{T}$ *for all* $x \neq y \in \mathcal{T}$.

Proof. Property (Z0) holds by construction. We prove property (Z1). Let $x \neq y \in \mathcal{T}$. Then

$$\zeta_x(y) = \{z \in \mathcal{T} : x \notin [z,y]\} \quad \text{and} \quad \zeta_y(x) = \{z \in \mathcal{T} : y \notin [z,x]\}. \tag{15}$$

At the same time, the condition $x \neq y$ implies by Axiom (A2) that for each $z \in \mathcal{T}$, one of the relations $x \notin [z,y]$, $y \notin [z,x]$ holds true. This means that $\zeta_x(y) \cup \zeta_y(x) = \mathcal{T}$, if we take (15) into account. The lemma is proved. □

It turns out that relations (Z0) and (Z1) can also be used as axioms for an alternative definition of pretrees via partitions: we can consider a pretree as a set $\mathcal{T}$ with a system of partitions $\{\zeta_x\}_{x \in \mathcal{T}}$ (to a point $x \in \mathcal{T}$, the partition ζ_x is assigned) having properties (Z0) and (Z1).

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[17] We use the term from [37, p. 7]. An alternative term from the convexity theory is *copoints*: maximal convex sets in the complement of one or another point (the convexity of branches is proved in Lemma 5.10).

5.9. Theorem. *The map assigning to a pretree the system of its nodal partitions establishes a one-to-one correspondence between the pretrees and the systems of the form* $\{\zeta_x\}_{x\in X}$ *consisting of the partitions of a set* X *that have properties* (Z0) *and* (Z1).

Proof. In order to prove this theorem, it suffices to show that:

1) the system of nodal partitions of an arbitrary pretree has properties (Z0) and (Z1);

2) the structure of an arbitrary pretree is uniquely determined by its system of nodal partitions[18];

3) if a system $\{\xi_x\}_{x\in X}$ of partitions ξ_x of a set X satisfies (Z0) and (Z1), then $\{\xi_x\}_{x\in X}$ is the system of nodal partitions of some pretree structure on X.

Assertion 1 was proved in Lemma 5.8.

Assertion 2 follows from (14).

We prove assertion 3. Assume that a system $\{\xi_x\}_{x\in X}$ of partitions ξ_x of a set X satisfies (Z0) and (Z1). We define a ternary relation $\mathcal{S}$ on X by the rule (cf. (14))

$$\mathcal{S}prq \Leftrightarrow (p \neq r \neq q) \wedge (\xi_r(p) \neq \xi_r(q)). \tag{16}$$

We show that $\mathcal{S}$ is the structure of a pretree, i.e., satisfies Axioms (T0)–(T3).

The fact that Axioms (T0) and (T1) are satisfied is obvious from (16).

In order to show that Axiom (T2) is satisfied, we assume that there exist points $x, y, z \in X$ such that $y \neq z$, $\mathcal{S}xyz$, and $\mathcal{S}xzy$. Then $x \neq y \neq z \neq x$, $\xi_y(x) \neq \xi_y(z)$, and $\xi_z(x) \neq \xi_z(y)$ (by (16)). In particular, $x \notin \xi_y(z)$ and $x \notin \xi_z(y)$. However, the condition $z \neq y$ implies by (Z1) that $\xi_y(z) \cup \xi_z(y) = X$. Consequently, $x \notin X$. This contradiction shows that, indeed, (T2) is satisfied.

Axiom (T3): we show that if $\mathcal{S}xyz$ and $w \neq y$, then either $\mathcal{S}xyw$ or $\mathcal{S}wyz$. The condition $\mathcal{S}xyz$ means that $x \neq y \neq z$ and $\xi_y(x) \neq \xi_y(z)$. The condition $\xi_y(x) \neq \xi_y(z)$ implies that either $\xi_y(w) \neq \xi_y(z)$ or $\xi_y(w) \neq \xi_y(x)$. The conditions $x \neq y \neq z$ and $w \neq y$ imply that $x \neq y \neq w$ and $w \neq y \neq z$. Consequently, we have $\mathcal{S}xyw$ if $\xi_y(w) \neq \xi_y(x)$ and $\mathcal{S}wyz$ if $\xi_y(w) \neq \xi_y(z)$.

Thus, $(X, \mathcal{S})$ is a pretree. In order to complete the proof, it remains to show that, for every point $r \in X$, the corresponding nodal partition ζ_r of the pretree $(X, \mathcal{S})$ coincides with ξ_r. We compare the partitions ζ_r and ξ_r. The element $\zeta_r(r)$ coincides with $\{r\}$ by the definition of a nodal partition, while $\xi_r(r) = \{r\}$ due to the assumption that property (Z0) is satisfied in the system $\{\xi_x\}_{x\in X}$. Next, we observe that from (14) and (16) (respectively) it follows that for any $p, q \in X \setminus \{r\}$ (i.e., whenever $p \neq r \neq q$) we have

$$\mathcal{S}prq \Leftrightarrow \zeta_r(p) \neq \zeta_r(q), \tag{17}$$

$$\mathcal{S}prq \Leftrightarrow \xi_r(p) \neq \xi_r(q). \tag{18}$$

Since $\zeta_r(r) = \{r\} = \xi_r(r)$, (17) and (18) imply that $\zeta_r = \xi_r$. Assertion 3, and with it the theorem, is proved. ☐

The following lemmas contain some properties of branches.

5.10. Lemma. *In every pretree* $\mathcal{T}$, *the following properties hold true.*

(1) *Each branch has a unique pin.*
(2) *Each branch is convex.*
(3) *If* $(B_i)_{i\in I}$ *is a collection of branches with a common pin* r, *then the union* $U_* := U \cup \{r\}$, *where* $U := \bigcup_{i\in I} B_i$, *is convex.*
(4) *Each shadow is convex.*
(5) *If* S *is a convex subset in* $\mathcal{T}$ *and* $r \in \mathcal{T} \setminus S$, *then* S *is contained in some branch of the partition* ζ_r.

[18] In fact, the structure of a pretree is uniquely determined by the totality of its nodal partitions (when we forget about indexation). See Corollary 5.11 and its proof.

Proof. (1) Suppose that B is a branch in $\mathcal{T}$. By definition, B has a pin (say b). Assume that a point a is also a pin of B. Then $a \notin B$ and $b \notin B$ by the definition of pins and branches (Definition 5.1). Take an arbitrary point $t \in B$. Since $t \in B$, $b \notin B$, and a is a pin of B, Definition 5.1 shows that $a \in [b,t]$. In a similar way, we check that $b \in [a,t]$, whence $a = b$ by (A2). The assertion is proved.

(2) Suppose that B is a branch in $\mathcal{T}$, and let r be its pin. If $x, y \in B$, then $r \notin [x,y]$ by the definition of branches, while property (A4) implies that for each $t \in [x,y]$ we have $[x,t] \subset [x,y]$ and hence $r \notin [x,t]$, whence, again by the definition of branches, it follows that $t \in B$. Therefore, if $x, y \in B$, then $[x,y] \subset B$, i.e., B is convex.

(3) Assume that U_* is not convex. Then there exist $p, q \in U_*$ such that $[p,q] \not\subset U_*$. We take x in $[p,q] \setminus U_*$. Observe that $r \in [p,x]$ (if $p = r$, this is true by (A0); if $p \in U$, this is implied by Definition 5.1). Similarly, it can be shown that $r \in [x,q]$. By property (A5), the relations $x \in [p,q]$ and $r \in [p,x]$ imply that $x \in [r,q]$. By Axiom (A2), the relations $r \in [x,q]$ and $x \in [r,q]$ imply that $x = r$. However, $r \in U_*$ and $x \notin U_*$. This contradiction proves the claim.

(4) This follows from (3) by the definitions.

(5) Assume to the contrary that S is not contained in a branch of the partition ζ_r. Then, since $r \notin S$, it follows that S intersects at least two distinct branches (say B_1 and B_2) of ζ_r. For any points $a_1 \in S \cap B_1$ and $a_2 \in S \cap B_2$, we have $r \in [a_1, a_2]$ (by Definition 5.1). Then $r \in S$ because S is convex, a contradiction. □

5.11. Corollary. *A proper subset in a pretree is convex if and only if it is the intersection of branches in this pretree. The structure of a pretree is uniquely determined by the totality of its branches.*

Proof. The first assertion of this corollary is an obvious consequence of Lemma 5.10(5). The second assertion follows from the first because the structure of an arbitrary pretree is uniquely determined by the totality of its convex subsets (Theorem 2.2(3)). □

5.12. Lemma. *Suppose that A and B are branches in a pretree $\mathcal{T}$, and let a and b be their pins, respectively. Then*:

1) $(a \in B) \wedge (b \in A) \Leftrightarrow A \cup B = \mathcal{T}$;
2) $(a \in B) \wedge (b \notin A) \Leftrightarrow A \subsetneq B$;
3) $(a \notin B) \wedge (b \notin A) \wedge (A \neq B) \Leftrightarrow (A \cap B = \varnothing) \wedge (A \cup B \neq \mathcal{T})$.

Proof. 1. If $a \in B$ and $b \in A$, then $B = \zeta_b(a)$, $A = \zeta_a(b)$, and $a \neq b$ (because $a \in B$, while b is not contained in B, being its pin), whence by (Z1) it follows that $A \cup B = \zeta_a(b) \cup \zeta_b(a) = \mathcal{T}$. Conversely, if $A \cup B = \mathcal{T}$, then $a \in B$ (because $a \notin A$) and $b \in A$ (because $b \notin B$).

2. If $a \in B$ and $b \notin A$, then $B = \zeta_b(a)$, $A \neq \zeta_a(b)$, and $a \neq b$ (because $a \in B$, while b is not contained in B, being its pin). By (Z1), $\zeta_b(a) \cup \zeta_a(b) = \mathcal{T}$. Therefore, the branch $B = \zeta_b(a)$ contains all elements of the partition ζ_a with the exception of $\zeta_a(b)$. In particular, B contains A. Since $B \setminus A$ contains a (and hence, is nonempty), it follows that $A \subsetneq B$.

If $A \subsetneq B$, then $b \notin A$ (because $b \notin B$). In order to prove that $a \in B$, we assume to the contrary that $a \notin B$ and consider the branch $\zeta_b(a)$. By arguments of the first part of this proof, the conditions $a \in \zeta_b(a)$ and $b \notin A$ imply that $A \subset \zeta_b(a)$. This contradicts the condition $A \subset B$ because $\zeta_b(a)$ and B are distinct elements of ζ_b. Therefore, a lies in B. The assertion is proved.

3. Suppose $a \notin B$, $b \notin A$, and $A \neq B$. Then $A \cup B \neq \mathcal{T}$ by items 1, 2 of this lemma. In order to prove that $A \cap B = \varnothing$, we consider the cases where $a = b$ and $a \neq b$. In the case where $a = b$, the branches A and B are elements of one and the same partition

$\zeta_a = \zeta_b$, showing that the relation $A \cap B = \varnothing$ follows from the relation $A \neq B$. In the case where $a \neq b$, we have $\zeta_b(a) \cup \zeta_a(b) = \mathcal{T}$ by (Z1). At the same time, A and $\zeta_a(b)$ are distinct elements of ζ_a (because $b \notin A$), which implies that $A \cap \zeta_a(b) = \varnothing$. Similarly, the condition $a \notin B$ implies (in the case where $a \neq b$) that $B \cap \zeta_b(a) = \varnothing$. Obviously, the relations $\zeta_b(a) \cup \zeta_a(b) = \mathcal{T}$, $A \cap \zeta_a(b) = \varnothing$, and $B \cap \zeta_b(a) = \varnothing$ imply that $A \cap B = \varnothing$.

In order to prove the reverse implication, we observe that the relations $a \notin B$ and $b \notin A$ follow from the condition $A \cup B \neq \mathcal{T}$ by items 1, 2 of this lemma, while the condition $A \cap B = \varnothing$ implies that $A \neq B$. □

5.13. Lemma. *Let $y \neq x \neq z$ be points in a pretree $\mathcal{T}$. Then*

$$y \in \langle x, z] \iff \zeta_y(x) \subset \zeta_z(x).$$

Proof. Suppose that $y \in \langle x, z]$. In this case, if $y = z$, then $\zeta_y(x) = \zeta_z(x)$ so that the inclusion $\zeta_y(x) \subset \zeta_z(x)$ is fulfilled. If $y \neq z$, then $z \notin [x, y]$ by (A2) because $y \in \langle x, z] \subset [x, z]$. The condition $z \notin [x, y]$ implies by the definition of branches that $y \in \zeta_z(x)$. The condition $y \in [x, z]$ implies by the definition of branches that $z \notin \zeta_y(x)$. The conditions $y \in \zeta_z(x)$ and $z \notin \zeta_y(x)$ imply by Lemma 5.12(2) that $\zeta_y(x) \subset \zeta_z(x)$.

Suppose that $\zeta_y(x) \subset \zeta_z(x)$. In the case where $\zeta_y(x) = \zeta_z(x)$, Lemma 5.10(1) yields $y = z$, whence $y \in \langle x, z]$. In the case where $\zeta_y(x) \subsetneq \zeta_z(x)$, Lemma 5.12(2) yields $z \notin \zeta_y(x)$, whence it follows by the definition of branches that $y \in [x, z]$, so that $y \in \langle x, z]$ because $x \neq y$. □

5.14. Lemma. *Let a and b be two distinct points in a pretree $\mathcal{T}$. Assume that $[a, b] = \{a, b\}$ and that at least one of a and b is regular*[19]*. Then the branches $\zeta_a(b)$ and $\zeta_b(a)$ are mutually complementary, i.e., $\mathcal{T} \setminus \zeta_a(b) = \zeta_b(a)$.*

Proof. We assume without loss of generality that b is regular. Let x be a point in $\zeta_a(b)$. Then $a \notin [x, b]$ by the definition of branches. Therefore, $[x, b] \cap [b, a] = \{b\}$. By the regularity of b, we have $b \in [a, x]$. This means that $x \notin \zeta_b(a)$. Consequently, $\zeta_a(b) \cap \zeta_b(a) = \varnothing$. From this, using the fact that $\zeta_a(b) \cup \zeta_b(a) = \mathcal{T}$ (property (Z1)), we obtain the required condition $\mathcal{T} \setminus \zeta_a(b) = \zeta_b(a)$. □

5.15. Notation. For a point x and a subset S in a pretree $\mathcal{T}$, we introduce the following notation:

$$U_S(x) := \{t \in \mathcal{T} \,:\, [t, x] \cap S = \varnothing\}. \tag{19}$$

5.16. Lemma. *Let x, y be points and S, Q subsets in a pretree $\mathcal{T}$. Then:*

(1) $U_S(x) = \mathcal{T} \Leftrightarrow S = \varnothing$;
(2) $U_S(x) = \varnothing \Leftrightarrow x \in S$;
(3) $U_S(x) \ni x \Leftrightarrow S \not\ni x$;
(4) $U_S(x) \cap U_Q(x) = U_{S \cup Q}(x)$;
(5) *if* $x \neq y$*, then* $U_{\{y\}}(x) = \zeta_y(x)$;
(6) *if* $x \notin S$*, then* $U_S(x) = \bigcap_{r \in S} \zeta_r(x)$;
(7) *if* $y \in U_S(x)$*, then* $U_S(x) = U_S(y)$.

Proof. (1) If $U_S(x) = \mathcal{T}$, then $[t, x] \cap S = \varnothing$ for all $t \in \mathcal{T}$. Since $t \in [t, x]$ by Axiom (A0), we have $t \notin S$, so that $S = \varnothing$.

The reverse implication follows directly from the definition.

(2) If $U_S(x) = \varnothing$, then $x \notin U_S(x)$, which means by definition that $[x, x] \cap S \neq \varnothing$. It remains to recall that $[x, x] = \{x\}$ (Property (A$\frac{1}{2}$)).

If $x \in S$, then for any $t \in \mathcal{T}$ the set $[t, x] \cap S$ contains x (because $x \in [t, x]$ by Axiom (A0)) and hence is not empty. This means by definition that $U_S(x) = \varnothing$.

[19] The definition of the regularity property was given in Subsection 4.8.

(3) If $x \in U_S(x)$, then $U_S(x) \neq \varnothing$, whence $x \notin S$ by (2).

If $x \notin S$, then $[x,x] \cap S = \varnothing$ because $[x,x] = \{x\}$ by Property (A$\frac{1}{2}$), whence we deduce by definition that $x \in U_S(x)$.

(4) This follows directly from the definition (19).

(5) This follows from the definition of branches.

(6) This follows from (4) and (5).

(7) We define a relation R_S on the set $\mathcal{T} \setminus S$ by setting xR_Sy if $[x,y] \cap S = \varnothing$ (cf. Definition 5.1). Observe that R_S is an equivalence relation (it is reflexive by (A$\frac{1}{2}$), symmetric by (A1), and transitive by (A3)). Directly from the definitions, it follows that for each $x \in \mathcal{T} \setminus S$ the set $U_S(x)$ is the class of R_S that contains x. This implies the required assertion immediately. □

§6. The shadow topology

In this section, we define the shadow topology and prove some of its properties.

6.1. Definition. We recall that the *branches* of a pretree $\mathcal{T}$ are the sets of the form

$$\zeta_a(b) = \{t \in \mathcal{T} \,:\, a \notin [t,b]\},$$

where $a \neq b$ (Definition 5.1). We define the *shadow topology* on a pretree $\mathcal{T}$ to be the smallest topology containing all the branches of $\mathcal{T}$. In other words, the branches form a subbase of the shadow topology.

The shadows (i.e., the complements of branches; Definition 5.1) form a subbase of closed sets for the shadow topology.

6.2. Examples. 1. On a pretree with the empty ternary relation, the shadow topology coincides with the finite complement topology.

2. On a linearly ordered set viewed as a pretree, the shadow topology coincides with the usual order topology.

3. On any dendritic space (see the definition in Example 1.2(3)), the shadow topology is contained in the original one (see [39, Theorem 20]) and coincides with it if and only if the space is peripherally finite[20] (see [39, Theorem 21] and also [29, Corollary 5.12]).

In particular, on a dendron, the shadow topology coincides with the original one (see, e.g., [26, Corollary 2.2]).

The passage to the shadow topology turns any dendritic space into a regular locally connected and arcwise connected peripherally finite dendritic space (see [39, Theorems 20, 21]).

4. On a tree (whether it be a $\mathbb{Z}$-tree, CW-tree, or simplicial $\mathbb{R}$-tree; see the remark in footnote 1), the shadow topology is contained in the standard one and coincides with it if and only if the tree is locally finite. (See Example 3.)

5. The shadow topology on an $\mathbb{R}$-tree T is contained in the topology of the metric (see Proposition 6.6) and coincides with it if and only if the (metric) completion of T is locally compact, i.e., if every bounded infinite sequence of points in T has a fundamental subsequence. (This is derived from the statements of Example 3.) In particular, these two topologies coincide in the following cases:

— T is simplicial and locally finite,

— T is locally compact in the topology of the metric,

— T is "of finite length", i.e., it is covered by a set of intervals of finite total length.

6. Let T be a tree (as in Example 4) and let $\mathrm{Ends}(T)$ be the set of its ends. The union $T \cup \mathrm{Ends}(T)$ has a natural pretree structure (it is described in §11), which contains T as

[20] A topological space is said to be *peripherally finite* if it has a base of open sets each of whose elements has finite boundary.

an embedded pretree. The presence of a natural pretree structure on $T \cup \mathrm{Ends}(T)$ makes it possible to consider the shadow topology on this union. At the same time, there is a certain standard topology on $T \cup \mathrm{Ends}(T)$[21]. It can be shown that the standard topology is the smallest topology that contains the shadow topology and induces the initial topology on the tree T. The shadow topology on the pretree $T \cup \mathrm{Ends}(T)$ is contained in the standard one and coincides with it if and only if T is locally finite (see Proposition 12.3).

7. A pseudotree (see the definition in Example 1.2(7)) is a semilattice if and only if the corresponding pretree is median. In this case, the shadow topology coincides with the topology $T'_{\leq}$ from [31].

6.3. Lemma. *Let $\mathcal{T}$ be a pretree. Then the collection $\mathfrak{U}$ of all the subsets of the form*

$$U_F(x) := \{t \in \mathcal{T} \,:\, [t, x] \cap F = \varnothing\},$$

where $x \in \mathcal{T}$ and $F \subset \mathcal{T}$ is at most finite, is a base of the shadow topology.

Proof. Lemma 5.16 implies that the collection $\mathfrak{U}$ consists of all finite intersections of the branches of $\mathcal{T}$ together with the empty set and the set $\mathcal{T}$ itself. It remains to notice that, by definition, the set of all branches is a subbase of the shadow topology and that, in any topological space X with a subbase $\mathcal{P}$, the set of all finite intersections of elements in $\mathcal{P}$ together with the empty set and the set X is a base of X. □

6.4. Lemma. *A sequence $(x_i)_{i\in\mathbb{N}}$ of points in a pretree $\mathcal{T}$ converges to a point $x \in \mathcal{T}$ in the shadow topology if and only if for each point $z \in (\mathcal{T}\setminus\{x\})$, the set $\{j \in \mathbb{N} \,:\, z \in [x, x_j]\}$ is at most finite, i.e., if*

$$\bigcap_{k\in\mathbb{N}} \bigcup_{j>k} [x, x_j] = \{x\}.$$

Proof. By definition, a sequence $(x_i)_{i\in\mathbb{N}}$ of points in a topological space X converges to a point $x \in X$ if every neighborhood of x contains all but finitely many elements of $(x_i)_{i\in\mathbb{N}}$. If $\mathcal{P}$ is a subbase for X, then for every neighborhood U of x there exists a finite collection of elements of $\mathcal{P}$ whose intersection contains x and is contained in U. Obviously, this implies that $(x_i)_{i\in\mathbb{N}}$ converges to x if and only if each element of $\mathcal{P}$ containing x also contains all but finitely many elements of $(x_i)_{i\in\mathbb{N}}$.

We pass to the proof of the lemma. Let $(x_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{T}$, and let x be a point in $\mathcal{T}$. By Definition 5.1, the family $\{\zeta_z(x)\}_{z\in(\mathcal{T}\setminus\{x\})}$ is precisely the set of all branches in $\mathcal{T}$ that contain x. Then, since the branches form a subbase of the shadow topology, the above fact implies that $(x_i)_{i\in\mathbb{N}}$ converges to x if and only if for each point $z \in (\mathcal{T} \setminus \{x\})$, the branch $\zeta_z(x)$ contains all but finitely many elements of $(x_i)_{i\in\mathbb{N}}$, i.e., if for each point $z \in (\mathcal{T} \setminus \{x\})$, the set $\{j \in \mathbb{N} \,:\, x_j \notin \zeta_z(x)\}$ is at most finite. It remains to observe that, by the definitions, the conditions $x_j \notin \zeta_z(x)$ and $z \in [x, x_j]$ are equivalent whenever $z \neq x$. □

6.5. Proposition (cf. [5, Lemma 7.8])**.** *Let S be a subset in a pretree $\mathcal{T}$. Then the shadow topology on S (regarded as a pretree with the structure induced by that of $\mathcal{T}$) is contained in the relativization of the shadow topology on $\mathcal{T}$ to S. If S is convex in $\mathcal{T}$, then the above topologies coincide.*

[21]There are several techniques developed for defining the set $\mathrm{Ends}(T)$ of ends of a tree and the standard topology on the union $T \cup \mathrm{Ends}(T)$. The set $\mathrm{Ends}(T)$ can be regarded, for example, as

A) the set of equivalence classes of *cofinal rays* (a generalization of this approach to the case of an arbitrary pretree is considered in §11);

B) the *set of ends* in the sense of Freudenthal (see, e.g., [4, 8]);

C) the *boundary at infinity* in the of sense of Gromov hyperbolic spaces (see, e.g., [17]);

D) the *visual boundary* of CAT(0) space, with the so-called *cone topology* (see, e.g., [4]).

Proof. We denote by ζ_x^S and $\zeta_x^{\mathcal{T}}$ the nodal partitions in S and $\mathcal{T}$, respectively (see Definition 5.1). The definition of branches (the same Definition 5.1) shows that each branch $\zeta_x^S(y)$, $x \neq y \in S$, of the pretree S is the intersection of the set S with the branch $\zeta_x^{\mathcal{T}}(y)$ of the pretree $\mathcal{T}$. Since the branches generate the shadow topology, this implies the first assertion of the proposition. Thus, in order to prove the second assertion, it only remains to check that for any points $a \neq b \in \mathcal{T}$, the intersection $S \cap \zeta_a^{\mathcal{T}}(b)$ is open in the shadow topology of the pretree S in the case where S is convex in $\mathcal{T}$. If $a \in S$, then $S \cap \zeta_a^{\mathcal{T}}(b)$ is either empty or a branch of ζ_a^S (and therefore, is open in both cases). If $a \notin S$, then, by Lemma 5.10(5), the intersection $S \cap \zeta_a^{\mathcal{T}}(b)$ is either empty or coincides with S (and hence is open). □

6.6. Proposition. *On a metric pretree*[22]*, the shadow topology is contained in the topology of the metric.*

Proof. Since the branches generate the shadow topology, in order to prove the proposition it suffices to show that all of the branches are open in the topology of the metric. Let B be a branch in a metric pretree (T, ρ), and let r be the pin of B. By the definition of the branches, for any points $x \in B$ and $y \in T \setminus B$, we have $r \in [x, y]$, which is equivalent to the identity $\rho(x, y) = \rho(x, r) + \rho(r, y)$. This means that for any $x \in B$, the open ball centered at x with radius $\rho(x, r)$ is contained in B. This obviously implies the required assertion. □

6.7. Proposition. *In a regular* (*in particular, median*) *pretree, the convex hull of a closed* (*with respect to the shadow topology*) *set is closed. In particular, in such a pretree, the closed intervals are closed subsets.*

6.8. *Remark.* A little more detailed analysis shows that all closed intervals (and with them, all rays and lines) in a pretree $\mathcal{T}$ are closed subsets with respect to the shadow topology if and only if $\mathcal{T}$ has neither singularities of type Y_1 nor singularities of type Y_2 (see Definition 4.3).

Proof of Proposition 6.7. Let K be a closed (with respect to the shadow topology) subset in a regular pretree $\mathcal{T}$. The case with K empty is trivial, and we assume that K is not empty.

First, we show that for every point $x \in \mathcal{T} \setminus K$ there is a finite subset F_x in $\mathcal{T} \setminus \{x\}$ such that

(i) the neighborhood[23] $U_{F_x}(x)$ of x does not intersect K;
(ii) for all $p \neq q \in F_x$, we have $[x, p] \cap [x, q] = \{x\}$.

Indeed, since K is closed and $x \notin K$, while the sets of the form $U_F(z)$, where $z \in \mathcal{T}$ and $F \subset \mathcal{T}$ is at most finite, form the base of the shadow topology by Lemma 6.3, it follows that there exists a point $z \in \mathcal{T}$ and a finite set $F \subset \mathcal{T}$ such that $x \in U_F(z) \subset \mathcal{T} \setminus K$. Then Lemma 5.16(7) implies that $U_F(z) = U_F(x)$, whence

$$x \in U_F(x) \subset \mathcal{T} \setminus K. \tag{20}$$

Let F_x be a subset of minimal cardinality among all the finite subsets F satisfying condition (20). (We note that $x \notin F_x$ by Lemma 5.16(3).) Then F_x satisfies the requirement (i) by construction. We show that F_x satisfies the requirement (ii). Assume to the contrary that F_x does not satisfy (ii), i.e., F_x contains a pair of distinct points $p \neq q$ such that $[x, p] \cap [x, q] \supsetneq \{x\}$. Then there exists a point $r \in [x, p] \cap [x, q]$ that does not coincide

[22] See Definition 4.17.
[23] We use the notation introduced in Subsection 5.15.

with x. We show that the set $F := F_x \setminus \{p,q\} \cup \{r\}$ satisfies (i). Indeed, $U_F(x)$ contains x because $x \notin F$ (see Lemma 5.16). By 5.16(4), we have

$$U_{F_x}(x) = U_{F_x \setminus \{p,q\}}(x) \cap \zeta_p(x) \cap \zeta_q(x), \tag{21}$$

$$U_F(x) = U_{F_x \setminus \{p,q\}}(x) \cap \zeta_r(x). \tag{22}$$

Since $r \in [x,p] \cap [x,q]$ and $r \neq x$, we have $r \in \langle x,p]$ and $r \in \langle x,q]$. It then follows by Lemma 5.13 that $\zeta_r(x) \subset \zeta_p(x)$ and $\zeta_r(x) \subset \zeta_q(x)$. From this, by (21) and (22), we deduce that $U_F(x) \subset U_{F_x}(x) \subset \mathcal{T} \setminus K$. Thus, the assumption that F_x does not satisfy (ii) implies that the set $F = F_x \setminus \{p,q\} \cup \{r\}$ satisfies (i), which is impossible by the choice of F_x. Therefore, F_x satisfies (ii). The auxiliary assertion is proved.

Now we consider hull(K). Let $x \in \mathcal{T} \setminus \text{hull}(K)$. Let F_x be a subset of minimal cardinality among all the finite subsets in $\mathcal{T} \setminus \{x\}$ that satisfy the requirements (i) and (ii). We show that F_x is a singleton (F_x is not empty due to the assumed nonemptiness of K). Indeed, should there be two distinct points p and q in F_x, then, because of the assumed regularity of $\mathcal{T}$, condition (ii) would imply that $x \in [p,q]$, while by condition (i), the assumption that the cardinality of F_x is minimal would imply that the shadows $\mathcal{T} \setminus \zeta_p(x)$ and $\mathcal{T} \setminus \zeta_q(x)$ both intersect K, so that $p \in [a,x]$ and $q \in [x,b]$ for some $a,b \in K$. From the relations $x \in [p,q]$, $p \in [a,x]$, and $q \in [x,b]$, by applying (A8) twice and recalling that $p \neq x \neq q$, we deduce that $x \in [a,q\rangle$ and then that $x \in [a,b]$, which is impossible because $x \notin \text{hull}(K)$.

Thus, we have shown that for any point $x \in \mathcal{T} \setminus \text{hull}(K)$, $\mathcal{T}$ has a branch (F_x) that contains x and does not intersect K (whence, by the convexity of the shadows (Lemma 5.10(4)), it follows that F_x does not intersect hull(K) either). This means that the set $\mathcal{T} \setminus \text{hull}(K)$ is open (in the shadow topology). The proposition is proved. $\square$

We present results of the type of the Krein–Milman theorem.

6.9. Proposition. *In a bounded pretree, the convex hull of the terminal*[24] *points coincides with the entire pretree.*

Proof. Let $\mathcal{T}$ be a bounded pretree, and let ex($\mathcal{T}$) be the set of its terminal points. Let $a \in \mathcal{T}$ be an arbitrary point. We show that $a \in \text{hull}(\text{ex}(\mathcal{T}))$. It suffices to consider the case where a is not terminal. Then, in accordance with the definitions, there are points $p,q \in \mathcal{T}$ such that $a \in \langle p,q\rangle$. On $\mathcal{T}$, let $<_a$ be the order relation defined by the rule

$$x <_a y \Leftrightarrow [a,x] \subsetneq [a,y].$$

We show that every chain C in $(\mathcal{T}, <_a)$ has an upper bound. Indeed, it is easily seen that C is a linear set in $\mathcal{T}$, whence it follows that C lies in a closed interval $[v,w]$ because $\mathcal{T}$ is bounded. By (A3), $[v,w] \subset [a,v] \cup [a,w]$, which implies that each element c in C is contained in $[a,v] \cup [a,w]$. If $c \in [a,v]$ (respectively, $c \in [a,w]$), then all of the elements of C that are $<_a$-less than c also lie in $[a,v]$ (respectively, in $[a,w]$). It is easy to check that this implies that C is contained either in $[a,v]$ or in $[a,w]$, which means that there is an upper bound for C in the set $\{v,w\}$.

Now, applying the Kuratowski–Zorn lemma, we see that there are maximal elements p' and q' in $(\mathcal{T}, <_a)$ such that $p \leq_a p'$ and $q \leq_a q'$, i.e., $p \in [a,p']$ and $q \in [a,q']$. Applying (A8) twice, from the conditions $p \in [a,p']$, $a \in \langle p,q\rangle$, and $q \in [a,q']$ we deduce that $a \in \langle p,q'\rangle$ and then that $a \in \langle p',q'\rangle$.

The maximal elements of the order $<_a$ are terminal points in $\mathcal{T}$. (Indeed, if a point $s \in \mathcal{T}$ is not terminal, then there exist $v,w \in \mathcal{T}$ such that $s \in \langle v,w\rangle$, whence by (A3) it follows that s lies in $[a,v\rangle \cup [a,w\rangle$ so that by (A4) we have either $s <_a v$ or $s <_a w$, i.e., s is not a maximal element.)

[24] See Definition 2.11.

Thus, we have shown that $a \in [p', q']$, where $p', q' \in \mathrm{ex}(\mathcal{T})$, whence $a \in \mathrm{hull}(\mathrm{ex}(\mathcal{T}))$. The proposition is proved. □

6.10. Corollary. *Let a convex subset K in a pretree $\mathcal{T}$ be compact in the shadow topology. Then K coincides with the convex hull of its extreme points.*

Proof. Let $\mathcal{S}_K$ be the restriction of the pretree structure of $\mathcal{T}$ to K. Then, by Proposition 6.5, the shadow topology on the pretree $(K, \mathcal{S}_K)$ is compact, whence by Theorem 8.2 it follows that $(K, \mathcal{S}_K)$ is bounded.

It remains to observe that the set of extreme points of the subset K in the pretree $\mathcal{T}$ coincides with the set of terminal points of the pretree $(K, \mathcal{S}_K)$, and then use Proposition 6.9. □

§7. Separation

This section is devoted to the separation properties of the shadow topology on a pretree. We use the patterns of proofs in [31].

Recall that a topological space X is said to be

— a *T_1-space* if all of its points are closed;

— *Hausdorff* if any two distinct points in X have disjoint neighborhoods;

— *regular* if every closed subset in X and any point that is not in this subset have disjoint neighborhoods;

— *normal* if any two disjoint closed subsets in X have disjoint neighborhoods;

— *hereditarily normal* (*completely normal*) if all its subspaces are normal in the induced topology;

— *perfectly normal* if for any two disjoint closed subsets A and B in X there exists a continuous map $f\colon X \to \mathbb{R}$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$;

— *monotonically normal* if X is a T_1-space and there exists an operator H which assigns to each pair (p, C), where $C \subset X$ is a closed set and $p \in X \setminus C$, an open set $H(p, C) \subset X$ such that

(1) $p \in H(p, C) \subset X \setminus C$,

(2) if $D \subset X$ is closed and $p \notin C \supset D$, then $H(p, C) \subset H(p, D)$,

(3) if $p, q \in X$ and $p \neq q$, then $H(p, \{q\}) \cap H(q, \{p\}) = \varnothing$.

7.1. *Remark.* A monotonically normal space is completely normal and Hausdorff. A perfectly normal space is completely normal. The monotone normality and complete normality are incomparable.

There exist pretrees whose shadow topology is monotonically normal but not perfectly normal. (Example: the set of all countable ordinals with the usual order topology.)

7.2. Lemma. *A pretree with the shadow topology is a T_1-space.*

Proof. This follows from the fact that for each pair of distinct points $x \neq y$ in a pretree, the branch $\zeta_x(y)$ (see Definition 5.1) contains y, does not contain x, and is open in the shadow topology by definition. □

The notions of a star and of a supremum of an arc, which appear in the following Theorem 7.3, are introduced in Subsections 4.5 and 3.1, respectively. The definitions of the median and regular pretrees are given in §4. The definition of a dendron is given in Example 1.2 (3).

7.3. Theorem. *Let $\mathcal{T}$ be a pretree. Then the following properties are equivalent*:

(i) *in $\mathcal{T}$, there are neither infinite stars nor directed arcs having more than one supremum* (*e.g., $\mathcal{T}$ is median or, more generally, regular*);

(ii) *the shadow topology on $\mathcal{T}$ is Hausdorff*;

(iii) *the shadow topology on $\mathcal{T}$ is regular*;
(iv) *the shadow topology on $\mathcal{T}$ is normal*;
(v) *the shadow topology on $\mathcal{T}$ is completely normal*;
(vi) *the shadow topology on $\mathcal{T}$ is monotonically normal*;
(vii) *the pretree $\mathcal{T}$ with the shadow topology embeds* (*as a topological space*) *in a dendron.*

Proof. • (i) $\Rightarrow$ (ii).

Let x and y be distinct points in $\mathcal{T}$. We show that x and y have disjoint neighborhoods.

In the case where $\langle x, y\rangle \neq \varnothing$, for an arbitrary point $r \in \langle x, y\rangle$, the branches

$$\zeta_r(x) \stackrel{\text{def}}{=} \{t \in \mathcal{T} : r \notin [t, x]\} \quad \text{and} \quad \zeta_r(y) \stackrel{\text{def}}{=} \{t \in \mathcal{T} : r \notin [t, y]\}$$

are disjoint neighborhoods for x and y. Indeed, we have $x \in \zeta_r(x)$ and $y \in \zeta_r(y)$ by definition. Furthermore, $\zeta_r(x) \neq \zeta_r(y)$ (because $r \in [x, y]$, whence by the definition of branches we see that $x \notin \zeta_r(y)$ and $y \notin \zeta_r(x)$), so that $\zeta_r(x)$ and $\zeta_r(y)$ are disjoint (being distinct elements of the partition ζ_r).

In the case where $\langle x, y\rangle = \varnothing$ (i.e., $[x, y] = \{x, y\}$), let S be a maximal star in $\mathcal{T}$ containing the star $\{x, y\}$. By the assumption of item (i), S is finite. We observe that for each $r \in S$, the set $S \setminus \{r\}$ is contained in a single branch of ζ_r (this easily follows from definitions). We denote this branch for $r \in S$ by D_r and set

$$P_x := \bigcap_{r \in S\setminus\{x\}} D_r \quad \text{and} \quad P_y := \bigcap_{r \in S\setminus\{y\}} D_r.$$

(Then $P_x \cap P_y = \bigcap_{r\in S} D_r$.) Since S is finite, it follows that P_x and P_y are open (in the shadow topology). By construction, $x \in P_x$ and $y \in P_y$. We show that $P := P_x \cap P_y = \varnothing$. Indeed, arguing by contradiction, assume that there exists a point $p \in P$. Then $p \notin S$ (because $p \in D_r$ and $r \notin D_r$ for all $r \in S$). Moreover, for any points $r \neq s \in S$, we have $r \notin [p, s]$ (this follows from the definition of branches, because $p, s \in D_r$ by construction). This implies that $[p, r\rangle = [p, s\rangle$ for all $r, s \in S$ (indeed, we have $[p, r] \subset [p, s] \cup [s, r]$ and $[p, s] \subset [p, r] \cup [s, r]$ by (A3), while $[s, r] = \{s, r\}$ and $\{s, r\} \not\subset [p, r\rangle \cup [p, s\rangle$ as was shown above). We denote the interval $[p, r\rangle$, which does not depend on the choice of $r \in S$, by $[p, S\rangle$. Observe that $[p, S\rangle \subset P$, because the definition of branches implies that $[p, r\rangle \subset D_r$ for each $r \in S$. Let $<$ be that direction on $[p, S\rangle$ in which p is the minimal element (see Lemma 3.3(4)). In this case, if $([p, S\rangle, <)$ has a largest element, then from the construction it easily follows that $S \cup \{m\}$ is a star, which contradicts the maximality of the star S, while if $([p, S\rangle, <)$ has no largest element, then each point of S is a supremum in $([p, S\rangle, <)$, which contradicts the assumption of item (i). This contradiction shows that $P_x \cap P_y = \varnothing$, i.e., P_x and P_y are disjoint neighborhoods for x and y. This proves that the shadow topology on $\mathcal{T}$ is Hausdorff.

• $\neg$(i) $\Rightarrow$ $\neg$(ii).

On a pretree that is a star, the shadow topology coincides with the topology of finite complements (see Example 6.2(1)) and hence is Hausdorff if and only if this star is finite. This implies by Proposition 6.5 and due to the convexity of the stars that the shadow topology on a pretree with an infinite star is not Hausdorff because the subspaces of a Hausdorff space are Hausdorff.

If a pretree $\mathcal{T}$ has two distinct points a and b that are suprema of one and the same directed arc $(A, <)$, then the shadow topology on $\mathcal{T}$ is not Hausdorff because a and b have no disjoint neighborhoods. (Indeed, if a point s is a supremum of a directed arc $(A, <)$, then every branch $\zeta_r(s)$ containing s includes either the entire arc A (if $r \notin A$) or its upper section $\langle r, s\rangle$ (if $r \in A$). This implies that any neighborhood of s contains a nonempty upper section of $(A, <)$.)

• (ii) ⇒ (vii).

Lemma 5.12 implies that the collection of all the branches of an arbitrary pretree is syzygetic (see Definition in Subsection 1.8). As was shown in [27] (see also [26, Theorem 6.6] and [31, Corollary 3.13]), a Hausdorff space with syzygetic (closed[25]) subbase embeds in some dendron.

• (vii) ⇒ (vi).

The dendrons are monotonically normal (see, for example, the proof of Corollary 3.14 in [31]). From the definition it is clear that the monotone normality is a hereditary property, i.e., a subset of a monotonically normal space with the induced topology is monotonically normal.

• (vi) ⇒ (v) ⇒ (iv).

These implications are valid for any spaces by definition.

• (iv) ⇒ (iii) ⇒ (ii).

This follows directly from Lemma 7.2. □

§8. Compactness

In this section, we prove Theorem 8.2 on necessary and sufficient conditions of compactness for the shadow topology on a pretree. This theorem generalizes similar compactness results obtained for various classes of tree-like structures in [5, Proposition 7.13], and [28, Proposition 4.2] (the case of the union of a countable tree with the space of its ends), [15, Proposition 7.13], and [12, Proposition 1.13] (the case of separable $\mathbb{R}$-trees).

The notions of boundedness and weak completeness of a pretree, which appear in Theorem 8.2, were introduced in Subsections 3.6 and 4.12. The definitions of a directed arc and its supremum were given in Subsection 3.1. The notion of a minimal upper bound for a chain of branches, which is similar to the notion of a supremum of a directed arc and is also used in Theorem 8.2, is defined below.

8.1. Definition. We agree to view the totality $\mathcal{B}(\mathcal{T})$ of all the branches of a pretree $\mathcal{T}$ as a partially ordered set with the subset inclusion partial order. Accordingly, we say that a family $\mathcal{C}$ of branches in $\mathcal{B}(\mathcal{T})$ is a *chain* if in each pair $B_1, B_2 \in \mathcal{C}$ one of the branches B_1, B_2 contains the other one. A branch $D \in \mathcal{B}(\mathcal{T})$ is an *upper bound* of a collection $\mathcal{P} \subset \mathcal{B}(\mathcal{T})$ if $B \subset D$ for all $B \in \mathcal{P}$. A branch $D \in \mathcal{B}(\mathcal{T})$ is a *minimal upper bound* of $\mathcal{P}$ if $\bigcup_{B\in\mathcal{P}} B \subset D$ and there is no $D' \in \mathcal{B}(\mathcal{T})$ such that $\bigcup_{B\in\mathcal{P}} B \subset D' \subsetneq D$. We remark that a collection of branches without *least upper bound* either has no minimal upper bound or has several ones.

8.2. Theorem. *Let $\mathcal{T}$ be a pretree. Then the following properties are equivalent*:

(i) *$\mathcal{T}$ is bounded and weakly complete*;

(ii) *all arcs in $\mathcal{T}$ are intervals*;

(iii) *each directed arc in $\mathcal{T}$ has a supremum*;

(iv) *each chain of branches in $\mathcal{T}$ has a minimal upper bound*;

(v) *the shadow topology on $\mathcal{T}$ is compact.*

Proof. • (i) ⇒ (ii).

In a bounded weakly complete pretree, each arc is bounded (due to the boundedness of the pretree) and hence is an interval (due to the weak completeness).

• (ii) ⇒ (i).

If all arcs in $\mathcal{T}$ are intervals, then $\mathcal{T}$ is weakly complete (by the definition of weak completeness). Furthermore, $\mathcal{T}$ is also bounded in this case, because each linear subset

[25]The collection of complements of subsets forming an open subbase is a closed subbase. The collection of complements of subsets forming a syzygetic collection is also syzygetic.

is contained in its convex hull, which is an arc by Lemma 3.2, and each arc is an interval by assumption, while each interval is contained in a closed interval by definition.

• (ii) ⇔ (iii).

The equivalence of (ii) and (iii) was proved in [5, Lemma 3.13]. (In [5], the pretrees with property (ii) were called *complete*.)

• (iii) ⇒ (iv).

Assume that each directed arc in $\mathcal{T}$ has a supremum. We show that an arbitrary chain $\mathcal{C}$ of branches in $\mathcal{T}$ has a minimal upper bound. Note that the largest element of a chain is its minimal upper bound, and therefore it suffices to consider the case where $\mathcal{C}$ has no largest element.

We denote by $R_{\mathcal{C}}$ the subset in $\mathcal{T}$ formed by the pins of the branches of $\mathcal{C}$. Since each branch has a unique pin (Lemma 5.10), while distinct branches with nonempty intersection have distinct pins (this follows directly from the definitions), it follows that the transition from a branch to its pin gives a one-to-one correspondence between the elements of $\mathcal{C}$ and $R_{\mathcal{C}}$. We denote by $<$ the linear order on $R_{\mathcal{C}}$ corresponding to the order on $\mathcal{C}$ (for a pair of points $r, s \in R_{\mathcal{C}}$, we set $r < s$ if $B_r \subsetneq B_s$, where B_r and B_s are the branches in $\mathcal{C}$ whose pins are the points r and s, respectively).

We observe that the order $<$ is compatible with the pretree structure of $\mathcal{T}$ in the sense that every triple r, s, t of points in $R_{\mathcal{C}}$ satisfying $r < s < t$ satisfies also the relation $s \in \langle r, t\rangle$. Indeed, by definition, the condition $r < s < t$ is equivalent to the condition $B_r \subsetneq B_s \subsetneq B_t$, where B_r, B_s, and B_t are the branches in $\mathcal{C}$ whose pins are the points r, s, and t, respectively. The condition $B_r \subsetneq B_s \subsetneq B_t$ implies by Lemma 5.12(2) that $r \in B_s$ and $t \notin B_s$. This means by the definition of branches that $s \in [r, t]$, which implies that $s \in \langle r, t\rangle$ because $r \neq s \neq t$. This observation shows, by the definition of directions and due to the linearity of the order $<$, that $R_{\mathcal{C}}$ is a linear subset in $\mathcal{T}$, while the order $<$ is a direction on $R_{\mathcal{C}}$.

We consider the convex hull $\bar{R}_{\mathcal{C}} := \mathrm{hull}(R_{\mathcal{C}})$. By Lemma 3.2, from the linearity of $R_{\mathcal{C}}$ it follows that $\bar{R}_{\mathcal{C}}$ is an arc. Let $<'$ be the direction on $\bar{R}_{\mathcal{C}}$ induced[26] by the direction $<$ on $R_{\mathcal{C}}$. By the hypothesis, the directed arc $(\bar{R}_{\mathcal{C}}, <')$ has a supremum (say s) in $\mathcal{T}$. Then the definitions show that the set $\bar{R}_{\mathcal{C}} \cup \{s\}$ is an arc, while s is the largest element of this arc (with respect to the direction $<''$ on $\bar{R}_{\mathcal{C}} \cup \{s\}$ induced by $<'$). This implies that $s \notin \bar{R}_{\mathcal{C}}$, because we consider the case where $\mathcal{C}$ has no largest element. (Indeed, the condition that $\mathcal{C}$ has no largest element implies immediately that $(R_{\mathcal{C}}, <)$ has no largest element. Consequently, $(\bar{R}_{\mathcal{C}}, <')$ also has no largest element, because for each $x \in \bar{R}_{\mathcal{C}} \setminus R_{\mathcal{C}}$ there exist $a, b \in R_{\mathcal{C}}$ such that $x \in \langle a, b\rangle$, whence it follows that either $x <' a$ or $x <' b$.) Since $s \notin \bar{R}_{\mathcal{C}}$ and $\bar{R}_{\mathcal{C}}$ is convex, Lemma 5.10(5) shows that there exists a branch B_* such that s is the pin of B_* and $\bar{R}_{\mathcal{C}}$ is contained in B_*.

We check that B_* is a minimal upper bound of $\mathcal{C}$.

First, we demonstrate that B_* is an upper bound of $\mathcal{C}$, i.e., $\bigcup_{B\in\mathcal{C}} B \subset B_*$. Let B_r be a branch in $\mathcal{C}$, where $r \in R_{\mathcal{C}}$ is the pin of B_r. Since $\mathcal{C}$ has no largest element, there exists a branch $B_{r'} \in \mathcal{C}$ with a pin $r' \in R_{\mathcal{C}}$ such that $B_r \subsetneq B_{r'}$. The condition $B_r \subsetneq B_{r'}$ implies that $r < r'$ and $r <'' r'$. We also have $r' <'' s$, because s is the largest element in $(\bar{R}_{\mathcal{C}} \cup \{s\}, <'')$ and $r' \neq s$. This implies that $r' \in \langle r, s]$ because $<''$ is a direction. By Lemma 5.12(2), the condition $B_r \subsetneq B_{r'}$ implies that $r \in B_{r'}$, whence $B_{r'} = \zeta_{r'}(r)$. We observe that r is in B_* (because B_* contains $\bar{R}_{\mathcal{C}}$), so that $B_* = \zeta_s(r)$. The obtained relations $B_{r'} = \zeta_{r'}(r)$, $B_* = \zeta_s(r)$, and $r' \in \langle r, s]$ imply by Lemma 5.13 that $B_r \subsetneq B_{r'} \subset B_*$. Thus, it is proved that B_* is an upper bound of $\mathcal{C}$.

Now we show that the upper bound B_* is minimal. Arguing by contradiction, assume that it is not minimal, i.e., there exists a branch B' such that $\bigcup_{B\in\mathcal{C}} B \subset B' \subsetneq B_*$. Then

[26] This term for directions was defined in Subsection 3.4.

B' contains $R_{\mathcal{C}}$. (Indeed, since we consider the case where $\mathcal{C}$ has no largest element, for each point $r \in R_{\mathcal{C}}$ the corresponding branch $B_r \in \mathcal{C}$ with the pin r should be contained as a proper subset in some branch $D \in \mathcal{C}$. Then r also lies in D by Lemma 5.12(2), so that $r \in D \subset B'$.) Consequently, since B' is convex (see Lemma 5.10(2)), it contains also the convex hull $\bar{R}_{\mathcal{C}} = \operatorname{hull}(R_{\mathcal{C}})$. Let s' be the pin of B'. Observe that for any point $r \in R_{\mathcal{C}}$ we have $r \in R_{\mathcal{C}} \subset B' \subsetneq B_*$, whence $s' \in [r, s] \subset \bar{R}_{\mathcal{C}} \cup \{s\}$ by Lemma 5.13. At the same time, $s' \notin \bar{R}_{\mathcal{C}}$ because $\bar{R}_{\mathcal{C}} \subset B'$, while B' does not contain its pin s'. Consequently, $s' = s$. However, by Lemma 5.12(2), this contradicts the assumption $B' \subsetneq B_*$. The contradiction obtained shows that B_* is the minimal upper bound of $\mathcal{C}$. Thus, we have shown that each chain of branches has a minimal upper bound.

• (iv) ⇒ (iii).

Assume that each chain of branches in $\mathcal{T}$ has a minimal upper bound. We show that each directed arc $(J, <)$ in $\mathcal{T}$ has a supremum. Obviously, in the case where $(J, <)$ has the largest element, this element is a supremum of $(J, <)$. We consider the case where $(J, <)$ has no largest element. For a point $z \in J$, let J_z denote the corresponding lower section[27] $\{t \in J : t < z\}$ of $(J, <)$. Since J is an arc, the definition of directions implies that, for any $z \in J$, J_z is a convex (and linear) subset of $\mathcal{T}$. If a point $z \in J$ is not the smallest element in $(J, <)$, then J_z is not empty (and is thus an arc). Lemma 5.10(5) implies that if a point $z \in J$ is not the smallest element in $(J, <)$, then the arc J_z is contained in some branch $D_z := D_z(J, <)$ of the partition ζ_z.

We denote by $\mathcal{C}_J := \mathcal{C}(J, <)$ the set of all branches of the form D_z described above (where the point $z \in J$ is not the smallest element in $(J, <)$). Let D_x and D_y be two arbitrary branches in $\mathcal{C}_J$ with $x < y \in J$. We take an arbitrary point $w \in J_x \subset D_x$. Then, since $w < x < y$, we have $x \in \langle w, y]$. Moreover, D_x and D_y both contain w because $D_x \supset J_x \ni w$ and $D_y \supset J_y \ni w$. Applying Lemma 5.13, we see that $D_x \subsetneq D_y$. Thus, we have proved that $\mathcal{C}_J$ is a chain and that if $D_x, D_y \in \mathcal{C}_J$ and $x < y$, then $D_x \subsetneq D_y$.

Since it is assumed that each chain of branches in $\mathcal{T}$ has a minimal upper bound, there exists a branch B_m that is a minimal upper bound for the chain $\mathcal{C}_J$. We prove that the pin m of B_m is a supremum for $(J, <)$. For this, it suffices to show that the set $J \cup \{m\}$ is an arc, and m is its largest element with respect to the direction $<'$ (on this arc) that is induced by the direction $<$ on J. (Indeed, by demonstrating that $(J \cup \{m\}, <')$ is a directed arc and m is its largest element, we thereby prove that for each point $x \in J$, we have

$$\{t \in J : x \le t\} = \{t \in (J \cup \{m\}) : x \le' t <' m\} = [x, m\rangle,$$

which means precisely that m is a supremum for $(J, <)$.)

Observe that, since we consider the case where $(J, <)$ has no largest element, m is not in J. (Indeed, if m is in J, then there exists a point $p \in J$ such that $m < p$, due to the assumption that $(J, <)$ has no largest element. This gives the inclusion $m \in J_p \subset D_p \subset B_m$, which is impossible because no branch can contain its pin.)

We show that $J \cup \{m\}$ is linear. Since J is linear, in order to check the linearity of $J \cup \{m\}$ it suffices to verify that $y \in [x, m]$ for arbitrary $x < y \in J$. The condition $x < y$ yields $x \in D_y$. We observe furthermore that $m \notin D_y$ (because $D_y \subset B_m$ and $m \notin B_m$). This implies by (11) that $y \in [x, m]$. Thus, the linearity of $J \cup \{m\}$ is proved.

It remains to verify that $J \cup \{m\}$ is convex. Suppose that $J \cup \{m\}$ is not convex and denote its convex hull by K. By Lemma 3.2, K is an arc. We denote by $<'$ and $<''$ the directions on (the linear set) $J \cup \{m\}$ and on (the arc) K, respectively, induced by the direction $<$ on J. The definition of directions implies that a subset of a directed arc in a pretree is convex if and only if it is order-convex with respect to the given direction. In particular, this implies that the convex hull of a subset in a directed arc coincides

[27]The definition of a lower section was given in footnote 12.

with the order-convex hull of this subset (with respect to the direction). Thus, the sets J and $\{m\}$ are order-convex in the linearly ordered set $(K, <'')$, while the order-convex hull of $J \cup \{m\}$ in $(K, <'')$ coincides with the entire set K. Obviously, this means that one of the sets J and $\{m\}$ is a lower section and the other one is an upper section in $(K, <'')$. The above proof of the linearity of $J \cup \{m\}$ clearly demonstrates that m is the largest element in $(J \cup \{m\}, <')$, and hence also in $(K, <'')$. Consequently, $\{m\}$ is an upper section and J is a lower section in $(K, <'')$. Clearly, this means that for any $x \in J$ and $p \in K \setminus (J \cup \{m\})$ we have $x <'' p <'' m$. (The set $K \setminus (J \cup \{m\})$ is not empty because we assume that K is the convex hull of $J \cup \{m\}$, which is nonconvex.)

Now we define the set of branches $\mathcal{C}_K := \mathcal{C}(K, <'')$ for the directed arc $(K, <'')$ in exactly the same way as the set $\mathcal{C}_J$ was constructed for $(J, <)$. (We denote by K_z the set $\{t \in K : t <'' z\}$ for a point $z \in K$. Then for any point $z \in K$ that is not the smallest element in $(K, <)$, there exists a unique branch $D''_z := D''_z(K, <'')$ such that z is the pin of D''_z and K_z is contained in D''_z. The set $\mathcal{C}_K$ is defined as the set of all the branches of the form D''_z, where the point $z \in K$ is not the smallest element in $(K, <'')$.) By an almost word-for-word repetition, in application to $\mathcal{C}_K$, of the above proof that $\mathcal{C}_J$ is a chain, we deduce that $\mathcal{C}_K$ is a chain and that if $D''_x, D''_y \in \mathcal{C}_K$ and $x <'' y$, then $D''_x \subsetneq D''_y$. We observe that $B_m = D''_m$ (because these branches have one and the same pin m and both contain J) and that $D''_z = D_z$ for each point $z \in J \subset K$ that is nonminimal with respect to $<''$ (because D''_z and D_z have one and the same pin z and both contain J_z). Now, since for all $x \in J$ and $p \in K \setminus (J \cup \{m\})$ we have $x <'' p <'' m$ (as proved above) and since $D''_x \subsetneq D''_y$ whenever $D''_x, D''_y \in \mathcal{C}_K$ and $x <'' y$, it follows that for each nonminimal $x \in J$ we have

$$D_x = D''_x \subsetneq D''_p \subsetneq D''_m = B_m,$$

which means that D''_p is an upper bound for $\mathcal{C}_J$ and that the upper bound B_m is not minimal. We have reached a contradiction. Consequently, $J \cup \{m\}$ is convex. Thus, m is a supremum for $(J, <)$.

• (iv) ⇒ (v).

Assume that each chain of branches in $\mathcal{T}$ has a minimal upper bound. We show that, in this case, the shadow topology on $\mathcal{T}$ is compact. Recall that the set of branches $\mathcal{B}(\mathcal{T})$ is a *subbase* of this topology. By the Alexander lemma, a topological space with a subbase $\mathcal{P}$ is compact if and only if each cover by elements of $\mathcal{P}$ has a finite subcover. Thus, in order to prove that the shadow topology on $\mathcal{T}$ is compact, it suffices to show that every cover of $\mathcal{T}$ by branches has a finite subcover. Assume to the contrary that there exists a set $\mathcal{E}$ of branches that covers $\mathcal{T}$ and has no finite subcover. We shall show that, under this assumption, $\mathcal{E}$ has the following two mutually exclusive properties.

P1. For each branch $B \in \mathcal{E}$, there is a branch $B' \in \mathcal{E}$ such that $B \subsetneq B'$.

P2. Each chain of branches in $\mathcal{E}$ has an upper bound in $\mathcal{E}$.

Property P1 follows from Lemma 5.12. Indeed, suppose $B \in \mathcal{E}$ and let $b \in \mathcal{T}$ be the pin of B. Then $b \notin B$ by definition. Since $\mathcal{E}$ covers $\mathcal{T}$, there exists a branch $B' \in \mathcal{E}$ containing b. Then Lemma 5.12 implies that either $B' \supsetneq B$ (if B does not contain the pin of B') or $B' \cup B = \mathcal{T}$ (otherwise). The assumption that $\mathcal{E}$ has no finite subcover excludes the possibility that $B' \cup B = \mathcal{T}$. Therefore, $B' \supsetneq B$, as required.

We prove property P2. Let $\mathcal{C}$ be a chain in $\mathcal{E}$. By the hypothesis, there is a minimal upper bound $B_m \in \mathcal{B}(\mathcal{T})$ for $\mathcal{C}$. Let m denote the pin of B_m. Since $\mathcal{E}$ covers $\mathcal{T}$, there exists a branch $B_r \in \mathcal{E}$, with a pin r, containing m. We show that B_r is an upper bound for $\mathcal{C}$.

For this, first we prove that the union $\bigcup_{B \in \mathcal{C}} B$ does not contain the pin r of B_r. Indeed, assuming to the contrary that $r \in D$ for some branch $D \in \mathcal{C}$ and applying Lemma 5.12, we see that either $D \supset B_r$ or $D \cup B_r = \mathcal{T}$. The assumption that $\mathcal{E}$ has no finite subcover excludes the possibility that $D \cup B_r = \mathcal{T}$. The case where $D \supset B_r$ is also impossible because, by construction, B_r contains m, which is not in D (because D, being an element of $\mathcal{C}$, is contained in the upper bound B_m and hence does not contain its pin m). Thus, $r \notin \bigcup_{B\in\mathcal{C}} B$. Then, by Lemma 5.10, there exists a branch $B'_r \in \zeta_r$ containing $\bigcup_{B\in\mathcal{C}} B$ (which is convex as the union of convex sets forming a chain). We show that $B'_r = B_r$. Indeed, assume to the contrary that $B'_r \neq B_r$. Then $B'_r \cap B_r = \varnothing$, because B'_r and B_r are elements of one and the same partition ζ_r. Since $m \in B_r$ (by construction), Lemma 5.12 shows that either $B_m \supset B_r$ or $B_m \cup B_r = \mathcal{T}$. The conditions $B_m \supset \bigcup_{B\in\mathcal{C}} B$, $B'_r \supset \bigcup_{B\in\mathcal{C}} B$, and $B'_r \cap B_r = \varnothing$ imply $B_m \not\supset B_r$. Consequently, $B_m \cup B_r = \mathcal{T}$. The conditions $B_m \cup B_r = \mathcal{T}$ and $B'_r \cap B_r = \varnothing$ imply that $B'_r \subset B_m$. Moreover, $B'_r \subsetneq B_m$ (because the conditions $B_m \cup B_r = \mathcal{T}$ and $r \notin B'_r$ imply that $r \in (B_m \setminus B'_r)$). Therefore, $B_m \supsetneq B'_r \supset \bigcup_{B\in\mathcal{C}} B$, which contradicts the assumed minimality of the upper bound B_m. Hence, $B_r = B'_r \supset \bigcup_{B\in\mathcal{C}} B$. Thus, the branch $B_r \in \mathcal{E}$ is an upper bound for $\mathcal{C}$. Property P2 for $\mathcal{E}$ is proved.

Now we observe that, by the Kuratowski–Zorn lemma, property P2 implies that $\mathcal{E}$ has a maximal element, while property P1 implies that $\mathcal{E}$ has no maximal elements. This contradiction proves that $\mathcal{T}$ is compact.

• (v) ⇒ (iv).

We assume that the set of branches of $\mathcal{T}$ contains a chain $\mathcal{C}$ that has no minimal upper bound, and show that the shadow topology on $\mathcal{T}$ is not compact in this case. Let $\mathcal{D}$ be the set of all branches in $\mathcal{T}$ that do not intersect the union $\bigcup_{B\in\mathcal{C}} B$. We show that the collection $\mathcal{D} \cup \mathcal{C}$ covers $\mathcal{T}$. For this, we assume to the contrary that there exists a point $x \in \mathcal{T}$ covered by no branch of $\mathcal{D} \cup \mathcal{C}$ and consider the partition ζ_x. Since the set $\bigcup_{B\in\mathcal{C}} B$ is convex (this follows immediately from the convexity of branches; see Lemma 5.10(2)), from the assumption that $x \notin \bigcup_{B\in\mathcal{C}} B$ we obtain by Lemma 5.10(5) that ζ_x has a branch, B_x, containing the set $\bigcup_{B\in\mathcal{C}} B$. Since $\mathcal{C}$ has no minimal upper bound, its upper bound B_x is not minimal. This means that there exists a branch B_y (with a pin y) such that $\bigcup_{B\in\mathcal{C}} B \subset B_y \subsetneq B_x$. Observe that $y \neq x$ (because $B_y \subsetneq B_x$). Observe also that $\zeta_y(x)$ and B_y are distinct elements of the partition ζ_y (because $x \notin B_y$ due to the conditions $x \notin B_x$ and $B_y \subsetneq B_x$), whence it follows that $\zeta_y(x) \cap B_y = \varnothing$. The conditions $\zeta_y(x) \cap B_y = \varnothing$ and $\bigcup_{B\in\mathcal{C}} B \subset B_y$ imply that $\zeta_y(x) \cap \bigl(\bigcup_{B\in\mathcal{C}} B\bigr) = \varnothing$. Therefore, $\zeta_y(x) \in \mathcal{D}$, i.e., x is contained in a branch of $\mathcal{D}$. Thus, we obtain a contradiction, which proves that $\mathcal{D} \cup \mathcal{C}$ covers $\mathcal{T}$.

It is easily seen that the cover $\mathcal{D} \cup \mathcal{C}$ has no finite subcover. Indeed, by construction, none of the branches in $\mathcal{D}$ intersects the set $\bigcup_{B\in\mathcal{C}} B$, while no finite subset of $\mathcal{C}$ covers $\bigcup_{B\in\mathcal{C}} B$ (because $\mathcal{C}$ has no minimal upper bound and hence has no maximal element). Thus, the set $\mathcal{D} \cup \mathcal{C}$ is an open cover without finite subcover, which means that $\mathcal{T}$ is not compact in the shadow topology. The theorem is proved. □

§9. Sequential compactness

In this section, we prove that the shadow topology on a pretree is sequentially compact if and only if it is countably compact.

Recall that a topological space X is said to be *countably compact* if any countable open cover (i.e., a cover by open sets) of X has a finite subcover. A topological space X is said to be *sequentially compact* if every sequence of points in X has a convergent subsequence. Every sequentially compact topological space is countably compact (see, e.g., [34]). A metric space is sequentially compact if and only if it is compact. There exist sequentially compact spaces that are not compact, and also there are compact but not sequentially compact spaces. There exist pretrees whose shadow topology is sequentially compact but not compact. (Example: the set of all countable ordinals with the usual order topology.) The following theorem shows that a pretree with the compact shadow topology is also sequentially compact.

9.1. Theorem. *The shadow topology on a pretree is sequentially compact if and only if it is countably compact.*

In order to prove this theorem, we need the following lemma.

9.2. Lemma. *Let a and b be points in a pretree $\mathcal{T}$. We denote by Q_x, where $x \in \mathcal{T}$, the intersection $[a, b] \cap [a, x]$. Then*:

(1) *if $x, y \in \mathcal{T}$, then one of the sets Q_x, Q_y contains the other*;
(2) *every finite subset $S \subset \mathcal{T}$ has a point $s' \in S$ such that $Q_{s'} = \bigcap_{s \in S} Q_s$.*

Proof. Let $<$ denote the direction on the interval $[a, b]$ in which a is the minimal element (see Lemma 3.3(4)). Then Lemma 3.3(5) implies that for each point $x \in \mathcal{T}$, the set Q_x is a lower section of $([a, b], <)$. This immediately yields the two assertions of the lemma, because the set of lower sections of any linearly ordered set is linearly ordered by inclusion. □

9.3. *Remark.* Obviously, Lemma 9.2 implies that if $t \in \mathcal{T}$ and $S \subset \mathcal{T}$ is a finite subset, then there are elements $s_1, s_2 \in S$ such that $\bigcap_{s \in S} [t, s] = [t, s_1] \cap [t, s_2]$; moreover, for every $s_1 \in S$ there exists $s_2 \in S$ such that $\bigcap_{s \in S} [t, s] = [t, s_1] \cap [t, s_2]$ (set $a := t$ and $b := s_1$).

Proof of Theorem 9.1. Since every sequentially compact topological space is countably compact (see, e.g., [34]), we only need to show that countable compactness implies sequential compactness for pretrees (throughout this proof, by a pretree we mean a pretree endowed with the shadow topology). Let $\mathcal{T}$ be a countably compact pretree, and let $(x_i)_{i \in \mathbb{N}}$ be a sequence of points in $\mathcal{T}$. We recall that a topological space X is countably compact if and only if every sequence in X has an accumulation point in X (see, e.g., [34]). Therefore, $(x_i)_{i \in \mathbb{N}}$ has an accumulation point (say x) in $\mathcal{T}$. We show that $(x_i)_{i \in \mathbb{N}}$ has a subsequence converging to x (in the shadow topology).

Obviously, it suffices to consider the case where $x_i \neq x$ for each $i \in \mathbb{N}$ and $x_i \neq x_j$ if $i \neq j$. Recall that we denote by R'_x the equivalence relation on $\mathcal{T} \setminus \{x\}$ such that $y R'_x z$ if and only if $[x, y] \cap [x, z] \neq \{x\}$ (see Remark 5.4). Let $\alpha \in \{1, 2, \dots, \infty\}$ be the number of classes of R'_x that contain points of $(x_i)_{i \in \mathbb{N}}$.

If $\alpha = \infty$, then $(x_i)_{i \in \mathbb{N}}$ has an infinite subsequence $(p_i)_{i \in \mathbb{N}}$ consisting of pairwise non-equivalent (with respect to R'_x) elements. This means that $[x, p_i] \cap [x, p_j] = \{x\}$ for all $i \neq j \in \mathbb{N}$, which implies by Lemma 6.4 that $(p_i)_{i \in \mathbb{N}}$ converges to x.

If α is finite, we consider the finite family $Y_1, \dots, Y_\alpha$ of all classes of R'_x that contain points of $(x_i)_{i \in \mathbb{N}}$. We show that there is a class $Y_* \in \{Y_1, \dots, Y_\alpha\}$ that satisfies the condition $\bigcap_{i \in N_*} \langle x, x_i] = \varnothing$, where N_* stands for the set $\{i : x_i \in Y_*\}$. Indeed, should the intersections $\bigcap_{i \in N_k} \langle x, x_i]$, where $N_k := \{i : x_i \in Y_k\}$, be nonempty for all $k \in \{1, \dots, \alpha\}$, then, by taking for each $k \in \{1, \dots, \alpha\}$ the point $z_k \in \bigcap_{i \in N_k} \langle x, x_i]$, the intersection $\bigcap_{k \in \{1, \dots, \alpha\}} \zeta_{z_k}(x)$ of the branches $\zeta_{z_k}(x)$ would have been an open neighborhood of x containing no point of $(x_i)_{i \in \mathbb{N}}$, which is impossible because x is an accumulation point for $(x_i)_{i \in \mathbb{N}}$.

Observe that for each finite set $S \subset \mathcal{T} \setminus \{x\}$ of R'_x-equivalent points, the intersection $\bigcap_{s \in S} \langle x, s]$ is not empty. (This follows from Lemma 9.2(2), because for any pair s_1, s_2 of R'_x-equivalent points the intersection $\langle x, s_1] \cap \langle x, s_2]$ is nonempty by the definition of the R'_x-equivalence.) Consequently, since the intersection $\bigcap_{i \in N_*} \langle x, x_i]$ is empty, the class Y_* contains an infinite number of elements of $(x_i)_{i \in \mathbb{N}}$. Let $(t_j)_{j \in \mathbb{N}}$ be the infinite subsequence in $(x_i)_{i \in \mathbb{N}}$ consisting of all x_i's that are in Y_*. (Note that $\bigcap_{j \in \mathbb{N}} \langle x, t_j] = \bigcap_{i \in N_*} \langle x, x_i] = \varnothing$.) We show that $(t_j)_{j \in \mathbb{N}}$ has a subsequence converging to x.

For each $j \in \mathbb{N}$, we set

$$P_j := \bigcap_{i \leq j} \langle x, t_i] \quad \text{and} \quad I_j := \langle x, t_1] \cap \langle x, t_j].$$

Observe that (for each $j \in \mathbb{N}$) we have the following properties:

$$P_j \supset P_{j+1}, \tag{23}$$

$$P_j \neq \varnothing, \tag{24}$$

$$\bigcap_{i \in \mathbb{N}} P_i = \varnothing, \tag{25}$$

$$\bigcap_{i \leq j} I_i = P_j, \tag{26}$$

$$(P_{j+1} \neq P_j) \Rightarrow (P_{j+1} = I_{j+1}). \tag{27}$$

Properties (23) and (26) are obvious, property (24) follows from Lemma 9.2(2) by the above argument, because $\{t_1, \dots, t_j\}$ is a finite set of R'_x-equivalent points. Property (25) is proved by the following chain of identities:

$$\bigcap_{i \in \mathbb{N}} P_i = \bigcap_{j \in \mathbb{N}} \langle x, t_j] = \bigcap_{i \in N_*} \langle x, x_i] = \varnothing.$$

We prove property (27). By (26), the inequality $P_{j+1} \neq P_j$ is equivalent to

$$\bigcap_{i \leq j+1} I_i \neq \bigcap_{i \leq j} I_i. \tag{28}$$

Lemma 9.2 implies that the sets of the form $Q_p := \langle x, t_1] \cap \langle x, p]$ are linearly ordered by inclusion. In particular, the intersections $\bigcap_{i \leq j+1} I_i$ and $\bigcap_{i \leq j} I_i$ are equal to the minimal I_i's in the sets $\{I_i\}_{i \leq j+1}$ and $\{I_i\}_{i \leq j}$. Consequently, condition (28) implies that the set I_{j+1} is minimal (by inclusion) in $\{I_i\}_{i \leq j+1}$, which implies that $\bigcap_{i \leq j+1} I_i = I_{j+1}$ and $P_{j+1} = I_{j+1}$.

Let $M := \{1\} \cup \{j \in \mathbb{N} \setminus \{1\} : P_j \neq P_{j-1}\}$. Properties (23)–(25) imply that M is infinite. We show that the sequence $(t_j)_{j \in M}$ converges to x. Suppose to the contrary that it does not. Then, by Lemma 6.4, there exists a point $z \in \mathcal{T} \setminus \{x\}$ that is contained in an infinite number of intervals from the set $\{\langle x, t_j] : j \in M\}$. In particular, this implies that $z \in Y_*$ (because for each $t \in Y_*$, the interval $\langle x, t]$ is contained in Y_* by the definition of the R'_x-equivalence). Hence, the set $I_z := \langle x, t_1] \cap \langle x, z]$ is nonempty and is contained in an infinite number of intervals from the set $\{\langle x, t_1] \cap \langle x, t_j] : j \in M\}$. By the definition of M, from (27) it follows that $\langle x, t_1] \cap \langle x, t_j] = P_j$ for each $j \in M$. Therefore, an infinite number of P_j's contain I_z, whence by property (23) it follows that all P_j's contain I_z, which contradicts (25). This proves that $(t_j)_{j \in M}$ converges to x. It remains to recall that $(t_j)_{j \in M}$ is a subsequence of $(x_i)_{i \in \mathbb{N}}$. The theorem is proved. □

§10. Metrizability

In this section, we prove a series of assertions concerning the metrizability of the shadow topology.

10.1. Definitions. We say that a subset S in a pretree $\mathcal{T}$ is *quasidense* in $\mathcal{T}$ if for each pair of distinct points $a \neq b \in \mathcal{T}$, the interval $[a, b\rangle$ contains a point of S. A pretree containing an at most countable quasidense subset is *quasiseparable*.

We say that a pretree is *quasi-Hausdorff* if it contains neither infinite stars nor directed arcs having more than one supremum (see item (i) in Theorem 7.3). Theorem 7.3 implies that a pretree is quasi-Hausdorff if and only if its shadow topology is Hausdorff. Thus, every regular (in particular, median) pretree is quasi-Hausdorff.

We say that a pretree is *quasicompact* if it is bounded and weakly complete (Theorem 8.2 implies that a pretree is quasicompact if and only if its shadow topology is compact.)

10.2. Theorem. (1) *A pretree is quasiseparable if and only if its shadow topology is second-countable.* (2) *The shadow topology of a quasi-Hausdorff quasiseparable pretree is metrizable.* (3) *The shadow topology of a quasicompact quasi-Hausdorff pretree $\mathcal{T}$ is metrizable if and only if $\mathcal{T}$ is quasiseparable.*

10.3. *Remark.* A metric on a pretree can be compatible with the shadow topology but incompatible with the pretree structure (in the sense of Definition 4.17), and *vice versa*. Theorem 10.2 concerns the metrizability of the shadow topology and does not address the issue of compatibility of the metric with the pretree structure. Generally speaking, it can be shown that each quasi-Hausdorff quasiseparable pretree has a metric that is compatible with both the pretree structure and the shadow topology.

In order to prove Theorem 10.2, we need the following lemma.

10.4. Lemma. *A subset S of a pretree $\mathcal{T}$ is quasidense if and only if the family of branches $\mathcal{B}_S := \{\zeta_a(b) : a \neq b \in S\}$ is a subbase of the shadow topology.*

Proof. Let S be a quasidense subset in $\mathcal{T}$. Since, by definition, the set $\mathcal{B}(\mathcal{T})$ of all the branches of $\mathcal{T}$ is a subbase for the shadow topology and $\mathcal{B}_S \subset \mathcal{B}(\mathcal{T})$, in order to prove that $\mathcal{B}_S$ is a subbase for the shadow topology it suffices to show that each branch in $\mathcal{T}$ is a union of branches from $\mathcal{B}_S$. In other words, it suffices to show that for any branch $B \in \mathcal{B}(\mathcal{T})$ and any its point $x \in B$ there exists a branch $D \in \mathcal{B}_S$ such that $x \in D \subset B$. Given $B \in \mathcal{B}(\mathcal{T})$ and $x \in B$, we denote by r the pin of B (then $B = \zeta_r(x)$). Since S is quasidense, there exist points $a, b \in S$ such that $a \in [r, x\rangle$ and $b \in [x, a\rangle$. We set $D := \zeta_a(b)$ and show that $x \in D \subset B$. The relation $x \in D = \zeta_a(b)$, which is equivalent to $a \notin [x, b]$ by definition, follows from the condition $b \in [x, a\rangle$ by (A2). In order to show that $\zeta_a(b) = D \subset B = \zeta_r(x)$ it suffices to check that $a \in [b, t]$ for all $t \in \mathcal{T}$ such that $r \in [x, t]$. Let $<$ denote the direction on $[x, t]$ in which x is the minimal element (see Lemma 3.3(4)). The conditions $r \in [x, t]$, $a \in [r, x\rangle$, and $b \in [x, a\rangle$ imply by property (A4) that $a, b \in [x, t]$. The same conditions imply by the definition of directions that

$$x \leq b < a \leq r \leq t,$$

i.e., $a \in [b, t]$. Thus, we have shown that $\mathcal{B}_S$ is a subbase for the shadow topology.

Let S be a subset in $\mathcal{T}$ such that $\mathcal{B}_S$ is a subbase for the shadow topology. Then, since $\mathcal{T}$ with the shadow topology is a T_1-space (Theorem 7.2), it follows that for each pair x, y of distinct points in $\mathcal{T}$ there exists a branch $B \in \mathcal{B}_S$ such that $x \notin B$ and $y \in B$.

Let $r \in S$ be the pin of B. The conditions $x \notin B$ and $y \in B$ imply that $r \in [x, y]$ and $r \neq x$ (because $r \notin B$ by definition). Therefore, $r \in [y, x\rangle$. This proves that S is quasidense in $\mathcal{T}$. □

Proof of Theorem 10.2. (1) Lemma 10.4 implies that the shadow topology of every quasiseparable pretree has a countable subbase and hence is second-countable.

Conversely, suppose that the shadow topology of a pretree $\mathcal{T}$ is second-countable. We recall that *in a second-countable topological space, every subbase contains an* (*at most*) *countable subcollection that is also a subbase*[28]. This implies (because the set of all the

[28]Assume that A is a subbase in a second-countable topological space X. Then (by definition) the collection B of all finite intersections of members of A is a base of X. Since X is second-countable, the base B contains a countable base $B' \subset B$ (see, for instance, [2, Chapter I, §1, Theorem 2] or [38,

branches is by definition a subbase of the shadow topology) that there exists an at most countable collection $\mathcal{D}$ of branches that is a subbase for the shadow topology on $\mathcal{T}$. We denote by S_1 the set of all the pins of the branches from $\mathcal{D}$. Then S_1 is at most countable because each branch has a unique pin (Lemma 5.10). Let S_2 be any at most countable subset in $\mathcal{T}$ that intersects all of the branches from $\mathcal{D}$. Then the set $S := S_1 \cup S_2$ is at most countable, while the set $\mathcal{B}_S = \{\zeta_a(b) : a \neq b \in S\}$ contains $\mathcal{D}$ by construction and hence is a subbase for the shadow topology. Then Lemma 10.4 shows that S is quasidense in $\mathcal{T}$, which implies by definition that the shadow topology on $\mathcal{T}$ is quasiseparable.

(2) By assertion (1), the shadow topology of every quasiseparable pretree is second-countable. The shadow topology of a quasi-Hausdorff pretree is regular and Hausdorff by Theorem 7.3. It remains to recall the Tychonoff–Urysohn metrization theorem, which states that *every second-countable regular Hausdorff space is metrizable*.

(3) Let $\mathcal{T}$ be a quasicompact quasi-Hausdorff pretree. Then the shadow topology on $\mathcal{T}$ is Hausdorff and compact by Theorems 7.3 and 8.2. We recall that *a compact Hausdorff space is metrizable if and only if it is second-countable* (see, e.g., [2, Chapter V, §2, Theorem 4]). Therefore, the shadow topology on $\mathcal{T}$ is metrizable if and only if it is second-countable, while assertion (1) says that the last condition is equivalent to quasiseparability. □

§11. Space of ends

In this section, we define the ends of a pretree and extend the pretree structure to the set of ends. There are several approaches to defining the ends of (pre)trees. We exploit the classical definition of the ends via the classes of cofinal rays. An equivalent definition can be obtained, for example, via the notion of *flows* introduced in [5] if we define the ends as the flows of a certain type.

11.1. Definition. Let $\mathcal{T}$ be a pretree. Recall that a *ray* in a pretree is an unbounded arc having an extreme point (Definition 3.7). On the set of all rays in $\mathcal{T}$, we define a binary relation E by setting $(R, R') \in E$ for two rays R and R' if the intersection $R \cap R'$ is a half-line. Then E is an equivalence relation (obviously, E is reflexive and symmetric; Lemma 3.8(10) implies that E is transitive). The *ends* of $\mathcal{T}$ are the E-classes of rays. We denote by $\mathrm{Ends}(\mathcal{T})$ the set of all the ends of $\mathcal{T}$.

11.2. Definition. Let $\mathcal{T}$ be a pretree, let $\mathcal{S} \subset \mathcal{T}^3$ be its ternary structure, and let $\mathrm{Ends}(\mathcal{T})$ be the set of its ends. On the set $\mathcal{T} \cup \mathrm{Ends}(\mathcal{T})$, we define a ternary structure $\widehat{\mathcal{S}}$ as follows.

- R1. If $x, y, z \in \mathcal{T}$, then $\widehat{\mathcal{S}}xyz \Leftrightarrow \mathcal{S}xyz$.
- R2. If $x, y \in \mathcal{T}$ and $\omega \in \mathrm{Ends}(\mathcal{T})$, then $\widehat{\mathcal{S}}xy\omega \Leftrightarrow \widehat{\mathcal{S}}\omega yx \Leftrightarrow$ there is a ray $R \in \omega$ such that $\mathcal{S}xyr$ for all $r \in R$.
- R3. If $x \in \mathcal{T}$ and $\omega, \tau \in \mathrm{Ends}(\mathcal{T})$, then $\widehat{\mathcal{S}}\omega x\tau \Leftrightarrow$ there are rays $R \in \omega$ and $Q \in \tau$ such that $\mathcal{S}rxq$ for all $r \in R$ and $q \in Q$.
- R4. If $\omega \in \mathrm{Ends}(\mathcal{T})$ and $x, y \in \mathcal{T} \cup \mathrm{Ends}(\mathcal{T})$, then $(x, \omega, y) \notin \widehat{\mathcal{S}}$.

11.3. Theorem. *The set $\mathcal{T} \cup \mathrm{Ends}(\mathcal{T})$ with the structure $\widehat{\mathcal{S}}$ described above is a pretree.*

Proof. The fact that $\widehat{\mathcal{S}}$ satisfies Axioms (T0), (T1), and (T2) easily follows from the fact that $\mathcal{S}$ satisfies these axioms; in order to make this conclusion, we need only the definition of $\widehat{\mathcal{S}}$ and the fact that a pair of rays representing one and the same end have a nonempty intersection (this concerns Axioms (T0) and (T2) in the case of R3).

Problem 16(15).10]). Obviously, the members of A involved in the construction of members of B' constitute the required countable subbase $A' \subset A$.

In order to verify that Axiom (T3) is satisfied, we use the system of notions described in §5 and related to the branches and nodal partitions of pretrees. Let x, y, z, w be a quadruple of points in $\mathcal{T} \cup \mathrm{Ends}(\mathcal{T})$ such that $\widehat{\mathcal{S}}xyz$ and $y \neq w$. The condition $\widehat{\mathcal{S}}xyz$ implies by the definition of $\widehat{\mathcal{S}}$ that $y \in \mathcal{T}$. We recall that the element of a partition ζ containing a point $t \in \mathcal{T}$ is denoted by $\zeta_y(t)$.

If $\omega \in \mathrm{Ends}(\mathcal{T})$, we denote by $\zeta_y(\omega)$ the branch of ζ_y in which there are rays representing ω. (The existence of $\zeta_y(\omega)$ follows from Lemmas 3.8(6) and 5.10(5): Lemma 3.8(6) implies that every ray $R \in \omega$ contains a ray R' such that $R' \in \omega$ and $y \notin R'$, while Lemma 5.10(5) implies that the ray R', being a convex set, is contained in a branch of ζ_y. The uniqueness of $\zeta_y(\omega)$ follows, because distinct branches of a partition do not intersect, while any two rays representing one and the same end have nonempty intersection by definition.)

The definitions of $\widehat{\mathcal{S}}$ and ζ_y imply directly that the conditions $\widehat{\mathcal{S}}pyq$ and $\zeta_y(p) \neq \zeta_y(q)$ are equivalent whenever $p, q \in \mathcal{T} \cup \mathrm{Ends}(\mathcal{T}) \setminus \{y\}$. Since it is assumed that $\widehat{\mathcal{S}}xyz$ and $y \neq w$, we have $\{x, z, w\} \not\ni y$ and $\zeta_y(x) \neq \zeta_y(z)$. Hence, either $\zeta_y(w) \neq \zeta_y(x)$ and $\widehat{\mathcal{S}}xyw$ or $\zeta_y(w) \neq \zeta_y(z)$ and $\widehat{\mathcal{S}}wyz$. Therefore, Axiom (T3) is satisfied by $\widehat{\mathcal{S}}$. □

11.4. Definition. In what follows, given a pretree $\mathcal{T}$, we view the set $\widehat{\mathcal{T}} := \mathcal{T} \cup \mathrm{Ends}(\mathcal{T})$ as the pretree endowed with the pretree structure described in Subsection 11.2.

For the intervals of the pretree $\widehat{\mathcal{T}}$, we use the notation from Subsection 1.12. This leads to no confusion because, by the definition of the pretree $\widehat{\mathcal{T}}$, its system of intervals contains that of the pretree $\mathcal{T}$. In the cases where the distinction between the intervals of the pretrees $\mathcal{T}$ and $\widehat{\mathcal{T}}$ is appropriate in the notation, the intervals will be equipped with respective indices: for example, $[a, b]_{\mathcal{T}}$ and $[x, y\rangle_{\widehat{\mathcal{T}}}$ (in this case, $[a, b]_{\mathcal{T}} = [a, b]_{\widehat{\mathcal{T}}}$ whenever $a, b \in \mathcal{T}$).

11.5. Lemma. 1. *Let p be a point in a pretree $\mathcal{T}$, and let ω be an end in $\mathrm{Ends}(\mathcal{T})$. Then there exists a unique ray emanating from p and representing ω. This ray coincides with the interval $[p, \omega\rangle$ in the pretree $\mathcal{T} \cup \mathrm{Ends}(\mathcal{T})$.*

2. *Let ω and τ be two distinct ends of a pretree $\mathcal{T}$. Then there exists a unique line containing some rays of ω and of τ. This line coincides with the interval $\langle\omega, \tau\rangle$ in the pretree $\mathcal{T} \cup \mathrm{Ends}(\mathcal{T})$.*

Proof. 1. Let R be a ray representing ω, and let x be the extreme point of R. We set $H := R \setminus [p, x]$ and show that the convex hull of the set $\{p\} \cup H$ is a ray emanating from p and representing ω.

First, we verify that $\{p\} \cup H$ is linear. Since H is linear (as a subset of a ray), it suffices to check that for arbitrary $a, b \in H$ one of the relations $a \in [p, b]$ and $b \in [p, a]$ holds. Indeed, since a and b are in the ray R emanating from x, we have one of the relations $a \in [x, b]$ and $b \in [x, a]$. In the case where $a \in [x, b]$, we have $a \in [p, b]$ by Axiom (A3) (because $H = R \setminus [p, x]$, whence $a, b \notin [p, x]$). In the case where $b \in [x, a]$, we have $b \in [p, a]$. Thus, $\{p\} \cup H$ is linear.

From the above considerations it follows, moreover, that p is extreme in $\{p\} \cup H$. We also observe that H is unbounded (otherwise, the ray R is contained in the union of the bounded sets H and $[p, x]$, which is impossible by Lemma 3.8(2)). Therefore, the set $\{p\} \cup H$ is linear and unbounded, and p is its extreme point. This implies by Lemmas 3.2 and 2.13 that the convex hull $R_p := \mathrm{hull}(\{p\} \cup H)$ is an unbounded arc and p is its extreme point, which means that R_p is a ray emanating from p.

The ray R_p represents the end ω, because the intersection $R_p \cap R$ is a half-line ($R_p \cap R$ is an unbounded arc because it is the intersection of arcs and contains H; but $R_p \cap R$ is not a line, because it is contained in a ray).

Now we prove the identity $R_p = [p, \omega\rangle$, which will imply, in particular, that R_p is the only ray with the required properties.

We show that $R_p \subset [p, \omega\rangle$. The point p is in $[p, \omega\rangle$ because $p \neq \omega$. Let z be a point in $R_p \setminus \{p\}$, and let y be a point in $R_p \setminus [p, z]$. Then the set $R_y := R_p \setminus [p, y\rangle$ is a ray (Lemma 3.8(8)) representing ω, while the definition of directions easily implies that $z \in \langle p, t\rangle$ for any $t \in R_y$. This implies by the definition of the pretree structure on $\mathcal{T} \cup \mathrm{Ends}(\mathcal{T})$ that $z \in [p, \omega\rangle$. Thus, $R_p \subset [p, \omega\rangle$.

We show that $[p, \omega\rangle \subset R_p$. Observe that $[p, \omega\rangle = \{p\} \cup \langle p, \omega\rangle$ (because $p \neq \omega$). The point p is in R_p by definition. Let z be a point in $\langle p, \omega\rangle$. Then, by the definition of the pretree structure on $\mathcal{T} \cup \mathrm{Ends}(\mathcal{T})$, there exists a ray $R' \in \omega$ such that $z \in \langle p, t\rangle$ for all $t \in R'$. However, R intersects R_p. Taking $t \in R' \cap R_p$, we see that $z \in \langle p, t\rangle \subset R_p$ (because the rays are convex).

The assertion is proved.

2. Let $x \in \mathcal{T}$ be a point. By assertion 1, there are rays $R_1 \in \omega$ and $R_2 \in \tau$ emanating from x. We set $H_1 := R_1 \setminus R_2$, $H_2 := R_2 \setminus R_1$ and show that the convex hull of $H_1 \cup H_2$ is a line containing some rays of ω and of τ.

First, we check that the set $H_1 \cup H_2$ is linear. For this, since H_1 and H_2 are linear (being subsets of rays), it suffices to check that for arbitrary $a \in H_i$ and $b, c \in H_j$, where $\{i, j\} = \{1, 2\}$, one of the relations $b \in [a, c]$ and $c \in [a, b]$ holds. Indeed, we observe that $b, c \notin [a, x]$ because $[a, x] \subset R_i$, while $b, c \in H_j = R_j \setminus R_i$. Since b and c are in the ray R_j emanating from x, it follows that we have one of the relations $b \in [x, c]$ and $c \in [x, b]$. In the case where $b \in [x, c]$, the condition $b \notin [a, x]$ implies by Axiom (A3) that $b \in [a, c]$. In the case where $c \in [x, b]$, the condition $c \notin [a, x]$ implies by (A3) that $c \in [a, b]$. Thus, $H_1 \cup H_2$ is linear.

Next, we show that H_1 contains a ray representing ω and H_2 contains a ray representing τ. Indeed, since $\omega \neq \tau$, Definition 11.1 shows that the intersection $K := R_1 \cap R_2$ is not a line. However, K is convex (as the intersection of two convex sets). Obviously, if a convex subset of a ray is not a half-line, then it is bounded. It easily follows that $H_i = R_i \setminus K$ $(i = 1, 2)$ contains a ray R_i'. Since $R_i \cap R_i' = R_i'$, the ray R_1' is in ω (together with R_1), while R_2' is in τ.

Therefore, the set $H_1 \cup H_2$ is linear and contains (disjoint) rays representing ω and τ. This implies by Lemmas 3.2 and 3.8(9) that the convex hull $L := \mathrm{hull}(H_1 \cup H_2)$ is a line (and contains some rays of ω and of τ, as required).

Now we prove that $L = \langle \omega, \tau\rangle$, which will imply, in particular, that L is the only line with the required properties.

We show that $L \subset \langle \omega, \tau\rangle$. Let z be a point in L. We take an arbitrary direction $<$ on L and a pair of points $a, b \in L$ such that $a < z < b$; then, by Lemma 3.8, one of the rays

$$R_a := \{t \in L : t \leq a\} \quad \text{and} \quad R_b := \{t \in L : b \leq t\}$$

is in ω and the other one is in τ. It also follows that for any $r \in R_a$ and $s \in R_b$ we have $r < z < s$, which implies by the definition of directions that $z \in \langle r, s\rangle$. By the definition of the pretree structure on $\mathcal{T} \cup \mathrm{Ends}(\mathcal{T})$, this means that $z \in \langle \omega, \tau\rangle$.

We show that $\langle \omega, \tau\rangle \subset L$. Let z be a point in $\langle \omega, \tau\rangle$. Then the definition of the pretree structure on $\mathcal{T} \cup \mathrm{Ends}(\mathcal{T})$ implies that there are rays $R \in \omega$ and $Q \in \tau$ such that $z \in \langle r, q\rangle$ for all $r \in R$ and $q \in Q$. However, R and Q both intersect L because L contains rays representing τ and ω. Taking $r \in R \cap L$ and $q \in Q \cap L$, we see that $z \in \langle r, q\rangle \subset L$ (because all lines are convex).

The lemma is proved. $\square$

11.6. Proposition. *Let $\mathcal{T}$ be a pretree. Then*:

(1) *the set* $\mathrm{Ends}(\mathcal{T})$ *is empty if and only if* $\mathcal{T}$ *is bounded*;
(2) *all points of* $\mathrm{Ends}(\mathcal{T})$ *are terminal in the pretree* $\widehat{\mathcal{T}}$;
(3) *in the pretree* $\widehat{\mathcal{T}}$, *the set* $\mathcal{T}$ *is convex*;
(4) *the pretree* $\widehat{\mathcal{T}}$ *is bounded.*

Proof. (1) We show that the following properties are equivalent:

(a) $\mathrm{Ends}(\mathcal{T}) = \varnothing$,
(b) $\mathcal{T}$ contains no rays,
(c) $\mathcal{T}$ contains no half-lines,
(d) $\mathcal{T}$ contains no unbounded arcs,
(e) $\mathcal{T}$ contains no unbounded linear sets ($\overset{\text{def}}{\Leftrightarrow}$ $\mathcal{T}$ is bounded).

Properties (a) and (b) are equivalent because by definition, $\mathrm{Ends}(\mathcal{T})$ is the set of classes of equivalent rays in $\mathcal{T}$.

Properties (b) and (c) are equivalent because each ray is a half-line, while each half-line is contained in a ray.

Properties (c) and (d) are equivalent because all unbounded arcs are lines and half-lines, while each line contains half-lines (see Lemma 3.8).

Properties (d) and (e) are equivalent because the convex hull of an unbounded linear set is an arc by Lemma 3.2.

(2) This follows from item R4 in the definition of the structure $\widehat{\mathcal{S}}$ (Definition 11.2).

(3) If $x, y \in \mathcal{T}$, then $[x, y]_{\widehat{\mathcal{T}}} = [x, y]_{\mathcal{T}} \subset \mathcal{T}$ by the definition of the pretree $\widehat{\mathcal{T}}$. This means by definition that $\mathcal{T}$ is convex in $\widehat{\mathcal{T}}$.

(4) Assume that the pretree $\widehat{\mathcal{T}}$ is unbounded. Then the proof of assertion (1) implies that $\widehat{\mathcal{T}}$ contains a ray R. Let r be the extreme point of R. Since all points of $\mathrm{Ends}(\mathcal{T})$ are terminal in $\widehat{\mathcal{T}}$ (assertion (2)), it follows that each of these points is extreme in any subset of $\widehat{\mathcal{T}}$ containing it (in the sense of Definition 2.11). Consequently, since a ray has a unique extreme point (Lemma 3.8), the set $R \setminus \{r\}$ is contained in $\mathcal{T}$. We take an arbitrary point $x \in R \setminus \{r\}$ and consider the set $R_x := R \setminus [r, x\rangle$. Lemma 3.8 implies that R_x is a ray in $\widehat{\mathcal{T}}$ and x is the extreme point of R_x. Observe that R_x is contained in $\mathcal{T}$. It is easily seen that R_x is a ray in the pretree $\mathcal{T}$ as well. Let $\omega \in \mathrm{Ends}(\mathcal{T})$ be the end represented by R_x. Then Lemma 11.5 implies that the set R_x is contained in the interval $[x, \omega]$ of the pretree $\widehat{\mathcal{T}}$, i.e., R_x is bounded in $\widehat{\mathcal{T}}$ and, hence, is not a ray there. This contradiction proves that $\widehat{\mathcal{T}}$ is bounded. □

11.7. Proposition. *Let $\mathcal{T}$ be a pretree, and let $\widehat{\mathcal{T}} = \mathcal{T} \cup \mathrm{Ends}(\mathcal{T})$. Then*:

(1) *the pretree* $\widehat{\mathcal{T}}$ *is median if and only if* $\mathcal{T}$ *is median*;
(2) *the pretree* $\widehat{\mathcal{T}}$ *is weakly complete if and only if* $\mathcal{T}$ *is weakly complete*;
(3) *the pretree* $\widehat{\mathcal{T}}$ *is quasiseparable if and only if* $\mathcal{T}$ *is quasiseparable.*

11.8. *Remark.* The pretrees $\mathcal{T}$ and $\mathcal{T} \cup \mathrm{Ends}(\mathcal{T})$ are also indistinguishable by such properties as regularity, presence of singularities (Definition 4.3), Dedekind completeness, existence of a metric compatible with the pretree structure (see Definition 4.17), etc.

Proof of Proposition 11.7. (1) We use the following assertion.

11.9. Assertion. *If $p, t, q \in \widehat{\mathcal{T}}$ and $t \notin \{p, q\}$, then*

$$[t, p] \cap [t, q] \cap \mathcal{T} \neq \varnothing.$$

Proof. In the case where $t \in \mathcal{T}$, the set $[t, p] \cap [t, q] \cap \mathcal{T}$ contains t. In the case where $t \in \mathrm{Ends}(\mathcal{T})$, Lemma 11.5 implies that each of the intervals $[t, p]$ and $[t, q]$ contains a ray representing t, whence it follows that $[t,p] \cap [t,q]$ contains the intersection of two such rays, which is a half-line in $\mathcal{T}$ by Definition 11.1. □

From the definition of a median pretree, it easily follows that a convex subset of a median pretree, when viewed as an independent pretree with the induced structure, is median. Consequently, if $\widehat{\mathcal{T}}$ is median, then $\mathcal{T}$ is also median, because $\mathcal{T}$ is convex in $\widehat{\mathcal{T}}$ (Proposition 11.6) and its ternary structure is embedded in the structure of $\widehat{\mathcal{T}}$.

We show that $\widehat{\mathcal{T}}$ is median whenever $\mathcal{T}$ is median. We use the notation

$$Y(a,b,c) := [a,b] \cap [b,c] \cap [a,c].$$

Let a,b,c be a triple of points in $\widehat{\mathcal{T}}$. In the case where the points a, b, and c are not all distinct, one of these point is the median of the triple by Axiom (A0). In the case where $a \neq b \neq c \neq a$, Assertion 11.9 implies that there are points $a',b',c' \in \mathcal{T}$ such that

$$a' \in [b,a] \cap [a,c], \quad b' \in [a,b] \cap [b,c], \quad \text{and} \quad c' \in [a,c] \cap [c,b].$$

We observe that

$$[a',b'] \subset [a,b], \quad [a',c'] \subset [a,c], \quad \text{and} \quad [b',c'] \subset [b,c]$$

by Lemma 1.18, whence $Y(a',b',c') \subset Y(a,b,c)$. It remains to remark that $Y(a',b',c')$ is nonempty because $\mathcal{T}$ is median.

(2) Assume that there exists a weakly complete pretree $\mathcal{T}$ such that the pretree $\widehat{\mathcal{T}}$ is not weakly complete. Then, since $\widehat{\mathcal{T}}$ is bounded (Proposition 11.6), Theorem 8.2 shows that $\widehat{\mathcal{T}}$ contains a directed arc $(A,<)$ that has no supremum. In particular, $(A,<)$ has no largest element. We observe that the intersection $A \cap \mathcal{T}$ is nonempty, because the subset $\mathrm{Ends}(\mathcal{T})$ of the pretree $\widehat{\mathcal{T}}$ contains only one-point arcs (this follows, e.g., from Lemma 11.5). Let x be a point in $A \cap \mathcal{T}$. We denote by A_x the upper section $\{t \in A : x \leq t\}$ and let $<_x$ be the restriction of $<$ to A_x. It is clear that A_x is an arc and that the directed arc $(A_x,<_x)$ has no supremum (because $(A,<)$ has no supremum). In particular, $(A_x,<_x)$ has no largest element. This implies that A_x is contained in $\mathcal{T}$ (because $\mathcal{T}$ contains x, which is the only extreme point of A_x, and all points of $\mathrm{Ends}(\mathcal{T})$ are terminal in $\widehat{\mathcal{T}}$ (Proposition 11.6), whence each of these points is extreme in any subset of $\widehat{\mathcal{T}}$ containing it). Clearly, this means that $(A_x,<_x)$ is a directed arc in the pretree $\mathcal{T}$. The proof now splits into two cases:

(i) the arc A_x is bounded in the pretree $\mathcal{T}$;

(ii) the arc A_x is unbounded in the pretree $\mathcal{T}$.

In the case (i), the arc A_x is an interval in the pretree $\mathcal{T}$, because we assume that $\mathcal{T}$ is weakly complete. Since $(A_x,<_x)$ has the smallest element x, and has no largest element, it follows that $A_x = [x,y\rangle$ for some $y \in \mathcal{T}$. However, in this case, y is a supremum of the directed arc $(A_x,<_x)$ in the pretree $\mathcal{T}$, and since $\mathcal{T}$ is convex in $\widehat{\mathcal{T}}$, it then follows that y is a supremum of the directed arc $(A_x,<_x)$ in the pretree $\widehat{\mathcal{T}}$ as well.

In the case (ii), by definition, A_x is a ray in $\mathcal{T}$ emanating from x. Let $\omega \in \mathrm{Ends}(\mathcal{T}) \subset \widehat{\mathcal{T}}$ be the end of $\mathcal{T}$ containing the ray A_x. Then $A_x = [x,\omega\rangle$ by Lemma 11.5, so that ω is a supremum of the directed arc $(A_x,<_x)$ in the pretree $\widehat{\mathcal{T}}$.

Thus, the two possible cases lead to a contradiction. Therefore, $\widehat{\mathcal{T}}$ is weakly complete.

Let us show that $\mathcal{T}$ is weakly complete whenever so is $\widehat{\mathcal{T}}$. Suppose L is a bounded arc in the pretree $\mathcal{T}$. Then L is a bounded arc in the pretree $\widehat{\mathcal{T}}$ either (because $\mathcal{T}$ is convex in $\widehat{\mathcal{T}}$). If $\widehat{\mathcal{T}}$ is weakly complete, then L is an interval in $\widehat{\mathcal{T}}$, so that there are $a,b \in \widehat{\mathcal{T}}$ such that $L \in \{[a,b],[a,b\rangle,\langle a,b\rangle\}$. In the case where either a or b is in $\mathrm{Ends}(\mathcal{T})$, the intervals $[a,b]$, $[a,b\rangle$, and $\langle a,b\rangle$ are unbounded arcs in the pretree $\mathcal{T}$ by Lemma 11.5.

Consequently, both a and b are in $\mathcal{T}$, so that the intervals $[a,b]$, $[a,b\rangle$, and $\langle a,b\rangle$ of the pretree $\widehat{\mathcal{T}}$ are also intervals of the pretree $\mathcal{T}$. The assertion is proved.

(3) If a subset $S \subset \mathcal{T}$ is quasidense in the pretree $\mathcal{T}$, then it is quasidense also in the pretree $\widehat{\mathcal{T}}$, i.e., for any points $a \neq b \in \widehat{\mathcal{T}}$, the interval $[a,b\rangle$ contains a point of S. Indeed, if both a and b are in $\mathcal{T}$, then $[a,b\rangle$ contains a point of S because S is quasidense in $\mathcal{T}$. If either a or b is in $\operatorname{Ends}(\mathcal{T})$, then Lemma 11.5 implies that $[a,b\rangle$ contains an unbounded arc of $\mathcal{T}$, while it is clear that each unbounded arc in $\mathcal{T}$ contains infinitely many points of S.

The fact proved above implies that the pretree $\widehat{\mathcal{T}}$ is quasiseparable whenever the pretree $\mathcal{T}$ is quasiseparable. The reverse implication is obvious. □

11.10. Corollary. *Let $\mathcal{T}$ be a pretree. Then the following conditions are equivalent*:

(i) *the pretree $\mathcal{T}$ is weakly complete*;
(ii) *the pretree $\mathcal{T} \cup \operatorname{Ends}(\mathcal{T})$ with the shadow topology is compact.*

Proof. If $\mathcal{T}$ is weakly complete, then so is $\mathcal{T} \cup \operatorname{Ends}(\mathcal{T})$ (Proposition 11.7). Next, $\mathcal{T} \cup \operatorname{Ends}(\mathcal{T})$ is bounded (Proposition 11.6). Consequently, $\mathcal{T} \cup \operatorname{Ends}(\mathcal{T})$ with the shadow topology is compact by Theorem 8.2.

Conversely, if $\mathcal{T} \cup \operatorname{Ends}(\mathcal{T})$ with the shadow topology is compact, then it is weakly complete by Theorem 8.2. Then $\mathcal{T}$ is weakly complete by Proposition 11.7. □

§12. Trees

In this section, we discuss the properties of the shadow topology on an ordinary $\mathbb{Z}$-tree and on the union of a $\mathbb{Z}$-tree with the set of its ends. (The shadow topology in this case was considered in [28, 18].)

In terms of the theory of pretrees, $\mathbb{Z}$-trees are the median pretrees (Definition 4.1) all of whose intervals are finite sets. The "equivalence" between this definition and the classical definitions of $\mathbb{Z}$-trees was proved, e.g., in [1, Lemma 29.1] and in [5, Lemma 3.34].

In what follows, we treat $\mathbb{Z}$-trees as a class of pretrees and apply to them the notions and constructions given above for the pretrees (such as intervals, rays, branches, completeness, etc.). A significant part of these notions are direct generalizations of the corresponding standard notions in the theory of trees.

Furthermore, hereinafter we use a series of concepts, not described above, from the classical theory of trees. Thus, the points of a $\mathbb{Z}$-tree are called *vertices*, the (unordered) pairs of adjacent vertices (i.e., vertices with no other vertices lying between them) are called *edges*. We recall that the *degree* of a vertex is the number of edges containing this vertex. A $\mathbb{Z}$-tree is said to be *locally finite* if all of its vertices have finite degrees. The standard integer-valued metric on a $\mathbb{Z}$-tree will be denoted by dist: if a and b are vertices in a $\mathbb{Z}$-tree, then $\operatorname{dist}(a,b)$ equals the number of edges contained in the interval $[a,b]$ (which is less by 1 than the number of points in that interval).

Recall that the *set of ends* $\operatorname{Ends}(T)$ of a $\mathbb{Z}$-tree T is defined as the set of equivalence classes of *cofinal* rays in T, where two rays are equivalent (cofinal) if their intersection is a ray (see §11, where the set of ends is described for the case of an arbitrary pretree). Theorems 7.3, 8.2, 9.1, and 10.2 yield the following result concerning the shadow topology on the pretree $T \cup \operatorname{Ends}(T)$ (see Definition 6.1).

12.1. Corollary. *If T is a $\mathbb{Z}$-tree, then the shadow topology on the pretree $\widehat{T} := T \cup \operatorname{Ends}(T)$ is normal, Hausdorff, compact, and sequentially compact. The shadow topology on $\widehat{T}$ is metrizable if and only if T is at most countable.*

Proof. Since all intervals in a $\mathbb{Z}$-tree are finite sets, Definition 4.12 shows that all $\mathbb{Z}$-trees are Dedekind complete (hence weakly complete by Lemma 4.13). All $\mathbb{Z}$-trees are median (see the definition). Therefore, since T is median and weakly complete, by Proposition 11.7 it follows that $\widehat{T}$ is also median and weakly complete. Moreover, $\widehat{T}$ is bounded (see Proposition 11.6). Since $\widehat{T}$ is median, Theorem 7.3 implies that the shadow topology of $\widehat{T}$ is normal and Hausdorff. The weak completeness and boundedness of $\widehat{T}$ imply by Theorem 8.2 that the shadow topology of $\widehat{T}$ is compact (cf. Corollary 11.10). The sequential compactness follows from Theorem 9.1.

As for the metrizability, since the shadow topology on $\widehat{T}$ is compact and Hausdorff, it is metrizable if and only if it is second-countable (see, e.g., [2, Chapter V, §2, Theorem 4]). Theorem 10.2 says that the shadow topology on a pretree is second-countable if and only if the pretree is quasiseparable (see Definition 10.1), while Proposition 11.7 says that $\widehat{T}$ is quasiseparable if and only if T is quasiseparable. Therefore, the shadow topology on $\widehat{T}$ is metrizable if and only if T is quasiseparable. By Definition 10.1, a pretree is quasiseparable if it has an at most countable quasidense subset. The definitions imply directly that the only quasidense subset of a $\mathbb{Z}$-tree is the set of all of its points (vertices). Consequently, the shadow topology on $\widehat{T}$ is metrizable if and only if T is at most countable. □

12.2. Definition. Strong topology**.** Let T be a $\mathbb{Z}$-tree. We define the *strong topology* on the pretree $T \cup \mathrm{Ends}(T)$ to be the smallest topology containing all of the branches of $T \cup \mathrm{Ends}(T)$ and all of the vertices of T. In other words, the strong topology is the smallest topology that contains the shadow topology and induces the discrete topology (i.e., the topology of the metric dist) on T.

The strong topology is usually regarded as the standard one for the union $T \cup \mathrm{Ends}(T)$. There are several known approaches to its description (see footnote 21). The following facts are not hard to prove.

12.3. Proposition. *Let T be a $\mathbb{Z}$-tree.*

I. *The shadow topology on $T \cup \mathrm{Ends}(T)$ is contained in the strong one and coincides with it if and only if T is locally finite.*
II. *The shadow and strong topologies coincide on* $\mathrm{Ends}(T)$*.*
III. *The strong topology on $T \cup \mathrm{Ends}(T)$ is metrizable* (*in particular, it is normal and Hausdorff*)*. It is compact if and only if T is locally finite.* (*Cf. Corollary* 12.1.)

12.4. *Remark.* We recall that a topological space is said to be *totally disconnected* if all its subsets containing more than one point are disconnected. For any $\mathbb{Z}$-tree T, the space $T \cup \mathrm{Ends}(T)$ is totally disconnected in both the strong and the shadow topology.

If each vertex in a locally finite $\mathbb{Z}$-tree T has degree at least 3, then the space of ends $\mathrm{Ends}(T)$ is a *Cantor space* (that is, a space homeomorphic to the Cantor set). (This follows from Brouwer's theorem, which states that any nonempty compact Hausdorff space without isolated points and having countable base consisting of clopen sets is a Cantor space.)

If each vertex in a countable tree T has infinite degree, then $T \cup \mathrm{Ends}(T)$ with the shadow topology is a Cantor space, while $\mathrm{Ends}(T)$ is homeomorphic to the *Baire space*[29] or, equivalently, to the set of all irrational numbers with the topology inherited from the real line.

The facts mentioned above show that if a $\mathbb{Z}$-tree T is countable, then both the shadow and strong topology on $T \cup \mathrm{Ends}(T)$ embeds in the real line.

[29]The *Baire space* is the set $\mathbb{N}^{\mathbb{N}}$ of all infinite sequences $(x_i)_{i\in\mathbb{N}}$ of positive integers, with a natural topology generated by the collections of the form

$$\{(x_i)_{i\in\mathbb{N}} : x_p = k\}, \quad p, k \in \mathbb{N}.$$

12.5. *Remark. Metrics on $\mathbb{Z}$-trees.* We present some (easily provable) facts concerning metrics on $\mathbb{Z}$-trees. Let T be a $\mathbb{Z}$-tree with the set of edges E, and let $f\colon E \to \mathbb{R}_+$ be an arbitrary real-valued positive function. We define the function $\operatorname{dist}_f\colon T^2 \to \mathbb{R}$ by setting

$$\operatorname{dist}_f(a,b) := \sum_{e\in E\,:\,e\subset[a,b]} f(e).$$

It is easy to check that dist_f is a metric on T and that this metric is compatible (in the sense of Definition 4.17) with the pretree structure on T. The standard metric dist on T coincides with the metric $\operatorname{dist}_{f_1}$, where $f_1(E) = \{1\}$.

If T is locally finite, then for any $f\colon E \to \mathbb{R}_+$, the metric dist_f induces the discrete topology on T. If T is not locally finite, then for certain functions $f\colon E \to \mathbb{R}_+$, the metric dist_f is not discrete. In order to construct a corresponding example, take a vertex $v \in T$ of infinite degree and consider a function $f\colon E \to \mathbb{R}_+$ such that

$$\inf\{f(e) \,:\, e \in E, v \in e\} = 0.$$

If T is countable, then so is E; in this case, there exist functions $f\colon E \to \mathbb{R}_+$ with $\sum_{e\in E} f(e) < \infty$ (it is natural to call such functions *summable*). If a function $f\colon E \to \mathbb{R}_+$ is summable, then the metric dist_f induces the shadow topology on T.

Let $f\colon E \to \mathbb{R}_+$ be an arbitrary positive function, and let $(T'_f, \operatorname{dist}'_f)$ be the metric completion of the space $(T, \operatorname{dist}_f)$. We denote by $\partial_f T$ the set $T'_f \setminus T$ (with the inherited metric). It is easily seen that for every function $f\colon E \to \mathbb{R}_+$ we have a natural canonical injection

$$\partial_f T \to \operatorname{Ends}(T) \tag{29}$$

and the corresponding map

$$T'_f \to T \cup \operatorname{Ends}(T). \tag{30}$$

The injection (29) is a topological embedding with respect to the standard[30] topology on $\operatorname{Ends}(T)$. It is easy to see that for every $\mathbb{Z}$-tree T, there exists a function $f\colon E \to \mathbb{R}_+$ such that the metric dist_f is bounded. (In particular, if T is countable and f is summable, then dist_f is bounded.) If dist_f is bounded, then the canonical injection (29) is a bijection and, moreover, a homeomorphism. If dist_f is bounded and induces the discrete topology, then the canonical bijection (30) is a homeomorphism with respect to the strong topology on $T \cup \operatorname{Ends}(T)$. If T is countable and $f\colon E \to \mathbb{R}_+$ is summable, then the bijection (30) is a homeomorphism with respect to the shadow topology on $T \cup \operatorname{Ends}(T)$.

12.6. *Remark.* In [8], D. I. Cartwright, P. M. Soardi, and W. Woess described compactifications for graphs that are not locally finite. In particular, they constructed a (metrizable) compactification $\widetilde{T}$ of a (countable and not locally finite) $\mathbb{Z}$-tree T by adding to $T \cup \operatorname{Ends}(T)$ the set T^* of *improper vertices* (the set T^* is a copy of the set $T^\infty \subset T$ of vertices of infinite degree).

It can be shown that the shadow topology on $T \cup \operatorname{Ends}(T)$ is the quotient topology of the Cartwright–Soardi–Woess compactification, with respect to the quotient map

$$T \cup \operatorname{Ends}(T) \cup T^* \to T \cup \operatorname{Ends}(T)$$

sending each improper vertex $x^* \in T^*$ to the corresponding vertex $x \in T^\infty$.

We remark that the Cartwright–Soardi–Woess compactification can be generalized to a much wider class of pretrees than that of $\mathbb{Z}$-trees, with preserving the described relationship with the shadow topology.

[30] Recall that the shadow and strong topologies induce one and the same topology on the set of ends; we refer to this topology as to the standard one.

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