# ABEL AND TAUBERIAN THEOREMS FOR INTEGRALS 

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#### Abstract

A new method is suggested for obtaining Abel and Tauberian Theorems for integrals of the form $\int_{0}^{\infty} K\left(\frac{t}{r}\right) d \mu(t)$. It is based on properties of limit sets for measures. Accordingly, a version of Azarin's cluster set theory for Radon measures on the half-line $(0, \infty)$ is created. Theorems of new sort are proved, in which the asymptotic behavior of the above integrals is described in terms of cluster sets for $\mu$. With the use of these results and a stronger version (also proved in the paper) of Karleman's well-known analytic continuation lemma, the second Tauberian theorem by Wiener is refined considerably.


## Contents

§1. Preface
§2. On proximate orders
§3. Measures; limit sets of measures
§4. Abel theorems for integrals
§5. Tauberian theorems for integrals
References

## §1. Preface

Since a proximate order occurs in the most part of the statements presented here, we begin with some notation and relevant facts.

Proximate orders $\rho(r)$ play an important part in the theory of Abel and Tauberian theorems, in the growth theory for subharmonic functions, and in probability theory. If $\rho(r)$ is a proximate order, then the function $V(r)=r^{\rho(r)}$ is a Karamata function of slow regular variation. Some properties of a proximate order were discussed in [1]. The more resent book [2] is an encyclopaedic treatise on regularly varying functions and their applications.

Before the formal definition of a proximate order, the reader may assume that $V(r)=$ $r^{\rho} a(r)$,

$$
a(r)= \begin{cases}\ln ^{\alpha}(e r), & r \geq 1 \\ \ln ^{\alpha} \frac{e}{r}, & r \in(0,1),\end{cases}
$$

where $\rho$ and $\alpha$ are arbitrary real numbers.
Let $\rho=\rho(\infty)=\lim _{r \rightarrow \infty} \rho(r)$. Putting

$$
\begin{equation*}
\gamma(t)=\sup _{r>0} \frac{V(r t)}{t^{\rho} V(r)} \tag{1.1}
\end{equation*}
$$

we clearly have $V(r t) \leq t^{\rho} \gamma(t) V(r)$ for $r>0$ and $t>0$.

[^0]In Theorem 9 (in the Preface we use the same enumeration of statements as in the main text) it will be proved that $\gamma(t)$ is a continuous function with

$$
\lim _{t \rightarrow \infty} \frac{\ln \gamma(t)}{\ln t}=0, \quad \lim _{t \rightarrow \infty} \frac{\ln \gamma\left(\frac{1}{t}\right)}{\ln t}=0 .
$$

This theorem is a slight refinement of Potter's result in [3] and lightens its statement. It is an important tool for handling proximate orders. In what follows, we do not denote any other functions by $\gamma(t)$.

Now, we define two classes of Radon measures on $(0, \infty)$. We shall mainly deal with measures of these classes here.

Let $\mathfrak{M}_{\infty}(\rho(r))$ denote the class of Radon measures on $(0, \infty)$ that obey the inequality

$$
\sup _{r \geq 1} \frac{|\mu|([r, e r])}{V(r)}<\infty .
$$

Next, $\mathfrak{M}(\rho(r))$ will denote the class of Radon measures on $(0, \infty)$ that obey the inequality

$$
\sup _{r>0} \frac{|\mu|([r, e r])}{V(r)}<\infty .
$$

If

$$
\limsup _{r \rightarrow \infty} \frac{|\mu|([r, e r])}{V(r)} \in(0, \infty)
$$

then the proximate order $\rho(r)$ will be called a proximate order for the measure $\mu$, and the number $\rho=\rho(\infty)$ will be called the order of $\mu$. Thus, Radon measures of an arbitrary real order can be considered.

In accordance with Azarin's ideas, the cluster set $\operatorname{Fr}[\mu]=\operatorname{Fr}[\rho(r), \mu]$ of a Radon measure is defined to be the set of all measures $\nu$ of the form $\nu=\lim _{n \rightarrow \infty} \mu_{t_{n}}$, where $t_{n} \rightarrow \infty$ and $\mu_{t}$ is defined by the formula

$$
\mu_{t}(E)=\frac{\mu(t E)}{V(t)} .
$$

The relation $\nu=\lim _{n \rightarrow \infty} \mu_{t_{n}}$ means that the sequence $\mu_{t_{n}}$ converges coarsely to $\nu$. The coarse convergence will be defined in $\S 3$.

It can be shown that if $\mu \in \mathfrak{M}_{\infty}(\rho(r))$, then the set $\operatorname{Fr}[\mu]$ possesses the properties listed in the corresponding theorem by Azarin, see [4]. These properties are also listed in Theorem 20 of the present paper. Azarin only considered positive measures on $\mathbb{R}^{n}$; he defined a proximate order of a measure in a somewhat different way, applicable only when $\rho>0$. In this case, our definition of a proximate order is equivalent to Azarin's definition. It should also be noted that Azarin considered a convergence of measures different from that used here.

A measure $\mu$ is said to be (Azarin) regular if the cluster set $\operatorname{Fr}[\mu]$ consists of a unique measure $\nu$. In this case, necessarily $d \nu(t)=c t^{\rho-1} d t$ (Theorem [27).

The following notation will be adopted throughout the paper.

1. The function $\Psi(r)$ is defined by

$$
\begin{equation*}
\Psi(r)=\int_{0}^{\infty} K\left(\frac{t}{r}\right) d \mu(t) . \tag{1.2}
\end{equation*}
$$

Sometimes a more informative notation $\Psi(K, r)$ will be used in place of $\Psi(r)$.
2. $J(r)=\frac{1}{V(r)} \Psi(r)$.
3. The measure $s$ is defined by $d s(t)=\Psi(t) d t$.

The symbol $\mu(t)$ will denote the distribution function of a measure $\mu$, so that $\mu((a, b])=$ $\mu(b)-\mu(a)$.

The cluster set of a function $f$ in the direction $r \rightarrow \infty$ (i.e., the set of limits $\lim _{n \rightarrow \infty} f\left(r_{n}\right)$ as $\left.r_{n} \rightarrow \infty\right)$ will be denoted by $L(f, \infty)$.

A function $f(r)$ is said to be compactly supported on $(0, \infty)$ if $\operatorname{supp} f \subset[a, b] \subset(0, \infty)$.
The properties of $\operatorname{Fr}[\mu]$ have permitted us to prove several new theorems of Abel and Tauberian type for integrals of the form (1.2).

Theorems of Abel type are those describing properties of $\Psi$ if $\mu$ is given.
Theorems of Tauberian type are those describing properties of $\mu$ on the basis of known properties of $\Psi$.

Many well-known theorems of Abel type claim that $\Psi(r) \sim B V(r)$ whenever $\mu^{\prime}(t) \sim$ $A \frac{V(t)}{t}(t \rightarrow \infty)$. See, for instance, the books [2, 5, 6, 7].

We state a simplest Abel type theorem of the present paper.
Theorem 31. Let $\mu \in \mathfrak{M}_{\infty}(\rho(r))$, and let $K$ be a continuous compactly supported kernel on $(0, \infty)$. Then

$$
L(J, \infty)=\left\{\int_{0}^{\infty} K(t) d \nu(t): \nu \in \operatorname{Fr}[\mu]\right\} .
$$

Other Abel type theorems obtained here are analogs of Theorem 31, which are proved under various restrictions on $K$ and $\mu$. Sometimes we lift the requirement that $K$ be compactly supported. In other, more complicated cases, we also lift the requirement that $K$ be continuous. In the general case, $K$ is a Borel function on $(0, \infty)$.

An important distinction of the above results from the results known before should be mentioned. In the latter, the case of a regular measure $\mu$ was treated. In our results, a much wider class of measures is studied. In particular, this is $\mathfrak{M}_{\infty}(\rho(r))$ in Theorem 31,

Without the assumption of continuity for $K$, the cluster set $\operatorname{Fr}[\mu]$ with an arbitrary Radon measure $\mu$ does not determine the set $L(J, \infty)$ any longer, as can be seen from Theorem 32,

An important new result is the statement that, for discontinuous $K$, the cluster set $\operatorname{Fr}[\mu]$ determines the set $\operatorname{Fr}[s]$ uniquely. The involvement of the cluster set $\operatorname{Fr}[s]$ for the measure $s$ should be treated as an achievement of the paper. The next statement is among our principal results. Recall that $\gamma(t)$ is defined by (1.1).
Theorem 38. Let $\rho(r)$ be an arbitrary proximate order, and let $\mu \in \mathfrak{M}(\rho(r))$. Suppose that $K$ is a Borel function on $(0, \infty)$ such that $t^{\rho-1} \gamma(t) K(t) \in L_{1}(0, \infty)$, and let $\Psi$ be defined by (1.2). Then the measure $s$, $d s(t)=\Psi(t) d t$, belongs to $\mathfrak{M}(\rho(r)+1)$, and its cluster set $\operatorname{Fr}[\rho(r)+1, s]$ consists of absolutely continuous measures whose densities constitute the set

$$
\left\{\int_{0}^{\infty} K\left(\frac{t}{u}\right) d \nu(t): \nu \in \operatorname{Fr}[\mu]\right\} .
$$

In the case where our results yield $L(J, \infty)=\{0\}$, the question about the order of growth of $\Psi(r)$ at infinity remains open.

Consider the case where the kernel $K$ is infinitely differentiable on $(0, \infty)$ and compactly supported. Then, along with (1.2), we have the following formulas for $\Psi(r)$ :

$$
\begin{equation*}
(-1)^{n+1} r^{n+1} \Psi(r)=\int_{0}^{\infty} K^{(n+1)}\left(\frac{t}{r}\right) F_{n}(t) d t, \quad n=0,1, \ldots, \tag{1.3}
\end{equation*}
$$

where $F_{0}(t)=\mu(t), F_{n+1}^{\prime}(t)=F_{n}(t)$.
The question arises as to whether Theorem 31) and formulas (1.3) allow us to determine the order of growth of $\Psi(r)$ at infinity. This question will be treated in $\S 4$. The answer is as follows. Often, the order can be determined, but there are various exceptional cases in which the question lies beyond the scope of the present paper.

Now, we turn to Tauberian theorems. We recall the important Tauberian theorems proved by Wiener.

Theorem 1. Suppose $F(x) \in L_{1}(-\infty, \infty)$ with $\int_{-\infty}^{\infty} F(x) e^{-i \lambda x} d x \neq 0$ for $\lambda \in(-\infty, \infty)$, and let $g(x)$ be a bounded measurable function on the real axis. Let

$$
\lim _{x \rightarrow+\infty} \int_{-\infty}^{\infty} F(x-y) g(y) d y=A \int_{-\infty}^{\infty} F(y) d y
$$

Then for every $H \in L_{1}(-\infty, \infty)$ we have

$$
\lim _{x \rightarrow+\infty} \int_{-\infty}^{\infty} H(x-y) g(y) d y=A \int_{-\infty}^{\infty} H(y) d y
$$

Denote by $M$ the space of continuous functions $F$ on $(-\infty, \infty)$ with the mixed norm

$$
\|F\|_{M}=\sum_{n=-\infty}^{\infty} \max \{|F(x)|: x \in[n, n+1]\} .
$$

The next statement is often called Wiener's second Tauberian theorem.
Theorem 2. Suppose that $F \in M$ and $\int_{-\infty}^{\infty} F(x) e^{-i \lambda x} d x \neq 0$ for $\lambda \in(-\infty, \infty)$. Let $\nu$ be a Radon measure on the real axis such that $|\nu|([n, n+1]) \leq B$ for every $n$, with $B$ independent of $n$. Let

$$
\lim _{x \rightarrow+\infty} \int_{-\infty}^{\infty} F(x-y) d \nu(y)=A \int_{-\infty}^{\infty} F(y) d y
$$

Then for every $H \in M$ we have

$$
\lim _{x \rightarrow+\infty} \int_{-\infty}^{\infty} H(x-y) d \nu(y)=A \int_{-\infty}^{\infty} H(y) d y
$$

It will be convenient for us to deal with the multiplicative version of Theorem 2 resulting from it by the change of variables $y=\ln t$ in the integrals. We use also the notation $x=\ln r, K(t)=\frac{1}{t} F(-\ln t), d \mu(t)=t d \nu(\ln t)$.

Let $M_{1}$ denote he space of function $K$ continuous on $(0, \infty)$ and such that the series $\sum_{n=-\infty}^{\infty} K_{n} e^{n}$ converges, where $K_{n}=\max \left\{|K(x)|: x \in\left[e^{n}, e^{n+1}\right]\right\}$.
Theorem 3. Suppose that $K(t) \in M_{1}$ satisfies $\int_{0}^{\infty} K(t) t^{i \lambda} d t \neq 0, \lambda \in(-\infty, \infty)$. Let $\mu$ be a Radon measure on $(0, \infty)$ such that $|\mu|\left(\left[e^{n}, e^{n+1}\right]\right) \leq B e^{n}$ for all integers $n$, with $B$ independent of $n$. Suppose that

$$
\lim _{r \rightarrow \infty} \frac{1}{r} \int_{0}^{\infty} K\left(\frac{t}{r}\right) d \mu(t)=A \int_{0}^{\infty} K(t) d t
$$

Then for every $Q \in M_{1}$ we have

$$
\lim _{r \rightarrow \infty} \frac{1}{r} \int_{0}^{\infty} Q\left(\frac{t}{r}\right) d \mu(t)=A \int_{0}^{\infty} Q(t) d t
$$

The wording of Theorem 3 inherits Wiener's wording of Tauberian theorems. However, Theorem 3 is equivalent to a statement in a more standard form specific for Tauberian theorems.
Theorem 4. Suppose that $K(t) \in M_{1}$ satisfies $\int_{0}^{\infty} K(t) t^{i \lambda} d t \neq 0$ for $\lambda \in(-\infty, \infty)$. Let $\mu$ be a Radon measure on $(0, \infty)$ such that $|\mu|\left(\left[e^{n}, e^{n+1}\right]\right) \leq B e^{n}$ for all integers $n$, with $B$ independent of $n$. Suppose also that the following limit exists:

$$
\lim _{r \rightarrow \infty} \frac{1}{r} \int_{0}^{\infty} K\left(\frac{t}{r}\right) d \mu(t)=c
$$

Then the cluster set $\operatorname{Fr}[\mu]$ consists of a unique measure $\nu$, specifically, $d \nu(x)=\frac{c}{c_{1}} d x$, $c_{1}=\int_{0}^{\infty} K(t) d t$.

Theorem 4 is a trivial consequence of Theorems 3 (it suffices to look at the precise definition of the cluster set in the present paper, namely, at the explanation of the meaning of the relation $\left.\mu_{t_{n}} \rightarrow \nu\right)$. At the same time, Theorem 3 is a consequence of Theorem 4 and the corresponding Abel-type theorem.

Now, we analyze a Tauberian theorem proved in this paper.
Theorem 52. Let $\rho(r)$ be an arbitrary proximate order. Suppose that $\mu$ is a Radon measure on $(0, \infty)$ belonging to $\mathfrak{M}(\rho(r))$ and $K$ a Borel function on $(0, \infty)$ with $t^{\rho-1} \gamma(t) K(t)$ $\in L_{1}(0, \infty)$. Suppose also that the function $\int_{0}^{\infty} K(t) t^{\rho-1+i \lambda} d t$ does not vanish on the real axis. Let $\Psi(r)$ be defined by (1.2). If the measure $s, d s(t)=\Psi(t) d t$, is regular with respect to the proximate order $\rho(r)+1$, then $\mu$ is regular with respect to $\rho(r)$. Moreover, if $\operatorname{Fr}[s]$ consists of only one measure whose density is $c^{\rho}$, then $\operatorname{Fr}[\mu]$ reduces to the measure whose density is $\frac{c}{c_{1}} t^{\rho-1}$, where $c_{1}=\int_{0}^{\infty} K(t) t^{\rho-1} d t$.

1. Theorem 4 is about the case of $\rho(r) \equiv 1$, Theorem 52 is about a general proximate order. However, even the partial case of Theorem 52 with $\rho(r) \equiv 1$ is much stronger than Theorem 4. We comment on this case in detail.
2. The assumptions about $K$ are relaxed considerably. We permit discontinuities and local unboundedness of $K$. In Theorem 4 it was required that $K$ be continuous and the series $\sum_{n=-\infty}^{\infty} K_{n} e^{n}$ converge. In the partial case in question of Theorem 52, the last requirement is relaxed to $K(t) \in L_{1}(0, \infty)$.
3. The requirement that the limit $\lim _{r \rightarrow \infty} \frac{1}{r} \Psi(r)$ should exist is replaced in the version in quastion of Theorem 52 by, roughly speaking, the requirement that the limit

$$
\lim _{r \rightarrow \infty} \frac{1}{r^{2}} \int_{1}^{r} \Psi(t) d t
$$

should exist (see Theorem 29 and 30 for precise statements).
However, the conclusions of Theorem 4 and of the version in question of Theorem 52 are the same.

Also, Theorem 52 is a refinement of the result by Bingham, Goldie, and Teugels (see [2, Subsection 4.9, Theorem 4.9.1]; it was also quoted in [7. Chapter 4, Theorem 9.3]). Like Theorem 52, the theorem of Bingham, Goldie, and Teugels treats an arbitrary proximate order. But, unlike Theorem 52, it treats only positive measures and requires the finiteness of the mixed norm for $K$.

We say a few words about the proof of Theorem 52 and the proof of the uniqueness of a solution of the integral equation

$$
\int_{0}^{\infty} K\left(\frac{t}{r}\right) d \nu(t)=c r^{\rho}
$$

with an unknown measure $\nu$. This is done by the Carleman method, but we must improve Carleman's analytic continuation lemma.

Also, a version of Theorem 52 will be proved where the function $\int_{0}^{\infty} K(t) t^{\rho-1+i \lambda} d t$ is permitted to vanish on a finite set on the real axis.

Resorting to some repetition, we emphasize once again the following. We consider integrals of the form

$$
\int_{0}^{\infty} K\left(\frac{t}{r}\right) f(t) d t, \quad \int_{0}^{\infty} K\left(\frac{t}{r}\right) d \mu(t)
$$

Such integrals occur in the theory of growth of entire and subharmonic functions, in probability theory, in the proofs Abel-type and Tauberian theorems, in operator theory.

In the majority of the preceding Abel-type theorems for such integrals, it was assumed that $f(t) \sim \frac{c}{t} V(t), \mu(t) \sim C V(t)(t \rightarrow \infty)$. We prove Abel-type theorems under the basic assumption that $\mu \in \mathfrak{M}_{\infty}(\rho(r))$.

In the previous results of Abel and Tauberian type for the integrals $\int_{0}^{\infty} K\left(\frac{t}{r}\right) d \mu(t)$, the kernel $K$ was subject to fairly strong restrictions. For instance, in Theorems 4.4.2 and 4.9.1 in [2] (which are most close to our results), it was required that $K$ be continuous and

$$
\sum_{n=-\infty}^{\infty} \min \left(e^{-\sigma n}, e^{-\tau n}\right) \max _{\left[e^{n}, e^{n+1}\right]}|K(t)|<\infty,
$$

where $\sigma$ and $\tau$ satisfy $\sigma<\rho<\tau$ and $\rho$ is the order of $\mu$. We relax this requirement to $K(t) t^{\rho-1} \gamma(t) \in L_{1}(0, \infty)$. In their comments to Theorem 4.9.1 (Subsection. 4.9, item 2), the authors of [2] wrote: "As to the continuity of $K$, no clear general method of lifting this condition is apparent".

In the majority of our proofs, we use the properties of cluster sets for measures. Earlier, similar properties were used in the theory of growth for subharmonic functions.

## §2. On PROXIMATE ORDERS

Let $f(r)$ be a positive function on $(0, \infty)$. Suppose we want to describe its asymptotic behavior at infinity. An important numerical characteristic of $f$ is its order $\rho$ defined by the formula

$$
\rho=\limsup _{r \rightarrow \infty} \frac{\ln f(r)}{\ln r}
$$

In general, $\rho$ is an element of the extended real line $[-\infty, \infty]$. The relation $\rho \in(-\infty, \infty)$ distinguishes an important class of functions called the functions of finite order. In the sequel, we consider the functions of finite order only.

If $\rho$ is the order of $f$ and $\varepsilon$ is an arbitrary strictly positive number, then

$$
\begin{array}{ll}
f(r)<r^{\rho+\varepsilon}, & r \geq R(\varepsilon), \\
f(r)>r^{\rho-\varepsilon}, & r \in E, \tag{2.4}
\end{array}
$$

where $E$ is a certain unbounded set depending on $\varepsilon$ and $f$. If, in a specific problem, inequalities (2.4) are too rough, finer growth characteristics should be introduced.

The type of $f$ at the order $\rho$ is the quantity

$$
\sigma=\limsup _{r \rightarrow \infty} \frac{f(r)}{r^{\rho}}
$$

The example of $f(r)=A r^{\rho}(\ln (e+r))^{\beta}, A>0, \beta \in(-\infty, \infty)$, shows that, for functions of order $\rho$, the quantity $\sigma$ can be an arbitrary element of the set $[0, \infty]$.

The functions $f$ is said to be of minimal, normal, or maximal type at the order $\rho$ if, respectively, $\sigma=0, \sigma \in(0, \infty)$, and $\sigma=\infty$. If $\sigma<\infty$, then for every $\varepsilon>0$ we have

$$
\begin{array}{ll}
f(r)<(\sigma+\varepsilon) r^{\rho}, & r \geq R(\varepsilon), \\
f(r)>(\sigma-\varepsilon) r^{\rho}, & r \in E, \tag{2.5}
\end{array}
$$

where $E$ is a certain unbounded set depending on $f$ and $\varepsilon$. Inequalities (2.5) are much sharper than (2.4).

We say that a function $f(r)$ grows at infinity as $\varphi(r)$ if there are two constants $a$ and $b$, $0<a<b$, with

$$
\begin{array}{ll}
f(r)<b \varphi(r), & r>R, \\
f(r)>a \varphi(r), & r \in E,
\end{array}
$$

where $E$ is an unbounded set. If $f(r)$ is of minimal type at the order $\rho$, then inequalities (2.5) show that $f(r)$ grows at infinity as $r^{\rho}$.

The following problem, to be called Problem A, arises naturally in connection with the said above. Indicate a class $\mathfrak{A}$ that consists of fairly simple functions resembling $r^{\rho}$
and has the property that for every $f$ of finite order there exists a function $\varphi(r)$ in $\mathfrak{A}$ such that $f(r)$ grows at infinity as $\varphi(r)$.

We already know that the class consisting of only one function $r^{\rho}$ does not fit. Let $\ln _{k} r$ be the $k$ th iteration of the logarithm (for example, $\ln _{2} r=\ln \ln r$ ). We denote by $e_{k}$ the sequence defined by $e_{1}=e, e_{k+1}=e^{e_{k}}$. It can be shown that the class consisting of the functions

$$
\begin{equation*}
\varphi(r)=r^{\rho}(\ln (r+e))^{\alpha_{1}} \ldots\left(\ln _{k}\left(r+e_{k}\right)\right)^{\alpha_{k}}, \tag{2.6}
\end{equation*}
$$

where $k$ is an arbitrary positive integer, also does not fit. Clearly, the problem stated above is nontrivial. A way to its solution was indicated by Valiron, see 8 .

A locally absolutely continuous function $\rho(r)$ on $(0, \infty)$ is called a proximate order (in the sense of Valiron) if

1) $\lim _{r \rightarrow+\infty} \rho(r)=\rho(\infty)=\rho \in(-\infty, \infty)$;
2) $\lim _{r \rightarrow+\infty} r \ln r \rho^{\prime}(r)=0$.

Note that $\rho^{\prime}(r)$ is understood as a derivation number of maximal absolute value.
We shall use the following property of a proximate order (see, e.g., 1] Chapter 1, §12]).

Theorem 5. Let $\rho(r)$ be an arbitrary proximate order, and let $\rho=\rho(\infty)$. Then for every $t>0$ the following limit exists:

$$
\lim _{r \rightarrow \infty} \frac{V(t r)}{V(r)}=t^{\rho} .
$$

Moreover, the convergence is uniform on every segment $[a, b] \subset(0, \infty)$.
A proximate order $\rho(r)$ is called a proximate order of a function $f(r)$ if

$$
\limsup _{r \rightarrow \infty} \frac{f(r)}{V(r)}=\sigma \in(0, \infty)
$$

This relation is equivalent to the statement that $f(r)$ grows at infinity as $V(r)$. Note also that if $\rho(r)$ is a proximate order for $f(r)$, then $\rho=\rho(\infty)$ is the order of $f(r)$.

The notion of a proximate order is important because the class $\mathfrak{A}$ consisting of the functions $V(r)=r^{\rho(r)}$ is a solution of Problem $A$ stated above. This is a consequence of the following statement.

Theorem 6. Let $f(r)$ be a function of finite order $\rho$. Then there exists a proximate order $\rho(r)$ such that the following conditions are satisfied:

1) $\lim _{r \rightarrow \infty} \rho(r)=\rho$;
2) $\rho(r)$ is a monotonic function on $[1, \infty)$;
3) we have

$$
(r+e) \ln (r+e)\left|\rho^{\prime}(r)\right| \leq|\rho(r)-\rho|, \quad r \geq 1 ;
$$

4) we have

$$
\limsup _{r \rightarrow \infty} \frac{f(r)}{V(r)}=\sigma \in(0, \infty)
$$

The proof of this theorem can be found in [9. However, in 9] the continuity of $f(r)$ was assumed. We are going to show that this additional requirement is inessential.

Let $f(r)$ be an arbitrary function of order $\rho$. There is no loss of generality in assuming that $f(r)$ is bounded on any segment $[0, N]$. Taking an integer $n \geq 0$, we denote $m_{n}=$ $\inf \{f(x): x \in[n, n+1]\}, M_{n}=\sup \{f(x): x \in[n, n+1]\}, \alpha_{n}=n+\frac{1}{3}, \beta_{n}=n+\frac{2}{3}$. We construct a function $f_{1}(r)$ in the following way. It is linear on every segment $\left[n, \alpha_{n}\right]$, $\left[\alpha_{n}, \beta_{n}\right],\left[\beta_{n}, n+1\right]$ and $f_{1}(n)=f(n), f_{1}\left(\alpha_{n}\right)=m_{n}, f_{1}\left(\beta_{n}\right)=M_{n}$. Then $f_{1}(r)$ is continuous on $[0, \infty)$. Clearly, every proximate order for $f_{1}(r)$ is also a proximate order for $f(r)$.

We note that the function $\rho(r)$ whose existence is claimed in Theorem 6 possesses some additional properties that may be absent for arbitrary proximate orders. First, this is the condition for $\rho(r)$ to be monotonic, and second, condition 3) in the theorem is a stronger restriction on $\rho(r)$ than the requirement $\lim _{r \rightarrow+\infty} r \ln r \rho^{\prime}(r)=0$ in the definition of a proximate order. For this reason, Theorem 6 does not follow from similar statements in [1, 2].

We also note the relationship of proximate orders with regularly varying functions in the sense of Karamata.

A positive function $f$ on $(0, \infty)$ is said to be regularly varying in the sense of Karamata if for every $\lambda>0$ the limit

$$
\lim _{r \rightarrow \infty} \frac{f(\lambda r)}{f(r)}
$$

exists and is finite. We have the following statement.
Theorem 7. If $f(r)$ is a measurable function regularly varying in the sense of Karamata, then there exists a function $C(r) \rightarrow 1($ as $r \rightarrow \infty)$ and a proximate order $\rho(r)$ with $f(r)=C(r) V(r)$.

This is a well-known theorem about representation of regularly varying functions (see, e.g., [2, Theorem 1.3.1]).

A proximate order $\rho(r)$ is said to be a zero proximate order if $\lim _{r \rightarrow \infty} \rho(r)=0$.
If $\rho(r)$ is an arbitrary proximate order, then $\rho(r)=\rho+\widehat{\rho}(r)$, where $\rho=\lim _{r \rightarrow \infty} \rho(r)$ and $\widehat{\rho}(r)$ is a zero proximate order.

The introduction of proximate orders is aimed at the possibility for every function $f(r)$ of finite order to find a function $V(r)=r^{\rho(r)}$ such that $f(r)$ grows at infinity as $V(r)$. In this setting, the behavior of $\rho(r)$ near zero plays no role. However, specific problems of various origins often lead to integrals of the form $\int_{0}^{\infty} K(t, r) V(t) d t$. In the study of such integrals, the behavior of $\rho(t)$ near zero is as important as its behavior near infinity. Thus, in this paper we shall assume in addition to the above that any zero proximate order $\rho(t)$ satisfies $\rho\left(\frac{1}{r}\right)=-\rho(r)$, which is equivalent to the relation $V\left(\frac{1}{r}\right)=V(r)$. For instance, the function $V(r)=1+|\ln r|^{\alpha}, \alpha>1$, is such. The corresponding proximate order is given by the formula

$$
\rho(r)=\frac{\ln \left(1+|\ln r|^{\alpha}\right)}{\ln r} .
$$

Note that for $\alpha \leq 1$ the function given by this formula is not a proximate order because, by definition, a proximate order $\rho(r)$ must be absolutely continuous on $(0, \infty)$.

We also note that the relation $\rho\left(\frac{1}{r}\right)=-\rho(r)$ singles out the point 1 among other points on $(0, \infty)$. In particular, we see that $\rho(1)=0$.

In the study of a proximate order, along with $\rho(r)$ it is useful to employ the function $\eta(r)=\rho(r)+r \ln r \rho^{\prime}(r)$. We have

$$
\begin{equation*}
V(r)=\exp \left(\int_{1}^{r} \frac{\eta(t)}{t} d t\right) \tag{2.7}
\end{equation*}
$$

The proof of this relation reduces to taking the logarithms of the two sides followed by differentiation.

If $\rho(r)$ is zero proximate order, then $\eta(r) \rightarrow 0$ as $r \rightarrow \infty$. If, moreover, $\rho\left(\frac{1}{r}\right)=-\rho(r)$, then $\eta\left(\frac{1}{r}\right)=-\eta(r)$ and, by (2.7), it easily follows that

$$
\begin{equation*}
V(r) \leq M_{\varepsilon}\left(r^{\varepsilon}+r^{-\varepsilon}\right) \tag{2.8}
\end{equation*}
$$

on $(0, \infty)$ for every $\varepsilon>0$. Though rough, this inequality turns out to be useful sometimes.
If $\eta(t)$ is a locally integrable function on $[1, \infty)$ tending to zero at infinity, then there exists an infinitely differentiable function $\eta_{1}(t)$ on $[1, \infty)$ tending to zero at infinity
and such that the integral $\int_{1}^{\infty} \frac{\left|\eta(t)-\eta_{1}(t)\right|}{t} d t$ converges. Moreover, we can require that $\int_{1}^{\infty} \frac{\eta(t)-\eta_{1}(t)}{t} d t=0$. Then the function $V(r)$ defined by (2.7) satisfies

$$
V(r)=C(r) V_{1}(r)
$$

where

$$
V_{1}(r)=\exp \left(\int_{1}^{r} \frac{\eta_{1}(t)}{t} d t\right), \quad C(r)=\exp \left(\int_{r}^{\infty} \frac{\eta_{1}(t)-\eta(t)}{t} d t\right)
$$

These comments imply the following statement.
Theorem 8. Let $\rho(r)$ be an arbitrary zero proximate order such that $\rho\left(\frac{1}{r}\right)=-\rho(r)$. Then

$$
\begin{equation*}
V(r)=C(r) V_{1}(r) \tag{2.9}
\end{equation*}
$$

where $V_{1}(r)=r^{\rho_{1}(r)}, \rho_{1}(r)$ is a zero proximate order infinitely differentiable on $(0, \infty) \backslash\{1\}$ and satisfying $\rho_{1}\left(\frac{1}{r}\right)=-\rho_{1}(r)$, and $C(r)$ is a function continuous on $(0, \infty)$ and such that $C(r) \rightarrow 1$ as $r \rightarrow \infty$ or $r \rightarrow 0$. Moreover, the infinitesimal order of $C(r)-1$ as $r \rightarrow \infty$ and as $r \rightarrow 0$ can be prescribed arbitrarily.

Next, we shall explore the function $\gamma(t)$ mentioned in the Preface. The following lemma contains some easy properties of $\gamma(t)$.
Lemma 1. Let $\rho(r)$ be a zero proximate order satisfying $\rho\left(\frac{1}{r}\right)=-\rho(r)$, and let

$$
\gamma(t)=\gamma(\rho(\cdot), t)=\sup _{r>0} \frac{V(r t)}{V(r)}, \quad \underline{\gamma}(t)=\underline{\gamma}(\rho(\cdot), t)=\inf _{r>0} \frac{V(r t)}{V(r)} .
$$

Then the following statements hold true.

1) $\gamma(t), \underline{\gamma}(t) \in(0, \infty)$;
2) $\underline{\gamma}(t) \leq \gamma(t), \underline{\gamma}(1)=\gamma(1)=1$;
3) $\bar{\gamma}\left(\frac{1}{t}\right)=\frac{1}{\underline{\gamma}(t)}, \gamma\left(\frac{1}{t}\right)=\gamma\left(\rho(\cdot), \frac{1}{t}\right)=\gamma(-\rho(\cdot), t)$;
4) $\gamma\left(t_{1} t_{2}\right) \leq \gamma\left(t_{1}\right) \gamma\left(t_{2}\right), \underline{\gamma}\left(t_{1} t_{2}\right) \geq \underline{\gamma}\left(t_{1}\right) \underline{\gamma}\left(t_{2}\right)$;
5) $\gamma(t) \geq V(t), \underline{\gamma}(t) \leq \overline{V( }(t)$;
6) the functions $\gamma(t)$ and $\underline{\gamma}(t)$ are continuous on $(0, \infty)$.

Proof. Statement 2) is obvious. Next, we have

$$
\begin{aligned}
& \gamma\left(\frac{1}{t}\right)=\sup _{r>0} \frac{V\left(\frac{r}{t}\right)}{V(r)}=\sup _{R>0} \frac{V(R)}{V(t R)}=\frac{1}{\inf _{R>0} \frac{V(t R)}{V(R)}}=\frac{1}{\underline{\gamma}(t)} \\
& \gamma\left(\frac{1}{t}\right)=\sup _{R>0} \frac{V(R)}{V(t R)}=\sup _{R>0} \frac{R^{\rho(R)}}{(t R)^{\rho(t R)}}=\sup _{R>0} \frac{(t R)^{-\rho(t R)}}{R^{-\rho(R)}}=\gamma(-\rho(\cdot), t) .
\end{aligned}
$$

This proves 3 ).
Since $V(1)=1$, property 5 ) follows.
Since $\lim _{r \rightarrow \infty} \frac{V(r t)}{V(r)}=1$ and $\lim _{r \rightarrow 0} \frac{V(r t)}{V(r)}=1$, there exist $r_{1}$ and $r_{2}, 0<r_{1}<r_{2}$, such that $\frac{V(r t)}{V(r)} \leq 2$ for $r \in\left(0, r_{1}\right) \cup\left(r_{2}, \infty\right)$. Since the function $\frac{V(r t)}{V(r)}$ is continuous for $r \in\left[r_{1}, r_{2}\right]$, there exists $M>0$ such that $\frac{V(r t)}{V(r)} \leq M$ for $r \in\left[r_{1}, r_{2}\right]$. If follows that $\gamma(t) \leq \max (M, 2)$. Together with the inequality $\gamma(t) \geq V(t)$, this yields $\gamma(t) \in(0, \infty)$. Next, $\underline{\gamma}(t) \in(0, \infty)$ by 3$)$. This proves statement 1$)$.

Statement 4) is obvious.
Put

$$
\gamma(r, t)=\frac{V(t r)}{V(r)}
$$

Let $[a, b]$ be an arbitrary segment in $(0, \infty)$, and let $\varepsilon$ be an arbitrary strictly positive number. Since, by Theorem 5 the limits

$$
\lim _{r \rightarrow \infty} \frac{V(r t)}{V(r)}=1, \quad \lim _{r \rightarrow 0} \frac{V(r t)}{V(r)}=1
$$

are attained uniformly on $[a, b]$, we can find $r_{3}$ and $r_{4}, 0<r_{3}<r_{4}$, such that for all $t_{1}$, $t_{2} \in[a, b]$ and all $r \in\left(0, r_{3}\right) \cup\left(r_{4}, \infty\right)$ we have

$$
\begin{equation*}
-\varepsilon \leq \gamma\left(r, t_{2}\right)-\gamma\left(r, t_{1}\right) \leq \varepsilon \tag{2.10}
\end{equation*}
$$

Since $\gamma(r, t)$ is continuous on the rectangle $\left[r_{3}, r_{4}\right] \times[a, b]$, by the Cantor theorem it is uniformly continuous on this rectangle. Thus, there exists $\delta>0$ such that (2.10) is fulfilled whenever $t_{1}, t_{2} \in[a, b]$ and $\left|t_{1}-t_{2}\right|<\delta$. Together with the facts proved earlier, this implies that (2.10) is fulfilled with arbitrary $r>0$ whenever $t_{1}, t_{2} \in[a, b]$ and $\left|t_{2}-t_{1}\right|<\delta$.

Let $t_{1}, t_{2} \in[a, b]$ and $\left|t_{2}-t_{1}\right|<\delta$. Then for every $r>0$ we have

$$
\gamma\left(t_{2}\right)-\gamma\left(t_{1}\right) \leq \gamma\left(t_{2}\right)-\gamma\left(r, t_{1}\right)
$$

There exists $r>0$ such that $\gamma\left(t_{2}\right)<\gamma\left(r, t_{2}\right)+\varepsilon$. Together with (2.10), this yields $\gamma\left(t_{2}\right)-\gamma\left(t_{1}\right)<2 \varepsilon$. Interchanging the roles of $t_{1}$ and $t_{2}$, we arrive at $\left|\gamma\left(t_{2}\right)-\gamma\left(t_{1}\right)\right|<2 \varepsilon$. This implies the continuity of $\gamma(t)$ on the segment $[a, b]$ and, consequently, on $(0, \infty)$. Thus, statement 6) and, with it, the lemma, are proved.

Theorem 9. Let $\rho(r)$ be an arbitrary zero proximate order with $\rho\left(\frac{1}{r}\right)=-\rho(r)$, and let

$$
\begin{equation*}
\gamma(t)=\sup _{r>0} \frac{V(t r)}{V(t)} . \tag{2.11}
\end{equation*}
$$

Then

$$
\lim _{t \rightarrow \infty} \frac{\ln \gamma(t)}{\ln t}=0, \quad \lim _{t \rightarrow \infty} \frac{\ln \gamma\left(\frac{1}{t}\right)}{\ln t}=0
$$

Proof. We define $V_{1}(r)$ by (2.9). Then

$$
\frac{1}{M} \leq \frac{V_{1}(r)}{V(r)} \leq M
$$

Next, we have

$$
\gamma(t)=\sup _{r>0} \frac{V(r t)}{V(r)}=\sup _{r>0} \frac{V_{1}(r t)}{V_{1}(r)} \frac{V(r t)}{V_{1}(r t)} \frac{V_{1}(r)}{V(r)} \leq M^{2} \sup _{r>0} \frac{V_{1}(r t)}{V_{1}(r)}=M^{2} \gamma\left(\rho_{1}(\cdot), t\right)
$$

Therefore, it suffices to prove the theorem in the case where the proximate order $\rho(r)$ is differentiable on the set $(0, \infty) \backslash\{1\}$. In the rest of the proof, we assume this.

Put $h(x)=\ln V\left(e^{x}\right)$. Then $h(x)$ is a continuous even function differentiable everywhere except, maybe, at zero. The fact that $\rho(r)$ is a zero proximate order implies

$$
\begin{equation*}
\lim _{x \rightarrow+\infty} \frac{h(x)}{x}=0 \tag{2.12}
\end{equation*}
$$

The condition $\lim _{r \rightarrow \infty} r \ln r \rho^{\prime}(r)=0$, which is equivalent to $\lim _{r \rightarrow \infty} \frac{r V^{\prime}(r)}{V(r)}=0$, leads to the relation

$$
\begin{equation*}
\lim _{x \rightarrow+\infty} h^{\prime}(x)=0 \tag{2.13}
\end{equation*}
$$

We have

$$
\varphi(y)=\ln \gamma\left(e^{y}\right)=\sup _{x \in(-\infty, \infty)}(h(x+y)-h(x))
$$

The first statement of the theorem is equivalent to the relation

$$
\begin{equation*}
\lim _{y \rightarrow+\infty} \frac{\varphi(y)}{y}=0 \tag{2.14}
\end{equation*}
$$

If (2.14) fails, then there exists $a>0$, a sequence $y_{n} \rightarrow+\infty$, and a sequence $x_{n} \in$ $(-\infty, \infty)$ such that

$$
\begin{equation*}
\left|h\left(x_{n}+y_{n}\right)-h\left(x_{n}\right)\right| \geq a y_{n} \tag{2.15}
\end{equation*}
$$

Moreover, we may also assume that there exists $\alpha \in[-\infty, \infty]$ with $x_{n} \rightarrow \alpha$ as $n \rightarrow \infty$. Let $\varepsilon$ be an arbitrary number in the interval $\left(0, \frac{1}{2} a\right)$.

Suppose that $\alpha \in(-\infty, \infty)$. Then the inequalities

$$
\left|h\left(x_{n}+y_{n}\right)-h\left(x_{n}\right)\right| \leq \varepsilon\left(x_{n}+y_{n}\right)+|h(\alpha)|+1 \leq \varepsilon y_{n}+\varepsilon(|\alpha|+1)+|h(\alpha)|+1
$$

are fulfilled for all sufficiently large $n$. This contradicts (2.15).
Suppose that $\alpha=+\infty$. Then there exists $\xi_{n} \in\left(x_{n}, x_{n}+y_{n}\right)$ such that $h\left(x_{n}+y_{n}\right)-$ $h\left(x_{n}\right)=h^{\prime}\left(\xi_{n}\right) y_{n}$. Now, (2.13) shows that $\left|h\left(x_{n}+y_{n}\right)-h\left(x_{n}\right)\right| \leq \varepsilon y_{n}$ for all sufficiently large $n$. This contradicts (2.15).

Now, suppose that $\alpha=-\infty$. We may assume additionally that there exists $\beta \in$ $[-\infty, \infty]$ with $x_{n}+y_{n} \rightarrow \beta$ as $n \rightarrow \infty$.

Suppose that $\beta \in(-\infty, \infty)$. Then for all sufficiently large $n$ we have

$$
\begin{aligned}
\left|h\left(x_{n}+y_{n}\right)-h\left(x_{n}\right)\right| \leq|h(\beta)|+1+\varepsilon\left|x_{n}\right| & \leq|h(\beta)|+1+\varepsilon\left|x_{n}+y_{n}\right|+\varepsilon y_{n} \\
& \leq|h(\beta)|+1+\varepsilon(|\beta|+1)+\varepsilon y_{n} .
\end{aligned}
$$

This contradicts (2.15).
Suppose that $\beta=-\infty$. Then there exists $\xi_{n} \in\left(x_{n}, x_{n}+y_{n}\right)$ with $h\left(x_{n}+y_{n}\right)-h\left(x_{n}\right)=$ $h^{\prime}\left(\xi_{n}\right) y_{n}$. Since $h$ is even, (2.13) shows that $\left|h\left(x_{n}+y_{n}\right)-h\left(x_{n}\right)\right| \leq \varepsilon y_{n}$ for all sufficiently large $n$. This contradicts (2.15).

Now, suppose that $\beta=+\infty$. Then for all $n$ sufficiently large we have

$$
\left|h\left(x_{n}+y_{n}\right)-h\left(x_{n}\right)\right| \leq \varepsilon\left(x_{n}+y_{n}\right)+\varepsilon\left|x_{n}\right|=\varepsilon y_{n} .
$$

This contradicts (2.15).
The contradictions obtained above prove (2.14) and, with it, the formula

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} \frac{\ln \gamma(t)}{\ln t}=0 \tag{2.16}
\end{equation*}
$$

Together with statement 3) of Lemma this yields the relation

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} \frac{\ln \gamma\left(\frac{1}{t}\right)}{\ln t}=0 \tag{2.17}
\end{equation*}
$$

Let $\rho(r)$ be a proximate order representable in the form $\rho(r)=\rho+\hat{\rho}(r)$, where $\hat{\rho}(r)$ is a zero proximate order satisfying $\hat{\rho}\left(\frac{1}{r}\right)=-\rho\left(\frac{1}{r}\right)$. Then for $r>0$ and $t>0$ we have

$$
\begin{equation*}
V(r t) \leq t^{\rho} \gamma(t) V(r) \tag{2.18}
\end{equation*}
$$

where the continuous function $\gamma(t)$ satisfies (2.16) and (2.17). As was mentioned in the Preface, this inequality is a refinement of Potter's result 3 with a much simpler statement. Potter's result was cited and used in [2]. Inequality (2.18) will be employed fairly often in this paper. In particular, the proof of the following lemma shows why this inequality is important.
Lemma 2. Let $\rho(r)$ be an arbitrary zero proximate order satisfying $\rho\left(\frac{1}{r}\right)=-\rho(r)$, and let

$$
V_{1}(r)=\frac{2 r}{\pi} \int_{0}^{\infty} \frac{V(t)}{t^{2}+r^{2}} d t
$$

Then the following statements are true:

1) $V_{1}(r)$ admits a holomorphic extension from $(0, \infty)$ to the half-plane $\operatorname{Re} z>0$;
2) $V_{1}(r)=r^{\rho_{1}(r)}$, where $\rho_{1}(r)$ is a zero proximate order satisfying $\rho_{1}\left(\frac{1}{r}\right)=-\rho_{1}(r)$;
3) we have

$$
\lim _{r \rightarrow \infty} \frac{V_{1}(r)}{V(r)}=1
$$

Proof. The fact that the function

$$
V_{1}(z)=\frac{2 z}{\pi} \int_{0}^{\infty} \frac{V(t)}{t^{2}+z^{2}} d t
$$

is holomorphic in the half-plane $\operatorname{Re} z>0$ is an easy consequence of the inequality $V(t) \leq$ $M\left(t^{\frac{1}{2}}+t^{-\frac{1}{2}}\right)($ see $(2.8))$. This proves statement 1$)$.

We split the semiaxis $[0, \infty)$ into three parts, namely, $[0, \varepsilon r],[\varepsilon r, N r]$, and $[N r, \infty)$. Accordingly, $V_{1}(r)$ splits into the sum of three integrals:

$$
V_{1}(r)=I_{1}(r)+I_{2}(r)+I_{3}(r)
$$

We have

$$
\frac{I_{2}(r)}{V(r)}=\frac{2}{\pi} \int_{\varepsilon}^{N} \frac{V(u r)}{V(r)} \frac{d u}{u^{2}+1}
$$

Theorem 5 shows that

$$
\begin{equation*}
\lim _{r \rightarrow \infty} \frac{I_{2}(r)}{V(r)}=\frac{2}{\pi} \int_{\varepsilon}^{N} \frac{d u}{u^{2}+1} \tag{2.19}
\end{equation*}
$$

Next, we obtain

$$
I_{1}(r)=\frac{2}{\pi} \int_{0}^{\varepsilon} \frac{V(u r)}{u^{2}+1} d u \leq \frac{2 V(r)}{\pi} \int_{0}^{\varepsilon} \frac{\gamma(u)}{u^{2}+1} d u
$$

implying the inequality

$$
\begin{equation*}
\limsup _{r \rightarrow \infty} \frac{I_{1}(r)}{V(r)} \leq \frac{2}{\pi} \int_{0}^{\varepsilon} \frac{\gamma(u)}{u^{2}+1} d u \tag{2.20}
\end{equation*}
$$

Similarly, we see that

$$
\begin{equation*}
\limsup _{r \rightarrow \infty} \frac{I_{3}(r)}{V(r)} \leq \frac{2}{\pi} \int_{N}^{\infty} \frac{\gamma(u)}{u^{2}+1} d u \tag{2.21}
\end{equation*}
$$

Formulas (2.19)-(2.21) imply

$$
\limsup _{r \rightarrow \infty}\left|\frac{V_{1}(r)}{V(r)}-1\right| \leq \frac{2}{\pi} \int_{0}^{\varepsilon} \frac{\gamma(u)}{u^{2}+1} d u+1-\frac{2}{\pi} \int_{\varepsilon}^{N} \frac{d u}{u^{2}+1}+\frac{2}{\pi} \int_{N}^{\infty} \frac{\gamma(u)}{u^{2}+1} d u
$$

Passing to the limit as $\varepsilon \rightarrow 0, N \rightarrow \infty$, we prove statement 3 ) of the lemma.
We define $\rho_{1}(r)$ by the relation $V_{1}(r)=r^{\rho_{1}(r)}$. By statement 3 ), we obtain

$$
\begin{equation*}
\lim _{r \rightarrow \infty} \rho_{1}(r)=0 \tag{2.22}
\end{equation*}
$$

Next, we have

$$
r V_{1}^{\prime}(r)=\frac{2 r}{\pi} \int_{0}^{\infty} \frac{t^{2}-r^{2}}{\left(t^{2}+r^{2}\right)^{2}} V(t) d t
$$

Repeating the arguments that prove 3), we arrive at

$$
\begin{align*}
\lim _{r \rightarrow \infty} \frac{r V_{1}^{\prime}(r)}{V(r)} & =-\frac{2}{\pi} \int_{0}^{\infty} \frac{d}{d t} \frac{t}{t^{2}+1} d t=0 \\
\lim _{r \rightarrow \infty} \frac{r V_{1}^{\prime}(r)}{V_{1}(r)} & =0 \tag{2.23}
\end{align*}
$$

Formulas (2.22) and (2.23) show that $\rho_{1}(r)$ is a zero proximate order. Next,

$$
V_{1}\left(\frac{1}{r}\right)=\frac{2}{\pi r} \int_{0}^{\infty} \frac{V(t)}{t^{2}+\frac{1}{r^{2}}} d t=\frac{2}{\pi r} \int_{0}^{\infty} \frac{V\left(\frac{1}{u}\right)}{\frac{1}{u^{2}}+\frac{1}{r^{2}}} \frac{d u}{u^{2}}=\frac{2 r}{\pi} \int_{0}^{\infty} \frac{V(u)}{u^{2}+r^{2}} d u=V_{1}(r)
$$

This shows that $\rho_{1}\left(\frac{1}{r}\right)=-\rho_{1}(r)$. Thus, statement 2) and, with it, the Lemma 2 are proved.

In particular, Lemma 2 shows that, when solving Problem $A$ (see the beginning of this section), we need not consider the class of all functions $V(r)=r^{\rho(r)}$, where $\rho(r)$ is an arbitrary proximate order. Instead, we can take the much smaller class of all functions $V(r)$ for which $\rho(r)$ is analytic on $(0, \infty)$ and is representable in the form $\rho(r)=\rho+\hat{\rho}(r)$, where $\widehat{\rho}(r)$ is a zero proximate order satisfying $\widehat{\rho}\left(\frac{1}{r}\right)=-\widehat{\rho}(r)$.

Theorem 9 says that the function $\gamma(t)$ satisfies (2.16) and (2.17). Much sharper estimates hold true for a fairly large class of functions of the form $V(r)=r^{\rho(r)}$. Lemma says that $\gamma(t) \geq V(t)$. The following theorem provides us with a large class of proximate orders such that the corresponding functions $\gamma(t)$ satisfy $\gamma(t) \leq M V(t)$ for $t \geq 1$ with some constant.

Theorem 10. Let $\rho(r)$ be a zero proximate order differentiable two times on $(0, \infty) \backslash\{1\}$ and satisfying $\rho\left(\frac{1}{r}\right)=-\rho(r)$. Let $V(r)=r^{\rho(r)}, V(r) \rightarrow \infty(r \rightarrow \infty)$, and let the function $h(x)=\ln V\left(e^{x}\right)$ be concave in some neighborhood of infinity. Define $\gamma(t)$ by (2.11). Then there exists a constant $M$ such that $\gamma(t) \leq M V(t)$ for $t \geq 1$. If $h(x)$ is concave on $(0, \infty)$, then $\gamma(t)=V(t)$ for $t \geq 1$.

Proof. Consider the function $a(x)=h(x)-x h^{\prime}(x)$. We have $a^{\prime}(x)=-x h^{\prime \prime}(x)$. By assumption, $a(x)$ is monotone increasing in a neighborhood of infinity. For $x \geq 1$, we have

$$
\begin{equation*}
h(x)=-x\left(\int_{1}^{x} \frac{a(t)}{t^{2}} d t+c\right), \quad c=-h(1) . \tag{2.24}
\end{equation*}
$$

We claim that the integral

$$
\begin{equation*}
\int_{1}^{\infty} \frac{a(t)}{t^{2}} d t \tag{2.25}
\end{equation*}
$$

converges. Otherwise, if $a(t) \leq 0$ in a neighborhood of infinity, we arrive at a contradiction with (2.12), and the assumption that $a(t) \geq 0$ in a neighborhood of infinity contradicts the relation $V(r) \rightarrow \infty$ (as $r \rightarrow \infty$ ). Thus, the integral (2.25) does converge. Now, (2.24) can be rewritten in the form

$$
\begin{equation*}
h(x)=x \int_{x}^{\infty} \frac{a(t)}{t^{2}} d t \tag{2.26}
\end{equation*}
$$

By (2.12), the supplementary summand $c_{1} x$ on the right arising in the passage from (2.24) to (2.26) is absent.

We show that

$$
\begin{equation*}
\lim _{t \rightarrow \infty} a(t)=+\infty \tag{2.27}
\end{equation*}
$$

Indeed, otherwise, $h(x)$ and, with it, $V(r)$ are bounded in a neighborhood of $+\infty$, which contradicts the assumptions of the theorem.

Let $x_{0}>0$ be such that the function $h(x)$ is concave a for $x \geq x_{0}$. The equation of the tangent to the curve $y=h(x)$ at $x_{0}$ has the form

$$
Y(x)=h\left(x_{0}\right)-x_{0} h^{\prime}\left(x_{0}\right)+h^{\prime}\left(x_{0}\right) x .
$$

If $x_{0}$ is sufficiently large, we have $Y(0)>0$. Then there exists a function $h_{1}(x)$ with the
following properties:

1) $h_{1}(x)$ is an even function continuous on $(-\infty, \infty)$ that is concave and differentiable on ( $0, \infty$ );
2) $h_{1}(x) \geq 0$ for $x \geq 0, h_{1}(0)=0$;
3) $h_{1}(x)=h(x)$ for $x \geq x_{0}$.

These properties imply the following supplementary properties:
4) there exists a constant $M_{1}$ such that $\left|h(x)-h_{1}(x)\right| \leq M_{1}$ on the entire real axis;
5) $h_{1}(x)$ is monotone increasing on $[0, \infty)$;
6) $h_{1}(x+y) \leq h_{1}(x)+h_{1}(y)$ for $x \geq 0$ and $y \geq 0$.

Next, for $y>0$ we have

$$
\begin{aligned}
\ln \gamma\left(e^{y}\right) & =\sup _{x \in(-\infty, \infty)}(h(x+y)-h(x)) \leq 2 M_{1}+\sup _{x \in(-\infty, \infty)}\left(h_{1}(x+y)-h_{1}(x)\right) \\
& =2 M_{1}+\sup _{x \in(-\infty, \infty)}\left(h_{1}(|x+y|)-h_{1}(|x|)\right) \\
& \leq 2 M_{1}+\sup _{x \in(-\infty, \infty)}\left(h_{1}(|x|+y)-h_{1}(|x|)\right) \\
& \leq 2 M_{1}+h_{1}(y) \leq 3 M_{1}+h(y)=3 M_{1}+\ln V\left(e^{y}\right) .
\end{aligned}
$$

It follows that $\gamma(t) \leq e^{3 M_{1}} V(t)$ for $t>1$. If $h(x)$ is concave on the semiaxis $[0, \infty)$, we have $h_{1}(x)=h(x), M_{1}=0, \gamma(t) \leq V(t)$. Together with the inequality $\gamma(t) \geq V(t)$, this yields $\gamma(t)=V(t)$.

Quite often, the principal role is played by the function $V(r)=r^{\rho(r)}$ rather than by a proximate order $\rho(r)$. Sometimes, not only the properties of $V(r)$ near infinity are important, but also its behavior near zero (for instance, the knowledge of this behavior required in the study of the integral $\left.\int_{0}^{\infty} K(t, r) V(t) d t\right)$. So, as has already been said, in what follows we assume that zero proximate orders satisfy the relation $\rho\left(\frac{1}{r}\right)=-\rho(r)$. This is equivalent to the relation $V\left(\frac{1}{r}\right)=V(r)$. By Theorem 8 , the requirement that $\rho(r)$ be differentiable on $(0, \infty) \backslash\{1\}$ is often not an essential restriction. The relation

$$
\rho(r)=\frac{\ln V(r)}{\ln r}
$$

shows that the requirement of differentiability for $\rho(r)$ at 1 is fairly restrictive. For instance, this condition eliminates the functions $\rho(r)=\frac{A \mid \ln r r^{\alpha}}{\ln r}, \alpha \in(0,1)$, for which $V(r)$ has the form $V(r)=\exp \left(A|\ln r|^{\alpha}\right)$. Since these functions are simple, they are used as "patterns for comparison". Also, for them we have $\gamma(t)=V(t)$ for $t \geq 1$.

Surely, by Lemma 2 also the requirement of the existence of $\rho^{\prime}(1)$ is inessential: it suffices to replace $V(r)$ by $V_{1}(r)$. However, the use of $V_{1}(r)$ presents some difficulties because this function is somewhat complicated.

In the rest of the paper, by a zero proximate order we mean a function $\rho(r)$ with the following properties:

1) $\lim _{r \rightarrow \infty} \rho(r)=0$;
2) $\rho\left(\frac{1}{r}\right)=-\rho(r)$;
3) $\rho(r)$ is continuously differentiable on the set $(0, \infty) \backslash\{1\}$;
4) $\lim _{r \rightarrow \infty} r \ln r \rho^{\prime}(r)=0$;
5) the function $V(r)=r^{\rho(r)}$ extends by continuity to the point 1 and $V(1)=1$ (the function $\rho(r)$ itself may fail to be defined at 1 ).

Other proximate orders $\rho(r)$ have the form $\rho(r)=\rho+\hat{\rho}(r)$, whee $\hat{\rho}(r)$ is a zero proximate order satisfying the above conditions.

## §3. Measures; limit sets of measures

In this section we present the theory of limit sets for Radon measures of finite order on the semiaxis $(0, \infty)$. For the reader's convenience, we define the terms and state the results to be used. The omitted proofs can be found in [10, 11, 12 .

We start with some information from the "abstract" measure theory.
A measure space with a real measure $\mu$ is a triple $(X, \mathcal{A}, \mu)$, where $X$ is a set, $\mathcal{A}$ is a $\sigma$-algebra of subsets of $X$, and $\mu$ is a function on $\mathcal{A}$ with values in $[-\infty, \infty]$ that is countably additive: $\mu\left(\bigcup_{k=1}^{\infty} E_{k}\right)=\sum_{k=1}^{\infty} \mu\left(E_{k}\right)$ whenever the $E_{k} \in \mathcal{A}$ are pairwise disjoint ( $E_{k} \cap E_{j}=\varnothing$ if $k \neq j$ ).

The measure $\mu$ is said to be positive if $\mu(E) \geq 0$ for every $E \in \mathcal{A}$.
A positive measure is said to be finite if $\mu(X)<\infty$.
Let $A \in \mathcal{A}$. The restriction of $\mu$ to $A$ (denoted by $\mu_{A}$ ) is the measure defined by $\mu_{A}(E)=\mu(A \cap E)$.

A measure $\mu$ is said to be supported on $A$ if $\mu_{A}=\mu$.
Let $\mu_{1}$ and $\mu_{2}$ be two measures defined on the same $\sigma$-algebra $\mathcal{A}$. They are said to be mutually singular if they are supported on disjoint sets $A_{1}$ and $A_{2}$. The Hahn theorem says that if $\mu$ is a real measure, then there exist two sets $A_{1}$ and $A_{2}$ in $\mathcal{A}$ such that $A_{1} \cap A_{2}=\varnothing, X=A_{1} \cup A_{2}$, the restriction of $\mu$ to $A_{1}$ is positive and the restriction of $\mu$ to $A_{2}$ is negative.

This pair of sets $A_{1}, A_{2}$ is called a Hahn decomposition for $\mu$. The measure $\mu_{+}=\mu_{A_{1}}$ is called the positive component of $\mu$, and the measure $\mu_{-}=-\mu_{A_{2}}$ is called its negative component. Thus, an arbitrary real measure $\mu$ is the difference of two mutually singular positive measures.

The representation $\mu=\mu_{+}-\mu_{-}$of $\mu$ as a difference of two mutually singular positive measures is called the Jordan decomposition of $\mu$.

Though a Hahn decomposition $X=A_{1} \cup A_{2}$ is not unique, the Jordan decomposition $\mu=\mu_{+}-\mu_{-}$is unique. The measures $\mu_{+}$and $\mu_{-}$are uniquely determined by $\mu$. If $\mu=\mu_{+}-\mu_{-}$is the Jordan decomposition of a measure $\mu$, then at least one of the measures is finite (otherwise, the formula $\mu(X)=\mu_{+}(X)-\mu_{-}(X)$ has no sense).

The measure $|\mu|=\mu_{+}+\mu_{-}$is called the modulus or the total variation of $\mu$.
In the important case where $X$ is a topological space and $\mathcal{A}$ is the $\sigma$-algebra of its Borel sets, $\mu$ is called a Borel measure.

Let $X=K$ be a compact metric space. Then the set of all finite real Borel measures on $K$ is a Banach space. This space can be identified with the space dual to $C(K)$, i.e., to the space of real continuous functions on $K$ with the norm $\|f\|=\max _{x \in K}|f(x)|$. This follows from the Riesz theorem that says that every continuous linear functional $T$ on $C(K)$ has the form

$$
(T, f)=\int_{K} f(x) d \mu(x)
$$

where $\mu$ is a finite Borel measure on $K$.
In the sequel, we write $(\mu, f)$ in place of $(T, f)$. We have $\|\mu\|=|\mu|(K)(\|\mu\|$ is the norm of the linear functional $\mu,\|\mu\|=\sup _{\|f\| \leq 1}(\mu, f)$ ).

The Banach space of all finite real Borel measures on $K$ will be denoted by $\mathcal{M}_{r}(K)$. In $\mathcal{M}_{r}(K)$, convergence in norm is less important than weak convergence. By the generally adopted terminology, a sequence $\mu_{n}$ converges weakly to $\mu$ (in symbols: $\mu=$ $\mathrm{w}-\lim _{n \rightarrow \infty} \mu_{n}$ ) if the numerical sequence ( $\mu_{n}, f$ ) converges to ( $\mu, f$ ) for every $f \in C(K)$.

By the Alaogly theorem, if $H \subset \mathcal{M}_{r}(K)$ and $\sup \{|\mu|(K): \mu \in H\}<\infty$, then every sequence $\mu_{n} \in H$ has a weakly convergent subsequence.

Along with the real space $\mathcal{M}_{r}(K)$, the complex Banach space $\mathcal{M}_{c}(K)$ of all complex measures $\mu=\mu_{1}+i \mu_{2}\left(\mu_{1}, \mu_{2}\right.$ being finite real Borel measures) is also considered. The
space $\mathcal{M}_{c}(K)$ can be identified with the dual to the Banach space $C(K)$ of all complex continuous functions on $K$, where $\|f\|=\max _{x \in K}|f(x)|$. In the space $\mathcal{M}_{c}(K)$, we also have $\|\mu\|=\sup _{\|f\| \leq 1}|(\mu, f)|=|\mu|(K)$.

In the complex case, the definition of the measure $|\mu|$ is more involved, but we have the following simple inequalities. If $\mu=\mu_{1}+i \mu_{2}$, then $\left|\mu_{1}\right| \leq|\mu|,\left|\mu_{2}\right| \leq|\mu|$, and $|\mu| \leq\left|\mu_{1}\right|+\left|\mu_{2}\right|$.

Now, we pass to the analysis of an object important for our study, namely, the space of Radon measures on the semiaxis $(0, \infty)$.

First, we introduce the space $\Phi$ of test functions on $(0, \infty)$ (we have borrowed the term from the theory of distributions and the notation from Landkof's book [12]). A function $f$ belongs to $\Phi$ if $f$ is continuous on $(0, \infty)$ and there exists a segment $[a, b]$ such that $\operatorname{supp} f \subset[a, b] \subset(0, \infty)$. We shall consider either the real or the complex space $\Phi$.

The notion of convergence in $\Phi$ is introduced in a well-known way. A sequence $f_{n}$ is said to converge to $f$ in the space $\Phi$ if there exists a segment $[a, b] \subset(0, \infty)$ such that $\operatorname{supp} f_{n} \subset[a, b]$ for every $n$ and the sequence $f_{n}$ converges uniformly to $f(x)$ on $(0, \infty)$.

For a function $f \in \Phi$, we use the notation $\|f\|=\max |f(x)|$.
A Borel measure $\mu$ on $(0, \infty)$ is said to be locally finite if $|\mu|([a, b])<\infty$ for every segment $[a, b] \subset(0, \infty)$.

A set function $\mu$ is called a real Radon measure on $(0, \infty)$ if it is representable in the form $\mu=\mu_{1}-\mu_{2}$, where $\mu_{1}$ and $\mu_{2}$ are mutually singular positive Borel measures on $(0, \infty)$.

For every Borel set $A \subset(0, \infty)$, the restriction of $\mu$ to $A$ is defined by the formula $\mu_{A}=\left(\mu_{1}\right)_{A}-\left(\mu_{2}\right)_{A}$. Though the set function $\mu$ defined above is not a Borel measure on $(0, \infty)$ (it is not defined on the Borel sets $E$ for which either $\mu_{1}(E)=\mu_{2}(E)=+\infty$ or $\left.\mu_{1}(E)=\mu_{2}(E)=-\infty\right)$, the restriction $\mu_{[a, b]}$ is a finite Borel measure for every segment $[a, b] \subset(0, \infty)$.

Thus, for every $f \in \Phi$ the expression $(\mu, f)=\int_{0}^{\infty} f(x) d \mu(x)$ makes sense. Clearly, the function $(\mu, f)$ defined in this way is a linear functional on $\Phi$. It is also clear that this functional is continuous, i.e., $\left(\mu, f_{n}\right) \rightarrow(\mu, f)$ whenever $f_{n} \rightarrow f$ in $\Phi$.

The converse is also well known. If $T$ is a continuous linear functional on $\Phi$, then there exists a Radon measure $\mu$ such that $(T, f)=(\mu, f)$ for every $f \in \Phi$. Moreover, a Radon measure $\mu$ with this property is unique.

The initial definition of Radon measures was given by Bourbaki for an arbitrary locally compact space. In accordance with that definition, Radon measures are continuous linear functionals on the space of compactly supported continuous functions $f$. A functional $T$ is said to be positive if $(T, f) \geq 0$ for every $f \geq 0$. An adaptation of Bourbaki's result to metric locally compact underlying spaces says that in this case every continuous positive functional coincides with a positive locally finite Borel measure. Next, every continuous linear functional is representable as the difference of two positive continuous linear functionals.

Thus, if the locally compact space in question is the semiaxis $(0, \infty)$, the definition of a Radon measure given in this paper is equivalent to Bourbaki's definition.

The set of real Radon measures on $(0, \infty)$ is a real linear space, to be denoted by $\mathfrak{R}$.
We shall also consider the complex linear space $\mathfrak{R}_{c}$ of complex Radon measures $\mu=$ $\mu_{1}+i \mu_{2}$, where $\mu_{1}$ and $\mu_{2}$ are real Radon measures.

We introduce the notion of coarse convergence in the spaces $\mathfrak{R}$ and $\Re_{c}$. The terminology is due to Bourbaki. A sequence $\mu_{n}$ of Radon measures is said to converge coarsely to a Radon measure $\mu$ (notation: $\mu=\lim _{n \rightarrow \infty} \mu_{n}$ or $\mu_{n} \rightarrow \mu$ ) if for every $f \in \Phi$ the numerical sequence ( $\mu_{n}, f$ ) converges to $(\mu, f)$. The term "coarse convergence" plays a privileged role in our paper. The formula $\mu_{n} \rightarrow \mu$, when used without explanations, always means
coarse convergence. However, we also use other types of convergence for sequences $\mu_{n}$ of measures.

A set $E$ is said to be Jordan measurable with respect to a Radon measure $\mu$ if $|\mu|(\partial E)=0$.

The following statements are well known.
Theorem 11. Suppose that $\mu_{n} \rightarrow \mu$ and $\left|\mu_{n}\right| \rightarrow \hat{\mu}$. If $E$ is Jordan measurable with respect to $\hat{\mu}$, then $\left(\mu_{n}\right)_{E} \rightarrow \mu_{E}$.
Theorem 12. Suppose that $\mu_{n} \rightarrow \mu$ and $\left|\mu_{n}\right| \rightarrow \hat{\mu}$. Let $K \subset(0, \infty)$ be a compact set Jordan measurable with respect to $\hat{\mu}$. Then the sequence $\left(\mu_{n}\right)_{K}$ converges weakly to $\mu_{K}$.

For sequences of positive measures, the proofs of these statements can be found in [12, Introduction, $\S 1]$. In the general case, it suffices to apply these statements to the measures $\left(\mu_{n}\right)_{+}$and $\left(\mu_{n}\right)_{-}$separately. We shall need the following statement, which can be deduced from Theorem 11.
Theorem 13. Suppose that a sequence $\mu_{n}$ of Radon measures on $(0, \infty)$ converges coarsely to a Radon measure $\nu$. Suppose, moreover, that $\left|\mu_{n}\right| \rightarrow \widehat{\nu}$ and $\widehat{\nu}(\{\xi\})=0$. Then for every $\varphi \in \Phi$ we have

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \int_{0}^{\infty} \varphi(t) \chi_{(0, \xi]}(t) d \mu_{n}(t) & =\int_{0}^{\infty} \varphi(t) \chi_{(0, \xi]}(t) d \nu(t) \\
\lim _{n \rightarrow \infty} \int_{0}^{\infty} \varphi(t) \chi_{(\xi, \infty)}(t) d \mu_{n}(t) & =\int_{0}^{\infty} \varphi(t) \chi_{(\xi, \infty)}(t) d \nu(t)
\end{aligned}
$$

Proof. It suffices to take $(0, \xi]$ and $(\xi, \infty)$ for the role of $E$ in Theorem 11 .
A set $H \subset \mathfrak{R}_{c}(\mathfrak{R})$ is said to be coarsely bounded if for every $\varphi \in \Phi$ the set $\{(\mu, f)$ : $\mu \in H\}$ is bounded.

A set $H$ is said to be strongly bounded if for every segment $[a, b] \subset(0, \infty)$ the set $\{|\mu|([a, b]): \mu \in H\}$ is bounded.

A set $H$ is said to be compact if every sequence $\mu_{n} \in H$ has a coarsely convergent subsequence.

The next statement is the main result of measure theory to be used in this paper.
Theorem 14. The classes of coarsely bounded, strongly bounded, and compact subsets of $\mathfrak{R}\left(\mathfrak{R}_{c}\right)$ coincide.

The proof can be found in [10, Chapter 3, $\S 1$, Theorem 15, Remark].
Note that the following statement is an easy consequence of the above theorem.
Theorem 15. The mapping $(\mu, \varphi): \mathfrak{R}_{c} \times \Phi \rightarrow \mathbb{C}$ is a continuous mapping of two variables.

Proof. Let $\mu_{n} \rightarrow \mu, \varphi_{n} \rightarrow \varphi$. There is a segment $[a, b] \subset(0, \infty)$ such that $\operatorname{supp} \varphi_{n} \subset[a, b]$, $\operatorname{supp} \varphi \subset[a, b]$. Since $\mu_{n} \rightarrow \mu$, the sequence $\mu_{n}$ is coarsely bounded and, consequently, strongly bounded. Therefore, there exists a constant $M$ such that $\left|\mu_{n}\right|([a, b]) \leq M$. Next, we have
$\left|\left(\mu_{n}, \varphi_{n}\right)-(\mu, \varphi)\right| \leq\left|\mu_{n}\right|([a, b])\left\|\varphi_{n}-\varphi\right\|+\left|\left(\left(\mu-\mu_{n}\right), \varphi\right)\right| \leq M\left\|\varphi_{n}-\varphi\right\|+\left|\left(\mu_{n}-\mu\right), \varphi\right|$. This implies the claim.

Theorem 16. A sequence of positive Radon measures $\mu_{n}$ converges coarsely to a Radon measure $\mu$ if $\left(\mu_{n}, f\right) \rightarrow(\mu, f)$ for $f$ in a dense subset of $\Phi$.
Theorem 17. Suppose that Radon measures $\mu$ and $\nu$ are such that $(\mu, f)=(\nu, f)$ for $f$ in a dense subset of $\Phi$. Then $\mu=\nu$.

The proofs of Theorems 16 and 17 can be found in [12, Introduction, §1].
Though we usually employ coarse convergence, sometimes it is more convenient to argue in metric spaces. For this reason, we shall use the following well-known metrics in $\mathfrak{R}_{c}$. Let $\left\{\varphi_{n}: n=1,2, \ldots\right\}$ be a countable dense subset of $\Phi$. This means that for every $\varphi \in \Phi$ there exists a subsequence $\varphi_{n_{k}}$ such that $\varphi_{n_{k}} \rightarrow \varphi$. Next, we introduce the function

$$
\begin{equation*}
d\left(\mu_{1}, \mu_{2}\right)=\sum_{n=1}^{\infty} \frac{\left|\left(\mu_{1}-\mu_{2}\right)\left(\varphi_{n}\right)\right|}{2^{n}\left(1+\left|\left(\mu_{1}-\mu_{2}\right)\left(\varphi_{n}\right)\right|\right)}, \tag{3.28}
\end{equation*}
$$

$\mu_{1}, \mu_{2} \in \mathfrak{R}_{c}$. It can easily be checked that $d$ is a metric on $\Re_{c}$.
Moreover, it is easily seen that $d\left(\mu_{k}, \mu\right) \rightarrow 0$ whenever $\mu_{k} \rightarrow \mu$. However, the converse is not true. Let, for example, all functions $\varphi_{n}$ be continuously differentiable, and let

$$
\begin{aligned}
\mu_{k}=k & \left(\delta\left(x-1-\frac{1}{2 k}\right)-\delta\left(x-1+\frac{1}{2 k}\right)\right) \\
& -\sqrt{2} k\left(\delta\left(x-1-\frac{1}{2 \sqrt{2} k}\right)-\delta\left(x-1+\frac{1}{2 \sqrt{2} k}\right)\right),
\end{aligned}
$$

where $\delta(x-a)$ is the Dirac measure at $a$. Then $d\left(\mu_{k}, 0\right) \rightarrow 0$ but the relation $\mu_{k} \rightarrow 0$ fails. Yet, the following theorem holds.

Theorem 18. If $\mu_{k}$ is a compact sequence in $\mathfrak{R}_{c}$ and $d\left(\mu_{k}, \mu\right) \rightarrow 0$, then $\mu_{k}$ tends to $\mu$ coarsely.

Proof. If not, there exists $\varphi \in \Phi$ and two subsequences $\mu_{k_{p}^{1}}, \mu_{k_{p}^{2}}$ of the sequence $\mu_{k}$ such that

$$
\lim _{p \rightarrow \infty}\left(\mu_{k_{p}^{1}}, \varphi\right) \neq \lim _{p \rightarrow \infty}\left(\mu_{k_{p}^{2}}, \varphi\right) .
$$

Let $\nu_{p}=\mu_{k_{p}^{1}}-\mu_{k_{p}^{2}}$, and let $\varphi_{n}$ be the sequence of functions in $\Phi$ that determines the metric $d$. Since the sequence $\varphi_{n}$ is dense in $\Phi$, there exists its subsequence $\psi_{n}$ that converges to $\varphi$ in $\Phi$. There is a segment $[a, b]$ on the semiaxis $(0, \infty)$ such that $\operatorname{supp} \psi_{n} \subset[a, b]$ for every $n$. Therefore, $\lim \sup _{p \rightarrow \infty}\left|\nu_{p}(\varphi)\right| \leq M\left\|\varphi-\psi_{n}\right\|$ with some constant $M$ independent of $n$. Thus, $\nu_{p}(\varphi) \rightarrow 0$, which contradicts the choice of $\varphi$.

Thus, generally speaking, convergence in the metric $d$ is weaker than coarse convergence, whereas the two types of convergence coincide on compact sets. The metric $d$ is determined by a countable dense sequence $\varphi_{n}$. Thus, there are infinitely many metrics of this type. In general, convergence in one metric does not imply convergence in another one. However, on compact sets, convergence in any metric is equivalent to coarse convergence, so all they are equivalent.

Next, for $\mu \in \mathfrak{R}_{c}$, we shall consider the integrals $\int_{0}^{\infty} f(x) d \mu(x)$ not necessarily with $f \in \Phi$. In this connection we note that, if $f \in \Phi$, then the integral in question can be viewed as the Riemann-Stiltjes integral of $f$ against the distribution function $\mu(x)$ for $\mu$. However, we shall also need the case of an arbitrary Borel function $f$ on $(0, \infty)$. Suppose first that supp $f \subset[a, b] \subset(0, \infty)$. The restriction of $\mu$ to $[a, b]$ is a finite Borel measure, and $\int_{a}^{b} f(x) d \mu(x)$ is viewed as the Lebesgue integral of $f$ against $\mu$. Surely, not every such function is $\mu$-integrable.

Now, let $f(x)$ be an arbitrary Borel function on $(0, \infty)$ that is locally integrable against $\mu$. Than we put

$$
\int_{0}^{\infty} f(x) d \mu(x)=\lim _{\substack{a \rightarrow+0 \\ b \rightarrow+\infty}} \int_{a}^{b} f(x) d \mu(x)
$$

So, $\int_{0}^{\infty} f(x) d \mu(x)$ is viewed as an improper integral with singularities at zero and at infinity. For some $f$, this integral may converge.

Let $\rho(t)$ be an arbitrary proximate order. On the space $\mathfrak{R}_{c}$, we introduce the oneparameter family of Azarin transformations $A_{t}: \mathfrak{R}_{c} \rightarrow \mathfrak{R}_{c}, t \in(0, \infty)$, by the formulas

$$
\mu_{t}=A_{t} \mu, \quad \mu_{t}(E)=\frac{\mu(t E)}{V(t)}
$$

If $\mu(x)$ is the distribution function for a measure $\mu$, then the distribution function for $\mu_{t}$ is $\frac{1}{V(t)} \mu(t x)$.

Let $f \in \Phi$. A change of variables yields

$$
\int_{E} f(x) d \mu_{t}(x)=\frac{1}{V(t)} \int_{t E} f\left(\frac{y}{t}\right) d \mu(y)
$$

and, in particular,

$$
\begin{equation*}
\int_{0}^{\infty} f(x) d \mu_{t}(x)=\frac{1}{V(t)} \int_{0}^{\infty} f\left(\frac{y}{t}\right) d \mu(y) \tag{3.29}
\end{equation*}
$$

Formula (3.29) and Theorem 15 easily imply that the function $\mu_{t}: \mathfrak{R}_{c} \times(0, \infty) \rightarrow \mathfrak{R}_{c}$ is continuous in the totality of variables, that is, $\left(\mu_{n}\right)_{t_{n}} \rightarrow \mu_{\tau}$ whenever $t_{n} \rightarrow \tau$ and $\mu_{n} \rightarrow \mu$.

Classical dynamical systems in a metric space $X$ are defined (see [13]) as oneparametric families of mappings $B_{t}: X \rightarrow X, t \in(-\infty, \infty)$, satisfying the following conditions:

1) $B_{0} x=x$ (the initial condition);
2) the mapping $B_{t}: X \times(-\infty, \infty) \rightarrow X$ is continuous in the totality of variables (the continuity condition);
3) $B_{t_{1}} B_{t_{2}}=B_{t_{1}+t_{2}}$ (the group condition).

If $\rho(r) \equiv \rho$, then the Azarin system $A_{t}$ is a dynamical system in $\mathfrak{R}_{c}$ with the coarse convergence, where the additive group of reals is replaced by the multiplicative group of strictly positive reals, the initial condition reads as $A_{1} \mu=\mu$, and the group condition looks like this: $A_{t_{1}} A_{t_{2}} \mu=A_{t_{1} t_{2}} \mu$.

For positive measures $\mu$ on $\mathbb{R}^{m}$, the system $A_{t} \mu$ was introduced by Azarin (see [4, 14]), who efficiently applied it to the theory of subharmonic functions.

In the classical theory of dynamical systems, the set

$$
\left\{y \in X: y=\lim _{n \rightarrow \infty} B_{t_{n}} x, \lim _{n \rightarrow \infty} t_{n}=+\infty\right\}
$$

is called the $\omega$-cluster set of the trajectory $B_{t} x$.
To avoid terminological complications, we shall refer to a system $A_{t} \mu$ as an Azarin dynamical system even in the case of an arbitrary proximate order. The set

$$
\left\{\nu \in \mathfrak{R}_{c}: \nu=\lim _{n \rightarrow \infty} A_{t_{n}} \mu, \lim _{n \rightarrow \infty} t_{n}=+\infty\right\}
$$

will be called the Azarin cluster set for $\mu$, and Azarin's notation $\operatorname{Fr}[\mu]$ will be used for it. Nothing definite can be said about $\operatorname{Fr}[\mu]$ without knowledge of the relationship between $\mu$ and $\rho(r)$. So, we shall assume that $\mu \in \mathfrak{M}_{\infty}(\rho(r))$. The set $\mathfrak{M}_{\infty}(\rho(r))$ was defined in the Preface. Since a Radon measure is locally finite, we see that $\mu \in \mathfrak{M}_{\infty}(\rho(r))$ if and only if

$$
\limsup _{r \rightarrow \infty} \frac{|\mu|([r, e r])}{V(r)}<\infty
$$

As we shall see, if $\mu \in \mathfrak{M}_{\infty}(\rho(r))$, the properties of set $\operatorname{Fr}[\mu]$ are similar to those of cluster sets in the theory of dynamical systems.

The relation $\mu \in \mathfrak{M}_{\infty}(\rho(r))$ easily implies that the positive semitrajectory $\mu_{t}$ (the set $\left.\left\{\mu_{t}: t \in[1, \infty)\right\}\right)$ is compact. We state this as a separate lemma.
Lemma 3. Let $\rho(r)$ be an arbitrary proximate order, and let $\mu \in \mathfrak{M}_{\infty}(\rho(r))$. Then the semitrajectory $\mu_{t}, t \geq 1$, is a compact set of Radon measures.

Proof. Let $0<a<b<\infty$. By (2.11) and (2.18), we obtain

$$
\left|\mu_{t}\right|([a, b])=\frac{|\mu|([a t, b t])}{V(a t)} \frac{V(a t)}{V(t)} \leq \gamma(a) a^{\rho} \frac{|\mu|\left(\left[a t, \frac{b}{a} a t\right]\right)}{V(a t)} \leq M(a, b)
$$

The last inequality is true by the definition of $\mathfrak{M}_{\infty}(\rho(r))$ and because $\mu$ is locally finite. Now, the claim of the lemma follows by Theorem 14 .

Our next goal is a description of the properties of cluster sets for measures belonging to $\mathfrak{M}_{\infty}(\rho(r))$. We start with a theorem that reduces this problem to the simpler case where $\rho(r) \equiv \rho$.
Theorem 19. Let $\rho_{1}(r)$ and $\rho_{2}(r)$ be proximate orders such that $\lim \rho_{1}(r)=\rho_{1}$ and $\lim \rho_{2}(r)=\rho_{2}(r \rightarrow \infty)$. Taking $\mu \in \mathfrak{M}_{\infty}\left(\rho_{1}(r)\right)$, we put

$$
d \lambda(t)=\frac{V_{2}(t)}{V_{1}(t)} d \mu(t)
$$

Then $\lambda \in \mathfrak{M}_{\infty}\left(\rho_{2}(r)\right)$, and the relation $\mu_{t_{n}} \rightarrow \nu$ is equivalent to the relation $\lambda_{t_{n}} \rightarrow \nu_{1}$, where $d \nu_{1}(t)=t^{\rho_{2}-\rho_{1}} d \nu(t)$. Here

$$
\mu_{t}(E)=\frac{\mu(t E)}{V_{1}(t)}, \quad \lambda_{t}(E)=\frac{\lambda(t E)}{V_{2}(t)}, \quad V_{1}(r)=r^{\rho_{1}(r)}, \quad V_{2}(r)=r^{\rho_{2}(r)}, t_{n} \rightarrow \infty
$$

Proof. We have

$$
|\lambda|([r, e r])=\int_{r}^{e r} \frac{V_{2}(t)}{V_{1}(t)} d|\mu|(t)
$$

Theorem 5 shows that

$$
\limsup _{r \rightarrow \infty} \frac{|\lambda|([r, e r])}{V_{2}(r)} \leq\left(e^{\left(\rho_{2}-\rho_{1}\right)_{+}}\right) \limsup _{r \rightarrow \infty} \frac{|\mu|([r, e r])}{V_{1}(r)} \quad\left(a_{+}=\max \{a, 0\}\right),
$$

whence $\lambda \in \mathfrak{M}_{\infty}\left(\rho_{2}(r)\right)$. Suppose that $\mu_{t_{n}} \rightarrow \nu$, then for $\varphi \in \Phi$ we obtain

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \int_{0}^{\infty} \varphi(t) d \lambda_{t_{n}}(t)=\lim _{n \rightarrow \infty} \frac{1}{V_{2}\left(t_{n}\right)} \int_{0}^{\infty} \varphi\left(\frac{u}{t_{n}}\right) d \lambda(u) \\
& \quad=\lim _{n \rightarrow \infty} \frac{1}{V_{2}\left(t_{n}\right)} \int_{0}^{\infty} \varphi\left(\frac{u}{t_{n}}\right) \frac{V_{2}(u)}{V_{1}(u)} d \mu(u)=\lim _{n \rightarrow \infty} \int_{0}^{\infty} \varphi(\tau) \frac{V_{2}\left(\tau t_{n}\right)}{V_{1}\left(\tau t_{n}\right)} \frac{V_{1}\left(t_{n}\right)}{V_{2}\left(t_{n}\right)} d \mu_{t_{n}}(\tau)
\end{aligned}
$$

Since the sequence

$$
\varphi(\tau) \frac{V_{2}\left(\tau t_{n}\right)}{V_{1}\left(\tau t_{n}\right)} \frac{V_{1}\left(t_{n}\right)}{V_{2}\left(t_{n}\right)}
$$

converges to $\tau^{\rho_{2}-\rho_{1}} \varphi(\tau)$ in $\Phi$, and since $\mu_{t_{k}} \rightarrow \nu$, by Theorem 15 we obtain

$$
\lim _{n \rightarrow \infty} \int_{0}^{\infty} \varphi(t) d \lambda_{t_{n}}(t)=\int_{0}^{\infty} \tau^{\rho_{2}-\rho_{1}} \varphi(\tau) d \nu(\tau)
$$

Thus, we have proved that $\lambda_{t_{n}} \rightarrow \nu_{1}$. Finally, the roles of $\mu$ and $\nu$ can be interchanged in the above argument.

A similar theorem for measures on the plane can be found in [15, Theorem 4].
Remark 1. If $\rho_{1}=\rho_{2}$, then the measures $\nu$ and $\nu_{1}$ coincide.
In the case where $\rho(r) \equiv \rho$, the mapping $A_{t}, A_{t} \mu=\mu_{t}$, will be denoted by $F_{t}$ or $F_{t}(\rho)$.
In the following statement, some properties of the cluster set $\operatorname{Fr}[\mu]$ are collected.

Theorem 20. Let $\rho(r)$ be an arbitrary proximate order, and let $\mu \in \mathfrak{M}_{\infty}(\rho(r))$. Then:

1) $\operatorname{Fr}[\mu]$ is a nonempty compact set;
2) $\operatorname{Fr}[\mu]$ is a connected set in the metric space $\left(\Re_{c}, d\right)$;
3) $\operatorname{Fr}[\mu]$ is invariant under $F_{t}$, moreover, $F_{t}$ is a bijection of $\operatorname{Fr}[\mu]$ onto itself.

Proof. By the remark to Theorem [19] there is no loss of generality in assuming that $\rho(r) \equiv \rho$. By Lemma 3, the semitrajectory $\mu_{t}, t \geq 1$, is a compact set. In the language of the theory of dynamical systems, this can be restated as follows: the movement $\mu_{t}$ is positively Lagrange stable. On compact subsets in $\Re_{c}$, coarse convergence is equivalent to the convergence in the metric space $\left(\Re_{c}, d\right)$. Thus, the Azarin cluster set $\operatorname{Fr}[\mu]$ coincides with the $\omega$-cluster set of the trajectory $\mu_{t}$ in the metric space $\left(\Re_{c}, d\right)$. Now, we can employ the fairly well developed theory of dynamical systems in metric spaces. Theorem 10 in Chapter $5, \S 3$ of the book [13] says that $\operatorname{Fr}[\mu]$ is a nonempty compact set taken by $F_{t}$ onto itself bijectively, and Theorem 14 is the same book says that $\operatorname{Fr}[\mu]$ is connected.

The following theorems are a useful supplement to the preceding one.
Theorem 21. Suppose that $\mu \in \mathfrak{M}_{\infty}(\rho(r))$, a sequence $t_{n} \rightarrow \infty$ satisfies $\mu_{t_{n}} \rightarrow \nu$, and a sequence $\tau_{n}$ converges to $\tau>0$. Then the sequence $\mu_{\tau_{n} t_{n}}$ converges to $\nu_{\tau}$.

Proof. Let $\varphi$ be an arbitrary function belonging to $\Phi$. We have

$$
\left(\varphi, \mu_{t_{n} \tau_{n}}\right)=\frac{1}{V\left(t_{n} \tau_{n}\right)} \int_{0}^{\infty} \varphi\left(\frac{u}{t_{n} \tau_{n}}\right) d \mu(u)=\frac{V\left(t_{n}\right)}{V\left(t_{n} \tau_{n}\right)} \int_{0}^{\infty} \varphi\left(\frac{\xi}{\tau_{n}}\right) d \mu_{t_{n}}(\xi)
$$

Since $\mu_{t_{n}} \rightarrow \nu$ and the sequence $\frac{V\left(t_{n}\right)}{V\left(t_{n} \tau_{n}\right)} \varphi\left(\frac{\xi}{\tau_{n}}\right)$ converges to $\frac{1}{\tau^{\rho}} \varphi\left(\frac{\xi}{\tau}\right)$ in $\Phi$, by Theorem 15 we obtain

$$
\lim _{n \rightarrow \infty}\left(\varphi, \mu_{t_{n} \tau_{n}}\right)=\frac{1}{\tau^{\rho}} \int_{0}^{\infty} \varphi\left(\frac{\xi}{\tau}\right) d \nu(\xi)=\left(\varphi, \nu_{\tau}\right)
$$

Thus, $\mu_{t_{n} \tau_{n}} \rightarrow \nu_{\tau}$.
Theorem 22. Suppose that $\mu \in \mathfrak{M}_{\infty}(\rho(r))$ and a sequence $t_{n} \rightarrow \infty$ is such that $\mu_{t_{n}} \rightarrow \nu$ and $|\mu|_{t_{n}} \rightarrow \widehat{\nu}$. Then $|\nu| \leq \widehat{\nu}$.

Proof. For arbitrary $\varphi$ in $\Phi$, we have

$$
|(\nu, \varphi)|=\lim _{n \rightarrow \infty}\left|\left(\mu_{t_{n}}, \varphi\right)\right| \leq \lim _{n \rightarrow \infty}\left(|\mu|_{t_{n}},|\varphi|\right)=(\hat{\nu},|\varphi|),
$$

and it easily follows that $|\nu|(E) \leq \widehat{\nu}(E)$.
In the next theorem we show that some asymptotic estimates for $\mu$ imply certain global estimates for the measures $\nu$ in $\operatorname{Fr}[\mu]$. As a preliminary, we introduce some new notions.

If a Radon measure $\mu$ is real, then, along with the Azarin cluster set, there are two other important asymptotic characteristics of $\mu$, namely, its upper and lower densities $N(\alpha)$ and $\underline{N}(\alpha)$. These are functions on $[0, \infty)$, so, mathematically, these two objects are simpler than $\operatorname{Fr}[\mu]$. This is an advantage of density functions. On the other hand, as we shall see later, generally speaking, the set $\operatorname{Fr}[\mu]$ gives much more information about $\mu$ than $N(\alpha)$ and $\underline{N}(\alpha)$.

Let $\mu$ be a real Radon measure on $(0, \infty)$ whose distribution function is $\mu(r)$, and let $\rho(r)$ be a proximate order. The upper density of $\mu$ with respect to $\rho(r)$ is defined to be the quantity

$$
\begin{equation*}
N(\alpha)=\limsup _{r \rightarrow \infty} \frac{\mu(r+\alpha r)-\mu(r)}{V(r)} . \tag{3.30}
\end{equation*}
$$

For $\alpha>0$ we can also write

$$
\begin{equation*}
N(\alpha)=\underset{r \rightarrow \infty}{\limsup } \frac{\mu(r, r+\alpha r]}{V(r)} . \tag{3.31}
\end{equation*}
$$

Observe that formula (3.30) defines $N(\alpha)$ for $\alpha>-1$, and formula (3.31) defines it for $\alpha>0$. In the sequel, we agree that $N(0)=0$.

The lower density of $\mu$ with respect to $\rho(r)$ is defined similarly:

$$
\underline{N}(\alpha)=\liminf _{r \rightarrow \infty} \frac{\mu(r+\alpha r)-\mu(r)}{V(r)} .
$$

The properties of limits and those of a proximate order $\rho(r)$ imply the following inequalities for $N(\alpha)$ and $\underline{N}(\alpha)$ :

$$
\begin{align*}
& N(\alpha+\beta) \leq N(\alpha)+(1+\alpha)^{\rho} N\left(\frac{\beta}{1+\alpha}\right),  \tag{3.32}\\
& N(\alpha+\beta) \geq N(\alpha)+(1+\alpha)^{\rho} \underline{N}\left(\frac{\beta}{1+\alpha}\right),  \tag{3.33}\\
& \underline{N}(\alpha+\beta) \geq \underline{N}(\alpha)+(1+\alpha)^{\rho} \underline{N}\left(\frac{\beta}{1+\alpha}\right),  \tag{3.34}\\
& \underline{N}(\alpha+\beta) \leq \underline{N}(\alpha)+(1+\alpha)^{\rho} N\left(\frac{\beta}{1+\alpha}\right), \tag{3.35}
\end{align*}
$$

where $\rho=\rho(\infty)=\lim _{r \rightarrow \infty} \rho(r)$. We agree that if the right-hand side in some of these inequalities is $\infty-\infty$, then this particular inequality is a void statement. If $\mu \in \mathfrak{M}_{\infty}(\rho(r))$, we always have $-\infty<\underline{N}(\alpha) \leq N(\alpha)<\infty$. However, $\underline{N}(\alpha)$ and $N(\alpha)$ may happen to be finite for measures not belonging to $\mathfrak{M}_{\infty}(\rho(r))$. This emphasizes the importance of $N(\alpha)$ and $\underline{N}(\alpha)$ in the study of measures $\mu$. Recall that the theorem about the properties of $\operatorname{Fr}[\mu]$ was proved under the assumption that $\mu \in \mathfrak{M}_{\infty}(\rho(r))$.

Theorem 23. Let $\mu \in \mathfrak{M}_{\infty}(\rho(r))$ be a real measure, let $\operatorname{Fr}[\mu]$ be its cluster set, and let $N(\alpha)$ and $\underline{N}(\alpha)$ be its upper and lower densities. Then for every measure $\nu \in \operatorname{Fr}[\mu]$ there exists an at most countable set $E(\nu)$ such that for $a, b \notin E(\nu), 0<a<b<\infty$, we have

$$
\nu([a, b]) \leq a^{\rho} N\left(\frac{b}{a}-1\right), \quad \nu([a, b]) \geq a^{\rho} \underline{N}\left(\frac{b}{a}-1\right) .
$$

For every $a$ and $b, 0<a<b<\infty$, we have

$$
\nu([a, b]) \leq a^{\rho} \liminf _{\varepsilon \rightarrow+0} N\left(\frac{b}{a}-1+\varepsilon\right), \quad \nu([a, b]) \geq a^{\rho} \limsup _{\varepsilon \rightarrow+0} \underline{N}\left(\frac{b}{a}-1+\varepsilon\right) .
$$

Proof. Let $\nu=\lim _{n \rightarrow \infty} \mu_{t_{n}}$. Assume, moreover, that the limit $\hat{\nu}=\lim _{n \rightarrow \infty}|\mu|_{t_{n}}$. Put $E(\nu)=\{x \in(0, \infty): \widehat{\nu}(\{x\})>0\}$. The set $E(\nu)$ is at most countable. Now, let $[a, b] \subset(0, \infty), a, b \notin E(\nu)$. By Theorem 12, we obtain

$$
\nu([a, b])=\lim _{n \rightarrow \infty} \mu_{t_{n}}([a, b]) .
$$

Since $\nu(\{a\})=0$, the same theorem implies $\lim _{n \rightarrow \infty} \mu_{t_{n}}(\{a\})=0$. Therefore,

$$
\nu([a, b])=\lim _{n \rightarrow \infty} \mu_{t_{n}}((a, b]) \leq \limsup _{r \rightarrow \infty} \mu_{r}((a, b])=a^{\rho} N\left(\frac{b}{a}-1\right) .
$$

The inequality $\nu([a, b]) \geq a^{\rho} \underline{N}\left(\frac{b}{a}-1\right)$ is proved similarly. Consider the general case. Suppose that $a_{k} \rightarrow a, a_{k}<a$, and $b_{k} \rightarrow b, b_{k}>b$, where $a_{k}, b_{k} \notin E(\nu)$ and

$$
\lim _{k \rightarrow \infty} N\left(\frac{b_{k}}{a_{k}}-1\right)=\liminf _{\varepsilon \rightarrow+0} N\left(\frac{b}{a}-1+\varepsilon\right) .
$$

We have

$$
\nu([a, b])=\lim _{k \rightarrow \infty} \nu\left(\left[a_{k}, b_{k}\right]\right) \leq \lim _{k \rightarrow \infty} a_{k}^{\rho} N\left(\frac{b_{k}}{a_{k}}-1\right)=a^{\rho} \liminf _{\varepsilon \rightarrow+0} N\left(\frac{b}{a}-1+\varepsilon\right) .
$$

Similarly,

$$
\nu([a, b]) \geq a^{\rho} \limsup _{\varepsilon \rightarrow+0} \underline{N}\left(\frac{b}{a}-1+\varepsilon\right) .
$$

We observe the following consequence of the above statement.
Remark 2. Let $\mu \in \mathfrak{M}_{\infty}(\rho(r))$, and let $N(\alpha) \equiv \underline{N}(\alpha) \equiv 0$. Then $\operatorname{Fr}[\mu]=\{0\}$.
It should be noted that the relations $\underline{N}(\alpha) \equiv N(\alpha) \equiv 0$ and

$$
\liminf _{r \rightarrow \infty} \frac{|\mu|([a r, b r])}{V(r)}>0
$$

are compatible.
A measure $\mu$ is said to be continuous if $x \quad \nu(\{x\})=0$ for every $x$. Sometimes, it is important that the measures in the cluster set $\operatorname{Fr}[\mu]$ be continuous. We present a condition sufficient for continuity.

Theorem 24. Let $N(\alpha)$ and $\underline{N}(\alpha)$ be the density functions with respect to a proximate order $\rho(r)$ for a measure $\mu \in \mathfrak{M}_{\infty}(\rho(r))$. If $N(\alpha)$ and $\underline{N}(\alpha)$ are continuous on $[0, \infty)$, then every measure $\nu$ in the set $\operatorname{Fr}[\mu]$ is continuous.
Proof. Let $\nu \in \operatorname{Fr}[\mu]$, and let $x \in(0, \infty)$. Take a strictly monotone increasing sequence $a_{n}$ and a strictly monotone decreasing sequence $b_{n}$ that converge to $x$ and have the property that $a_{n}, b_{n} \notin E(\nu)$. Theorem 23 yields

$$
\begin{aligned}
& \nu(\{x\})=\lim _{n \rightarrow \infty} \nu\left(\left[a_{n}, b_{n}\right]\right) \leq \lim _{n \rightarrow \infty} a_{n}^{\rho} N\left(\frac{b_{n}}{a_{n}}-1\right)=0, \\
& \nu(\{x\})=\lim _{n \rightarrow \infty} \nu\left(\left[a_{n}, b_{n}\right]\right) \geq \lim _{n \rightarrow \infty} a_{n}^{\rho} \underline{N}\left(\frac{b_{n}}{a_{n}}-1\right)=0 .
\end{aligned}
$$

For positive measures $\mu$, we have a criterion for the continuity of all measures in $\operatorname{Fr}[\mu]$.
Theorem 25. Suppose that $\mu$ is a positive measure and $\mu \in \mathfrak{M}_{\infty}(\rho(r))$. Let $N(\alpha)$ be the upper density of $\mu$ with respect to the proximate order $\rho(r)$. All measures in $\operatorname{Fr}[\mu]$ are continuous if and only if $N(\alpha) \rightarrow 0$ as $\alpha \rightarrow+0$.

Proof. Suppose that $N(\alpha) \rightarrow 0$ as $\alpha \rightarrow+0$. Then also $\underline{N}(\alpha) \rightarrow 0$ as $\alpha \rightarrow+0$. In this case, the functions $N(\alpha)$ and $\underline{N}(\alpha)$ are continuous. By the preceding theorem, an arbitrary measure in $\operatorname{Fr}[\mu]$ is continuous.

To prove the necessity of the condition $N(\alpha) \rightarrow 0$ as $\alpha \rightarrow+0$, we suppose the contrary: $N(\alpha) \rightarrow 2 a$ with $a>0$ as $\alpha \rightarrow+0$. Then there exist sequences $r_{n} \rightarrow \infty$ and $\varepsilon_{n} \downarrow 0$ such that $\mu\left(\left(r_{n},\left(1+\varepsilon_{n}\right) r_{n}\right]\right)>a V\left(r_{n}\right)$. We may assume that, moreover, $\mu_{r_{n}} \rightarrow \nu$. Let $0<a_{k}<1<b_{k}, a_{k} \rightarrow 1, b_{k} \rightarrow 1, \nu\left(\left\{a_{k}\right\}\right)=0, \nu\left(\left\{b_{k}\right\}\right)=0$. By Theorem 12 we see that

$$
\nu(\{1\})=\lim _{k \rightarrow \infty} \nu\left(\left[a_{k}, b_{k}\right]\right)=\lim _{k \rightarrow \infty} \lim _{n \rightarrow \infty} \mu_{r_{n}}\left(\left(a_{k}, b_{k}\right]\right)=\lim _{k \rightarrow \infty} \lim _{n \rightarrow \infty} \frac{\mu\left(\left(a_{k} r_{n}, b_{k} r_{n}\right]\right)}{V\left(r_{n}\right)} .
$$

If $k$ is fixed and $n$ is sufficiently large, then $\left(r_{n},\left(1+\varepsilon_{n}\right) r_{n}\right] \subset\left(a_{k} r_{n}, b_{k} r_{n}\right)$. Therefore,

$$
\lim _{n \rightarrow \infty} \frac{\mu\left(\left(a_{k} r_{n}, b_{k} r_{n}\right]\right)}{V\left(r_{n}\right)} \geq \limsup _{n \rightarrow \infty} \frac{\mu\left(\left(r_{n},\left(1+\varepsilon_{n}\right) r_{n}\right]\right)}{V\left(r_{n}\right)} \geq a
$$

Consequently, $\nu(\{1\}) \geq a$. This contradicts the continuity of all measures in $\operatorname{Fr}[\mu]$.

We denote by $M(\rho, \sigma)$ the set of real or complex Radon measures $\mu$ with

$$
\begin{aligned}
&|\mu|((0, r]) \leq \sigma r^{\rho} \text { for } 0<r<\infty, \rho>0 \\
&|\mu|([a, b]) \leq \sigma \ln \frac{b}{a} \text { for } 0<a<b<\infty, \rho=0 \\
&|\mu|((r, \infty)) \leq \sigma r^{\rho} \\
& \text { for } 0<r<\infty, \rho<0
\end{aligned}
$$

Theorem 26. Let $\mu \in \mathfrak{M}_{\infty}(\rho(r))$. Then there exists $\sigma>0$ such that $\operatorname{Fr}[\mu] \subset M(\rho, \sigma)$.
Proof. Denote by $N_{1}(\alpha)$ the upper density of $|\mu|$ with respect to the proximate order $\rho(r)$. Since $\mu \in \mathfrak{M}_{\infty}(\rho(r))$, we have $N_{1}(\alpha)<\infty$. Take $q>1$ arbitrarily. Applying Theorems 22 and 23, we obtain

$$
\begin{aligned}
|\nu|((0, r]) \leq \widehat{\nu}((0, r]) & =\sum_{n=0}^{\infty} \widehat{\nu}\left(\left(\frac{r}{q^{n+1}}, \frac{r}{q^{n}}\right]\right) \\
& \leq r^{\rho} N_{1}(q-1+0) \sum_{n=0}^{\infty} \frac{1}{q^{(n+1) \rho}}=\frac{N_{1}(q-1+0)}{q^{\rho}-1} r^{\rho}
\end{aligned}
$$

provided $\rho>0$. Similarly, for $\rho<0$ we obtain

$$
\begin{aligned}
|\nu|((r, \infty)) \leq \widehat{\nu}((r, \infty)) & =\sum_{n=0}^{\infty} \widehat{\nu}\left(\left(q^{n} r, q^{n+1} r\right]\right) \\
& \leq r^{\rho} N_{1}(q-1+0) \sum_{n=0}^{\infty} q^{n \rho}=\frac{N_{1}(q-1+0)}{1-q^{\rho}} r^{\rho}
\end{aligned}
$$

The case of $\rho=0$ is treated in the same way.
Remark 3. The above proof shows that the relation $\operatorname{Fr}[\mu] \subset M(\rho, \sigma)$ is ensured with

$$
\widehat{\sigma}=\inf _{q>1} \frac{N_{1}(q-1+0)}{\left|q^{\rho}-1\right|}
$$

in the role of $\sigma$.
Note that, by Theorems 6 and 14 in [16], for $\rho>0$ we have

$$
\hat{\sigma}=\lim _{q \rightarrow \infty} \frac{N_{1}(q-1+0)}{q^{\rho}-1}=\limsup _{r \rightarrow \infty} \frac{|\mu|((1, r])}{V(r)} .
$$

The case of $\rho<0$ was also considered in [16.
We present some examples of calculation of $\operatorname{Fr}[\mu]$.
Lemma 4. Let $\rho(r)$ be a proximate order, and let $\mu$ be a measure on $(0, \infty)$ with the density $\frac{V(x)}{x}$. Then the cluster set $\operatorname{Fr}[\mu]$ for $\mu$ with respect to $\rho(r)$ consists of only one measure $\nu$ with $d \nu(x)=x^{\rho-1} d x$.

Proof. Let $\varphi \in \Phi$. Using Theorem 5, we obtain

$$
\begin{aligned}
\left(\varphi, \mu_{t}\right) & =\int_{0}^{\infty} \varphi(x) d \mu_{t}(x)=\frac{1}{V(t)} \int_{0}^{\infty} \varphi(x) \frac{V(x t)}{x} d x \\
\lim _{t \rightarrow \infty}\left(\varphi, \mu_{t}\right) & =\int_{0}^{\infty} \varphi(x) x^{\rho-1} d x
\end{aligned}
$$

Lemma 5. Let $\rho(r)$ be a proximate order, and let $\mu$ be a measure on $(0, \infty)$ with the density $x^{i \lambda_{0}} \frac{V(x)}{x}$. Then

$$
\operatorname{Fr}[\mu]=\left\{\nu: d \nu(u)=e^{i \lambda_{0} c} u^{i \lambda_{0}+\rho-1} d u: c \in(-\infty, \infty)\right\} .
$$

Proof. Let $\varphi \in \Phi$. We have

$$
\left(\varphi, \mu_{r}\right)=\frac{1}{V(r)} \int_{0}^{\infty} \varphi\left(\frac{x}{r}\right) x^{i \lambda_{0}} \frac{V(x)}{x} d x=r^{i \lambda_{0}} \int_{0}^{\infty} \varphi(u) u^{i \lambda_{0}} \frac{V(u r)}{V(r)} \frac{1}{u} d u .
$$

and the claim easily follows by Theorem 5
A measure $\mu$ is said to be periodic of order $\rho$ with period $T>1$ if for this $T$ and an arbitrary Borel set $E$ we have

$$
\begin{equation*}
\mu(T E)=T^{\rho} \mu(E) . \tag{3.36}
\end{equation*}
$$

Lemma 6. Let $\mu$ be a locally finite periodic measure of order $\rho$ and with period $T>1$. Then $\mu \in \mathfrak{M}_{\infty}(\rho)$ and $\operatorname{Fr}[\mu]=\left\{\mu_{t}: 1 \leq t<T\right\}$.
Proof. The relation $\mu \in \mathfrak{M}_{\infty}(\rho)$ is obvious. Let $t \in[1, T)$, and let $t_{n}=t T^{n}$. Then $\mu_{t_{n}}=\mu_{t}$ by (3.36). Consequently, $\mu_{t_{n}}=\mu_{t}$. Thus, we have proved the inclusion $\left\{\mu_{t}: t \in[1, T)\right\} \subset \operatorname{Fr}[\mu]$.

Now, if $t_{n}$ be a sequence with $\mu_{t_{n}} \rightarrow \nu$, then there is an integer $m(n)$ such that $t_{n}=\tau_{n} T^{m(n)}$ and $\tau_{n} \in[1, T)$. By (3.36), we have $\mu_{t_{n}}=\mu_{\tau_{n}}$. By the Bolzano-Weierstrass theorem, there is a convergent subsequence $\tau_{n_{k}}$ with $\lim _{k \rightarrow \infty} \tau_{n_{k}}=\tau$ and $\tau \in[1, T]$. Surely, $\mu_{\tau_{n_{k}}} \rightarrow \mu_{\tau}$. Then $\nu=\lim _{k \rightarrow \infty} \mu_{t_{n_{k}}}=\lim _{k \rightarrow \infty} \mu_{\tau_{n_{k}}}=\mu_{\tau}$. Together with the relations $\mu=\mu_{1}=\mu_{T}$, this yields $\operatorname{Fr}[\mu] \subset\left\{\mu_{t}: t \in[1, T)\right\}$.
Remark 4. Let $\alpha$ be an arbitrary strictly positive number, and let $n$ be an integer. Under the assumptions of Lemma [6 we have

$$
\operatorname{Fr}[\mu]=\left\{\mu_{t}: t \in\left[\alpha T^{n}, \alpha T^{n+1}\right)\right\} .
$$

The case of $\alpha=1, n=0$ corresponds to the claim of Lemma 6 ,
Lemma 7. Suppose that $\rho>0$ and $R_{n}$ is a strictly monotone increasing sequence such that $R_{1} \geq 1$ and $\lim _{n \rightarrow \infty} \frac{R_{n}}{R_{n-1}}=\infty$. Define a measure $\mu$ by the formula

$$
\mu=\sum_{n=1}^{\infty} R_{n}^{\rho} \delta\left(x-R_{n}\right)
$$

Then the cluster set of $\mu$ with respect to the proximate order $\operatorname{Fr}[\mu]$ has the form

$$
\begin{equation*}
\operatorname{Fr}[\mu]=\left\{t^{\rho} \delta(x-t): t \in(0, \infty)\right\} \cup\{0\} . \tag{3.37}
\end{equation*}
$$

Proof. By the Stolz theorem, from the condition $R_{n-1}=o\left(R_{n}\right)$ we easily deduce that

$$
\sum_{k=1}^{n} R_{k}^{\rho} \sim R_{n}^{\rho} \quad(n \rightarrow \infty)
$$

Let $r \geq R_{1}$ be arbitrary, and let $n$ be the maximal number with $R_{n} \leq r$. Then

$$
\mu((0, r])=\sum_{k=1}^{n} R_{k}^{\rho}=(1+o(1)) R_{n}^{\rho} \leq(1+o(1)) r^{\rho} \quad(r \rightarrow \infty) .
$$

Consequently, $\mu \in \mathfrak{M}_{\infty}(\rho)$.
Now, let $t>0$ be arbitrary, and let $t_{n}=\frac{1}{t} R_{n}$. Let $\varphi$ be an arbitrary function in $\Phi$. Then for all $n$ sufficiently large we have

$$
\left(\mu_{t_{n}}, \varphi\right)=\frac{t^{\rho}}{R_{n}^{\rho}} \int_{0}^{\infty} \varphi\left(\frac{u t}{R_{n}}\right) d \mu(u)=t^{\rho} \varphi(t)=\left(t^{\rho} \delta(x-t), \varphi(x)\right) .
$$

It follows that $\mu_{t_{n}} \rightarrow t^{\rho} \delta(x-t)$. Similarly, we prove that if $t_{n}=\frac{1}{\tau_{n}} R_{n}$, where $\tau_{n}$ is a sequence satisfying $\tau_{n} \rightarrow \infty, \frac{\tau_{n} R_{n-1}}{R_{n}} \rightarrow 0$, then $\mu_{t_{n}} \rightarrow 0$.

We denote by $H$ the right-hand side of (3.37). We have proved that $H \subset \operatorname{Fr}[\mu]$.

Now, let $\nu$ be an arbitrary measure in $\operatorname{Fr}[\mu]$. Then $\nu=\lim \mu_{t_{n}}$, where $t_{n} \rightarrow \infty$. We may assume that, moreover, every segment $\left[R_{k-1}, R_{k+1}\right]$ contains at most one point $t_{n}$. We find a number $m(n)$ such that $\ln R_{m(n)}$ is the closest point to $\ln t_{n}$ in the sequence $\ln R_{k}$ (if there are two points with this property, we take the smaller index for $m(n)$ ). The additional restriction on the sequence $t_{n}$ ensures that $m\left(n_{1}\right) \neq m\left(n_{2}\right)$ for $n_{1} \neq n_{2}$. Thus, there is a unique representation of $t_{n}$ in the form

$$
\begin{equation*}
t_{n}=\frac{1}{\tau_{n}} R_{m(n)} . \tag{3.38}
\end{equation*}
$$

In addition, we may assume that the sequence $\tau_{n}$ converges either to a finite limit or to infinity. Suppose that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \tau_{n}=\infty \tag{3.39}
\end{equation*}
$$

We have

$$
\begin{align*}
& \ln t_{n}-\ln R_{m(n)-1} \geq \ln R_{m(n)}-\ln t_{n}, \quad t_{n} \geq \sqrt{\frac{R_{m(n)}}{R_{m(n)-1}}}  \tag{3.40}\\
& \tau_{n} \leq \sqrt{R_{m(n)-1} R_{m(n)}}, \quad \frac{\tau_{n} R_{m(n)-1}}{R_{m(n)}} \leq \sqrt{\frac{R_{m(n)-1}}{R_{m(n)}}}, \quad \lim _{n \rightarrow \infty} \frac{\tau_{n} R_{m(n)-1}}{R_{m(n)}}=0 .
\end{align*}
$$

By (3.38)-(3.40), we see that $\mu_{t_{n}} \rightarrow 0$. Hence, $\nu=0$ in the case in question.
Similarly, we prove that if $\tau_{n} \rightarrow 0$, then also $\mu_{t_{n}} \rightarrow 0$. This follows from the easy relation

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{\tau_{n} R_{m(n)+1}}{R_{m(n)}}=\infty \tag{3.41}
\end{equation*}
$$

Now, let $\tau_{n} \rightarrow \tau \in(0, \infty)$. We have already proved that $\mu_{\tilde{t}_{n}} \rightarrow \tau^{\rho} \delta(x-\tau)$, where $\tilde{t}_{n}=\frac{1}{\tau} R_{m(n)}$. Applying Theorem 21 to the sequence $\tilde{t}_{n}$, we obtain $\mu_{t_{n}} \rightarrow \tau^{\rho} \delta(x-\tau)$. Consequently, $\nu=\tau^{\rho} \delta(x-\tau)$ in this case. Thus, we have proved that always $\nu \in H$. So, the inclusion $\operatorname{Fr}[\mu] \subset H$ is established and the lemma follows.

Note that the proof of this lemma can be extracted from [17.
From a cluster set theory viewpoint, the simplest measures $\mu \in \mathfrak{M}_{\infty}(\rho(r))$ are those for which $\operatorname{Fr}[\mu]$ consists of only one measure $\nu$. Such measures are said to be regular (or regular in the sense of Azarin). We study possible forms of the limit measure for a regular measure $\mu$.

Theorem 27. Let $\rho(r)$ be a proximate order, let $\rho=\lim _{n \rightarrow \infty} \rho(r)$, and let $\mu \in \mathfrak{M}_{\infty}(\rho(r))$. Suppose that $\mu$ is regular and $\{\nu\}=\operatorname{Fr}[\mu]$. Then there exists a complex number $c$ such that $d \nu(r)=c r^{\rho-1} d r$.

Proof. If a complex Radon measure $\mu=\mu_{1}+i \mu_{2}$ is regular, then $\mu_{1}$ and $\mu_{2}$ are also regular. So, it suffices to prove the theorem for real Radon measures. Since $\operatorname{Fr}[\mu]$ is invariant under $F_{t}$, we have $\nu=F_{t} \nu$. Let $0<a<b<\infty$. In particular, then $\nu((a, b])=$ $\left(F_{t} \nu\right)((a, b])=t^{-\rho} \nu((a t, b t])$. Taking $t=\frac{1}{a}$, we obtain $\nu((a, b])=a^{\rho} \nu\left(\left(1, \frac{b}{a}\right]\right)$. Denote $N(s)=\nu((1,1+s]), s>0$. Then

$$
N\left(s_{1}+s_{2}\right)=\nu\left(\left(1,1+s_{1}\right]\right)+\nu\left(\left(1+s_{1}, 1+s_{1}+s_{2}\right]\right)=N\left(s_{1}\right)+\left(1+s_{1}\right)^{\rho} N\left(\frac{s_{2}}{1+s_{1}}\right) .
$$

The functions $N$ satisfying this identity were called $\rho$-additive in [16. Since $\nu$ is a locally finite measure on $(0, \infty)$, we see that $N$ is bounded on $(0,1]$. By [16, Theorem 4], there exists a unique real number $c$ such that

$$
N(s)=\frac{c}{\rho}\left((1+s)^{\rho}-1\right), \quad \rho \neq 0, \quad N(s)=c \ln (1+s), \quad \rho=0 .
$$

We have proved that

$$
\nu((1, b])=\frac{c}{\rho}\left(b^{\rho}-1\right), \quad \rho \neq 0, \quad \nu((1, b])=c \ln b, \quad \rho=0
$$

for $b>1$. There identities easily imply the claim.
The next property of regular measures is an easy consequence of definitions.
We shall say that a net $\mu_{R}(R \in(0, \infty))$ of measures converges coarsely to $\mu$ as $R \rightarrow \infty$ if for every $\varphi \in \Phi$ we have $\lim _{R \rightarrow \infty}\left(\mu_{R}, \varphi\right)=(\mu, \varphi)$.

Theorem 28. Let $\rho(r)$ be a proximate order, and let $\mu \in \mathfrak{M}_{\infty}(\rho(r))$ be a regular measure with $\{\nu\}=\operatorname{Fr}[\mu]$. Then

$$
\begin{equation*}
\nu=\lim _{R \rightarrow \infty} \mu_{R} \tag{3.42}
\end{equation*}
$$

Proof. We start with proving that the limit

$$
\begin{equation*}
\lim _{R \rightarrow \infty} \int_{0}^{\infty} \varphi(x) d \mu_{R}(x)=a(\varphi) \tag{3.43}
\end{equation*}
$$

exists for every $\varphi \in \Phi$. If this is not the case, there exists a function $\varphi \in \Phi$ and two sequences $r_{n} \rightarrow \infty$ and $R_{n} \rightarrow \infty$ such that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \int_{0}^{\infty} \varphi(x) d \mu_{r_{n}}(x) \neq \lim _{n \rightarrow \infty} \int_{0}^{\infty} \varphi(x) d \mu_{R_{n}}(x) \tag{3.44}
\end{equation*}
$$

Since the family of measures $\mu_{R}, R \geq 1$, is compact, without loss of generality we may assume that the sequences $\mu_{r_{n}}$ and $\mu_{R_{n}}$ converge coarsely. Since there is only one measure $\nu$ in the cluster set, we have $\mu_{r_{n}} \rightarrow \nu$ and $\mu_{R_{n}} \rightarrow \nu$. The definition of coarse convergence shows that

$$
\begin{align*}
& \lim _{n \rightarrow \infty} \int_{0}^{\infty} \varphi(x) d \mu_{r_{n}}(x)=\int_{0}^{\infty} \varphi(x) d \nu(x)  \tag{3.45}\\
& \lim _{n \rightarrow \infty} \int_{0}^{\infty} \varphi(x) d \mu_{R_{n}}(x)=\int_{0}^{\infty} \varphi(x) d \nu(x)
\end{align*}
$$

This contradicts (3.44). Thus, (3.43) is proved, implying that $a(\varphi)=\int_{0}^{\infty} \varphi(x) d \nu(x)$.
In the following theorem, the positive regular measures are described.
Theorem 29. Let $\rho(r)$ be an arbitrary proximate order, and let $\mu$ be a positive measure on the semiaxis $(0, \infty)$ with $\mu \in \mathfrak{M}_{\infty}(\rho(r))$. Then $\mu$ is regular with respect to $\rho(r)$ if and only if one of the following conditions (depending on the sign of $\rho$ ) is fulfilled:

$$
\begin{align*}
\lim _{R \rightarrow \infty} \frac{\mu((1, R])}{V(R)} & =c, \quad \rho>0  \tag{3.46}\\
\lim _{R \rightarrow \infty} \frac{\mu([R, \infty))}{V(R)} & =c, \quad \rho<0  \tag{3.47}\\
\lim _{R \rightarrow \infty} \frac{\mu((a R, b R])}{V(R)} & =c \ln \frac{b}{a}, \quad \rho=0 \tag{3.48}
\end{align*}
$$

for every $a$ and $b, 0<a<b<\infty$.
Writing the corresponding Riemann-Stiltjes integral sums, it is easy to check that identities (3.46)-(3.48) imply the identity

$$
\begin{equation*}
\lim _{R \rightarrow \infty} \int_{0}^{\infty} \varphi(x) d \mu_{R}(x)=\int_{0}^{\infty} \varphi(x) d \nu(x) \tag{3.49}
\end{equation*}
$$

where $d \nu(x)=c|\rho| x^{\rho-1} d x$ if $\rho \neq 0$, and $d \nu(x)=\frac{c}{x} d x$ if $\rho=0$. Clearly, (3.49) implies the regularity of $\mu$.

Conversely, let $\mu$ be a regular measure. Then by Theorem 28 we have $\lim _{R \rightarrow \infty} \mu_{R}=\nu$. By Theorem 26, $\nu$ is a continuous measure. Theorem 12 shows that $\lim _{R \rightarrow \infty} \mu_{R}([a, b])=$ $\nu([a, b])$. This is equivalent to (3.46)-(3.48).

Theorem 29 fails for real Radon measures. By Theorem 27 and Lemma 6 , in order to describe the real regular measures, it suffices to describe the real regular measures with zero cluster set. This is done in the following theorem.

Theorem 30. Let $\mu \in \mathfrak{M}_{\infty}(\rho(r))$. Then $\operatorname{Fr}[\mu]=\{0\}$ if and only if there exists a monotone increasing sequence $r_{n}$ tending to infinity and such that

$$
\lim _{n \rightarrow \infty} \frac{r_{n+1}}{r_{n}}=1, \quad \lim _{R \rightarrow \infty} \frac{1}{V(R)} \sum_{\left(r_{n}, r_{n+1}\right] \cap[R, 2 R] \neq \varnothing}\left|\mu\left(\left(r_{n}, r_{n+1}\right]\right)\right|=0 .
$$

We know the proof, but it is much harder than that of Theorem 30 and falls out the scope of the present paper. (However, it should be noted that the "if" part is fairly easy.) We have presented the statement for completeness only.

## §4. Abel theorems for integrals

We proceed to the exposition of results that can be called Abel theorems for integrals. The main objects of our study are the functions $\Psi(r), J(r)$, the measure $s$, and the set $L(J, \infty)$ (see formula (1.2) and the text after it).

Our goal in this section is to describe how the properties of $J(r)$ depend on restrictions on the kernel $K$ and the measure $\mu$.

We start with a simple statement involving nevertheless the main idea of the method: to use the cluster set $\operatorname{Fr}[\mu]$ in the study of the properties of $J(r)$.

Theorem 31. Let $\rho(r)$ be a proximate order, let $\mu \in \mathfrak{M}_{\infty}(\rho(r))$, and let $K$ be a continuous compactly supported kernel on $(0, \infty)$. Then

$$
\begin{equation*}
L(J, \infty)=\left\{\int_{0}^{\infty} K(u) d \nu(u): \nu \in \operatorname{Fr}[\mu]\right\} . \tag{4.50}
\end{equation*}
$$

Proof. Let $H$ denote the right-hand side of (4.50), and let $\nu$ be an arbitrary measure in $\operatorname{Fr}[\mu]$. There exists a sequence $r_{n} \rightarrow \infty$ such that $\mu_{r_{n}} \rightarrow \nu$. By the definition of coarse convergence, we have $J\left(r_{n}\right) \rightarrow \int_{0}^{\infty} K(u) d \nu(u)$. We have proved the inclusion $H \subset L(J, \infty)$.

Now, let $r_{n} \rightarrow \infty$ be such that the sequence $J\left(r_{n}\right)$ converges (in a proper or an improper sense; the latter means that it converges to infinity). By Lemma 3, the semitrajectory $\mu_{r}, r \geq 1$, is compact, so we may assume without loss of generality that the sequence $\mu_{r_{n}}$ converges coarsely to a measure $\nu$. Now, $\nu \in \operatorname{Fr}[\mu]$ by the definition of $\operatorname{Fr}[\mu]$. By the above, $J\left(r_{n}\right) \rightarrow \int_{0}^{\infty} K(u) d \nu(u)$. We have proved the inclusion $L(J, \infty) \subset H$ and, with it, the theorem.

Many further results in this section will be certain versions of Theorem 31. We shall consider various restrictions on the kernel $K$ and the measure $\mu$. To begin with, we understand what happens if we renounce the continuity of $K$. In this case, the set $\operatorname{Fr}[\mu]$ does not determine $L(J, \infty)$ any longer.

Let $\mu \in \mathfrak{M}_{\infty}(\rho(r))$. The extended cluster set $\widehat{\operatorname{Fr}}[\mu]$ for $\mu$ is defined to be the set of pairs $\left(\nu_{1}, \nu_{2}\right)$ of measures such that there exists a sequence $r_{n} \rightarrow \infty$ with $\mu_{r_{n}} \rightarrow \nu, \mu^{1}{ }_{r_{n}} \rightarrow \nu_{1}$, and $\mu_{r_{n}}^{2} \rightarrow \nu_{2}$, where $\mu_{r_{n}}^{1}$ is the restriction of $\mu_{r_{n}}$ to ( 0,1$]$, and $\mu_{r_{n}}^{2}$ is the restriction of $\mu_{r_{n}}$ to $(1, \infty)$.

Theorem 32. Let $\rho(r)$ be an arbitrary proximate order, let $\mu \in \mathfrak{M}_{\infty}(\rho(r))$, and let $K$ be a compactly supported function on $(0, \infty)$ continuous everywhere except the point 1. Moreover, suppose that $K(t)$ has a jump and is continuous from the left at 1 . Then

$$
L(J, \infty)=\left\{\int_{(0,1]} K(t) d \nu_{1}(t)+\int_{[1, \infty)} \tilde{K}(t) d \nu_{2}(t):\left(\nu_{1}, \nu_{2}\right) \in \widehat{\operatorname{Fr}}[\mu]\right\}
$$

where $\widetilde{K}$ is the continuous extension of $K(t)$ from $(1, \infty)$ to $[1, \infty)$.
Note that, though the measure $\mu_{r_{n}}^{2}$ has no mass at 1 , the measure $\nu_{2}$ may have some mass at this point. This is why $\widetilde{K}(t)$ cannot be replaced by $K(t)$ in this statement.
Proof. We argue as in Theorem 32. The difference is that we must use identities of the following kind. Let $K_{1}(t)$ be a continuous compactly supported extension of $K(t)$ from $(0,1]$ to $(0, \infty)$. Then

$$
\int_{0}^{1} K(t) d \nu_{1}(t)=\int_{0}^{1} K_{1}(t) d \nu_{1}(t)=\int_{0}^{\infty} K_{1}(t) d \nu_{1}(t)=\lim _{n \rightarrow \infty} \int_{0}^{\infty} K_{1}(t) d \mu_{r_{n}}^{1}(t) .
$$

Under some restrictions on $\mu$, the requirement of continuity for $K$ becomes redundant. If $K$ is discontinuous and $\mu$ is absolutely continuous, the integral $\int_{0}^{\infty} K(t) d \mu(t)$ should be understood as $\int_{0}^{\infty} K(t) \mu^{\prime}(t) d t$.
Theorem 33. Suppose that $\rho(r)$ is an arbitrary proximate order, $\mu$ is a Radon measure on $(0, \infty)$ with density $\mu^{\prime}(r)$ such that $\left|\mu^{\prime}(r)\right| \leq M \frac{V(r)}{r}, r \in[1, \infty)$, and $K$ is a compactly supported kernel belonging to $L_{1}(0, \infty)$. Then

$$
\begin{equation*}
L(J, \infty)=\left\{\int_{0}^{\infty} K(u) d \nu(u): \nu \in \operatorname{Fr}[\mu]\right\} . \tag{4.51}
\end{equation*}
$$

Proof. Let $N_{1}(t)$ be the upper density for $|\mu|$. We have

$$
\begin{aligned}
N_{1}(\alpha)=\limsup _{r \rightarrow \infty} \frac{|\mu|((r,(1+\alpha) r])}{V(r)} & \leq \lim _{r \rightarrow \infty} M \int_{r}^{(1+\alpha) r} \frac{1}{t} \frac{V(t)}{V(r)} d t \\
& =M \lim _{r \rightarrow \infty} \int_{1}^{1+\alpha} \frac{1}{u} \frac{V(u r)}{V(r)} d u=M \frac{(1+\alpha)^{\rho}-1}{\rho} .
\end{aligned}
$$

By Theorem [25, all measures in the set $\operatorname{Fr}[|\mu|]$ are continuous. All measures in $\operatorname{Fr}[\mu]$ are also continuous. By Theorem [12] it follows that if $\nu=\lim _{n \rightarrow \infty} \mu_{t_{n}}$, then

$$
\nu([a, b])=\lim _{n \rightarrow \infty} \mu_{t_{n}}([a, b])
$$

for every segment $[a, b] \subset(0, \infty)$. Therefore,

$$
|\nu([a, b])|<\limsup _{r \rightarrow \infty} \int_{a}^{b} d|\mu|_{t_{n}}(t) \leq M \lim _{n \rightarrow \infty} \int_{a}^{b} \frac{1}{t} \frac{V\left(t_{n} t\right)}{V\left(t_{n}\right)} d t=M \frac{b^{\rho}-a^{\rho}}{\rho}
$$

This shows that every measure $\nu$ belonging to $\operatorname{Fr}[\mu]$ is absolutely continuous and $\left|\nu^{\prime}(x)\right| \leq$ $M x^{\rho-1}$. Let $r_{n} \rightarrow \infty$ be a sequence with $\mu_{r_{n}} \rightarrow \nu$. Let $\varepsilon$ be a strictly positive number and $K_{1}$ and compactly supported function on $(0, \infty)$ such that

$$
\int_{0}^{\infty}\left|K(t)-K_{1}(t)\right| d t<\varepsilon
$$

Then

$$
\begin{align*}
& \left|\int_{0}^{\infty} K(t) d \mu_{r_{n}}(t)-\int_{0}^{\infty} K(t) d \nu(t)\right| \leq\left|\int_{0}^{\infty}\left(K(t)-K_{1}(t)\right) d \mu_{r_{n}}(t)\right| \\
& \quad+\left|\int_{0}^{\infty}\left(K(t)-K_{1}(t)\right) d \nu(t)\right|+\left|\int_{0}^{\infty} K_{1}(t) d \mu_{r_{n}}(t)-\int_{0}^{\infty} K_{1}(t) d \nu(t)\right| \tag{4.52}
\end{align*}
$$

We have

$$
\begin{aligned}
& \left|\int_{0}^{\infty}\left(K(t)-K_{1}(t)\right) d \mu_{r_{n}}(t)\right| \leq M \int_{0}^{\infty}\left|K(t)-K_{1}(t)\right| \frac{V\left(r_{n} t\right)}{t V\left(r_{n}\right)} d t \\
& \quad \leq 2 M \int_{0}^{\infty}\left|K(t)-K_{1}(t)\right| t^{\rho-1} d t \leq 2 M \max \left\{t^{\rho-1}: t \in[a, b]\right\} \varepsilon
\end{aligned}
$$

where $[a, b] \subset(0, \infty)$ is a segment such that $\operatorname{supp} K, \operatorname{supp} K_{1} \subset[a, b]$. The second summand on the right in (4.52) admits a similar estimate. The third summand tends to zero as $n \rightarrow \infty$. If follows that

$$
\lim _{n \rightarrow \infty} J\left(r_{n}\right)=\int_{0}^{\infty} K(u) d \nu(u)
$$

Thus, we have proved the inclusion $H \subset L(J, \infty)$, where $H$ is the right-hand side of (4.51). The proof is finished by an argument similar to that in the proof of Theorem 31.

In what follows, we replace the assumption that $K$ is compactly supported in Theorem 31-33 by a weaker restriction on $K$, which will not influence the claims. However, stronger restrictions should be imposed on $\mu$ for this.

We start with some definitions.
A triple $(K, \rho(r), \mu)$ is said to admit neutralization of zero if

$$
\lim _{\varepsilon \rightarrow+0} \limsup _{r \rightarrow \infty}\left|\int_{0}^{\varepsilon} K(t) d \mu_{r}(t)\right|=0
$$

This triple is said to admit neutralization of infinity if

$$
\lim _{N \rightarrow \infty} \limsup _{r \rightarrow \infty}\left|\int_{N}^{\infty} K(t) d \mu_{r}(t)\right|=0
$$

Note that if $K$ is a compactly supported kernel, then the triple ( $K, \rho(r), \mu$ ) admits neutralization both of zero and of infinity for every proximate order $\rho(r)$ and every Radon measure $\mu$ on $(0, \infty)$.

These definitions make it possible to simplify many statements because they permit us not to detalize conditions on $K$ and $\mu$ that ensure neutralization of zero or of infinity. Detalization of such conditions can be described in separate statements.

Lemma 8. If a triple ( $K, \rho(r), \mu$ ) admits neutralization of zero, then

$$
\lim _{\substack{\varepsilon_{1} \rightarrow+0 \\ \varepsilon_{2} \rightarrow+0}} \limsup _{r \rightarrow \infty}\left|\int_{\varepsilon_{1}}^{\varepsilon_{2}} K(t) d \mu_{r}(t)\right|=0
$$

Lemma 9. If a triple $(K, \rho(r), \mu)$ admits neutralization of infinity, then

$$
\lim _{\substack{N_{1} \rightarrow \infty \\ N_{2} \rightarrow \infty}} \limsup _{r \rightarrow \infty}\left|\int_{N_{1}}^{N_{2}} K(t) d \mu_{r}(t)\right|=0
$$

Lemmas 8 and 9 are obvious.
Theorem 34. Let $\rho(r)$ be an arbitrary proximate order, let $\mu \in \mathfrak{M}(\rho(r))$, and let $K$ be a continuous kernel on $(0, \infty)$. If the triple $(K, \rho(r), \mu)$ admits neutralization of zero and of infinity, then for every measure $\nu$ the integral $\int_{0}^{\infty} K(t) d \nu(t)$ exists as an improper integral with singular points 0 and $\infty$.

Proof. Let $\nu$ be an arbitrary measure belonging to $\operatorname{Fr}[\mu]$. There is a sequence $r_{n} \rightarrow \infty$ such that $\mu_{r_{n}} \rightarrow \nu$. We may assume that, moreover, $|\mu|_{r_{n}} \rightarrow \widehat{\nu}$. Since the triple
$(K, \rho(r), \mu)$ admits neutralization of zero, Lemma 8 shows that for every $\delta>0$ there exists $\varepsilon_{0}>0$ such that

$$
\limsup _{r \rightarrow \infty}\left|\int_{\varepsilon_{1}}^{\varepsilon_{2}} K(t) d \mu_{r}(t)\right| \leq \delta
$$

whenever $0<\varepsilon_{1}<\varepsilon_{2} \leq \varepsilon_{0}$. In particular,

$$
\begin{equation*}
\limsup _{n \rightarrow \infty}\left|\int_{\varepsilon_{1}}^{\varepsilon_{2}} K(t) d \mu_{r_{n}}(t)\right| \leq \delta . \tag{4.53}
\end{equation*}
$$

Let $E$ be the set of points that carry a mass of $\hat{\nu}$. Then $E$ is at most countable. We may assume that $\varepsilon_{1}, \varepsilon_{2} \notin E$. Using Theorem (12, we can rewrite (4.53) as follows:

$$
\begin{equation*}
\left|\int_{\varepsilon_{1}}^{\varepsilon_{2}} K(t) d \nu(t)\right| \leq \delta . \tag{4.54}
\end{equation*}
$$

Here we assume that $\varepsilon_{1}, \varepsilon_{2} \notin E$. But the relation

$$
\left|\int_{\varepsilon_{1}}^{\varepsilon_{2}} K(t) d \nu(t)\right|=\lim _{h \rightarrow+0}\left|\int_{\varepsilon_{1}-h}^{\varepsilon_{2}+h} K(t) d \nu(t)\right|
$$

shows that this restriction can be dropped. Then (4.54) means that the Cauchy convergence condition at the singular point zero is fulfilled for the integral $\int_{0}^{\infty} K(t) d \nu(t)$. Consequently, this integral converges at zero. The proof of convergence at infinity is similar.

Now, we can prove a version of Theorem 31 for a not necessarily compactly supported kernel $K$.

Theorem 35. Let $\rho(r)$ be an arbitrary proximate order, let $\mu \in \mathfrak{M}_{\infty}(\rho(r))$, and let $K$ be a continuous kernel on $(0, \infty)$. If the triple $(K, \rho(r), \mu)$ admits neutralization of zero and of infinity, then

$$
\begin{equation*}
L(J, \infty)=\left\{\int_{0}^{\infty} K(u) d \nu(u): \nu \in \operatorname{Fr}[\mu]\right\} \tag{4.55}
\end{equation*}
$$

Proof. Let $\nu$ be an arbitrary measure belonging to the cluster set $\operatorname{Fr}[\mu]$. There exists a sequence $r_{n} \rightarrow \infty$ such that $\mu_{r_{n}} \rightarrow \nu$ and $|\mu|_{r_{n}} \rightarrow \hat{\nu}$. Let $\delta$ be an arbitrary positive number. The assumption about neutralization and Theorem 34 imply that there exist numbers $\varepsilon_{0}>0$ and $N_{0}>0$ such that for $\varepsilon \in\left(0, \varepsilon_{0}\right)$ and $N>N_{0}$ we have

$$
\begin{gather*}
\limsup _{r \rightarrow \infty}\left|\int_{0}^{\varepsilon} K(t) d \mu_{r}(t)\right|<\delta, \quad \limsup _{r \rightarrow \infty}\left|\int_{N}^{\infty} K(t) d \mu_{r}(t)\right|<\delta, \\
\left|\int_{0}^{\varepsilon} K(t) d \nu(t)\right|<\delta, \quad\left|\int_{N}^{\infty} K(t) d \nu(t)\right|<\delta . \tag{4.56}
\end{gather*}
$$

We assume that, moreover, the points $\varepsilon$ and $N$ do not carry a mass of $\hat{\nu}$. Then

$$
\begin{align*}
& \left|\int_{0}^{\infty} K(t) d \mu_{r_{n}}(t)-\int_{0}^{\infty} K(t) d \nu(t)\right| \leq\left|\int_{0}^{\varepsilon} K(t) d \mu_{r_{n}}(t)\right|+\left|\int_{0}^{\varepsilon} K(t) d \nu(t)\right|  \tag{4.57}\\
& \quad+\left|\int_{N}^{\infty} K(t) d \mu_{r_{n}}(t)\right|+\left|\int_{N}^{\infty} K(t) d \nu(t)\right|+\left|\int_{\varepsilon}^{N} K(t) d \mu_{r_{n}}(t)-\int_{\varepsilon}^{N} K(t) d \nu(t)\right|
\end{align*}
$$

By Theorem 12, the very last summand tends to zero as $n \rightarrow \infty$. Combining this with (4.56) and (4.57), we see that

$$
\underset{n \rightarrow \infty}{\limsup }\left|\int_{0}^{\infty} K(t) d \mu_{r_{n}}(t)-\int_{0}^{\infty} K(t) d \nu(t)\right| \leq 4 \delta
$$

In its turn, this relation implies the inclusion $H \subset L(J, \infty)$, where $H$ is the right-hand side of (4.55). The rest of the proof is the same as for Theorem 31

Next, we present versions of Theorems 32 and 33 for $K$ not necessarily having compact support. The proofs fit into the same pattern as that of Theorem 35,

Theorem 36. Let $\rho(r)$ be an arbitrary proximate order, and let $\mu \in \mathfrak{M}_{\infty}(\rho(r))$. Suppose that $K(t)$ is a kernel on $(0, \infty)$ continuous everywhere except the point 1 at which it has a jump and is continuous from the left. Suppose also that the triple $(K, \rho(r), \mu)$ admits neutralization of zero and of infinity. Then

$$
L(J, \infty)=\left\{\int_{0}^{1} K(t) d \nu_{1}(t)+\int_{1}^{\infty} \widetilde{K}(t) d \nu_{2}(t):\left(\nu_{1}, \nu_{2}\right) \in \widehat{\operatorname{Fr}}[\mu]\right\}
$$

where $\widetilde{K}(t)$ is the continuous extension of $K$ from $(1, \infty)$ to $[1, \infty)$.
Theorem 37. Let $\rho(r)$ be an arbitrary proximate order, and let $\mu$ be a Radon measure on $(0, \infty)$ with density $\mu^{\prime}(r)$ that satisfies the inequality $\left|\mu^{\prime}(r)\right| \leq M \frac{V(r)}{r}$. Suppose that $K(t)$ is a locally integrable kernel on $(0, \infty)$ and that the triple $(K, \rho(r), \mu)$ admits neutralization of zero and of infinity. Then

$$
L(J, \infty)=\left\{\int_{0}^{\infty} K(t) d \nu(t): \nu \in \operatorname{Fr}[\mu]\right\} .
$$

In connection with Theorems 35-37, it becomes important to know under what restrictions on $K$ and $\mu$ the triple ( $K, \rho(r), \mu)$ admits neutralization of zero or of infinity. We shall prove two results on this subject. Recall that the function $\gamma(t)$ occurring in the next statements was defined by formula (1.1).

Lemma 10. Suppose that $\rho(r)$ is an arbitrary proximate order, $\mu$ is a Radon measure on $(0, \infty)$ with a density $\mu^{\prime}(r)$ such that $\left|\mu^{\prime}(r)\right| \leq M \frac{V(r)}{r}(r \in(0, \infty))$. Suppose also that $t^{\rho-1} \gamma(t) K(t) \in L_{1}(0, \infty)$. Then the triple $(K, \rho(r), \mu)$ admits neutralization of zero and of infinity.

Proof. Using inequality (2.18), we obtain

$$
\left|\int_{0}^{\varepsilon} K(t) d \mu_{r}(t)\right| \leq M \int_{0}^{\varepsilon}|K(t)| \frac{V(r t)}{t V(r)} d t \leq M \int_{0}^{\varepsilon} t^{\rho-1} \gamma(t)|K(t)| d t .
$$

Since $t^{\rho-1} \gamma(t)|K(t)| \in L_{1}(0, \infty)$, it follows that the triple ( $\left.K, \rho(r), \mu\right)$ admits neutralization of zero. Neutralization of infinity is ensured similarly.

Remark 5. In the statement of Lemma 10 the function $\gamma(t)$ occurred, the study of which presents some difficulties. However, this study becomes redundant if $t^{\rho-1} \frac{1+t^{2 \varepsilon}}{t^{\varepsilon}} K(t) \in$ $L_{1}(0, \infty)$ for some $\varepsilon>0$. Combined with Theorem 9 this relation implies that

$$
t^{\rho-1} \gamma(t) K(t) \in L_{1}(0, \infty)
$$

The next result goes back to Wiener.
Let $K(t)$ be a kernel on $(0, \infty)$. Put

$$
K_{n}=\sup \left\{|K(t)|: t \in\left(e^{n}, e^{n+1}\right]\right\}, \quad n \in(-\infty, \infty) .
$$

The set $\mathfrak{M}(\rho(r))$ occurring in the following statement was defined in the Preface.
Lemma 11. Suppose that $\rho(r)$ is an arbitrary proximate order and $\mu \in \mathfrak{M}(\rho(r))$. Let $K(t)$ be a Borel function on $(0, \infty)$ such that the series

$$
\begin{equation*}
\sum_{n=-\infty}^{\infty} e^{n \rho} \gamma\left(e^{n}\right) K_{n}, \quad \rho=\rho(\infty) \tag{4.58}
\end{equation*}
$$

converges. Then the triple $(K, \rho(r), \mu)$ admits neutralization of zero and of infinity.

Proof. Since $\mu \in \mathfrak{M}(\rho(r))$, there exists $A>0$ such that $|\mu|((r, e r]) \leq A V(r)$ for every $A>0$. Next, we have

$$
\begin{aligned}
& \left|\frac{1}{V(r)} \int_{0}^{\varepsilon r} K\left(\frac{t}{r}\right) d \mu(t)\right| \leq \frac{1}{V(r)} \sum_{n=-\infty}^{n_{0}} \int_{e^{n} r}^{e^{n+1} r}\left|K\left(\frac{u}{r}\right)\right| d|\mu|(u) \\
& \quad \leq \frac{1}{V(r)} \sum_{n=-\infty}^{n_{0}} K_{n}|\mu|\left(\left(e^{n} r, e^{n+1} r\right]\right) \leq \frac{A}{V(r)} \sum_{n=-\infty}^{n_{0}} K_{n} V\left(e^{n} r\right) \leq A \sum_{n=-\infty}^{n_{0}} K_{n} e^{n \rho} \gamma\left(e^{n}\right),
\end{aligned}
$$

where $n_{0}=[\ln \varepsilon]$. Combined with the convergence of the series (4.58), this implies neutralization of zero for the triple $(K, \rho(r), \mu)$. Neutralization of infinity is ensured similarly.

Remark 6. The convergence of the series (4.58) can be replaced by a stronger restriction, namely, by the requirement that the series

$$
\sum_{n=-\infty}^{\infty} \frac{1+e^{2 \varepsilon n}}{e^{\varepsilon n}} e^{n \rho} K_{n}
$$

converge for some $\gamma(t)$. Then there will be no need in the study of the function $\gamma(t)$.
Consider the function

$$
v(z)=\int_{0}^{\infty} \ln \left|1-\frac{z}{t}\right| d \mu(t)
$$

where $\mu$ is a positive measure on $(0, \infty)$ that belongs to $\mathfrak{M}_{\infty}(\rho(r))$ with $\rho=\rho(\infty) \in(0,1)$ and $\mu$ is such that the above integral converges at zero. This function is well known in the growth theory for subharmonic functions and has been the subject of numerous investigations. If the measure $\mu$ is regular, $v(z)$ belongs to a special class of subharmonic functions of completely regular growth (with respect to the proximate order $\rho(r)$ ) in the sense of Levin and Pfluger. In this case, the limit of $v(r) / V(r)$ as $r \rightarrow \infty$ may fail to exist, but the limit

$$
\begin{equation*}
\lim _{r \rightarrow \infty} \frac{1}{r V(r)} \int_{0}^{r} v(t) d t \tag{4.59}
\end{equation*}
$$

exists. The proofs of the above statements about $v(z)$ can be found in [18].
The existence of the limit (4.59) does not follow from the theorems proved above, but will be a consequence of the statement below. Consequently, we shall obtain a result as strong as those in the theory of subharmonic functions, but valid for kernels much more general than the kernel $\ln \left|1-\frac{r}{t}\right|$ involved in the definition of $v(r)$. It should also be noted that, as is seen from Theorem 32 and the properties of $v(r)$ mentioned above, in the case of discontinuous kernels the cluster set $\operatorname{Fr}[\mu]$ does not determine the asymptotic behavior of the function $\Psi$. However, as we shall see, this set determines the cluster set of the measure $s$. In the Preface we already characterized the next theorem as a principal result of the paper.

Theorem 38. Let $\rho(r)$ be an arbitrary proximate order, and let $\mu \in \mathfrak{M}(\rho(r))$. Suppose that $K(t)$ is a Borel function on $(0, \infty)$ such that $t^{\rho-1} \gamma(t) K(t) \in L_{1}(0, \infty)(\rho=\rho(\infty))$. Then the measure $s, d s(u)=\Psi(u) d u$, where $\Psi$ was defined by (1.2), belongs to the class $\mathfrak{M}(\rho(r)+1)$, and its cluster set $\operatorname{Fr}[\rho(r)+1, s]$ consists of absolutely continuous measures whose densities constitute the set

$$
\left\{\int_{0}^{\infty} K\left(\frac{t}{u}\right) d \nu(t): \nu \in \operatorname{Fr}[\mu]\right\} .
$$

Proof. First, we prove this theorem under the additional assumption $\rho>0$. Let $\mu(t)$ and $\widehat{\mu}(t)$ be the distribution functions for $\mu$ and $|\mu|$ normalized by the conditions $\mu(+0)=$ $\widehat{\mu}(+0)=0$. Since $\mu \in \mathfrak{M}(\rho(r))$, there exists a constant $A_{1}$ such that $|\mu|([r, e r]) \leq A_{1} V(r)$ on $(0, \infty)$. If $\rho>0$, it follows that

$$
\begin{align*}
|\mu|((0, r]) & =\sum_{n=-\infty}^{0}|\mu|\left(\left(e^{n-1} r, e^{n} r\right]\right) \leq A_{1} \sum_{n=-\infty}^{0} V\left(e^{n-1} r\right) \\
& =A_{1} \sum_{n=-\infty}^{0} \int_{e^{n-1} r}^{e^{n} r} \frac{V\left(e^{n-1} r\right)}{V(t)} \frac{V(t)}{t} d t \leq A_{1} \max _{\left[\frac{1}{e}, 1\right]} \gamma(t) \sum_{n=-\infty}^{0} \int_{e^{n-1} r}^{e^{n} r} \frac{V(t)}{t} d t  \tag{4.60}\\
& =A_{2} \int_{0}^{r} \frac{V(t)}{t} d t=A_{2} \int_{0}^{1} \frac{V(u r)}{u} d u \leq A_{2} \int_{0}^{1} \frac{u^{\rho} \gamma(u) V(r)}{u} d u=A V(r) .
\end{align*}
$$

We have used (2.18) in these estimates. Next, we have

$$
\begin{aligned}
|s|([R, e R]) & =\int_{R}^{e R}|\Psi(r)| d r \leq \int_{R}^{e R} \int_{0}^{\infty}\left|K\left(\frac{t}{r}\right)\right| d|\mu|(t) d r \\
& =\int_{0}^{\infty} \int_{R}^{e R}\left|K\left(\frac{t}{r}\right)\right| d r d \widehat{\mu}(t)=\int_{0}^{\infty} t \int_{\frac{t}{e R}}^{\frac{t}{R}} \frac{1}{u^{2}}|K(u)| d u d \widehat{\mu}(t)
\end{aligned}
$$

Integration by parts yields

$$
\begin{align*}
|s|([R, e R]) \leq\left\{t \int_{\frac{t}{e R}}^{\frac{t}{R}}\right. & \left.\frac{1}{u^{2}}|K(u)| d u \widehat{\mu}(t)\right\}\left.\right|_{0} ^{\infty}-\int_{0}^{\infty} \int_{\frac{t}{e R}}^{\frac{t}{R}} \frac{1}{u^{2}}|K(u)| d u \widehat{\mu}(t) d t  \tag{4.61}\\
& \quad-R \int_{0}^{\infty} \frac{1}{t}\left|K\left(\frac{t}{R}\right)\right| \widehat{\mu}(t) d t+e R \int_{0}^{\infty} \frac{1}{t}\left|K\left(\frac{t}{e R}\right)\right| \widehat{\mu}(t) d t
\end{align*}
$$

We need to estimate the positive summands on the right in (4.61). We shall use inequality (2.18). Observe that

$$
\begin{align*}
& \int_{\frac{t}{e R}}^{\frac{t}{R}} \frac{1}{u^{2}}|K(u)| d u=\frac{1}{\frac{t}{R} V\left(\frac{t}{R}\right)} \int_{\frac{t}{e R}}^{\frac{t}{R}} \frac{\frac{t}{R} V\left(\frac{t}{R}\right)}{u V(u)}|K(u)| \frac{V(u)}{u} d u  \tag{4.62}\\
& \leq \frac{1}{\frac{t}{R} V\left(\frac{t}{R}\right)} \int_{\frac{t}{e R}}^{\frac{t}{R}}\left(\frac{t}{R u}\right)^{\rho+1} \gamma\left(\frac{t}{R u}\right)|K(u)| \frac{V(u)}{u} d u \leq \frac{e^{\rho+1}}{\frac{t}{R} V\left(\frac{t}{R}\right)} \hat{\gamma} \int_{\frac{t}{e R}}^{\frac{t}{R}}|K(u)| \frac{V(u)}{u} d u
\end{align*}
$$

where $\hat{\gamma}=\max \{\gamma(x): x \in[1, e]\}$. Since $V(t) \leq t^{\rho} \gamma(t)$ and $t^{\rho-1} \gamma(t) K(t) \in L_{1}(0, \infty)$, it follows that $\frac{V(t)}{t} K(t) \in L_{1}(0, \infty)$. Now (4.62) implies that the integrated term in (4.61) vanishes.

We estimate the fourth term on the right in (4.61):

$$
\begin{aligned}
& e R \int_{0}^{\infty} \frac{1}{t}\left|K\left(\frac{t}{e R}\right)\right| \widehat{\mu}(t) d t \leq A e R \int_{0}^{\infty} \frac{1}{t}\left|K\left(\frac{t}{e R}\right)\right| V(t) d t \\
& \quad=A e R \int_{0}^{\infty} \frac{1}{u}|K(u)| V(e R u) d u \leq A e R V(e R) \int_{0}^{\infty} u^{\rho-1} \gamma(u)|K(u)| d u
\end{aligned}
$$

Thus, we have proved that

$$
|s|([R, e R]) \leq A \int_{0}^{\infty} u^{\rho-1} \gamma(u)|K(u)| d u e R V(e R)
$$

for $R>0$. This means that $s \in \mathfrak{M}(\rho(r)+1)$.

Let $\varphi \in \Phi$. We have

$$
\begin{align*}
\frac{1}{r V(r)} \int_{0}^{\infty} \varphi\left(\frac{u}{r}\right) d s(u) & =\frac{1}{r V(r)} \int_{0}^{\infty} \varphi\left(\frac{u}{r}\right) \Psi(u) d u \\
& =\frac{1}{r V(r)} \int_{0}^{\infty} \int_{0}^{\infty} \varphi\left(\frac{u}{r}\right) K\left(\frac{t}{u}\right) d \mu(t) d u \\
& =\frac{1}{r V(r)} \int_{0}^{\infty} \int_{0}^{\infty} \varphi\left(\frac{u}{r}\right) K\left(\frac{t}{u}\right) d u d \mu(t)  \tag{4.63}\\
& =\frac{1}{r V(r)} \int_{0}^{\infty} \int_{0}^{\infty} \frac{1}{x^{2}} \varphi\left(\frac{t}{x r}\right) K(x) d x d \mu(t) \\
& =\frac{1}{r V(r)} \int_{0}^{\infty} \frac{1}{x^{2}} K(x) \int_{0}^{\infty} t \varphi\left(\frac{t}{x r}\right) d \mu(t) d x
\end{align*}
$$

From the above arguments it easily follows that all integrals in (4.63) converge absolutely. This justifies the interchanges of the order of integration done in the course of the proof of (4.63).

Now, let $r_{n} \rightarrow \infty$ be a sequence such that $\mu_{r_{n}} \rightarrow \nu$. We have

$$
\lim _{n \rightarrow \infty} \frac{1}{r_{n} V\left(r_{n}\right)} \int_{0}^{\infty} t \varphi\left(\frac{t}{x r_{n}}\right) d \mu(t)=\lim _{n \rightarrow \infty} \int_{0}^{\infty} u \varphi\left(\frac{u}{x}\right) d \mu_{r_{n}}(u)=\int_{0}^{\infty} u \varphi\left(\frac{u}{x}\right) d \nu(u)
$$

Suppose that $\operatorname{supp} \varphi \subset[a, b] \subset(0, \infty)$. Then

$$
\begin{aligned}
\left|\frac{1}{r_{n} V\left(r_{n}\right)} \int_{0}^{\infty} t \varphi\left(\frac{t}{x r_{n}}\right) d \mu(t)\right| & \leq \frac{b x}{V\left(r_{n}\right)}\|\varphi\| \widehat{\mu}\left(b x r_{n}\right) \leq \frac{A b\|\varphi\| x}{V\left(r_{n}\right)} V\left(b x r_{n}\right) \\
& \leq A\|\varphi\|(b x)^{\rho+1} \gamma(b x) \leq A\|\varphi\| b^{\rho+1} \gamma(b) x^{\rho+1} \gamma(x)
\end{aligned}
$$

We have used the estimate $\gamma(b x) \leq \gamma(b) \gamma(x)$. It follows that

$$
\left|\frac{1}{x^{2}} K(x) \frac{1}{r_{n} V\left(r_{n}\right)} \int_{0}^{\infty} t \varphi\left(\frac{t}{x r_{n}}\right) d \mu(t)\right| \leq A\|\varphi\| b^{\rho+1} \gamma(b) x^{\rho-1} \gamma(x)|K(x)| .
$$

Now, the Lebesgue dominated convergence theorem implies

$$
\begin{aligned}
\lim _{n \rightarrow \infty} & \frac{1}{r_{n} V\left(r_{n}\right)} \int_{0}^{\infty} \frac{1}{x^{2}} K(x) \int_{0}^{\infty} t \varphi\left(\frac{t}{x r_{n}}\right) d \mu(t) d x \\
& =\int_{0}^{\infty} \frac{1}{x^{2}} K(x) \int_{0}^{\infty} u \varphi\left(\frac{u}{x}\right) d \nu(u) d x=\int_{0}^{\infty} u \int_{0}^{\infty} \frac{1}{x^{2}} \varphi\left(\frac{u}{x}\right) K(x) d x d \nu(u) \\
& =\int_{0}^{\infty} \int_{0}^{\infty} K\left(\frac{u}{t}\right) \varphi(t) d t d \nu(u)=\int_{0}^{\infty} \int_{0}^{\infty} K\left(\frac{u}{t}\right) d \nu(u) \varphi(t) d t
\end{aligned}
$$

Putting $r=r_{n}$ in (4.63) and passing to the limit as $n \rightarrow \infty$, we obtain

$$
\lim _{n \rightarrow \infty} \int_{0}^{\infty} \varphi(t) d s_{r_{n}}(t)=\int_{0}^{\infty} \int_{0}^{\infty} K\left(\frac{u}{t}\right) d \nu(u) \varphi(t) d t
$$

This means that the sequence $s_{r_{n}}$ of measures converges and its limit is the absolutely continuous measure with the density $\int_{0}^{\infty} K\left(\frac{u}{t}\right) d \nu(u)$. So, we have proved that $\operatorname{Fr}[s]$ contains all absolutely continuous measures with densities of the form $\int_{0}^{\infty} K\left(\frac{u}{t}\right) d \nu(u)$, where $\nu \in \operatorname{Fr}[\mu]$.

Now, let $\nu_{1}$ be an arbitrary measure in $\operatorname{Fr}[s]$, and let $R_{n} \rightarrow \infty$ be a sequence such that $s_{R_{n}} \rightarrow \nu_{1}$. We put $r=R_{n}$ in (4.63). There is no loss of generality in assuming that $\mu_{R_{n}} \rightarrow \nu \in \operatorname{Fr}[\mu]$. Then the above proof shows that $\nu_{1}$ is absolutely continuous and its density is $\int_{0}^{\infty} K\left(\frac{u}{t}\right) d \nu(u)$. This proves the theorem for $\rho>0$.

Let $\rho(r)$ be an arbitrary proximate order, and let $\rho=\rho(\infty)$. We take a real number $p$ such that $\rho_{1}=\rho+p>0$ and put $\rho_{1}(r)=\rho(r)+p$. Along with the formula

$$
\Psi(r)=\int_{0}^{\infty} K\left(\frac{t}{r}\right) d \mu(t)
$$

we also have the formula

$$
\Psi_{1}(r)=r^{p} \Psi(r)=\int_{0}^{\infty}\left(\frac{r}{t}\right)^{p} K\left(\frac{t}{r}\right) t^{p} d \mu(t)=\int_{0}^{\infty} K_{1}\left(\frac{t}{r}\right) d \mu_{1}(t)
$$

where $K_{1}(t)=t^{-p} K(t), d \mu_{1}(t)=t^{p} d \mu(t)$. Next,

$$
t^{\rho_{1}-1} \gamma(t) K_{1}(t)=t^{\rho+p-1} \gamma(t) t^{-p} K(t)=t^{\rho-1} \gamma(t) K(t)
$$

Applying Theorem [19] it is easy to show that the proximate order $\rho_{1}(r)$, the kernel $K_{1}$, and the measures $\mu_{1}$ and $s_{1}, d s_{1}(t)=\Psi_{1}(t) d t$, satisfy the assumptions of Theorem 38 and, furthermore, $\rho_{1}>0$. By what has already been proved, the set $\operatorname{Fr}\left[s_{1}\right]$ consists of absolutely continuous measures whose densities are of the form

$$
\left\{\int_{0}^{\infty} K_{1}\left(\frac{u}{t}\right) d \nu_{1}(t): \nu_{1} \in \operatorname{Fr}\left[\mu_{1}\right]\right\}
$$

Now, it remains to apply Theorem 19,
In Theorem 35, the behavior of the function

$$
\begin{equation*}
\Psi(r)=\int_{0}^{\infty} K\left(\frac{t}{r}\right) d \mu(t) \tag{4.64}
\end{equation*}
$$

was described in terms of the cluster set $\operatorname{Fr}[\mu]$. The applicability of that theorem is restricted by the possibility to verify neutralization of zero and of infinity for the triple ( $K, \rho(r), \mu)$.

However, there are other obstructions to the study of $\Psi(r)$. We shall consider the case of an infinitely differentiable compactly supported kernel $K$ on $(0, \infty)$ in detail. This restriction on $K$ is very strong. In particular, for such $K$ the triple ( $K, \rho(r), \mu)$ admits neutralization of zero and of infinity for every proximate order $\rho(r)$ and every Radon measure $\mu$ on $(0, \infty)$.

Suppose that $K$ is an infinitely differentiable kernel on the semiaxis $(0, \infty)$ and $\mu$ is a Radon measure on this semiaxis. Then we have infinitely many identities

$$
\begin{equation*}
(-1)^{n+1} r^{n+1} \Psi(r)=\int_{0}^{\infty} K^{(n+1)}\left(\frac{t}{r}\right) F_{n}(t) d t, \quad n=0,1, \ldots \tag{4.65}
\end{equation*}
$$

where $F_{0}(t)=\mu(t)$ is the distribution function for $\mu$ and $F_{n+1}^{\prime}(t)=F_{n}(t), n=0,1, \ldots$.
The question arises as to whether identities (4.65) and Theorem 31 allow us to determine the order of growth for $\Psi(r)$ at infinity. The answer given below is: "Yes in many cases, but there are some exceptions".

Consider a Radon measure $\mu_{1}$ on $(0, \infty)$ supported on $(0,1]$ and a compactly supported kernel $K$ on $(0, \infty)$. We put

$$
u(r)=\int_{0}^{\infty} K\left(\frac{t}{r}\right) d \mu_{1}(t)
$$

Then $u(r)$ vanishes in a neighborhood of infinity. So, if we are interested in the behavior of the function $\Psi(r)$ given by (4.64) near infinity, we may assume without loss of generality that the measure $\mu$ in (4.64) is supported on $(1, \infty)$. In the sequel, we assume this tacitly. Then the function $F_{0}(t)$ is bounded on every interval $(0, N)$, and all functions $F_{k}(t), k \geq 1$, can be taken continuous on $[0, \infty)$. At least, we may assume that the $F_{k}(t)$ are locally integrable on $(0, \infty)$ and the integral $\int_{0}^{1} F_{k}(t) d t$ exists for every $k \geq 0$.

Suppose that a function $f(t)$ is locally integrable on $(0, \infty)$ and the integral $\int_{0}^{1} f(t) d t$ exists, at least as an improper integral. The function $F(t)$ defined by $F(t)=-\int_{t}^{\infty} f(x) d x$ if this integral converges and by $F(t)=\int_{0}^{t} f(x) d x$ otherwise will be called the canonical primitive for $f$ on $(0, \infty)$. The canonical primitive is determined by $f$ uniquely. Every locally integrable function $f$ on $(0, \infty)$ possesses a primitive $F$, but the canonical primitive may fail to exist. For example, the canonical primitive does not exist for the function $f(t)=\frac{1}{t}$ on $(0, \infty)$.

The function $F_{0}(t)$ in formula (4.65) will be defined uniquely in the following way. We put $F_{0}(t)=-\mu((t, \infty))$ if $F_{0}(t)$ is finite (we remind the reader that $\mu$ is a Radon measure on $(0, \infty)$, so the quantity $\mu(E)$ may fail to exist for some Borel sets $E$ ), and $F_{0}(t)=\mu((0, t])$ otherwise. This definition always makes sense because $\mu$ is supported on $(1, \infty)$. After that, we define $F_{n+1}(t)$ to be the canonical primitive for $F_{n}(t)$. Then the sequence $F_{n}(t)$ in (4.65) is defined uniquely.

In $\S 2$, an order and a proximate order were introduced for positive functions. Now we extend these definitions to complex-valued functions.

Let $f(t)$ be a complex-valued function on $(0, \infty)$. The order $\rho$ of $f$ is defined by the formula

$$
\rho=\underset{r \rightarrow \infty}{\limsup } \frac{\ln |f(r)|}{\ln r}
$$

If $\rho$ is a real number, then $f$ is called a function of finite order $\rho$.
We shall say that a proximate order $\rho(r)$ is a proximate order for the function $f(r)$ if

$$
\limsup _{r \rightarrow \infty} \frac{|f(r)|}{V(r)}=\sigma \in(0, \infty)
$$

If $\rho(r)$ is a proximate order for the function $f(r)$, we say that $f(r)$ grows at infinity as $V(r)$.

Theorem 6 implies that if $f(r)$ is of finite order, then it also possesses a proximate order $\rho(r)$ (surely, there are infinitely many such orders).

In many problems (in particular, in connection with formula (4.65)) the question about estimates of a primitive $F$ for $f$ arises. We dwell on some details of this.

Lemma 12. Let $f$ be a locally integrable function of finite order on $(0, \infty)$ such that the integral $\int_{0}^{1} f(t) d t$ exists. Let $\rho(r)$ be a proximate order for $f$, and let $F$ be the canonical primitive for $F$. Then there exist constants $M_{k}$ and $r_{0}$ such that

1) $|F(r+\alpha r)-F(r)| \leq M_{1} \alpha r V(r), r \geq r_{0}, \alpha \in[0,1]$;
2) $|F(r)| \leq M_{2}\left(1+\int_{1}^{r} V(t) d t\right), r \geq 1$;
3) $|F(r)| \leq M_{3} \int_{r}^{\infty} V(t) d t, r \geq 1$;
4) $|F(r)| \leq M_{4} r V(r), r \geq 1$, if $\rho=\rho(\infty) \neq-1$;
5) the orders $\rho(f)$ and $\rho(F)$ for $f$ and $F$ satisfy the inequality $\rho(F) \leq \rho(f)+1$.

Note that if the integral $\int_{1}^{\infty} V(t) d t$ diverges, then 3$)$ is the trivial inequality $|F(r)| \leq \infty$.
Proof. Let $\sigma$ be the type of $f$ with respect to the proximate order $\rho(r)$. Then for all sufficiently large $r$ and $\alpha \in[0,1]$ we have

$$
\begin{aligned}
|F(r+\alpha r)-F(r)|=\left|\int_{r}^{(1+\alpha) r} f(t) d t\right| & \leq 2 \sigma \int_{r}^{(1+\alpha) r} V(t) d t=2 \sigma r \int_{1}^{1+\alpha} V(u r) d u \\
& \leq 2 \sigma r V(r) \int_{1}^{1+\alpha} u^{\rho} \gamma(u) d u \leq 2 \sigma \gamma_{1} \alpha r V(r)
\end{aligned}
$$

where $\gamma_{1}=\max \left\{u^{\rho} \gamma(u): u \in[1,2]\right\}$. This proves statement 1). Statements 2)-4) are easy consequences of 1 ) and the properties of a proximate order. Indeed, let us verify 2 )
for instance. Let $r \geq 2 r_{0}$. We define $n_{0}$ by the condition $2^{-n_{0}} r \in\left[r_{0}, 2 r_{0}\right)$ and put $r_{1}=2^{-n_{0}} r$. Then

$$
\begin{aligned}
|F(r)| & =\left|F\left(r_{1}\right)+\sum_{n=1}^{n_{0}}\left(F\left(2^{n} r_{1}\right)-F\left(2^{n-1} r_{1}\right)\right)\right| \\
& \leq\left|F\left(r_{1}\right)\right|+M_{1} \sum_{n=1}^{n_{0}} 2^{n-1} r_{1} V\left(2^{n-1} r_{1}\right)=\left|F\left(r_{1}\right)\right|+M_{1} \sum_{n=1}^{n_{0}} \int_{2^{n-1} r_{1}}^{2^{n} r_{1}} V\left(2^{n-1} r_{1}\right) d t \\
& \leq\left|F\left(r_{1}\right)\right|+M_{5} \sum_{n=1}^{n_{0}} \int_{2^{n-1} r_{1}}^{2^{n} r_{1}} V(t) d t=\left|F\left(r_{1}\right)\right|+M_{5} \int_{r_{1}}^{r} V(t) d t .
\end{aligned}
$$

This implies 2) immediately. The inequality $\rho(F) \leq \rho(f)+1$ is an easy consequence of statement 4) if $\rho \neq-1$ and of statement 2 if $\rho=-1$.

Lemma 12 motivates the following definition.
Let $f(r)$ be a locally integrable function of finite order on $(0, \infty)$. Suppose that the integral $\int_{0}^{1} f(t) d t$ exists, and let $F(r)$ be the canonical primitive for $f$ on $(0, \infty)$.

A proximate order $\rho(r)$ for $f$ is said to be stable if $\rho=\rho(\infty) \neq-1$ and

$$
\limsup _{r \rightarrow \infty} \frac{|F(r)|}{r V(r)}>0
$$

We observe that the functions $\cos r, \sin r$, and $e^{i r}$ do not possess a stable proximate order.

Theorem 39. Let $f(r)$ be a locally integrable function of finite order on $(0, \infty)$. Suppose that the integral $\int_{0}^{1} f(t) d t$ exists, and let $F(r)$ be the canonical primitive for $f$ on $(0, \infty)$. Let $\rho(r), \rho(\infty) \neq-1,-2$, be a stable proximate order for $f$. Then $\rho(r)+1$ is a stable proximate order for $F$.

Proof. The fact that $\rho(r)+1$ is a proximate order for $F$ follows from the definition and statement 4) in Lemma 12, It remains to prove that this proximate order is stable for $F$. But if not, then, by Lemma 12 and the definition of a stable proximate order, we have

$$
\begin{equation*}
\lim _{r \rightarrow \infty} \frac{F_{1}(r)}{r^{2} V(r)}=0 \tag{4.66}
\end{equation*}
$$

where $F_{1}(r)$ is the canonical primitive for $F(r)$ on $(0, \infty)$.
Suppose first that $\rho(\infty)>-2$. We show that then the integral $\int_{r}^{\infty} F(t) d t$ diverges. Since $\rho(r)+1$ is a proximate order for $F$, we may assume that there exists a number $m>0$ and a sequence $r_{n} \rightarrow \infty$ such that $\operatorname{Re} F\left(r_{n}\right) \geq 2 m r_{n} V\left(r_{n}\right)$. This can be ensured by replacement of $f$ with $-f$ or $\pm i f$ if necessary. Statement 1 ) of Lemma 12 implies that there exists $\alpha_{0}>0$ such that $\operatorname{Re} F(r) \geq m r_{n} V\left(r_{n}\right)$ for all $n$ sufficiently large and all $r \in\left[r_{n},\left(1+\alpha_{0}\right) r_{n}\right]$. Then

$$
\left|\int_{r_{n}}^{\left(1+\alpha_{0}\right) r_{n}} F(t) d t\right| \geq\left|\int_{r_{n}}^{\left(1+\alpha_{0}\right) r_{n}} \operatorname{Re} F(t) d t\right|=\int_{r_{n}}^{\left(1+\alpha_{0}\right) r_{n}} \operatorname{Re} F(t) d t \geq m \alpha_{0} r_{n}^{2} V\left(r_{n}\right)
$$

Since $\lim _{r \rightarrow \infty} r^{2} V(r)=\infty$, the Cauchy criterion shows that the integral $\int_{r}^{\infty} F(t) d t$ diverges. Thus, by the definition of a canonical primitive, formula (4.66) takes the form

$$
\begin{equation*}
\lim _{r \rightarrow \infty} \frac{1}{r^{2} V(r)} \int_{0}^{r} F(t) d t=0 \tag{4.67}
\end{equation*}
$$

A function $g(r)$ is said to be of slow variation relative to a proximate order $\rho(r)$ if

$$
\lim _{\substack{r \rightarrow \infty \\ \alpha \rightarrow 0}} \frac{g(r+\alpha r)-g(r)}{V(r)}=0 .
$$

Statement 1) of Lemma 12 implies that $F(r)$ is of slow variation relative to $\rho(r)+1$.
The following statement is a consequence of [19, Theorem 2].
Let $g$ be of show variation relative to a proximate order $\rho(r)$ with $\rho(\infty)>-1$. If

$$
\lim _{r \rightarrow \infty} \frac{1}{r V(r)} \int_{0}^{r} g(t) d t=a
$$

then

$$
\lim _{r \rightarrow \infty} \frac{g(r)}{V(r)}=a(\rho+1)
$$

Together with (4.66), this statement yields

$$
\begin{equation*}
\lim _{r \rightarrow \infty} \frac{F(r)}{r V(r)}=0 \tag{4.68}
\end{equation*}
$$

This contradicts the fact that $\rho(r)$ is a proximate order for $f$.
Thus, (4.66) leads to a contradiction, which proves the theorem for $\rho(\infty)>-2$. Now, assume that $\rho(\infty)<-2$. Then (4.66) takes the form

$$
\lim _{r \rightarrow \infty} \frac{1}{r^{2} V(r)} \int_{r}^{\infty} F(t) d t=0
$$

Now, the additional statement for the justification of (4.68) looks like this.
Let $g$ be a function of slow variation relative to a proximate order $\rho(r)$ with $\rho(\infty)<-1$. If

$$
\lim _{r \rightarrow \infty} \frac{1}{r V(r)} \int_{r}^{\infty} g(t) d t=a
$$

then

$$
\lim _{r \rightarrow \infty} \frac{g(r)}{V(r)}=-a(\rho+1)
$$

This is also a consequence of the results of [19. So, we have proved that, also for $\rho(\infty)<-2$, formula (4.66) implies (4.68), which contradicts the assumption of the theorem. In any case, the assumption that $\rho(r)+1$ is not a stable proximate order for $F(r)$ leads to a contradiction.

The theorem proved above is fairly peculiar. It can be viewed as a statement about the stabilization under consecutive integration for the property to have a stable proximate order. Let $F_{k}(r)$ be the consecutive canonical primitives on $(0, \infty)$ for $f(r)$. The theorem shows that if $F_{k}(r)$ has a stable proximate order $\rho_{k}(r)$ and $\rho_{k}(\infty)$ is not a negative integer, then the proximate order $\rho_{k}(r)+m$ is stable for $F_{k+m}(r)$ for every $m \geq 1$.

Theorem 40. Suppose that $f$ is a locally integrable function of finite order on $(0, \infty)$ and the integral $\int_{0}^{1} f(t) d t$ exists. Suppose that $f$ does not possess a stable proximate order, and let $\rho(r)$ be a proximate order for $f$ with $\rho(\infty) \neq-1$. Let $\lambda$ be the measure whose density is $f$. Then $\lambda \in \mathfrak{M}_{\infty}(\rho(r)+1)$ and $\operatorname{Fr}[\rho(r)+1, \lambda]=\{0\}$.

Proof. Since $\rho(r)$ is a proximate order for $f(r)$, it follows that $\lambda \in \mathfrak{M}_{\infty}(\rho(r)+1)$. Let $F$ be the canonical primitive for $f$ on $(0, \infty)$. Since $f$ has no stable proximate order, we have

$$
\begin{equation*}
\lim _{r \rightarrow \infty} \frac{F(r)}{r V(r)}=0 \tag{4.69}
\end{equation*}
$$

Let $\varphi$ be an arbitrary infinitely differentiable compactly supported function on $(0, \infty)$ with $\operatorname{supp} \varphi \subset[a, b] \subset(0, \infty)$. Then

$$
\begin{aligned}
\frac{1}{r V(r)} \int_{0}^{\infty} \varphi\left(\frac{t}{r}\right) d \lambda(t) & =\frac{1}{r V(r)} \int_{0}^{\infty} \varphi\left(\frac{t}{r}\right) f(t) d t \\
& =-\frac{1}{r^{2} V(r)} \int_{0}^{\infty} \varphi^{\prime}\left(\frac{t}{r}\right) F(t) d t=-\frac{1}{r V(r)} \int_{a}^{b} \varphi^{\prime}(u) F(u r) d u
\end{aligned}
$$

Together with (4.69), this yields

$$
\begin{equation*}
\lim _{r \rightarrow \infty} \frac{1}{r V(r)} \int_{0}^{\infty} \varphi\left(\frac{t}{r}\right) d \lambda(t)=0 \tag{4.70}
\end{equation*}
$$

Let $\nu$ be an arbitrary measure in $\operatorname{Fr}[\lambda]$. There is a sequence $r_{n} \rightarrow \infty$ such that $\lambda_{r_{n}} \rightarrow \nu$. Taking $r=r_{n}$ in (4.70), we obtain $\int_{0}^{\infty} \varphi(u) d \nu(u)=0$. Therefore, $\nu=0$ by Theorem [17] Thus, $\operatorname{Fr}[\lambda]=\{0\}$.

Theorem 41. Suppose that $f$ is a locally integrable function of finite order on $(0, \infty)$ and the integral $\int_{0}^{1} f(t) d t$ exists. Let $F$ be the canonical primitive for $f$ on $(0, \infty)$. Suppose that $f$ does not possess a stable proximate order. Let $\rho(r)$ be some proximate order for $f$ with $\rho(\infty) \neq-1$. If the order of $F$ is not equal to $-\infty$, then there exists a proximate order $\rho_{1}$ for $F$ with

$$
\lim _{r \rightarrow \infty} \frac{V_{1}(r)}{r V(r)}=0
$$

Proof. The claim is obvious if the order of $F$ satisfies $\rho_{1}<\rho(\infty)+1$, so we assume that $\rho_{1}=\rho(\infty)+1$. Since there is no stable proximate order for $f$, (4.69) is true. In combination with [9, Theorem 5], this yields a proximate order $\rho_{2}(r)$ with

$$
\lim _{r \rightarrow \infty} \frac{F(r)}{V_{2}(r)}=0, \quad \lim _{r \rightarrow \infty} \frac{V_{2}(r)}{r V(r)}=0
$$

The function $F_{1}(r)=\frac{F(r)}{V_{2}(r)}$ is of zero order and it tends to zero at infinity. By Theorem 55 it has a proximate order $\rho_{3}(r)$ that grows monotonically on $(1, \infty)$ and satisfies $\rho_{3}(\infty)=0$ (the case of $\rho_{3}(r) \equiv 0$ is not excluded). The proximate order $\rho_{1}(r)=\rho_{2}(r)+\rho_{3}(r)$ is a proximate order for $F(r)$. We have

$$
\lim _{r \rightarrow \infty} \frac{V_{2}(r) V_{3}(r)}{r V(r)}=0
$$

Thus, $\rho_{1}(r)$ is a required proximate order.
Now, we return to determining the order of growth of a function $\Psi(r)$ defined as in (4.64). If $\mu \in \mathfrak{M}_{\infty}(\rho(r))$, then $L(J, \infty)$ has the form

$$
L(J, \infty)=\left\{\int_{0}^{\infty} K(u) d \nu(u): \nu \in \operatorname{Fr}[\mu]\right\}
$$

by Theorem 31. If $L(J, \infty) \neq\{0\}$, then $\Psi$ grows as $V(r)$ at infinity. But if $L(J, \infty)=\{0\}$, we have $\Psi(r)=o(V(r))$, which does not determine the order of growth of $\Psi(r)$ at infinity.

The relation $L(J, \infty)=\{0\}$ may be fulfilled for various reasons. First, we consider the case where $\operatorname{Fr}[\mu] \neq\{0\}$, but $K$ is such that $\int_{0}^{\infty} K(u) d \nu(u)=0$ for every $\nu \in \operatorname{Fr}[\mu]$. In this case, Theorem 31does not allow us to determine the order of growth for $\Psi(r)$ at infinity. For example, it may happen that $\operatorname{Fr}[\mu]=\left\{u^{\rho-1} d u\right\}, \int_{0}^{\infty} K(u) u^{\rho-1} d u=0$. In the case in question, not merely our method fails, but in general the available information about $K$ and $\mu$ (we know that $\int_{0}^{\infty} K(u) d \nu(u)=0$ for $\nu \in \operatorname{Fr}[\mu]$ ) is insufficient for determining the order of growth of $\Psi$ at infinity. Some additional information is required, and the
complexity of the problem depends on this information. In the particular case where $d \mu(t)=t^{\rho-1} d t$, we have $\Psi(r) \equiv 0$.

Now, consider the case where $\operatorname{Fr}[\mu]=\{0\}$. Then we can use (4.65) to improve the estimate $\Psi(r)=o(V(r))$.

Let $F_{m}(t)$ have a proximate order $\rho_{m}(r)$. Then the measure $\lambda_{m}, d \lambda_{m}(t)=F_{m}(t) d t$, belongs to $\mathfrak{M}_{\infty}\left(\rho_{m}(r)+1\right)$. Put

$$
H_{m}=\left\{\int_{0}^{\infty} K^{(m+1)}(u) d \nu(u): u \in \operatorname{Fr}\left[\rho_{m}(r)+1, \lambda_{m}\right]\right\} .
$$

By Theorem 31, the cluster set for the function $(-1)^{m+1} r^{m+1} \Psi(r) / r V_{m}(r)$ in the direction $r \rightarrow \infty$ coincides with $H_{m}$. If $H_{m} \neq\{0\}$, if follows that $\Psi(r)$ grows at infinity as $\frac{V_{m}(r)}{r^{m}}$. But if $H_{m}=\{0\}$, we obtain $\Psi(r)=o\left(\frac{V_{m}(r)}{r^{m}}\right)$.

If $H_{m}=\{0\}$ and $\operatorname{Fr}\left[\lambda_{m}\right] \neq\{0\}$, we again obtain a case where Theorem 31 does not answer the question about the order of growth of $\Psi(r)$ at infinity.

Now, suppose that $F_{m}(r)$ does not possess a stable proximate order. Then, application of Theorems 31 and 40 to function $F_{m+1}(r)$ and the proximate order $\rho_{m}(r)+1$ shows that

$$
\Psi(r)=o\left(\frac{V_{m}(r)}{r^{m}}\right) .
$$

By Theorem 41, there exists a proximate order $\rho_{m+1}(r)$ for $F_{m+1}(r)$ with $V_{m+1}(r)=$ $o\left(r V_{m}(r)\right)$. Applying Theorem 31 to $F_{m+1}(r)$ and $\rho_{m+1}(r)$, we obtain the relation $\Psi(r)=$ $O\left(r^{-m-1} V_{m+1}(r)\right)$, which is stronger than the relation $\Psi(r)=o\left(r^{-m} V_{m}(r)\right)$.

Denote by $\rho_{n}$ a proximate order for $F_{n}(r)$. The following theorem holds.
Theorem 42. Let $K(t)$ be a compactly supported infinitely differentiable kernel, and let $\mu$ be a Radon measure on $(0, \infty)$ having no mass on $(0,1]$. We define $\Psi(r)$ by (4.64) and define uniquely the functions $F_{n}(t)$ in (4.65) by the algorithm described above. Let $\rho_{n}$ be the order of $F_{n}$. If

$$
\liminf _{m \rightarrow \infty}\left(\rho_{m}-m\right)=-\infty,
$$

then $\Psi(r)$ decays at infinity faster than an arbitrary power of $r$.
Proof. When applied to $F_{m}(r)$ and $\rho_{m}(r)$, Theorem 31 yields

$$
\Psi(r)=O\left(\frac{V_{m}(r)}{r^{m}}\right) .
$$

This proves the claim.
It is possible that $\rho_{n}=-\infty$ for some $n$ (for example, this is so if $d \mu(t)=T e^{i e^{t}} d t$ ). Then (4.65) and Theorem 31 imply that $\Psi(r)$ decays at infinity faster than an arbitrary power of $r$.

Now, we consider the case where

$$
\begin{equation*}
\liminf _{m \rightarrow \infty}\left(\rho_{m}-m\right)>-\infty \tag{4.71}
\end{equation*}
$$

Since $\rho_{n+1} \leq \rho_{n}+1$ by Lemma 12, it follows that $\rho_{n+1}=\rho_{n}+1-\varepsilon_{n}$, where $\varepsilon_{n} \geq 0$. Inequality (4.71) shows that the series $\sum_{n=0}^{\infty} \varepsilon_{n}$ converges. In its turn, this implies the convergence of the sequence $\rho_{n}-n$. Then $\rho_{n}=n+p+\delta_{n}$, where $\delta_{n} \rightarrow 0, \rho_{n}>0$ for $n>n_{0}$. If $F_{m}(r)$ has a stable proximate order $\rho_{m}(r)$ for some $m>n_{0}$, then Theorem 39 shows that for every $k \geq 0$ the function $\rho_{m+k}(r)=\rho_{m}(r)+k$ is a stable proximate order for $\frac{V_{m}(r)}{r^{m}}$ if $E_{m} \neq\{0\}$, but otherwise the method gives nothing beyond the relation $\Psi(r)=o\left(\frac{V_{m}(r)}{r^{m}}\right)$.

But if for all $m>n_{0}$ there is no stable proximate order for $F_{m}(r)$, then, by Theorem41, the functions $F_{m}(r)$ possess some proximate orders $\rho_{m}(r)$ such that $\Psi(r)=o\left(\frac{V_{m+1}(r)}{r^{m+1}}\right)$,
which is a refinement of the preceding relation $\Psi(r)=o\left(\frac{V_{m}(r)}{r^{m}}\right)$. The method gives nothing beyond this in the case in question.

We have discussed the possibilities of our method for a compactly supported infinitely differentiable kernel $K$. Under some conditions, these results can be extended to not necessarily compactly supported kernels.

In Theorem 35, it was required that the triple $(K, \rho(r), \mu)$ admit neutralization of zero and of infinity. There are also results about the asymptotic behavior of $\Psi$ in which this assumption is violated.

Theorem 43. Let $\rho(r)$ be a zero proximate order, and let $\mu \in \mathfrak{M}(\rho(r))$ be such that the limit $\lim _{\varepsilon \rightarrow 0} \mu([\varepsilon, 1])=\mu((0,1])$ exists. Let $K(t)$ be a continuous function on $[0, \infty)$. Put $K_{1}(t)=K(t)-K(0) \chi_{[0,1]}(t)$ and suppose that the triple $\left(K_{1}, \rho(r), \mu\right)$ admits neutralization of zero and of infinity. Then

$$
\int_{0}^{\infty} K\left(\frac{t}{r}\right) d \mu(t)=K(0) \mu((0, r])+\varphi(r)
$$

moreover, the cluster set of the function $\frac{\varphi(r)}{V(r)}$ as $r \rightarrow \infty$ has the form

$$
\left\{\int_{0}^{1}(K(t)-K(0)) d \nu_{1}(t)+\int_{1}^{\infty} K(t) d \nu_{2}(t):\left(\nu_{1}, \nu_{2}\right) \in \widehat{\operatorname{Fr}}(\mu)\right\} .
$$

Proof. We have

$$
\int_{0}^{\infty} K\left(\frac{t}{r}\right) d \mu(t)=K(0) \mu([0, r])+\int_{0}^{\infty} K_{1}\left(\frac{t}{r}\right) d \mu(t) .
$$

The function $K_{1}$ and the measure $\mu$ obey all assumptions of Theorem 36. Applying it, we complete the proof.

Many theorems involve the following assumption:

$$
\begin{equation*}
\Psi(r)=\int_{0}^{\infty} K\left(\frac{t}{r}\right) d \mu(t) \sim M V(r), \quad r \rightarrow \infty \tag{4.72}
\end{equation*}
$$

Under some restrictions on $K$ and $\mu$, we have $\Psi(r)=r^{\rho_{1}(r)}$, where $\rho_{1}(r)$ is a proximate order.

Theorem 44. Let $\rho(r)$ be a zero proximate order, $\mu$ a positive locally finite measure on $(0, \infty)$, and $K(t)$ a continuously differentiable function strictly monotone decreasing on $(0, \infty)$ and such that the triple $\left(t K^{\prime}(t), \rho(t), \mu\right)$ admits neutralization of zero and of infinity. Assume that (4.72) holds true and that the derivative $\Psi^{\prime}(r)$ can be calculated by the Leibnitz rule:

$$
\Psi^{\prime}(r)=-\frac{1}{r^{2}} \int_{0}^{\infty} t K^{\prime}\left(\frac{t}{r}\right) d \mu(t)
$$

If we define $\rho_{1}(r)$ by the formulas $r^{\rho_{1}(r)}=\Psi(r)$ for $r \geq 1$ and $\rho_{1}\left(\frac{1}{r}\right)=-\rho_{1}(r)$, then $\rho_{1}(r)$ is a zero proximate order.
Proof. We remind the reader that the relation $\rho_{1}\left(\frac{1}{r}\right)=-\rho_{1}(r)$ is a part of our definition of a zero proximate order (see $\S 2$ ). We have

$$
\begin{align*}
\Psi(2 r)-\Psi(r) & =\int_{0}^{\infty}\left(K\left(\frac{t}{2 r}\right)-K\left(\frac{t}{r}\right)\right) d \mu(t)  \tag{4.73}\\
& \geq \int_{r}^{2 r}\left(K\left(\frac{t}{2 r}\right)-K\left(\frac{t}{r}\right)\right) d \mu(t) \geq m \mu([r, 2 r]),
\end{align*}
$$

where $m=\min \left\{K\left(\frac{u}{2}\right)-K(u): u \in[1,2]\right\}>0$. By (4.72), it follows that $\operatorname{Fr}[\rho(r), \mu]=$ $\{0\}$.

Now, by assumption and by Theorem 35 we obtain

$$
\lim _{r \rightarrow \infty} \frac{r \Psi^{\prime}(r)}{V(r)}=0, \quad \lim _{r \rightarrow \infty} \frac{r \Psi^{\prime}(r)}{\Psi(r)}=0
$$

This argument shows also that the function $\rho_{1}(r)$ defined in the theorem is a zero proximate order.
Remark 7. Let us dwell on Theorem 108 in Hardy's book [6]. It claims that if $\rho(r)$ is a zero proximate order and $\mu$ is a positive locally finite measure on $[0, \infty)$, then the relations

$$
\Psi(r)=\int_{0}^{\infty} e^{-\frac{t}{r}} d \mu(t) \sim V(r), \quad \mu([0, r]) \sim V(r) \quad(r \rightarrow \infty)
$$

are equivalent.
We want to apply our theorems to this case and to look at the results. Theorem 44 implies that $\Psi(r)$ is representable in the form $\Psi(r)=r^{\rho_{1}(r)}=V_{1}(r)$, where $\rho_{1}(r)$ is a zero proximate order. There exists a zero proximate order $\rho_{2}(r)$ such that

$$
\limsup _{r \rightarrow \infty} \frac{V_{1}(2 r)-V_{1}(r)}{V_{2}(r)}<\infty, \quad \lim _{r \rightarrow \infty} \frac{V_{2}(r)}{V_{1}(r)}=0 .
$$

The simplest choice is to define $\rho_{2}(r)$ by the relation $r^{\rho_{2}(r)}=V_{1}(2 r)-V_{1}(r)$, provided this function is a proximate order.

We have

$$
\begin{aligned}
V_{1}(2 r)-V_{1}(r) & =\int_{0}^{\infty} e^{-\frac{t}{2 r}}\left(1-e^{-\frac{t}{2 r}}\right) d \mu(t) \\
& \geq \int_{r}^{2 r} e^{-\frac{t}{2 r}}\left(1-e^{-\frac{t}{2 r}}\right) d \mu(t) \geq e^{-1}\left(1-e^{-\frac{1}{2}}\right) \mu((r, 2 r])
\end{aligned}
$$

It follows that $\mu \in \mathfrak{M}_{\infty}\left(\rho_{2}(r)\right)$. Now, applying Theorem 36 to the second summand on the right in the formula

$$
\Psi(r)=\mu([0, r])+\int_{0}^{\infty}\left(e^{-\frac{t}{r}}-\chi_{[0, r]}(t)\right) d \mu(t)
$$

we obtain

$$
\mu([0, r])=\Psi(r)+O\left(V_{2}(r)\right) .
$$

This is stronger than Theorem 108.

## §5. Tauberian theorems for integrals

We start with a definition.
Let $\mu$ be a Radon measure on the real axis satisfying the condition $|\mu|([-t, t]) \leq$ $M\left(1+t^{\alpha}\right)$ with some $\alpha \geq 0$. The Carleman transform of $\mu$ is defined to be the pair $G(z)=\left(G_{+}(z), G_{-}(z)\right)$ of functions given by

$$
\begin{aligned}
& G_{+}(z)=\int_{0}^{\prime \infty} e^{i t z} d \mu(t), \quad \operatorname{Im} z>0, \\
& G_{-}(z)=-\int_{-\infty}^{\prime 0} e^{i t z} d \mu(t), \quad \operatorname{Im} z<0 .
\end{aligned}
$$

Primes near the integral signs means that we integrate against the measure $\mu-$ $\frac{1}{2} \mu(\{0\}) \delta(\delta$ is the Dirac unit point mass at zero) rather than against $\mu$. If $\mu(\{0\})=0$, the primes can be omitted.

Clearly, $G(z)$ is a locally holomorphic function on $\mathbb{C} \backslash \mathbb{R}(\mathbb{C}$ is the complex plane and $\mathbb{R}$ is the real axis).

Our exposition of Tauberian theorems depends rather essentially on the following statement about analytic continuation.

Theorem 45. Let $M>0$ be a fixed number and $\mu$ a Radon measure on the real axis such that $|\mu|([\alpha, \beta]) \leq M$ if $\beta-\alpha \leq 1$. Suppose that a Borel function $K(t)$ belongs to $L_{1}(-\infty, \infty)$ and $\widehat{K}(\lambda)=\int_{-\infty}^{\infty} K(t) e^{-i \lambda t} d t \neq 0$ for $\lambda \in(a, b)$. Suppose also that

$$
\begin{equation*}
\int_{-\infty}^{\infty} K(t-u) d \mu(t)=0, \quad u \in(-\infty, \infty) \tag{5.74}
\end{equation*}
$$

and let $G(z)=\left(G_{+}(z), G_{-}(z)\right)$ be the Carleman transform of $\mu$. Then the following is true:
1)

$$
\begin{equation*}
|G(z)| \leq M\left(1+\frac{1}{|y|}\right), \quad z=x+i y \tag{5.75}
\end{equation*}
$$

2) $G_{+}(z)$ admits analytic continuation through $(a, b)$ to $G_{-}(z)$.

Proof. First, we show that if $d-c=1$, then there exists $M_{1}$ such that

$$
\begin{equation*}
I(c, d)=\int_{c}^{d} \int_{-\infty}^{\infty}|K(t-u)| d|\mu|(t) d u \leq M_{1} \tag{5.76}
\end{equation*}
$$

for every $c \in(-\infty, \infty)$. We have

$$
\begin{aligned}
I(c, d) & =\int_{-\infty}^{\infty} \int_{c}^{d}|K(t-u)| d u d|\mu|(t)=\int_{-\infty}^{\infty} \int_{t-d}^{t-c}|K(\tau)| d \tau d|\mu|(t) \\
& =\sum_{n=-\infty}^{\infty} \int_{n}^{n+1} \int_{t-d}^{t-c}|K(\tau)| d \tau d|\mu|(t)
\end{aligned}
$$

By the mean value theorem, there exist points $t_{n} \in[n, n+1]$ with

$$
I(c, d)=\sum_{n=-\infty}^{\infty} \int_{t_{n}-d}^{t_{n}-c}|K(\tau)| d \tau \int_{n}^{n+1} d|\mu|(t) .
$$

By the restrictions on $\mu$, we obtain

$$
I(c, d) \leq M \sum_{n=-\infty}^{\infty} \int_{t_{n}-d}^{t_{n}-c}|K(\tau)| d \tau=M \int_{-\infty}^{\infty} a(\tau)|K(\tau)| d \tau
$$

where $a(\tau)$ is the number of segments $\left[t_{n}-d, t_{n}-c\right]$ that contain $\tau$. Now, (5.76) follows because $a(\tau) \leq 3$.

Next, we estimate $G(z)$. Let $\check{\mu}(t)$ be the distribution function for $|\mu|$ normalized by the condition $\breve{\mu}(0)=0$. By assumption, $|\breve{\mu}(t)| \leq M(1+|t|)$. Therefore,

$$
\left|G_{+}(z)\right| \leq \int_{0}^{\infty} e^{-t y} d \breve{\mu}(t)=\left.e^{-t y} \breve{\mu}(t)\right|_{0} ^{\infty}+y \int_{0}^{\infty} \breve{\mu}(t) e^{-t y} d t
$$

Using the estimate for $\breve{\mu}(t)$, we arrive at the inequality $\left|G_{+}(z)\right| \leq M\left(1+\frac{1}{y}\right)$. The function $G_{-}(z)$ is estimated similarly. This proves statement 1).

We denote $T_{\varepsilon}(t)=\left(\sin \frac{\varepsilon}{2} t / \frac{\varepsilon}{2} t\right)^{2}$ and define a measure $\mu_{\varepsilon}$ by $d \mu_{\varepsilon}(t)=T_{\varepsilon}(t) d \mu(t)$. Let $G^{\varepsilon}(z)$ be the Carleman transform of $\mu_{\varepsilon}$. We observe that the measure $\mu_{\varepsilon}$ is finite and that, by the inequality $T_{\varepsilon}(t) \leq 1$, the function $G$ can be replaced by $G_{\varepsilon}$ in (5.75).

Assuming that $\varepsilon$ is sufficiently small, let $\left[a_{1}, b_{1}\right]$ be a segment such that $a+\varepsilon<a_{1}<$ $b_{1}<b-\varepsilon$. Let $\xi$ be an arbitrary point of the segment $\left[a_{1}, b_{1}\right]$. We find a function
$K_{1} \in L_{1}(-\infty, \infty)$ such that

$$
\begin{equation*}
\int_{-\infty}^{\infty} K_{1}(u) K(t-u) d u=T_{\varepsilon}(t) e^{i \xi t} \tag{5.77}
\end{equation*}
$$

If such a function exists, we have

$$
\begin{equation*}
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_{1}(u) K(t-u) e^{-i t x} d u d t=\widehat{T}(\xi, x) \tag{5.78}
\end{equation*}
$$

where

$$
\widehat{T}(\xi, x)=\int_{-\infty}^{\infty} T_{\varepsilon}(t) e^{i \xi t} e^{-i x t} d t
$$

It is well known that the integral on the left in (5.78) converges absolutely, so we can change the order of integration. This yields

$$
\begin{align*}
\widehat{T}(\xi, x) & =\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(t-u) e^{-i t x} d t K_{1}(u) d u=\widehat{K}(x) \widehat{K}_{1}(x) \\
\hat{K}_{1}(x) & =\frac{\widehat{T}(\varepsilon, x)}{\widehat{K}(x)} \tag{5.79}
\end{align*}
$$

Now we forget the way in which (5.79) was deduced and consider the function $\hat{K}_{1}(x)$ determined by this formula. The support of $\widehat{T}(\varepsilon, x)$ is included in $[\xi-\varepsilon, \xi+\varepsilon]$, which is a part of $(a, b)$. The function $\hat{K}(x)$ does not vanish on $(a, b)$. So, $\hat{K}_{1}(x)$ is well defined on $[\xi-\varepsilon, \xi+\varepsilon]$. We agree that $\widehat{K}_{1}(x)$ vanishes outside this segment. Thus, $\widehat{K}_{1}(x)$ is a continuous compactly supported function.

By the Wiener division theorem, it follows that if $\widehat{K}_{1}(x)$ is defined by (5.79), then there exists a function $K_{1} \in L_{1}(-\infty, \infty)$ whose Fourier transform coincides with $\widehat{K}_{1}(x)$. Now, (5.77) follows from the identity $\widehat{K}_{1}(x) \widehat{K}(x)=\widehat{T}(\varepsilon, x)$ by taking inverse Fourier transforms. The existence of $K_{1}$ is proved.

We multiply the two sides of (5.74) by $K_{1}(u)$ and then integrate over $(-\infty, \infty)$, obtaining

$$
\begin{equation*}
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_{1}(u) K(t-u) d \mu(t) d u=0 \tag{5.80}
\end{equation*}
$$

We want to estimate the integral

$$
I=\int_{-\infty}^{\infty} \int_{-\infty}^{\infty}\left|K_{1}(u)\right||K(t-u)| d u d|\mu|(t)
$$

We have

$$
\begin{aligned}
I & =\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{n}^{n+1}\left|K_{1}(u)\right||K(t-u)| d u d|\mu|(t) \\
& \leq \sum_{n=-\infty}^{\infty} \max \left\{\left|K_{1}(u)\right|: u \in[n, n+1]\right\} \int_{-\infty}^{\infty} \int_{n}^{n+1}|K(t-u)| d u d|\mu|(t)
\end{aligned}
$$

Applying (5.76), we arrive at the inequality

$$
I \leq M_{1} \sum_{n=-\infty}^{\infty} \max \left\{\left|K_{1}(u)\right|: u \in[n, n+1]\right\} .
$$

The function $K_{1} \in L_{1}(-\infty, \infty)$ is the Fourier transform of a continuous compactly supported function. By Lemma $6_{7}$ in [20, Chapter 2, $\left.\S 11\right]$, the above series converges for
such functions. Thus, $I<\infty$. Now by the Tonelli and Fubini theorems, we see that the order of integration on the left in (5.80) can be interchanged. By (5.77), this yields

$$
\begin{equation*}
\int_{-\infty}^{\infty} e^{i \xi t} T_{\varepsilon}(t) d \mu(t)=0, \quad \xi \in\left[a_{1}, b_{1}\right] . \tag{5.81}
\end{equation*}
$$

Since $\mu_{\varepsilon}$ is finite, we see that $G_{+}^{\varepsilon}(z)$ is holomorphic in the upper half-plane and is continuous up to the boundary. Then (5.81) can be rewritten in the form $G_{+}^{\varepsilon}(\xi)=G_{-}^{\varepsilon}(\xi)$, $\xi \in\left[a_{1}, b_{1}\right]$. By the theorem of the elimination of singularities (see [21, Theorem 2.2, Chapter 4]), the functions $G_{+}^{\varepsilon}(z)$ and $G_{-}^{\varepsilon}(z)$ are analytic continuation for one another through ( $a_{1}, b_{1}$ ).

Next, in the complex plane, we consider the square $Q_{1}$ for which the segment $\left[a_{1}, b_{1}\right]$ is a diagonal. Consider also the family of functions

$$
F_{\eta}(z)=\left(z-a_{1}\right)\left(z-b_{1}\right) G^{\eta}(z), \quad \eta \in(0, \varepsilon],
$$

in this square. Some edge of $Q_{1}$ has the parametrization $z=a_{1}+t e^{\frac{i \pi}{4}}, t \in\left[0, \frac{b_{1}-a_{1}}{\sqrt{2}}\right]$. On this edge, we have

$$
\left|F_{\eta}(z(t))\right| \leq M t\left|a_{1}-b_{1}+t e^{\frac{i \pi}{4}}\right|\left|1+\frac{\sqrt{2}}{t}\right| \leq M_{2}
$$

Clearly, $\left|F_{\eta}(z)\right| \leq M_{2}$ on the boundary of the square.
Let $\delta \in\left(0, \frac{b_{1}-a_{1}}{4}\right)$, and let $Q_{2}$ be the square whose diagonal is $\left[a_{1}+\delta, b_{1}-\delta\right]$. On the boundary of $Q_{2}$, we have $\left|G^{\eta}(z)\right| \leq M_{3}(\delta)$ with some quantity $M_{3}$ depending on $\delta$.

By the Montel theorem, the family $G^{\eta}(z)$ is compact inside $Q_{2}$. So, there exists a function $H(z)$ holomorphic in $Q_{2}$ and a sequence $\eta_{n} \rightarrow 0$ such that the sequence $G^{\eta_{n}}(z)$ converges to $H(z)$ uniformly on compact sets inside $Q_{2}$. For $\operatorname{Im} z>0$, the sequence $G^{\eta_{n}}(z)$ converges to $G_{+}(z)$, and for $\operatorname{Im} z<0$ it converges to $G_{-}(z)$. It follows that $G_{-}(z)$ is an analytic continuation for $G^{+}(z)$ through $\left(a_{1}+\delta, b_{1}-\delta\right)$, and, consequently, through ( $a, b$ ) because $\varepsilon$ and $\delta$ are arbitrary.

Remark 8. Theorem 45 is a refinement of Carleman's analytic continuation lemma. In Carleman's original statement (Carleman's proof can also be found in [23]), measures with bounded density were considered. The above proof of Theorem 45 involves some arguments of Carleman.

Let $\mu$ be a Radon measure of finite order on the real axis, and let $G(z)=\left(G_{+}(z), G_{-}(z)\right)$ be the Carleman transform of $\mu$. We remove from the real axis all intervals $(a, b)$ such that $G_{+}(z)$ continues analytically into $G_{-}(z)$ through $(a, b)$. The remaining part of the real axis is called the Carleman spectrum of $\mu$.

It should be noted that it may happen that $G_{+}(z)$ is continuable analytically through $(a, b)$, but the result is not $G_{-}(z)$. In this case, $(a, b)$ is included in the Carleman spectrum of $\mu$.

In connection with Theorem 45 and the definition of the Carleman spectrum, it is interesting to mention the following statement pertaining to harmonic synthesis (see 24, [23] for the discussion of this subject).

Theorem 46. Let $\mu$ be a Radon measure on the real axis with bounded Carleman spectrum. Then $\mu$ is absolutely continuous and its density $g(t)$ is the restriction of an entire function of exponential type to the real axis.

Proof. Let $G(z)=\left(G_{+}(z), G_{-}(z)\right)$ be the Carleman transform of $\mu$. Since $\mu$ is of finite order, there exist numbers $M>0$ and $\alpha>0$ such that $\breve{\mu}(r)$ (the distribution function
for $|\mu|)$ satisfies $|\breve{\mu}(r)| \leq M\left(1+r^{\alpha}\right)$. Assuming that $\breve{\mu}(r)$ is normalized by the condition $\breve{\mu}(+0)=0$, we obtain

$$
\begin{aligned}
\left|G_{+}(z)\right| & \leq \frac{1}{2}|\mu(\{0\})|+\int_{(0, \infty)} e^{-t y} d \breve{\mu}(t)=\frac{1}{2}|\mu(\{0\})|+y \int_{0}^{\infty} \check{\mu}(t) e^{-t y} d t \\
& \leq \frac{1}{2}|\mu|(\{0\})+M y \int_{0}^{\infty}\left(1+t^{\alpha}\right) e^{-t y} d t \leq M_{1}\left(1+\frac{\Gamma(1+\alpha)}{y^{\alpha}}\right)
\end{aligned}
$$

The function $G_{-}(z)$ is estimated similarly. Therefore, $G(z)$ satisfies the estimate

$$
\begin{equation*}
|G(z)| \leq M_{1}\left(1+\frac{\Gamma(1+\alpha)}{|y|^{\alpha}}\right) \tag{5.82}
\end{equation*}
$$

Since the spectrum of $\mu$ is bounded, the point $\infty$ is an isolated singularity for $G(z)$. Arguing as in the proof of Theorem 45 (see the text involving the squares $Q_{1}$ and $Q_{2}$ ) for the function $\left(z-a_{1}\right)^{n}\left(z-b_{1}\right)^{n} G(z)$, we arrive at the conclusion that there exist $\alpha>0$, $\beta>0$, and $M_{2}>0$ such that $|G(z)| \leq M_{2}$ on the set $\{z=x+i y:|x| \geq \alpha,|y| \leq \beta\}$. Together with (5.82), this implies that $\infty$ is a removable singularity for $G(z)$.

It is easily seen that

$$
\begin{aligned}
\lim _{y \rightarrow+\infty} G_{+}(i y) & =\lim _{y \rightarrow+\infty} \int_{0}^{\prime \infty} e^{-t y} d \mu(t)=\frac{1}{2} \mu(\{0\}) \\
\lim _{y \rightarrow-\infty} G_{-}(i y) & =-\lim _{y \rightarrow-\infty} \int_{\infty}^{\prime 0} e^{t y} d \mu(t)=-\frac{1}{2} \mu(\{0\})
\end{aligned}
$$

These identities allow us to conclude that the relation $\mu(\{0\}) \neq 0$ would contradict the fact that $\infty$ is a removable singularity for $G$. Thus, $\mu(\{0\})=0$, and we have proved that $G(\infty)=0$. Consequently, there exist numbers $M_{3}>0$ and $R_{1}>0$ such that

$$
\begin{equation*}
|G(z)|<\frac{M_{3}}{|z|} \tag{5.83}
\end{equation*}
$$

for $|z|>R_{1}$.
Now, we define the function

$$
\begin{equation*}
g(t)=-\frac{1}{2 \pi} \int_{\mathfrak{L}} G(w) e^{-i t w} d w \tag{5.84}
\end{equation*}
$$

where $\mathfrak{L}$ is a closed smooth positively oriented Jordan curve encompassing the spectrum of $\mu$. Note that $G(w)$ is holomorphic in the closure of the unbounded domain whose boundary is $\mathfrak{L}$ and that $g(t)$ is an entire function of exponential type.

Next, for $\operatorname{Im} z>0$ we introduce the function

$$
G_{1}(z)=\int_{0}^{\infty} g(t) e^{i t z} d t=-\frac{1}{2 \pi} \int_{\mathfrak{L}} \int_{0}^{\infty} e^{i t(z-w)} d t G(w) d w
$$

Let $\operatorname{Im} z \geq 2 h$, where $h>0$ is arbitrary. Since the spectrum of $\mu$ lies on the real axis, the contour $\mathfrak{L}$ can be chosen in such a way that $\operatorname{Im} w<h$ for every $w \in \mathfrak{L}$. Then

$$
G_{1}(z)=-\frac{1}{2 \pi i} \int_{\mathfrak{L}} \frac{G(w)}{w-z} d w
$$

Now, inequality (5.83) and the Cauchy theorem imply that $G_{1}(z)=G(z)$ for $\operatorname{Im} z \geq 2 h$, and, consequently, for $\operatorname{Im} z>0$.

We have proved that the Carleman transforms of the measures $\mu_{1}, d \mu_{1}(t)=g(t) d t$, and $\mu$ coincide in the half-plane $\operatorname{Im} z>0$. In a similar way, it can be verified that they coincide for $\operatorname{Im} z<0$. Thus, $\mu$ and $\mu_{1}$ have the same Carleman transforms, therefore $\mu=\mu_{1}$.

Theorem 47. Let $\mu$ be a Radon measure of order $\rho$ on the real axis. Suppose that the Carleman spectrum of $\mu$ is finite. Then there exist polynomials $P_{\lambda}(t), \lambda \in \Lambda, \operatorname{deg} P_{\lambda}(t) \leq$ $\rho-1$, such that

$$
d \mu(t)=\sum_{\lambda \in \Lambda} P_{\lambda}(t) e^{-i \lambda t} d t
$$

Proof. Let $G(z)$ be the Carleman transform of $\mu$. The spectrum of $\mu$ is finite and, consequently, bounded. The proof of Theorem 46 shows that in this case $G(z)$ is holomorphic in a neighborhood of infinity and $G(\infty)=0$. It was also proved that $d \mu(t)=g(t) d t$ in the case in question, where $g$ is an entire function of exponential type. Denote $g_{1}(t)=\int_{0}^{t}|g(u)| d u$. Integration by parts shows that

$$
\left|G_{+}(z)\right|=\left|\int_{0}^{\infty} e^{i t z} g(t) d t\right| \leq \int_{0}^{\infty} e^{-t y}|g(t)| d t=\left.e^{-t y} g_{1}(t)\right|_{0} ^{\infty}+y \int_{0}^{\infty} e^{-t y} g_{1}(t) d t
$$

If $\rho<0$, we have $\left|g_{1}(t)\right| \leq M$. If $\rho \geq 0$, for every $\varepsilon>0$ we have $\left|g_{1}(t)\right| \leq M_{\varepsilon}|t|^{\rho+\varepsilon}$ for $t \geq 1$. Thus, $|G(z)| \leq M$ if $\rho<0$, and $|G(z)| \leq M_{\varepsilon}\left(\frac{1}{|y|}\right)^{\rho+\varepsilon}$ if $|y| \leq 1$ and $\rho \geq 0$.

Since the spectrum is finite, its points are proper isolated singularities for $G(z)$, and the estimates for $G(z)$ obtained above show that for $\rho<1$ these singularities are removable. In this case, $G(z)=0$ and $\mu=0$. But if $\rho \geq 1$, then the points of the spectrum of $G$ can be poles of order at most $\rho$ for $G(z)$. Therefore,

$$
G(z)=\sum_{\lambda \in \Lambda} \sum_{n=1}^{\rho} \frac{a_{n, \lambda}}{(z-\lambda)^{n}}
$$

Now, the theorem follows from (5.84) and the formula

$$
\int_{\mathfrak{L}} \frac{1}{(w-\lambda)^{n}} e^{-i t w} d w=2 \pi i(-i t)^{n} e^{-i \lambda t}
$$

Theorem 48. Let $M>0$ be a fixed number, and let $\mu$ be a Radon measure on the real axis satisfying $|\mu|([\alpha, \beta]) \leq M$ whenever $\beta-\alpha \leq 1$. Let $K$ be a Borel function in $L_{1}(-\infty, \infty)$ such that the set $\bar{\Lambda}=\{\lambda \in(-\infty, \infty): \widehat{K}(\lambda)=0\}$ is finite. Suppose that

$$
\int_{-\infty}^{\infty} K(t-u) d \mu(t)=0, \quad u \in(-\infty, \infty)
$$

Then there exist members $c_{\lambda}, \lambda \in \Lambda$, such that $d \mu(t)=\sum_{\lambda \in \Lambda} c_{\lambda} e^{-i \lambda t} d t$.
Proof. By assumption and by Theorem [45, the Carleman spectrum of $\mu$ is included in the finite set $\Lambda$. Now, Theorem 47 and the inequality $\rho \leq 1$ for the order $\rho$ of $\mu$ imply the claim.

Statements similar to theorems 46 48 were earlier proved by Korenblum, see [25].
Next, we state a multiplicative version of the last theorem.
Theorem 49. Let $M>0$ be a fixed number, and let $\mu$ be a Radon measure on $(0, \infty)$ satisfying $|\mu|([\alpha, \beta]) \leq M$ whenever $\frac{\beta}{\alpha} \leq e$. Let $K$ be a Borel function on $(0, \infty)$ such that $\frac{1}{t} K(t) \in L_{1}(0, \infty)$. Suppose that the set $\Lambda=\left\{\lambda \in(-\infty, \infty): \int_{0}^{\infty} \frac{1}{t} K(t) t^{-i \lambda} d t=0\right\}$ is finite. Suppose also that

$$
\int_{0}^{\infty} K\left(\frac{t}{r}\right) d \mu(t)=0, \quad r \in(0, \infty)
$$

Then there exist numbers $c_{\lambda}, \lambda \in \Lambda$, with $d \mu(t)=\frac{1}{t} \sum_{\lambda \in \Lambda} c_{\lambda} t^{-i \lambda} d t$.

Proof. We introduce a Radon measure $\nu$ on the real axis in the following way: $\nu([a, b])=$ $\mu\left(\left[e^{a}, e^{b}\right]\right)$. If $b \leq a+1$, we have

$$
|\nu|([a, b])=|\mu|\left(\left[e^{a}, e^{b}\right]\right) \leq|\mu|\left(\left[e^{a}, e^{a+1}\right]\right) \leq M .
$$

Put $K_{1}(x)=K\left(e^{x}\right)$. Then $K_{1}$ is a Borel function on the real axis belonging to the space $L_{1}(-\infty, \infty)$. We have

$$
\begin{aligned}
\int_{0}^{\infty} \frac{1}{t} K(t) t^{-i \lambda} d t & =\int_{-\infty}^{\infty} K_{1}(x) e^{-i x \lambda} d x \\
\int_{0}^{\infty} K\left(\frac{t}{r}\right) d \mu(t) & =\int_{-\infty}^{\infty} K_{1}(x-u) d \nu(x)=0, \quad u=\ln r
\end{aligned}
$$

The kernel $K_{1}$ and the measure $\nu$ enjoy all the assumptions of Theorem 48. Therefore,

$$
d \nu(x)=\sum_{\lambda \in \Lambda} c_{\lambda} e^{-i \lambda x} d x, \quad d \mu(t)=\frac{1}{t} \sum_{\lambda \in \Lambda} c_{\lambda} t^{-i \lambda} d t
$$

In the next statement, we describe the measures $\mu$ for which the function $\Psi(r)$ defined by (1.2) is the density of a regular measure. We remind the reader that $\gamma(t)$ was defined in (1.1).

Theorem 50. Let $\rho(r)$ be an arbitrary proximate order, $\mu$ a Radon measure on $(0, \infty)$ of class $\mathfrak{M}(\rho(r))$, and $K$ a Borel function on $(0, \infty)$ such that $t^{\rho-1} \gamma(t) K(t) \in L_{1}(0, \infty)$ and $c_{1}=\int_{0}^{\infty} K(t) t^{\rho-1} d t \neq 0$. Suppose that the set

$$
\Lambda=\left\{\lambda \in(-\infty, \infty): \int_{0}^{\infty} K(t) t^{\rho-1-i \lambda} d t=0\right\}
$$

is finite. Define $\Psi$ by (1.2). If the measure $s, d s(t)=\Psi(t) d t$, is regular with respect to the proximate order $\rho(r)+1$ and, moreover, $\operatorname{Fr}[s]=\{\sigma\}$, where $d \sigma(t)=c t^{\rho} d t$, then the cluster set $\operatorname{Fr}[\mu]$ consists of absolutely continuous measures $\nu$ and the density $h(t)$ of every such measure is of the form

$$
h(t)=\left(\frac{c}{c_{1}}+\sum_{\lambda \in \Lambda} c_{\lambda} t^{-i \lambda}\right) t^{\rho-1}
$$

where the $c_{\lambda}$ are some complex numbers.
Proof. By Theorem 38 for every $\nu \in \operatorname{Fr}[\mu]$ we have

$$
\begin{equation*}
\int_{0}^{\infty} K\left(\frac{t}{u}\right) d \nu(t)=c u^{\rho} . \tag{5.85}
\end{equation*}
$$

If $\nu_{2}(t)=\nu(t)-\nu_{1}(t)$, where $d \nu_{1}(t)=\frac{c}{c_{1} \rho} t^{\rho-1} d t$ for $\rho \neq 0$ and $d \nu_{1}(t)=\frac{c}{c_{1}} \frac{1}{t} d t$ for $\rho=0$, then

$$
\begin{equation*}
\int_{0}^{\infty} K\left(\frac{t}{u}\right) d \nu_{2}(t)=0 \tag{5.86}
\end{equation*}
$$

Let $K_{1}(t)=t^{\rho} K(t)$, and let $\nu_{3}$ be the measure defined by $d \nu_{3}(t)=t^{-\rho} d \nu_{2}(t)$. Then formula (5.86) can be rewritten in the form

$$
\int_{0}^{\infty} K_{1}\left(\frac{t}{u}\right) d \nu_{3}(t)=0
$$

Since $\mu \in \mathfrak{M}(\rho(r))$, we see that the function

$$
N(\alpha)=\limsup _{r \rightarrow \infty} \frac{\breve{\mu}(r+\alpha r)-\breve{\mu}(r)}{V(r)}
$$

where $\breve{\mu}(r)$ is the distribution function for $|\mu|$, is bounded on $[0, e]$. By Theorem 25], it follows that for any $\nu \in \operatorname{Fr}[\mu]$, on the set $(0, \infty) \times[1, e]$ we have $|\nu|([r, q r]) \leq M r^{\rho}$ with some constant $M$. Clearly, a similar estimate (possibly, with a different $M$ ) is fulfilled for $\nu_{2}$.

There exists $\xi \in[\alpha, \beta]$ such that

$$
\int_{\alpha}^{\beta} d\left|\nu_{3}\right|(t)=\int_{\alpha}^{\beta} t^{-\rho} d\left|\nu_{2}\right|(t)=\xi^{-\rho}\left|\nu_{2}\right|([\alpha, \beta])
$$

Now if $\beta \leq e \alpha$, then

$$
\left|\nu_{3}\right|([\alpha, \beta])=\xi^{-\rho}\left|\nu_{2}\right|([\alpha, \beta]) \leq M\left(\frac{\alpha}{\xi}\right)^{\rho} \leq M e^{|\rho|} .
$$

Also, we have $\frac{1}{t} K_{1}(t) \in L_{1}(0, \infty)$. Thus, the kernel $K_{1}$ and the measure $\nu_{3}$ satisfy the assumptions of Theorem 49, Therefore, $d \nu_{3}(t)=\frac{1}{t} \sum_{\lambda \in \Lambda} c_{\lambda} t^{-i \lambda} d t$, which is equivalent to the claim of the theorem.

There is also a version of Theorem 50 in which the restriction $c_{1} \neq 0$ is absent.
Theorem 51. Let $\rho(r)$ be an arbitrary proximate order, $\mu$ a Radon measure on $(0, \infty)$ of class $\mathfrak{M}(\rho(r))$, and $K$ a Borel function on $(0, \infty)$ with $t^{\rho-1} \gamma(t) K(t) \in L_{1}(0, \infty)$. Suppose that the set

$$
\Lambda=\left\{\lambda \in(-\infty, \infty): \int_{0}^{\infty} K(t) t^{\rho-1-i \lambda} d t=0\right\}
$$

is finite and $0 \in \Lambda$. Define $\Psi$ by formula (1.2). If the measure $s, d s(t)=\Psi(t) d t$, is regular with respect to the proximate order $\rho(r)+1$, then $\operatorname{Fr}[s]=\{0\}$, the set $\operatorname{Fr}[\mu]$ consists of absolutely continuous measures $\nu$, and the density of every such measure is of the form

$$
h(t)=\left(\sum_{\lambda \in \Lambda} c_{\lambda} t^{-i \lambda}\right) t^{\rho-1}
$$

where the $c_{\lambda}$ are some complex numbers.
Proof. Since the measure $s$ is regular, we obtain (5.85) as in the proof of the preceding theorem. After that we argue as follows. Let $\nu$ and $\sigma$ be two measures in $\operatorname{Fr}[\mu]$, and let $\nu_{3}=\nu-\sigma$. We have $\int_{0}^{\infty} K\left(\frac{t}{u}\right) d \nu_{3}(t)=0$. Repeating the arguments in the previous theorem, we obtain $d \nu_{3}(t)=\left(\sum_{\lambda \in \Lambda} c_{\lambda} t^{-i \lambda}\right) t^{\rho-1} d t$. We have proved that for every $\nu \in \operatorname{Fr}[\mu]$ there exists a collection of complex numbers $c_{\lambda}$ (which, generally speaking, depend on $\nu$ ) such that $d \nu=d \sigma+d \tau$, where $\sigma$ is an arbitrary fixed measure in $\operatorname{Fr}[\mu]$ and $d \tau(t)=\sum_{\lambda \in \Lambda}\left(c_{\lambda} t^{-i \lambda}\right) t^{\rho-1} d t$.

For every $r>0$ we have $d \nu_{r}=d \sigma_{r}+d \tau_{r}$. Observe that

$$
d \tau_{r}(t)=\left(\sum_{\lambda \in \Lambda} c_{\lambda} r^{-i \lambda} t^{-i \lambda}\right) t^{\rho-1} d t
$$

On the other hand, since $\nu_{r} \in \operatorname{Fr}[\mu]$, we have

$$
d \nu_{r}(t)=d \sigma(t)+\left(\sum_{\lambda \in \Lambda} c_{\lambda}(r) t^{-i \lambda}\right) t^{\rho-1} d t
$$

If $\Lambda=\left\{\lambda_{1}, \ldots, \lambda_{n}\right\}$, from the above it follows that

$$
\begin{equation*}
d \sigma_{r}(t)-d \sigma(t)=\left(\sum_{k=1}^{n}\left(c_{k}(r)-c_{k} r^{-i \lambda_{k}}\right) t^{-i \lambda_{k}}\right) t^{\rho-1} d t \tag{5.87}
\end{equation*}
$$

From (5.87) we deduce the formula

$$
\begin{equation*}
\int_{0}^{\infty} f(t)\left(d \sigma_{r}(t)-d \sigma(t)\right)=\sum_{k=1}^{n}\left(c_{k}(r)-c_{k} r^{-i \lambda_{k}}\right) \int_{0}^{\infty} f(t) t^{\rho-1-i \lambda_{k}} d t \tag{5.88}
\end{equation*}
$$

The choice of $f(t)$ will be specified later. Next, we have

$$
\int_{0}^{\infty} f(t) t^{\rho-1-i \lambda_{k}} d t=\int_{-\infty}^{\infty} e^{\rho x} f\left(e^{x}\right) e^{-i \lambda_{k} x} d x
$$

Now, choose $f(x)=f_{m}(x)$ in such a way that the function $\varphi_{m}(x)=e^{\rho x} f_{m}\left(e^{x}\right)$ satisfy

$$
\int_{-\infty}^{\infty} \varphi_{m}(x) e^{-i \lambda x} d x=\prod_{k \neq m}\left(\lambda-\lambda_{k}\right)\left(\frac{\sin \alpha \lambda}{\lambda}\right)^{p}
$$

where $p$ is a sufficiently large integer, and the real number $\alpha$ is such that $\sin \alpha \lambda_{m} \neq 0$ if $\lambda_{m} \neq 0$ and $\alpha=1$ if $\lambda_{m}=0$. The definition of $\varphi_{m}$ and the Paley-Wiener theorem show that $\varphi_{m}$ is compactly supported. The inversion formula for the Fourier transformation easily shows that $\varphi_{m}$ has $p-n-3$ continuous derivatives. In its turn, this implies that $\operatorname{supp} f_{m}$ is a compact subset of $(0, \infty)$ and $f_{m}$ has continuous derivative up to the order $p-n-3$ (inclusive). Under this choice of $f$, formula (5.88) takes the form

$$
\frac{1}{r^{\rho}} \int_{0}^{\infty} f_{m}\left(\frac{t}{r}\right) d \sigma(t)-\int_{0}^{\infty} f_{m}(t) d \sigma(t)=A_{m}\left(c_{m}(r)-c_{m} r^{-i \lambda_{m}}\right)
$$

where $A_{m} \neq 0$. This identity easily implies that every function $c_{k}(r)$ is infinitely differentiable on $(0, \infty)$ and $c_{m}(1)=c_{m}$. In what follows, we assume for definiteness that $\rho>0$. We also agree to normalize the distribution function $\sigma(t)$ for the measure $\sigma$ by the condition $\sigma(0)=0$. Then $\sigma_{r}(t)=\frac{\sigma(r t)}{r^{\rho}}$. Next, integrating the two sides of (5.87) over $[0,1]$, we arrive at the formula

$$
\frac{\sigma(r)}{r^{\rho}}-\sigma(1)=\sum_{k=1}^{n}\left(c_{k}(r)-c_{k} r^{-i \lambda_{k}}\right) \frac{1}{\rho-i \lambda_{k}} .
$$

This shows that $\sigma$ has an infinitely differentiable density, to be denoted by $h(t)$. Then (5.87) can be rewritten in the form

$$
\frac{h(r t)}{(r t)^{\rho-1}}-\frac{h(t)}{t^{\rho-1}}=\sum_{k=1}^{n}\left(c_{k}(r)-c_{k} r^{-i \lambda_{k}}\right) t^{-i \lambda_{k}}
$$

Denoting by $H(r, t)$ the right-hand side of this identity, we deduce that $H(r, t)$ satisfies the differential equation

$$
r H_{r}^{\prime}(r, t)-t H_{t}^{\prime}(r, t)=t\left(\frac{h(t)}{t^{\rho-1}}\right)^{\prime}
$$

This yields

$$
\sum_{k=1}^{n}\left(r c_{k}^{\prime}(r)+i \lambda_{k} c_{k}(r)\right) t^{-i \lambda_{k}}=t\left(\frac{h(t)}{t^{\rho-1}}\right)^{\prime}
$$

Since the functions $t^{-i \lambda_{k}}$ are linearly independent, it follows that there exist numbers $d_{k}$ such that $r c_{k}^{\prime}(r)+i \lambda_{k} c_{k}(r)=d_{k}$. Then $\left(\frac{h(t)}{t^{\rho-1}}\right)^{\prime}=\sum_{k=1}^{n} d_{k} t^{-i \lambda_{k}-1}$. Assuming that $\lambda_{1}=0$, we obtain $h(t)=\left(d_{1} \ln t+\sum_{k=2}^{n} \frac{d_{k}}{-i \lambda_{k}} t^{-i \lambda_{k}}+d\right) t^{\rho-1}$. Since $h(t)$ is the density of a measure $\sigma \in \operatorname{Fr}[\mu]$, Theorem 26 shows that the measure $\sigma$ must satisfy the inequality $|\sigma([r, e r])| \leq \alpha r^{\rho}$ with some constant $\alpha$. Therefore, $d_{1}=0$. We have proved that the density $h(t)$ of an arbitrary measure $\sigma \in \operatorname{Fr}[\mu]$ has the form indicated in the theorem. Next, $\operatorname{Fr}[s]=\{0\}$ by Theorem 38

In connection with Theorem 51] we consider the following example. Let $K$ be a kernel on $(0, \infty)$ such that $\frac{1}{t} K(t), \frac{\ln t}{t} K(t) \in L_{1}(0, \infty)$, and $\int_{0}^{\infty} \frac{1}{t} K(t) d t=0$. We have

$$
\Psi(r)=\int_{0}^{\infty} K\left(\frac{t}{r}\right) \frac{\ln t}{t} d t=\int_{0}^{\infty} K(u) \frac{\ln u+\ln r}{u} d u=\int_{0}^{\infty} K(u) \frac{\ln u}{u} d u
$$

Let $\rho(r)=0$. The measure $s, d s(t)=\Psi(t) d t$, is regular with respect to the proximate order identically equal to 1 . The measure $\mu, d \mu(t)=\frac{\ln t}{t} d t$, does not belong to $\mathfrak{M}_{\infty}(0)$ and, a fortiori, to $\mathfrak{M}(0)$.

The particular case of Theorem 50 in which $\Lambda=\varnothing$ yields the following Tauberian theorem.

Theorem 52. Let $\rho(r)$ be an arbitrary proximate order, $\mu$ a Radon measure on $(0, \infty)$ of class $\mathfrak{M}(\rho(r))$, and $K$ a Borel function on $(0, \infty)$ such that $t^{\rho-1} \gamma(t) K(t) \in L_{1}(0, \infty)$. Suppose that the function $\int_{0}^{\infty} K(t) t^{\rho-1-i \lambda} d t$ does not vanish on the real axis. Define $\Psi$ by (1.2). If the measure $s, d s(t)=\Psi(t) d t$, is regular with respect to the proximate order $\rho(r)+1$, then $\mu$ is regular with respect to $\rho(r)$, and if $\operatorname{Fr}[s]$ consists of only one measure with density ct $t^{\rho}$, then $\operatorname{Fr}[\mu]$ consists of only one measure with density $\frac{c}{c_{1}} t^{\rho-1}$, where $c_{1}=\int_{0}^{\infty} K(t) t^{\rho-1} d t$.
Proof. By Theorem [50], the set $\operatorname{Fr}[\mu]$ consists of a unique measure. Therefore, this measure is regular. Its cluster set was described in Theorem 50.

As was shown by Theorem [29] a positive measure $\mu$ is regular with respect to a proximate order $\rho(r)$ if and only if that

$$
\lim _{r \rightarrow \infty} \frac{\mu([a r, b r])}{V(r)}=c \frac{b^{\rho}-a^{\rho}}{\rho}, \quad 0<a<b<\infty
$$

with some constant $c$. Moreover, $\rho$ can be an arbitrary real number (for $\rho=0$, the righthand side is defined by continuity and is equal to $c \ln \frac{b}{a}$ ). If $\rho>0$, the above identity is equivalent to the identity $\lim _{r \rightarrow \infty} \frac{\mu((1, r])}{V(r)}=\frac{c}{\rho}$, and if $\rho<0$, it is equivalent to the identity $\lim _{r \rightarrow \infty} \frac{\mu((r, \infty))}{V(r)}=-\frac{c}{\rho}$.

Regular signed measures were described in Theorem 30 .
In Theorem 52, an arbitrary proximate order is considered. As was mentioned in the Preface, already the particular case of Theorem 52 when $\rho(r) \equiv 1$ is a refinement of Wiener's second Tauberian theorems in various respects.

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[^0]:    2010 Mathematics Subject Classification. Primary 40E05; Secondary 30D20.
    Key words and phrases. Valiron's proximate order, Radon measure, Azarin's cluster set for a measure, Azarin's regular measure, Wiener Tauberian theorem.

