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A REMARK ON THE REPRODUCING KERNEL THESIS FOR HANKEL OPERATORS

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Abstract. A simple proof is given of the so-called reproducing kernel thesis for Hankel operators.

Notation

$:=$	equal by definition;
$\mathbb{C}$	the complex plane;
$\mathbb{D}$	the unit disk, $\mathbb{D} := \{z \in \mathbb{C} : \|z\| < 1\}$;
$\mathbb{T}$	the unit circle, $\mathbb{T} := \partial\mathbb{D} = \{z \in \mathbb{C} : \|z\| = 1\}$;
$\widehat{f}(n)$	Fourier coefficient of the function f, $\widehat{f}(n) := (2\pi)^{-1} \int_{\mathbb{T}} f(z) z^{-n} \, \|dz\|$;
$L^p = L^p(\mathbb{T})$	Lebesgue spaces with respect to the normalized Lebesgue measure $(2\pi)^{-1}\|dz\|$ on $\mathbb{T}$;
H^p	Hardy spaces, $H^p := \{f \in L^p(\mathbb{T}) : \widehat{f}(n) = 0 \ \forall n < 0\}$;
H^2_-	$H^2_- := L^2(\mathbb{T}) \ominus H^2 = \{f \in L^2(\mathbb{T}) : \widehat{f}(n) = 0 \ \forall n \geq 0\}$;
$H^p(E)$	vector-valued Hardy spaces with values in a separable Hilbert space E;
$H^2_-(E)$	vector-valued H^2_-;
$\mathbb{P}_+$, $\mathbb{P}_-$	orthogonal projections onto H^2 and H^2_-, respectively;
$\\|\cdot\\|$, $\vert\cdot\vert$	norm; when dealing with vector-valued functions we use the symbol $\\|\cdot\\|$ (usually with a subscript) for the norm in a function space, while $\vert\cdot\vert$ is used for the norm in the underlying vector space. Thus, for a vector-valued function f, the symbol $\\|f\\|_2$ denotes its L^2-norm, but the symbol $\vert f\vert$ stands for the scalar-valued function whose value at a point z is the norm of the vector $f(z)$.

§1. Introduction and main results

A Hankel operator is a bounded linear operator $\Gamma : H^2 \to H^2_-$ such that its matrix with respect to the standard bases $\{z^n\}_{n\geq 0}$ and $\{\overline{z}^{n+1}\}_{n\geq 0}$ in H^2 and H^2_- (respectively)

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depends on the sum of indices, i.e., has the form $\{\gamma_{j+k+1}\}_{j,k=0}^{\infty}$. If

$$\varphi_- := \Gamma\mathbf{1} = \sum_{k=1}^{\infty} \gamma_k \bar{z}^k, \quad z \in \mathbb{T},$$

then the action of Γ on polynomials f is given by

$$\Gamma g = \mathbb{P}_-(\varphi_- f). \tag{1.1}$$

The function φ_- is called the *antianalytic symbol* of the operator Γ.

In this paper we shall also be dealing with the vectorial Hankel operators $\Gamma\colon H^2 \to H^2_-(E)$, where E is an auxilary (separable) Hilbert space. In this case, the entries γ_k are operators $\gamma_k\colon \mathbb{C} \to E$ and are naturally identified with vectors in E. Then the symbol φ_- is a vector-valued function in $H^2_-(E)$.

Note that in (1.1) we can replace φ_- by $\varphi \in L^2(\mathbb{T} \to E)$ such that $\varphi - \varphi_- \in H^2(E)$ (so $\widehat{\varphi}(n) = \gamma_n$ for all $n < 0$). Such a function φ is called *a symbol* of the operator Γ. Unlike the antianalytic symbol φ_-, the symbol φ is not unique. Note also that for any symbol φ of the Hankel operator Γ we have $\|\Gamma\| \le \|\varphi\|_\infty$, and the famous Nehari theorem states that one can find a symbol φ such that $\|\Gamma\| = \|\varphi\|_\infty$.

In this paper we deal with the so-called (*pre*)*Hankel* operators (a non-standard term), which are not assumed to be bounded, but only defined on polynomials (and have the Hankel matrix $\{\gamma_{j+k+1}\}_{j,k=0}^{\infty}$). In this case, the anianalytic symbol φ_- is also in H^2_-, and the action of Γ on polynomials is still given by (1.1). Using uniform approximation by polynomials, we can easily show that a (pre)Hankel operator Γ can be defined on $H^2 \cap C(\mathbb{T})$ and that its action on $H^2 \cap C(\mathbb{T})$ is still given by (1.1).

We recall that the normalized reproducing kernel k_λ, $\lambda \in \mathbb{D}$, of H^2 is given by

$$k_\lambda(z) := \frac{(1 - |\lambda|^2)^{1/2}}{1 - \bar{\lambda} z}, \tag{1.2}$$

and that $\|k_\lambda\|_2 = 1$.

Our goal in this paper is to give an elementary proof of the following well-known result.

Theorem 1.1 (Reproducing kernel thesis for Hankel operators)**.** *Let Γ be a possibly vectorial* (*pre*)*Hankel operator such that*

$$\sup_{\lambda \in \mathbb{D}} \|\Gamma k_\lambda\|_2 \le A < \infty.$$

Then Γ is bounded and $\|\Gamma\| \le 2\sqrt{e}A$.

This theorem for the scalar-valued case (with some constant C instead of $2\sqrt{e}$) was published in [1], and is widely used in the theory of Hankel operators.

The proof presented in this paper is quite elementary and involves only Green's formula: the standard proof employs the Nehari theorem, H^1-BMO duality, and the fact that the so-called Garsia norm is an equivalent norm in BMO.

While the Nehari theorem is a basic fact in the theory of Hankel operators, and the other facts are standard and well-known results in harmonic analysis, it is still interesting to know that none of these results is needed for the proof of the reproducing kernel thesis for Hankel operators (Theorem 1.1).

Finally, let us emphasize that while the target space of our operator is a vector-valued space $H^2_-(E)$, the domain is the usual scalar-valued H^2 (more precisely, initially a dense subset of H^2). It is known that the reproducing kernel thesis fails for operator-valued Hankel operators: while the thesis is true for Hankel operators acting from $H^2(\mathbb{C}^d)$ to $H^2_-(E)$, the constant grows logaritmically in d, see, e.g., [3].

§2. Proof of the main result

We fix some notation. For $f \in L^1(\mathbb{T})$ and $z \in \mathbb{D}$, let $f(z)$ denote the Poisson (harmonic) extension of f at the point z. Thus, for $\varphi \in L^2(\mathbb{T} \to E)$ the symbol $\|\varphi(z)\|^2$ is the square of tne norm (in E) of the harmonic extension of φ at the point $z \in \mathbb{D}$, and $\|\varphi\|^2(z)$ is the harmonic extension of $\|\varphi\|^2\big|_{\mathbb{T}}$ at z.

2.1. Hankel operators and reproducing kernels. We recall that the reproducing kernel K_λ, $\lambda \in \mathbb{D}$, of the Hardy space H^2 is given by

$$K_\lambda(z) = \frac{1}{1 - \bar{\lambda} z}.$$

It is called the *reproducing kernel* because

$$(f, K_\lambda) = f(\lambda) \tag{2.1}$$

for all $f \in H^2$. Note that, since for each $\lambda \in \mathbb{D}$ the function K_λ is bounded, a simple approximation argument implies that the reproducing kernel identity (2.1) is valid for all $f \in H^1$.

Using the reproducing kernel property (2.1) with $f = K_\lambda$, we get

$$\|K_\lambda\|_2^2 = (K_\lambda, K_\lambda) = (1 - |\lambda|^2)^{-1},$$

so the normalized reproducing kernel $k_\lambda := \|K_\lambda\|_2^{-1} K_\lambda$ is given by (1.2).

The following lemma is well known, it can be found, for example, in [1] (in an implicit form). We present it here only for the reader's convenience.

Lemma 2.1. *Let Γ be a (pre)Hankel operator, and let $\varphi \in H^2_-(E)$ be its antianlytic symbol $\varphi = \sum_{k=1}^\infty \gamma_k \bar{z}^k$ (to simplify the notation we skip the subscript "$-$" and use φ instead of φ_-). Then for all $\lambda \in \mathbb{D}$ we have*

$$\|\Gamma k_\lambda\|_2^2 = \|\varphi\|^2(\lambda) - \|\varphi(\lambda)\|^2.$$

To prove the lemma we need the following well-known fact.

Lemma 2.2. *Let $\varphi \in H^2_-(E)$. Then, for all $\lambda \in \mathbb{D}$,*

$$\mathbb{P}_+(\varphi k_\lambda) = k_\lambda \varphi(\lambda).$$

Proof. First, we prove this lemma for scalar-valued $\varphi \in H^2_-$.

Let $f := \mathbb{P}_+(\varphi K_\lambda)$, where K_λ is the reproducing kernel for H^2. Any $f \in H^2$ can be decomposed as

$$f = cK_\lambda + f_0,$$

where $f_0(\lambda) = 0$ and $c = (1 - |\lambda|^2) f(\lambda)$; note that $K_\lambda \perp f_0$.

First we show that $f_0 = 0$ for $f = \mathbb{P}_+(\varphi K_\lambda)$. Observe that $\bar{\varphi} f_0 \in H^1$ because $\bar{\varphi}, f_0 \in H^2$, so that we can use the reproducing kernel property (2.1) to show that

$$\|f_0\|_2^2 = (f_0, \mathbb{P}_+(\varphi K_\lambda)) = (f_0, \varphi K_\lambda) = (\bar{\varphi} f_0, K_\lambda) = (\bar{\varphi} f_0)(\lambda) = \overline{\varphi(\lambda)} f_0(\lambda) = 0.$$

On the other hand,

$$(K_\lambda, f) = (K_\lambda, \mathbb{P}_+(\varphi K_\lambda)) = (K_\lambda, \varphi K_\lambda) = (\bar{\varphi} K_\lambda, K_\lambda) = \overline{\varphi(\lambda)} K_\lambda(\lambda) = \overline{\varphi(\lambda)} (1 - |\lambda|^2)^{-1}.$$

Therefore,

$$(\mathbb{P}_+(\varphi K_\lambda), K_\lambda) = \varphi(\lambda)(1 - |\lambda|^2)^{-1} = \varphi(\lambda) \|K_\lambda\|_2^2,$$

whence

$$\mathbb{P}_+(\varphi K_\lambda) = \varphi(\lambda) K_\lambda.$$

Multiplying this identity by $(1 - |\lambda|^2)^{1/2}$, we get the conclusion of the lemma for scalar-valued φ.

The general vector-valued case can easily be obtained from the scalar-valued case by fixing an orthonormal basis $\{\mathbf{e}_k\}_k$ and applying the scalar-valued result to the coordinate functions φ_k, $\varphi_k(z) = (\varphi(z), \mathbf{e}_k)_E$. $\square$

Proof of Lemma 2.1. The function φk_λ can be written as the orthogonal sum

$$\varphi k_\lambda = \mathbb{P}_+(\varphi k_\lambda) + \mathbb{P}_-(\varphi k_\lambda),$$

so

$$\|\Gamma k_\lambda\|_2^2 = \|\mathbb{P}_-(\varphi k_\lambda)\|_2^2 = \|\varphi k_\lambda\|_2^2 - \|\mathbb{P}_+(\varphi k_\lambda)\|_2^2.$$

Observing that

$$|k_\lambda(z)|^2 = \frac{1-|\lambda|^2}{|1-\bar{\lambda}z|^2},$$

we can write

$$\|\varphi k_\lambda\|_2^2 = \frac{1}{2\pi}\int_{\mathbb{T}} \|\varphi(z)\|^2 |k_\lambda(z)|^2\,|dz| = \|\varphi\|^2(\lambda).$$

By Lemma 2.2, we have $\mathbb{P}_+(\varphi k_\lambda) = \varphi(\lambda)k_\lambda$, whence $\|\mathbb{P}_+(\varphi k_\lambda)\|_2^2 = \|\varphi(\lambda)\|^2$. $\square$

2.2. Green's formula and Littlewood–Paley identities. We need several well-known facts.

The first is Green's standard formula for the unit disk.

Lemma 2.3. *Let $U \in C^2(\mathbb{D}) \cap C(\bar{\mathbb{D}})$. Then*

$$\frac{1}{2\pi}\int_{\mathbb{T}} U(z)\,|dz| - U(0) = \frac{1}{2\pi}\int_{\mathbb{D}} \Delta U(z) \ln\frac{1}{|z|}\,dA(z).$$

Applying this lemma to $U(z) = \|f(z)\|^2$, $f \in H^2(E)$, and observing that $\Delta U = 4\partial\bar{\partial}U = 4\|f'\|^2$, we get the following Littlewood–Paley identity.

Lemma 2.4. *Let $f \in H^2(E)$. Then*

$$\|f\|_2^2 = \frac{2}{\pi}\int_{\mathbb{D}} \|f'(z)\|^2 \ln\frac{1}{|z|}\,dA(z) + \|f(0)\|^2.$$

Of course, first we have to apply Lemma 2.3 to $\|f(rz)\|^2$, $r < 1$, and then take the limit as $r \to 1$.

The following lemma is also well known, see for example Lemma 6 in Appendix 3 of the monograph [4].

Lemma 2.5. *Let u be a C^2 subharmonic function $(\Delta u \geq 0)$ in the unit disk $\mathbb{D}$, and let $0 \leq u(z) \leq 1$ for all $z \in \mathbb{D}$. Then, for all $f \in H^2(E)$,*

$$\frac{1}{2\pi}\int_{\mathbb{D}} \Delta u(z) \|f(z)\|^2 \ln\frac{1}{|z|}\,dA(z) \leq e\|f\|_2^2.$$

Proof. Replacing u and f by $u(rz)$ anf $f(rz)$, $r < 1$, and then taking the limit as $r \to 1$, we may always assume without loss of generality that u and f are continuous up to the boundary of $\mathbb{D}$, so that Green's formula (Lemma 2.3) applies to $U(z) = e^{u(z)}\|f(z)\|^2$. Direct computation using the fact that $\Delta = 4\partial\bar{\partial}$ shows that

$$\Delta\big(e^{u(z)}\|f(z)\|^2\big) = e^u(\Delta u)\|f\|^2 + 4e^u\|(\partial u)f + \partial f\|^2 \geq (\Delta u)\|f\|^2.$$

Then denoting $d\mu(z) = (2\pi)^{-1}\ln|z|^{-1}\,dA(z)$ and using Green's formula (Lemma 2.3), we can write

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{D}} \Delta u\,\|f\|^2\,d\mu \leq \int_{\mathbb{D}} \Delta\big(e^{\varphi}\|f\|^2\big)\,d\mu &= \frac{1}{2\pi}\int_{\mathbb{T}} e^{\varphi}\|f\|^2\,|dz| - e^{\varphi(0)}\|f(0)\|^2 \\ &\leq e\frac{1}{2\pi}\int_{\mathbb{T}} \|f\|^2\,|dz| = e\|f\|_2^2.\end{aligned}$$

$\square$

2.3. Proof of Theorem 1.1. By homogeneity, it suffices to prove the theorem only for $A = 1$, so we assume that

$$\sup_{\lambda\in\mathbb{D}} \|\Gamma k_\lambda\|_2 \le 1.$$

We introduce some notation. Fix an orthonormal basis $\{\mathbf{e}_k\}_k$ in E; for a vector $\mathbf{x} = \sum_k x_k \mathbf{e}_k \in E$ (of course $x_k = (\mathbf{x}, e_k)_E$), let $\bar{\mathbf{x}}$ be the "complex conjugate" vector, $\bar{\mathbf{x}} := \sum_k \bar{x}_k \mathbf{e}_k$. So for the function h with values in E the symbol $\overline{h}$ denotes the function obtained by taking the complex conjugates of the coordinate functions of h (the orthonormal basis $\{e_k\}_k$ is assumed to be fixed).

Let φ be the antianalytic symbol of the Hankel operator Γ, so that $\Gamma = \Gamma_\varphi$. Recall that for $z \in \mathbb{D}$ we use $\varphi(z)$ to denote the harmonic extension of φ to the unit disk; then $\bar{\varphi} \in H^2(E)$.

It suffices to estimate the operators Γ_{φ_r}, $\varphi_r(z) := \varphi(rz)$, $r \in (0,1)$, so without loss of generality we may assume that $\bar{\varphi}$ is analytic in some disk larger than $\mathbb{D}$.

We want to estimate

$$(\Gamma f, \bar{g}) = \frac{1}{2\pi}\int_{\mathbb{T}} (\varphi f, \bar{g})_E\, |dz|, \quad f \in H^2, \quad \bar{g} \in H^2_-(E) \ \text{(equivalently, } g \in zH^2(E)).$$

Since it suffices to check the boundedness on a dense set, we may assume that f and g are polynomials, so that we can apply Green's formula. Since f, g and $\bar{\varphi}$ are analytic in $\mathbb{D}$ and $\Delta = 4\partial\bar{\partial}$, we get $\bar{\partial}(\varphi f, g)_E = (f(\bar{\partial}\varphi), g)_E$ and

$$\Delta(\varphi f, g)_E = 4\left((f(\bar{\partial}\varphi), \bar{\partial}\bar{g})_E + (f'(\bar{\partial}\varphi), \bar{g})_E\right) = 4\left((f(\bar{\partial}\varphi), \bar{g}')_E + (f'(\bar{\partial}\varphi), \bar{g})_E\right).$$

Therefore, using Green's formula (Lemma 2.3) and the fact that the function $(\varphi f, \bar{g})_E$ vanishes at the origin, we obtain

$$(\Gamma f, \bar{g}) = \frac{1}{2\pi}\int_{\mathbb{T}} (\varphi f, \bar{g})_E\, |dz| = \frac{2}{\pi}\int_{\mathbb{D}} \left((f(\bar{\partial}\varphi), \bar{g}')_E + (f'(\bar{\partial}\varphi), \bar{g})_E\right) \ln\frac{1}{|z|}\, dA(z).$$

We can estimate by Cauchy–Schwarz:

$$\begin{aligned}
&\left|\frac{2}{\pi}\int_{\mathbb{D}} (f(\bar{\partial}\varphi), \bar{g}')_E \ln\frac{1}{|z|}\, dA(z)\right| \\
&\qquad \le \left(\frac{2}{\pi}\int_{\mathbb{D}} \boldsymbol{|}(\bar{\partial}\varphi)\boldsymbol{|}^2 |f|^2 \ln\frac{1}{|z|}\, dA(z)\right)^{1/2} \left(\frac{2}{\pi}\int_{\mathbb{D}} \boldsymbol{|}g'\boldsymbol{|}^2 \ln\frac{1}{|z|}\, dA(z)\right)^{1/2}.
\end{aligned}$$

By Lemma 2.4,

$$\frac{2}{\pi}\int_{\mathbb{D}} \boldsymbol{|}g'\boldsymbol{|}^2 \ln\frac{1}{|z|}\, dA(z) \le \|g\|_2^2. \tag{2.2}$$

To estimate the first integral, we define $u(z) = 1 + \boldsymbol{|}\varphi(z)\boldsymbol{|}^2 - \boldsymbol{|}\varphi\boldsymbol{|}^2(z)$, and observe that $\Delta u = 4\boldsymbol{|}\bar{\partial}\varphi\boldsymbol{|}^2$. From the assumption $\sup_{\lambda\in\mathbb{D}} \|\Gamma k_\lambda\|_2 \le 1$ and Lemma 2.1 it follows that $0 \le u(z) \le 1$, $z \in \mathbb{D}$. Therefore, by Lemma 2.5,

$$\frac{2}{\pi}\int_{\mathbb{D}} \boldsymbol{|}(\bar{\partial}\varphi)\boldsymbol{|}^2 |f|^2 \ln\frac{1}{|z|}\, dA(z) \le e\|f\|_2^2. \tag{2.3}$$

Gathering estimates (2.2) and (2.3) together, we get

$$\left|\frac{2}{\pi}\int_{\mathbb{D}} \left(f(\bar{\partial}\varphi), \bar{g}'\right)_E \ln\frac{1}{|z|}\, dA(z)\right| \le \sqrt{e}\|f\|_2\|g\|_2.$$

Similarly,

$$\left|\frac{2}{\pi}\int_{\mathbb{D}}(f'(\bar\partial\varphi),\bar g)_E \ln\frac{1}{|z|}\,dA(z)\right|$$
$$\le\left(\frac{2}{\pi}\int_{\mathbb{D}}\|(\bar\partial\varphi)\|^2\|g\|^2\ln\frac{1}{|z|}\,dA(z)\right)^{1/2}\left(\frac{2}{\pi}\int_{\mathbb{D}}|f'|^2\ln\frac{1}{|z|}\,dA(z)\right)^{1/2}.$$

Interchanging f and g in the above argument, we get the estimate

$$\left|\frac{2}{\pi}\int_{\mathbb{D}}\big(f'(\bar\partial\varphi),\bar g\big)_E\ln\frac{1}{|z|}\,dA(z)\right|\le\sqrt{e}\|f\|_2\|g\|_2,$$

whence $|(\Gamma f,\bar g)|\le 2\sqrt{e}\|f\|_2\|\bar g\|_2$. □

§3. Concluding remarks

The main idea of using only Green's formula (and Lemma 2.5) goes back to [5], where the reproducing kernel thesis for the Carleson embeding theorem for the disk and for the unit ball in $\mathbb{C}^n$ was proved by using a similar technique; for the disk the estimate $\sqrt{2e}$ for the norm of the embedding operator[1] was obtained, see Theorem 0.2 there.

However, the proof in the present paper is much simpler than in [5]. Namely, the proof in [5] required some not completely trivial comutations and estimates; in the present paper all the computations (modulo known facts such as Lemmas 2.1–2.5) can be done in one's head.

Using the estimate (mentioned above) for the Carleson embedding theorem from [5], B. Jacob, J. Partington, and S. Pott obtained in [2] the estimate $4\sqrt{2e}$ for the reproducing kernel thesis for Hankel operators. Their proof also involved Green's formula, but the proof presented here, besides giving a better constant, is significantly simpler and much more streamlined (in particular, because it does not employ the result from [5]).

Also, Theorem 1.1 here can be used to give an explicit constant in the reproducing kernel thesis for the so-called generalized embedding theorem, described below in Section 3.1, and in particular for the Carleson emebedding theorem, although for the Carleson embedding theorem it gives a constant worse than that obtained in [5].

3.1. Generalized embedding theorem. Let $\theta\in H^\infty$ be an inner function, and let $\mathbf{K}_\theta$ be the corresponding backward shift invariant subspace

$$\mathbf{K}_\theta := H^2\ominus\theta H^2.$$

It is well known (see, e.g., the projection lemma in [4, p. 34]) and is easy to prove that the orthogonal projection P_θ from H^2 onto $\mathbf{K}_\theta$ is given on the unit circle $\mathbb{T}$ by

$$P_\theta f = f-\theta\mathbb{P}_+(\bar\theta f)=\theta\mathbb{P}_-(\bar\theta f),\quad f\in H^2. \tag{3.1}$$

Let $(\mathcal{X},\mu)$ be a measure space, and let θ_λ, $\lambda\in\mathcal{X}$ be a measurable family of inner functions (meaning that the function $(z,\lambda)\mapsto\theta_\lambda(z)$ is measurable). Identity (3.1) implies that the projection-valued function $\lambda\mapsto P_{\theta_\lambda}$ is measurable (in the weak, and so in the strong sense), so one can ask under what conditions on the measure μ the following *generalized embedding theorem*

$$\int_{\mathcal{X}}\|\mathbb{P}_{\theta_\lambda}f\|^2_{H^2}\,d\mu(\lambda)\le C\|f\|^2_{H^2},\quad f\in H^2, \tag{3.2}$$

holds true.

[1] Compare with $2\sqrt{e}$ for Hankel operators.

Note that if θ is an elementary Blaschke factor,

$$\theta(z) = \frac{z-\lambda}{1-\bar{\lambda}z},$$

then the corresponding space $\mathbf{K}_\theta$ is spanned by the reproducing kernel k_λ, and

$$P_\theta f = (f, k_\lambda)k_\lambda = (1-|\lambda|^2)^{1/2} f(\lambda) k_\lambda,$$

so $\|P_\theta f\|_2^2 = (1-|\lambda|^2)|f(\lambda)|^2$.

Therefore, for $\mathcal{X} = \mathbb{D}$ and $\theta_\lambda(z) = (z-\lambda)/(1-\bar{\lambda}z)$, $\lambda \in \mathbb{D}$, estimate (3.2) reduces to the classical Carleson embedding theorem, and (3.2) is true if and only if the measure $(1-|\lambda|^2)\,d\mu(\lambda)$ is Carleson.

Define a Hankel operator $\Gamma\colon H^2 \to H^2_-(L^2(\mu))$ by

$$\Gamma f(z,\lambda) = \Gamma_{\overline{\theta_\lambda}} f(z) = \mathbb{P}_-(\bar{\theta}_\lambda f)(z), \quad f \in H^2, \ z \in \mathbb{T}, \ \lambda \in \mathbb{D}.$$

From (3.1), it follows that

$$\|\Gamma f(\,\cdot\,,\lambda)\|_2 = \|P_{\theta_\lambda} f\|_2,$$

showing that (3.2) is equivalent to the estimate $\|\Gamma\| \le \sqrt{C}$.

But for the Hankel operators the reproducing kernel thesis holds, and Theorem 1.1 implies that if

$$\int_{\mathbb{D}} \|P_{\theta_\lambda} k_a\|_2^2 \, d\mu(\lambda) \le A\|f\|_2^2, \quad a \in \mathbb{D},$$

then (3.2) is true with $C = 4eA$.

The fact that the reproducing kernel thesis is valid for the generalized embedding theorem (with some constant) was proved in [6]; the above argument connecting (3.2) and the boundedness of the Hankel operator Γ is essentially taken from there. Theorem 1.1 in the present paper simly gives us an explicit constant.

It also gives a simpler proof of Theorem 0.2 in [5] (reproducing kernel thesis for the Carleson emebedding theorem), but with a worse constant ($4e$ vs $2e$ in [5][2]).

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[2]If one considers the norms of the embedding operators, one should take square roots of the above constants.