SOLVABILITY OF LOADED LINEAR EVOLUTION EQUATIONS WITH A DEGENERATE OPERATOR AT THE DERIVATIVE

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ABSTRACT. The methods of degenerate semigroups of operators and the theorem on contraction mappings are used for the search of unique solvability conditions for the Cauchy problem and the generalized Showalter problem for a class of loaded linear differential operator equations of the first order with degenerate operator at the derivative. The general results obtained are applied to the study of initial boundary value problems for loaded partial differential equations not solvable with respect to the time derivative.

§1. INTRODUCTION

Initial boundary value problems for partial differential equations (PDE), which describe various processes in nature and techniques, are often convenient to study within the framework of the Cauchy problem for evolution equations in Banach spaces, see [1, 2, 3]. Some initial boundary value problems reduce to equations with a degenerate operator at the derivative (see [4, 5]), and involve the so-called loaded equations. Besides a differential part, such equations contain also a functional of the unknown function, presented, e.g., in the form of the integral of the solution over a set, see [6, 7, 8]. Such equations arise when one searches approximate solutions of differential equations, see [9], and in the mathematical modeling of nonlocal processes (see [10]), including fractal processes and mathematical biology phenomena (see [11]), in the theory of heat and mass transfer in composite environment with fractal organization (see [12]), and in economics.

Our purpose in this paper is to find solvability conditions for initial problems related to evolution equations with a degenerate operator at the derivative (often called the Sobolev type equations, see [5, 13, 14]) of the form

$$L\dot{u}(t) = Mu(t) + \int_0^T \mathcal{K}(t,s)u(s)\,d\mu(s), \quad t\in[0,T],$$

where $L \in \mathcal{L}(\mathfrak{U}; \mathfrak{V})$ is a continuous linear operator acting from a Banach space \mathfrak{U} to a Banach space \mathfrak{V} , ker $L \neq \{0\}$, $M \in \mathcal{C}l(\mathfrak{U}; \mathfrak{V})$ is a closed and densely defined linear operator in \mathfrak{U} acting to $\mathfrak{V}, T > 0, \mathcal{K}: [0,T] \times [0,T] \to \mathcal{L}(\mathfrak{U}; \mathfrak{V})$, and $\mu: [0,T] \to \mathbb{R}$ is a function of bounded variation. Presumably, loaded equations of the above type have not been studied so far.

The theory of degenerate operator semigroups and the contraction mapping theorem have been employed to study the solvability of loaded equations not solvable for the derivative, in particular, the form of the solution of

(1.1)
$$L\dot{u}(t) = Mu(t) + g(t),$$

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where $g: [0,T] \to \mathfrak{V}$, obtained in [15], was used for this purpose. This approach made it possible to lift the restrictions imposed in [16] on the kernel or range of the integral operator of the equation $L\dot{u}(t) = Mu(t) + Nu(t)$ with respect to the perturbing operator N.

The paper [17] by Rutkas was one of the first publications devoted to the study the solvability of (1.1) in the infinite-dimensional Banach space by the methods of the operator semigroups theory (see also the information on degenerate equations and references in [18, 19]). In [17], some conditions ensuring that the Cauchy problem for the homogeneous equation (1.1) (i.e., in the case of $g \equiv 0$) be well posed, uniformly normal, or uniformly well posed were found in terms of the operators L and M, in addition to the study of this equation by methods related to the Laplace transformation and in terms of pairs of dissipative operators in Hilbert spaces. At the same time, the existence and properties were discussed of the (C_0)-continuous semigroup for this equation, defined on the closure of initial values of solutions of the Cauchy problem on the positive real line. Later, Yagi and Favini [4], Sviridyuk and Fedorov [5, 15], and Melnikova [20] studied the existence of degenerate semigroups of various smoothness classes that resolve the homogeneous equation (1.1). A review of those results can be found in [15].

In $\S2$ we briefly formulate the results on the degenerate strongly continuous semigroups of operators to be employed in the main body of the paper.

§3 contains the main results. The Cauchy problem and the generalized Showalter problem are formulated for loaded equations of Sobolev type, and theorems on the unique solvability of these problems are proved in the case where the pair of operators in the leading part of the equation gives rise to a degenerate strongly continuous semigroup.

In §4, a loaded pseudo-parabolic equation arising in the theory of filtration is considered with various boundary and initial conditions, and with various integral operators. Some unique solvability conditions are established for all problems in question, on the basis of the abstract results obtained in the previous section. Possible improvements of the general results are presented in particular applications.

§2. Solution of a degenerate nonhomogeneous equation

We start with the conditions on the operators to be used in what follows, as well as the corresponding results of [15].

Let \mathfrak{U} and \mathfrak{V} be Banach spaces, let an operator $L: \mathfrak{U} \to \mathfrak{V}$ be linear and continuous $(L \in \mathcal{L}(\mathfrak{U}; \mathfrak{V}) \text{ in short})$, and let an operator $M: \text{dom } M \to \mathfrak{V}$ be linear, closed, and densely defined in \mathfrak{U} $(M \in \mathcal{Cl}(\mathfrak{U}; \mathfrak{V}))$. We introduce the notation

$$\mathbb{N}_{0} = \{0\} \cup \mathbb{N}, \quad \bar{\mathbb{R}}_{+} = \{0\} \cup \mathbb{R}_{+}, \quad \rho^{L}(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{V}, \mathfrak{U})\},\\ R^{L}_{\mu}(M) = (\mu L - M)^{-1}L, \quad L^{L}_{\mu} = L(\mu L - M)^{-1}.$$

Definition 1. Let $p \in \mathbb{N}_0$. *M* is called a strongly (L, p)-radial operator if

- (i) there exists $a \in \mathbb{R}$ such that $(a, +\infty) \subset \rho^L(M)$;
- (ii) there exists $K \in \mathbb{R}_+$ such that for all $\mu \in (a, +\infty)$ and all $n \in \mathbb{N}$ we have

$$\max\left\{\left\| (R^{L}_{\mu}(M))^{n(p+1)} \right\|_{\mathcal{L}(\mathfrak{U})}, \left\| (L^{L}_{\mu}(M))^{n(p+1)} \right\|_{\mathcal{L}(\mathfrak{V})} \right\} \le \frac{K}{(\mu-a)^{n(p+1)}};$$

(iii) there exists a dense subspace \mathfrak{V} in \mathfrak{V} such that

$$\|M(\mu L - M)^{-1} (L^{L}_{\mu}(M))^{p+1} f\|_{\mathfrak{V}} \leq \frac{\operatorname{const}(f)}{(\mu - a)^{p+2}}, \quad f \in \mathring{\mathfrak{V}}$$
$$\|(R^{L}_{\mu}(M))^{p+1} (\mu L - M)^{-1}\|_{\mathcal{L}(\mathfrak{V};\mathfrak{U})} \leq \frac{K}{(\mu - a)^{p+2}}$$

for all $\mu \in (a, +\infty)$.

Remark 1. In [21] it was proved that the conditions of Definition 1 are equivalent to the similar but more involved conditions used of [15] and [16].

We put $\mathfrak{U}^0 = \ker(R^L_\mu(M))^{p+1}$, $\mathfrak{V}^0 = \ker(L^L_\mu(M))^{p+1}$. Let \mathfrak{U}^1 be the closure of the range of the operator $(R^L_\mu(M))^{p+1}$ in the space \mathfrak{U} , and let \mathfrak{V}^1 be the closure of the range of $(L^L_\mu(M))^{p+1}$ in the space \mathfrak{V} . We denote by $L_k(M_k)$ the restriction of the operator L(M) to $\mathfrak{U}^k(\operatorname{dom} M_k = \operatorname{dom} M \cap \mathfrak{U}^k)$, k = 0, 1.

Theorem 1 (see [15]). Let an operator M be strongly (L, p)-radial. Then: (i) $\mathfrak{U} = \mathfrak{U}^0 \oplus \mathfrak{U}^1, \mathfrak{V} = \mathfrak{V}^0 \oplus \mathfrak{V}^1;$ (ii) $L_k \in \mathcal{L}(\mathfrak{U}^k; \mathfrak{V}^k), M_k \in \mathcal{Cl}(\mathfrak{U}^k; \mathfrak{V}^k), k = 0, 1;$ (iii) the operators $M_0^{-1} \in \mathcal{L}(\mathfrak{V}^0; \mathfrak{U}^0)$ and $L_1^{-1} \in \mathcal{L}(\mathfrak{V}^1; \mathfrak{U}^1)$ exist;

- (iv) $H = M_0^{-1}L_0$ is a nilpotent operator of degree at most p;
- (v) if $u_0 \in \operatorname{dom} M$, and $g \in C^1([0,T]; \mathfrak{V})$ is such that

$$(I-Q)g \in C^{p+1}([0,T];\mathfrak{V}^0)$$

and

(2.1)
$$(I-P)u_0 = -\sum_{k=0}^p H^k M_0^{-1} ((I-Q)g)^{(k)}(0),$$

then there exists a unique solution $u \in C^1([0,T];\mathfrak{U}) \cap C([0,T]; \operatorname{dom} M)$ of the Cauchy problem $u(0) = u_0$ for the equation

(2.2)
$$L\dot{u}(t) = Mu(t) + g(t), \quad t \in [0, T],$$

and

(2.3)
$$u(t) = U(t)u_0 + \int_0^t U(s)L_1^{-1}Qg(t-s)\,ds - \sum_{k=0}^p H^k M_0^{-1} \big((I-Q)g \big)^{(k)}(t),$$

where P is the projection along the subspace \mathfrak{U}^0 onto \mathfrak{U}^1 , and Q is projection along the subspace \mathfrak{V}^0 onto \mathfrak{V}^1 . The operators of a degenerate strongly continuous semigroup $\{U(t) \in \mathcal{L}(\mathfrak{U}): t \in \mathbb{R}_+\}$ satisfy the following condition:

$$||U(t)|| \leq Ke^{at}$$
 for all $t \in \overline{\mathbb{R}}_+$,

with a and K as in Definition 1;

(vi) if $u_0 \in \text{dom } M_1$ and $g \in C^1([0,T]; \mathfrak{V})$ is such that

$$(I-Q)g \in C^{p+1}([0,T];\mathfrak{V}^0),$$

then the problem $Pu(0) = u_0$ for equation (2.2) has a unique solution $u \in C^1([0,T];\mathfrak{U}) \cap C([0,T]; \operatorname{dom} M)$ of the form (2.3).

Remark 2. The projections can be calculated by the formulas

$$P = \operatorname{s-lim}_{\mu \to +\infty} (\mu R^L_\mu(M))^{p+1}, \quad Q = \operatorname{s-lim}_{\mu \to +\infty} (\mu L^L_\mu(M))^{p+1}.$$

Remark 3. The degeneracy of the semigroup occurring in Theorem 1 means that its unit U(0) is not an identity operator, but a nontrivial projection. In this case U(0) = P (for more information, see [15]).

§3. Loaded linear equation of Sobolev type

Let $\mathfrak{U}, \mathfrak{V}$ be Banach spaces, and let $L \in \mathcal{L}(\mathfrak{U}; \mathfrak{V}), M \in \mathcal{C}l(\mathfrak{U}; \mathfrak{V})$. Consider the Cauchy problem

(3.1)
$$u(0) = u_0$$

for a linear integro-differential equation

(3.2)
$$L\dot{u}(t) = Mu(t) + \int_0^T \mathcal{K}(t,s)u(s) \, d\mu(s), \quad t \in [0,T],$$

where T > 0, $\mathcal{K}: [0, T] \times [0, T] \to \mathcal{L}(\mathfrak{U}; \mathfrak{V})$, $\mu: [0, T] \to \mathbb{R}$ is a bounded variation function. A solution of problem (3.1), (3.2) is a function $u \in C^1([0, T]; \mathfrak{U})$ such that $u(t) \in \text{dom } M$ for all $t \in [0, T]$, relation (3.2) is fulfilled on the segment [0, T], and (3.1) is satisfied.

A loaded equation of the form

$$L\dot{u}(t) = Mu(t) + \sum_{l=1}^{l_0} C_l(t)u(t_l), \quad t \in [0, T],$$

where $l_0 \in \mathbb{N}$, $C_l(t) \in \mathcal{L}(\mathfrak{U}; \mathfrak{V})$ for $t \in [0, T]$, $l = 1, \ldots, l_0$, and $0 \le t_1 < t_2 < \cdots < t_{l_0} \le T$, is a particular case of equation (3.2).

Given a strongly (L, p)-radial operator M, an operator-valued function \mathcal{K} of class

$$C^{p+1,0}([0,T]\times[0,T];\mathcal{L}(\mathfrak{U};\mathfrak{V}))$$

(the functions whose partial derivatives with respect to the first argument of order at most p + 1 are continuous in both arguments), a function $\mu: [0,T] \to \mathbb{R}$ of bounded variation, and $T > 0, n = 0, 1, \dots, p + 1$, we denote

$$\frac{\partial^{n}\mathcal{K}}{\partial t^{n}} \equiv \mathcal{K}_{t}^{(n)}, \quad V_{0}^{T}(\mu) \max_{t,s \in [0,T]} \left\| \mathcal{K}_{t}^{(n)}(t,s) \right\|_{\mathcal{L}(\mathfrak{U},\mathfrak{V})} \equiv K_{n}(T),$$

$$V_{0}^{T}(\mu) \max_{t,s \in [0,T]} \left\{ s \left\| \mathcal{K}_{t}^{(n)}(t,s) \right\|_{\mathcal{L}(\mathfrak{U},\mathfrak{V})} \right\} \equiv K_{n,1}(T),$$

$$\left\| L_{1}^{-1}Q \right\|_{\mathcal{L}(\mathfrak{V};\mathfrak{U})} \equiv C_{1}, \quad \left\| H^{k}M_{0}^{-1}(I-Q) \right\|_{\mathcal{L}(\mathfrak{V};\mathfrak{U})} \equiv h_{k}, \quad k = 0, 1, \dots, p,$$

$$F(T) = \max \left\{ C_{1}K(T) \sum_{n=0}^{p+1} K_{n,1}(T) + h_{0} \sum_{n=0}^{p+1} K_{n}(T), h_{1} \sum_{n=0}^{p+1} K_{n}(T), \dots, h_{p} \sum_{n=0}^{p+1} K_{n}(T) \right\}.$$

Here $V_0^T(\mu)$ is the variation of μ on the segment [0, T],

$$K(T) = \max\{K, Ke^{aT}\} = \begin{cases} K, & a \le 0, \\ Ke^{aT}, & a > 0, \end{cases}$$

with K and a as in Definition 1. By Theorem 1 (v), $||U(t)||_{\mathcal{L}(\mathfrak{U})} \leq K(T)$ for all $t \in [0,T]$.

Theorem 2. Suppose M is a strongly (L, p)-radial operator, $u_0 \in \text{dom } M \cap \mathfrak{U}^1$, $\mathcal{K} \in C^{p+1,0}([0,T] \times [0,T]; \mathcal{L}(\mathfrak{U}; \mathfrak{V}))$,

(3.3)
$$\mathcal{K}_t^{(n)}(0,s) \equiv 0, \quad n = 0, 1, \dots, p,$$

and $\mu: [0,T] \to \mathbb{R}$ is a bounded variation function, F(T) < 1. Then problem (3.1), (3.2) admits a unique solution $u \in C^1([0,T];\mathfrak{U})$.

Proof. By Theorem 1 (v), a solution $u \in C^1([0,T]; \mathfrak{U})$ of problem (2.2), (3.1) exists, is unique, and has the form (2.3) for $t \in [0,T]$, provided g belongs to $C^{p+1}([0,T]; \mathfrak{V})$ and

satisfies (2.1) for $u_0 \in \text{dom } M$ given in (3.1). We introduce the operator

$$\begin{split} [\Phi g](t) &= \int_0^T \mathcal{K}(t,s) u(s) \, d\mu(s) \\ &= \int_0^T \mathcal{K}(t,s) U(s) u_0 \, d\mu(s) \, + \int_0^T \mathcal{K}(t,s) \int_0^s U(\tau) L_1^{-1} Q g(s-\tau) \, d\tau \, d\mu(s) \\ &- \int_0^T \mathcal{K}(t,s) \sum_{k=0}^p H^k M_0^{-1} (I-Q) g^{(k)}(s) \, d\mu(s). \end{split}$$

We show that

$$\frac{\partial}{\partial t} \int_0^T \mathcal{K}_t^{(n)}(t,s) f(s) \, d\mu(s) = \int_0^T \mathcal{K}_t^{(n+1)}(t,s) f(s) \, d\mu(s)$$

for n = 0, 1, ..., p, and $s \in [0, T]$, where $f \in C([0, T]; \mathfrak{U})$. By Lagrange's theorem, for some θ lying between t and $t + \delta$ we have

$$\begin{split} \left\| \frac{\mathcal{K}_{t}^{(n)}(t+\delta,s) - \mathcal{K}_{t}^{(n)}(t,s)}{\delta} f(s) - \mathcal{K}_{t}^{(n+1)}(t,s) f(s) \right\|_{\mathfrak{V}} \\ & \leq \left\| \mathcal{K}_{t}^{(n+1)}(\theta,s) - \mathcal{K}_{t}^{(n+1)}(t,s) \right\|_{\mathcal{L}(\mathfrak{U};\mathfrak{V})} \max_{\tau \in [0,T]} \|f(\tau)\|_{\mathfrak{U}} \to 0 \end{split}$$

as $\delta \to 0$, with convergence uniform with respect to $s \in [0,T]$, because the operatorvalued function $\mathcal{K}_t^{(n+1)}(t,s)$ is continuous on $[0,T] \times [0,T]$. Therefore, it is uniformly continuous on this compact set in the sense of the norm in $\mathcal{L}(\mathfrak{U};\mathfrak{V})$. Thus,

$$\frac{\partial^n}{\partial t^n} \int_0^T \mathcal{K}(t,s) f(s) \, d\mu(s) = \int_0^T \mathcal{K}_t^{(n)}(t,s) f(s) \, d\mu(s)$$

for $n = 1, 2, \ldots, p + 1$.

Consequently, under the conditions of the theorem,

$$\begin{split} [\Phi g]^{(n)}(t) &= \int_0^T \mathcal{K}_t^{(n)}(t,s) U(s) u_0 \, d\mu(s) + \int_0^T \mathcal{K}_t^{(n)}(t,s) \int_0^s U(\tau) L_1^{-1} Q g(s-\tau) \, d\tau \, d\mu(s) \\ &- \int_0^T \mathcal{K}_t^{(n)}(t,s) \sum_{k=0}^p H^k M_0^{-1} (I-Q) g^{(k)}(s) \, d\mu(s) \end{split}$$

for $t \in [0, T]$, n = 0, 1, ..., p + 1. Therefore, by (3.3), $[\Phi g]^{(n)}(0) = 0$ for n = 0, 1, ..., p. Since $(I-P)u_0 = 0$ by the assumptions of the theorem, the function Φg satisfies condition (2.1) with the given $u_0 \in \text{dom } M_1$.

Consider the Banach space $C^{p+1}([0,T];\mathfrak{V})$ with the norm

$$\|g\|_{p+1} = \sum_{k=0}^{p+1} \sup_{t \in [0,T]} \|g^{(k)}(t)\|_{\mathfrak{V}},$$

and the metric space

$$E = \left\{ g \in C^{p+1}([0,T];\mathfrak{V}) : \sum_{k=0}^{p} H^{k} M_{0}^{-1}(I-Q)g^{(k)}(0) = 0 \right\},\$$

with the metric $d(g,h) = ||g - h||_{p+1}$. This metric space is not empty: it contains the constant function $g \equiv 0$. We show that this space is complete. Let $\{g_n\} \subset E$, $\lim_{n,m\to\infty} \|g_n - g_m\|_{p+1} = 0$. By the completeness of $C^{p+1}([0,T];\mathfrak{V})$, there exists $g \in C^{p+1}([0,T];\mathfrak{V})$ such that $\lim_{n\to\infty} \|g_n - g\|_{p+1} = 0$. We have

$$\begin{split} \left\| \sum_{k=0}^{p} H^{k} M_{0}^{-1} (I-Q) g^{(k)}(0) \right\|_{\mathfrak{V}} &\leq \left\| \sum_{k=0}^{p} H^{k} M_{0}^{-1} (I-Q) \left(g^{(k)}(0) - g_{n}^{(k)}(0) \right) \right\|_{\mathfrak{V}} \\ &\leq \max_{k=0,\dots,p} h_{k} \|g - g_{n}\|_{p+1} \to 0 \end{split}$$

as $n \to \infty$. Thus, $g \in E$. We see that the operator Φ acts in E.

Furthermore,

$$\begin{split} [\Phi g_1](t) - [\Phi g_2](t) &= \int_0^T \mathcal{K}(t,s) \int_0^s U(s-\tau) L_1^{-1} Q(g_1(\tau) - g_2(\tau)) \, d\tau \, d\mu(s) \\ &- \int_0^T \mathcal{K}(t,s) \sum_{k=0}^p H^k M_0^{-1} (I-Q) (g_1^{(k)}(s) - g_2^{(k)}(s)) \, d\mu(s), \\ \|\Phi g_1 - \Phi g_2\|_{p+1} &\leq \sum_{n=0}^{p+1} \left(C_1 K(T) K_{n,1}(T) \|g_1 - g_2\|_0 + K_n(T) \sum_{k=0}^p h_k \|g_1^{(k)} - g_2^{(k)}\|_0 \right) \\ &\leq F(T) \|g_1 - g_2\|_{p+1}. \end{split}$$

Since F(T) < 1 by assumption, Φ is a contraction mapping on the complete metric space E. Consequently, by the fixed point theorem, there exists a unique element $g^* \in E$ such that $g^* = \Phi g^*$. In this case, since $L\dot{u}(t) - Mu(t) = g^*(t) = [\Phi g^*](t)$, the function

$$u(t) = U(t)u_0 + \int_0^t U(t-s)L_1^{-1}Qg^*(s)\,ds - \sum_{k=0}^p H^k M_0^{-1}(I-Q)g^{*(k)}(t)$$

solves of problems (2.2), (3.1) and (3.1), (3.2) simultaneously.

Let $u_1, u_2 \in C^1([0, T]; \mathfrak{U})$ be two solutions of problem (3.1), (3.2). Denoting

$$g_i(t) = \int_0^T \mathcal{K}(t,s) u_i(s) \, d\mu(s)$$

for i = 1, 2, we have

$$g_i^{(n)}(t) = \int_0^T \mathcal{K}_t^{(n)}(t,s) u_i(s) \, d\mu(s), \quad n = 1, \dots, p+1,$$

 $g_i^{(n)}(0) = 0$ for n = 0, 1, ..., p. Since g_i satisfies condition (2.1) with $u_0 \in \text{dom } M_1$, $L\dot{u}_i - Mu_i = g_i \in C^{p+1}([0, T]; \mathfrak{V})$ and $\Phi g_i = g_i$, it follows that

$$u_i(t) = U(t)u_0 + \int_0^t U(t-s)L_1^{-1}Qg_i(s)\,ds - \sum_{k=0}^p H^k M_0^{-1}(I-Q)g_i^{(k)}(t), \quad i = 1, 2.$$

We conclude that $g_1 \equiv g_2$, by the uniqueness of the fixed point of the mapping $\Phi: E \to E$. Denote $w(t) = u_1(t) - u_2(t)$, $t \in [0,T]$, then $L\dot{w}(t) - Mw(t) \equiv 0$, w(0) = 0. By Theorem 1(v), we have $w \equiv 0$, so that problem (3.1), (3.2) has a unique solution.

Now, consider a generalized Showalter problem (see [22])

$$Pu(0) = u_0,$$

which arises naturally for Sobolev type equations (see [16] and the examples below).

Theorem 3. Suppose that M is a strongly (L, p)-radial operator, $u_0 \in \text{dom } M \cap \mathfrak{U}^1$, $\mathcal{K} \in C^{p+1,0}([0,T] \times [0,T]; \mathcal{L}(\mathfrak{U}; \mathfrak{V})), \mu \colon [0,T] \to \mathbb{R}$ is a bounded variation function, and F(T) < 1. Then (3.2), (3.4) admits a unique solution $u \in C^1([0,T]; \mathfrak{U})$.

The difference between the proof of this theorem and that of the preceding theorem is the reference to Theorem 1 (vi) on the solvability of problem (2.2), (3.4). The entire space $E = C^{p+1}([0,T];\mathfrak{V})$ is viewed as the metric space where a contraction mapping of the above form acts.

Remark 4. The condition F(T) < 1 of Theorems 2 and 3 is determined by the operators L, M, the operator-valued function \mathcal{K} , and the variation of μ ; it does not depend on u_0 . Remark 5. Let $\mu: [0, \hat{T}] \to \mathbb{R}$. Note that the inequality F(T) < 1 can be ensured under some additional conditions if the remaining conditions of Theorem 2 (or 3) are assumed to be valid, by reducing T within a half-interval $(0, \hat{T}]$. Indeed,

$$\lim_{T \to 0+} K_{n,1}(T) = 0, \quad n = 0, 1, \dots, p+1,$$

by conditions (3.3), and

$$\lim_{T \to 0+} K_n(T) = 0, \quad n = 0, \dots, p.$$

Therefore,

$$F(0+) \equiv \lim_{T \to 0+} F(T) = \max_{k=0,\dots,p} h_k \big| \lim_{t \to 0+} \mu(t) - \mu(0) \big| \big\| \mathcal{K}_t^{(p+1)}(0,0) \big\|_{\mathcal{L}(\mathfrak{U};\mathfrak{V})}.$$

As the additional condition mentioned above we can take the inequality F(0+) < 1, which is valid obviously if the function μ is continuous from the right at zero, or $\mathcal{K}_t^{(p+1)}(0,0) = 0$. Then problem (3.1), (3.2) (or (3.2), (3.4)) is solvable on the interval [0,T] for small T.

Remark 6. The condition F(T) < 1 of Theorem 2 (3) may be replaced by a weaker one, $\widetilde{F}(T) < 1$, where

$$\widetilde{F}(T) = \max\left\{ C_1 K(T) \sum_{n=0}^{p+1} \widetilde{K}_{n,1}(T) + h_0 \sum_{n=0}^{p+1} \widetilde{K}_n(T), h_1 \sum_{n=0}^{p+1} \widetilde{K}_n(T), \dots, h_p \sum_{n=0}^{p+1} \widetilde{K}_n(T) \right\},\\ \widetilde{K}_n(T) \equiv \max_{t \in [0,T]} \int_0^T \left\| \mathcal{K}_t^{(n)}(t,s) \right\|_{\mathcal{L}(\mathfrak{U};\mathfrak{V})} d|\mu(s)|,\\ \widetilde{K}_{n,1}(T) \equiv \max_{t \in [0,T]} \int_0^T s \left\| \mathcal{K}_t^{(n)}(t,s) \right\|_{\mathcal{L}(\mathfrak{U};\mathfrak{V})} d|\mu(s)|$$

for T > 0, n = 0, 1, ..., p + 1, provided the function $\mu \colon [0, T] \to \mathbb{R}$ is monotone.

§4. Loaded pseudoparabolic equations

As an application of Theorem 2, we consider the initial boundary value problem

(4.1)
$$z(x,0) = z_0(x), \quad x \in \Omega,$$

(4.2)
$$z(x,t) = \Delta z(x,t) = 0, \quad (x,t) \in \partial \Omega \times [0,T],$$

for a modified Dzektser equation. This equation is of the form

(4.3)
$$(\lambda - \Delta)z_t(x, t) = \Delta z(x, t) - \beta \Delta^2 z(x, t) + \int_0^T \int_\Omega k(x, y, t, s) z(y, s) \, dy \, d\mu(s),$$
$$(x, t) \in \Omega \times [0, T],$$

and it arises in the theory of filtration, see [23]; the bounded domain $\Omega \subset \mathbb{R}^n$ is assumed to have a smooth boundary, $\lambda \in \mathbb{R}$, $\beta \in \mathbb{R}_+$.

We denote

$$\begin{aligned} H_0^2(\Omega) &= \left\{ u \in H^2(\Omega) \, : \, u(x) = 0, \, x \in \partial \Omega \right\}, \quad \|u\|_{H_0^2(\Omega)} = \|u\|_{L_2(\Omega)} + \|\Delta u\|_{L_2(\Omega)}, \\ H_\partial^4(\Omega) &= \left\{ u \in H^4(\Omega) \, : \, u(x) = \Delta u(x) = 0, \, \, x \in \partial \Omega \right\}. \end{aligned}$$

To reduce (4.1)–(4.3) to (3.1), (3.2), consider a family of integral operators $\mathcal{K}(t,s)$: $H_0^2(\Omega) \to L_2(\Omega), t, s \in [0,T]$, of the form

$$[\mathcal{K}(t,s)u](\,\cdot\,) = \int_{\Omega} k(\,\cdot\,,y,t,s)u(y)\,dy$$

By the Hölder inequality,

(4.4)

$$\begin{aligned} \|\mathcal{K}(t,s)u\|_{L_{2}(\Omega)}^{2} &= \int_{\Omega} \left| \int_{\Omega} k(x,y,t,s)u(y) \, dy \right|^{2} dx \\ &\leq \int_{\Omega} \int_{\Omega} |k(x,y,t,s)|^{2} \, dy \, dx \int_{\Omega} |u(y)|^{2} \, dy \\ &\leq \int_{\Omega} \int_{\Omega} |k(x,y,t,s)|^{2} \, dy \, dx \, \|u\|_{H_{0}^{2}(\Omega)}^{2}, \\ &\|\mathcal{K}(t,s)\|_{\mathcal{L}(H_{0}^{2}(\Omega);L_{2}(\Omega))} \leq \|k(\cdot,\cdot,t,s)\|_{L_{2}(\Omega\times\Omega)}, \quad t,s \in [0,T]. \end{aligned}$$

Let $\lambda_m, m \in \mathbb{N}$, denote the eigenvalues of the Laplace operator defined on $H_0^2(\Omega)$ and acting to $L_2(\Omega)$, numbered in the nonincreasing order with multiplicity counted. Next, let $\{\varphi_m : m \in \mathbb{N}\}$ be an orthonormal system of the corresponding eigenfunctions in the sense of the inner product in $L_2(\Omega)$. We assume that $\lambda_m = \lambda$ for some m, i.e., equation (4.3) is not solvable with respect to z_t .

Lemma 1. Suppose that $\beta > 0$, $\lambda - \beta \lambda^2 \neq 0$, $k(x, y, t, s) \in C([0, T] \times [0, T]; L_2(\Omega \times \Omega))$, $k_t(x, y, t, s) \in C([0, T] \times [0, T]; L_2(\Omega \times \Omega))$, and $\mu: [0, T] \to \mathbb{R}$ is a bounded variation function. Then

Proof. Consider the spaces $\mathfrak{U} = H_0^2(\Omega)$, $\mathfrak{V} = L_2(\Omega)$, and the operators $L = \lambda - \Delta$, $M = \Delta - \beta \Delta^2$ with dom $M = H_{\partial}^4(\Omega)$. It is known (see [16]) that M is a strongly (L, 0)-radial operator if $\lambda - \beta \lambda^2 \neq 0$.

We estimate the quantity F(T) for problem (4.1)–(4.3), using Theorem 2 and the results of [16]. We have

$$L_1^{-1}Q = \sum_{\lambda_m \neq \lambda}^{\infty} \frac{\langle \cdot, \varphi_m \rangle \varphi_m}{\lambda - \lambda_m},$$

and

$$\left\|L_1^{-1}Qv\right\|_{H^2(\Omega)}^2 = \sum_{\lambda_m \neq \lambda} \frac{(1+\lambda_m^2)|\langle v, \varphi_m \rangle|^2}{|\lambda - \lambda_m|^2}$$

for $v \in L_2(\Omega)$,

$$C_1 = \left\| L_1^{-1} Q \right\|_{\mathcal{L}(L_2(\Omega); H^2(\Omega))} = \sup_{\lambda \neq \lambda_m} \frac{\sqrt{1 + \lambda_m^2}}{|\lambda - \lambda_m|}.$$

Similarly,

$$\begin{split} M_0^{-1}(I-Q) &= \sum_{\lambda_m = \lambda} \frac{\langle \cdot, \varphi_m \rangle \varphi_m}{\lambda - \beta \lambda^2}, \quad h_0 = \left\| M_0^{-1}(I-Q) \right\|_{\mathcal{L}(L_2(\Omega); H_0^2(\Omega))} = \frac{\sqrt{1 + \lambda^2}}{|\lambda - \beta \lambda^2|}, \\ a &= \sup_{\lambda_m \neq \lambda} \frac{\lambda_m - \beta \lambda_m^2}{\lambda - \lambda_m}, \quad K = \max\left\{ 1, \sup_{\lambda \neq \lambda_m} \frac{\sqrt{1 + \lambda_m^2}}{|\lambda - \lambda_m|} \right\}, \\ K(T) &= \max\left\{ 1, \sup_{\lambda \neq \lambda_m} \frac{\sqrt{1 + \lambda_m^2}}{|\lambda - \lambda_m|}, e^{aT}, e^{aT} \sup_{\lambda \neq \lambda_m} \frac{\sqrt{1 + \lambda_m^2}}{|\lambda - \lambda_m|} \right\}, \\ K_0(T) &\leq V_0^T(\mu) \max_{t,s \in [0,T]} \| k(\cdot, \cdot, t, s) \|_{L_2(\Omega \times \Omega)}, \\ K_1(T) &\leq V_0^T(\mu) \max_{t,s \in [0,T]} \| k_t(\cdot, \cdot, t, s) \|_{L_2(\Omega \times \Omega)}, \\ K_{0,1}(T) &\leq V_0^T(\mu) \max_{t,s \in [0,T]} \left\{ s \| k(\cdot, \cdot, t, s) \|_{L_2(\Omega \times \Omega)} \right\}, \\ K_{1,1}(T) &\leq V_0^T(\mu) \max_{t,s \in [0,T]} \left\{ s \| k_t(\cdot, \cdot, t, s) \|_{L_2(\Omega \times \Omega)} \right\}. \end{split}$$

We have used inequality (4.4). The lemma is proved.

Let $\widetilde{F}(T)$ denote the right-hand side of (4.5).

Theorem 4. Suppose $\beta > 0$, $\lambda - \beta \lambda^2 \neq 0$, $z_0 \in H^4_{\partial}(\Omega)$, and $\langle z_0, \varphi_m \rangle = 0$ for $m \in \mathbb{N}$. Here $\lambda_m = \lambda$, $k(x, y, t, s) \in C([0, T] \times [0, T]; L_2(\Omega \times \Omega))$, $k_t(x, y, t, s) \in C([0, T] \times [0, T]; L_2(\Omega \times \Omega))$, $k(x, y, 0, s) \equiv 0$, and $\mu : [0, T] \to \mathbb{R}$ is a bounded variation function, $\widetilde{F}(T) < 1$. Then problem (4.1)–(4.3) admits a unique solution $u \in C^1([0, T]; H^2_0(\Omega))$.

Proof. Obviously, z_0 satisfies the condition imposed in Theorem 2 on the function u_0 , in terms of problem (4.1)–(4.3).

By (4.4), the functions $k(\cdot, \cdot, t, s), k_t(\cdot, \cdot, t, s) \in L_2(\Omega \times \Omega)$ give rise to the operators $\mathcal{K}(t, s), \mathcal{K}_t^{(1)}(t, s) \in \mathcal{L}(H_0^2(\Omega); L_2(\Omega))$ for $t, s \in [0, T]$. The other conditions on the operator-valued function \mathcal{K} of Theorem 2 follow from the conditions on k. It remains to apply Theorem 2.

For the same equation with a simpler integral operator

(4.6)
$$(\lambda - \Delta)z_t(x, t) = \Delta z(x, t) - \beta \Delta^2 z(x, t) + \int_0^T k(t, s) z(x, s) d\mu(s), \quad (x, t) \in \Omega \times [0, T],$$

consider the boundary conditions (4.2) and the initial condition

(4.7)
$$(\lambda - \Delta)(z(x,0) - z_0(x)) = 0, \quad x \in \Omega,$$

on the domain $\Omega \subset \mathbb{R}^n$. Then problem (4.2), (4.6), (4.7) is a special case of the generalized Showalter problem (3.2), (3.4), because the projection P has the form $P = \sum_{\lambda_m \neq \lambda} \langle \cdot, \varphi_m \rangle \varphi_m$ [16], and, therefore, $\ker(\lambda - \Delta) = \ker P$. The operators $\mathcal{K}(t,s)$: $H_0^2(\Omega) \to L_2(\Omega), t, s \in [0,T]$ are of the form

$$[\mathcal{K}(t,s)u](\cdot) = k(t,s)u(\cdot), \quad \|\mathcal{K}(t,s)\|_{\mathcal{L}(H^2_0(\Omega);L_2(\Omega))} \le |k(t,s)|.$$

For equation (4.6), we construct the function

$$\begin{split} \widetilde{F}(T) &= V_0^T(\mu) \left(\sup_{\lambda \neq \lambda_m} \frac{\sqrt{1 + \lambda_m^2}}{|\lambda - \lambda_m|} \max \left\{ 1, \sup_{\lambda \neq \lambda_m} \frac{\sqrt{1 + \lambda_m^2}}{|\lambda - \lambda_m|} \right\} \\ &\quad \times \max \left\{ 1, \exp \left(T \sup_{\lambda_m \neq \lambda} \frac{\lambda_m - \beta \lambda_m^2}{\lambda - \lambda_m} \right) \right\} \left(\max_{t, s \in [0, T]} \left\{ s | k(t, s) | \right\} + \max_{t, s \in [0, T]} \left\{ s | k_t(t, s) | \right\} \right) \\ &\quad + \frac{\sqrt{1 + \lambda^2}}{|\lambda - \beta \lambda^2|} \left(\max_{t, s \in [0, T]} | k(t, s) | + \max_{t, s \in [0, T]} | k_t(t, s) | \right) \right). \end{split}$$

The following result can easily be obtained, by arguing as in the proof of Theorem 4 and using Theorem 3.

Theorem 5. Suppose $\beta > 0$, $\lambda - \beta \lambda^2 \neq 0$, $z_0 \in H^4_{\partial}(\Omega)$, $k \in C([0,T] \times [0,T]; \mathbb{R})$, $k_t \in C([0,T] \times [0,T]; \mathbb{R})$, $\mu: [0,T] \to \mathbb{R}$ is a bounded variation function, and $\widetilde{F}(T) < 1$. Then problem (4.2), (4.6), (4.7) admits a unique solution $u \in C^1([0,T]; H^2_0(\Omega))$.

Note that $(\lambda - \Delta)z_0 \in \mathfrak{U}^1$ because $(I - P)(\lambda - \Delta)z_0 = 0$. Therefore, the conditions imposed in Theorem 3 on $u_0 = (\lambda - \Delta)z_0$ are fulfilled.

Suppose n = 1, $\Omega = (0, \pi)$, $\lambda = -1$, $\beta = 2$, T = 1. Consider the following special case of problem (4.2), (4.6), (4.7):

(4.8)
$$z(x,0) + z_{xx}(x,0) - z_0(x) - z_{0xx}(x,0) = 0, \quad x \in (0,\pi),$$

(4.9)
$$z(0,t) = z_{xx}(0,t) = z(\pi,t) = z_{xx}(\pi,t) = 0, \quad t \in [0,T],$$

(4.10)
$$\begin{aligned} -z_t(x,t) - z_{txx}(x,t) &= z_{xx}(x,t) - 2z_{xxxx}(x,t) + c(t)z(x,1), \\ (x,t) &\in (0,\pi) \times [0,1], \end{aligned}$$

 $c \in C^1([0,1];\mathbb{R}), c(0) = 0$. Then (see [16]) we have $\lambda_m = -m^2, \varphi_m = \sin mx$ for $m \in \mathbb{N}$,

$$C_{1} = \|L_{1}^{-1}Q\|_{\mathcal{L}(L_{2}(0,\pi);H^{2}(0,\pi))} = \sup_{m=2,3,\dots} \frac{\sqrt{1+m^{4}}}{m^{2}-1} = \frac{\sqrt{17}}{3},$$

$$h_{0} = \|M_{0}^{-1}(I-Q)\|_{\mathcal{L}(L_{2}(\Omega);H_{0}^{2}(\Omega))} = \frac{\sqrt{2}}{3},$$

$$a = \sup_{m=2,3,\dots} \frac{-m^{2}-2m^{4}}{m^{2}-1} = -12, \quad K(T) = K = \sup\left\{1,\frac{\sqrt{17}}{3}\right\} = \frac{\sqrt{17}}{3}.$$

Therefore,

$$\widetilde{F}(T) = \left(\frac{17}{9} + \frac{\sqrt{2}}{3}\right) \left(\max_{t \in [0,1]} |c(t)| + \max_{t \in [0,1]} |c'(t)|\right),$$

and problem (4.8)-(4.10) admits a unique solution on the segment [0, 1], provided

$$\left(\max_{t\in[0,1]}|c(t)|+\max_{t\in[0,1]}|c'(t)|\right)<\frac{9}{17+3\sqrt{2}}.$$

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