INDEPENDENT GENERATORS OF THE $K$-GROUP
OF A STANDARD TWO-DIMENSIONAL FIELD

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Abstract. It is proved that the $K$-group of any standard two-dimensional field possesses a system of independent generators. Sufficient conditions for generators to be independent are obtained. For a certain class of fields, such generators are described explicitly.

Introduction

In the present paper, we deal with the second topological Milnor $K$-group of a standard two-dimensional local field of mixed characteristic. In [2] Propositions 1.2 and 2.5, it was shown how to reduce the study of this $K$-group to the study of its subgroup generated by elements of the form $\{a, b\}$, where $a$ is a principal unit of the field; for a field $F$, this subgroup is denoted by $U(1)K^\top_2F$.

Here, we prove that for the field $F = k\{\{t\}\}$, where $k$ is a finite extension of $\mathbb{Q}_p$ for $p > 2$ that contains a primitive $p$th root of unity, the group $U(1)K^\top_2F$ has a collection of generators independent in the following sense.

Theorem 0.1. There exist countable collections $\{\varepsilon_i\} \in U(1)K^\top_2F$ and $\{\lambda_i\} \in \mathbb{N}$ such that any element $a \in U(1)K^\top_2F$ can be represented in the form

$$a = \{u, t\} + \sum \alpha_i \varepsilon_i,$$

where $u$ is a principal unit of the field $k$ and $\alpha_i \in \mathbb{Z}_p$; moreover, the unit $u$ is determined uniquely and the coefficients $\alpha_i$ are determined uniquely modulo $p^{\lambda_i}$.

The independent generators are described implicitly. They are constructed as ones “close” to the standard generators of the form $\{1+[\theta] \pi^it^j, t\}$ and $\{1+[\theta] \pi^it^j, \pi\}$, where $[\theta]$ is the lifting of an element $\theta \in \bar{k}$ and $\pi$ is a uniformizer of the field $k$. As was shown in [2] §10, generators are independent if their orders are minimal among the orders of elements “close” to standard generators. In §5 sufficient conditions are given for a collection of elements to satisfy the assumptions of Theorem 0.1 and it is proved that such a collection exists. In §4 estimates from below are obtained for the orders of generators; the proof of those estimates is based on a result from [7] concerning the embedding of a prime field extension in a cyclic extension of larger degree and on properties of the reciprocity map from [1].

In §7 we describe independent generators explicitly for a special type of fields, namely, for fields obtained from $\mathbb{Q}_p$ by adjoining an element $\sqrt[2n]{\zeta}$, where $\zeta$ is a primitive $p$th root of unity and $p \nmid l$.

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§1. Notation

1.1. Standard notation. Let \( p \) be a fixed prime number, \( p > 2 \).

We shall use the following notation:

- \( v_p(x) \) is the \( p \)-adic valuation of a \( p \)-adic number \( x \);
- \( \varphi \) is the map \( x \mapsto x^p - x \);
- \( \zeta_s \) denotes the \( s \)th primitive root of unity.

We assume that the set \( \mathbb{Z}^2 \) is lexicographically ordered in the following sense: \( (a, b) < (c, d) \) if \( b < a \) or \( b = d, a < c \).

For a discrete valuation field \( F \), we denote its valuation by \( v_F \) and its residue field by \( \overline{F} \); if \( \text{char} \, F = p \), we set \( e_F = v_F(p) \). By \( O_F \) and \( V_F \) we denote the ring of integers and the group of principal units of the field \( F \). An element the valuation of which is equal to 1 is called a uniformizer of a given field.

For a two-dimensional local field \( F \), we shall use the following notation:

\[
F^{(1)} = \overline{F}, \quad F^{(0)} = \overline{F}^{(1)};
\]

\( \tilde{v}_F = (v_F^{(1)}, v_F^{(2)}) \colon F \to \mathbb{Z}^2 \cup \{\infty\} \) is a valuation of rank 2 of the field \( F \); the valuation \( v^{(2)}_F \) will also be denoted by \( v_F \);

\( \mathfrak{M}_F = \{a \mid \tilde{v}_F(a) > 0\} \);

\( U_F(n_1, n_2) = \{1 + a \mid v_F(a) \geq n_1, n_2\} \);

\( U_F(n) = \{1 + a \mid v_F^{(2)}(a) \geq n\} \);

\( \mathfrak{M}_F \) is a canonical subgroup of \( F^* \) that consists of representatives of nonzero elements of \( F^{(0)} \);

\( [\theta] \) is the element of \( \mathfrak{M}_F \) that represents an element \( \theta \) in \( F^{(0)} \).

For a local or a two-dimensional local field \( F \), we denote by \( K_s F \) the \( s \)th Milnor \( K \)-group and by \( K_s^{\text{top}} F \) the topological group (see [6]). Usually, we shall consider the groups for \( s = 2 \). For a subgroup \( W \) of the group \( V \) we denote by \( WK_2^{\text{top}} F \) the subgroup of \( K_2^{\text{top}} F \) generated by the elements \( \{a, b\} \), where \( a \in W \).

1.2. Special notation. Suppose \( k \) is a local field, \( \text{char} \, k = 0 \), \( \text{char} \, \bar{k} = p > 2 \), and \( \zeta_p \in k \);

\( \pi \) is a uniformizer of the field \( k \);

\( K = k\{\ell\} \).

We set \( e = e_k, \bar{e} = (0; e) \); by \( O, \mathfrak{M}, U(n), U(n_1, n_2) \), and \( v \), we shall denote \( O_K, \mathfrak{M}_K, U_K(n), U_K(n_1, n_2) \), and \( v_K \), respectively.

Let \( \bar{v} = (v^{(1)}, v^{(2)}) \) denote a valuation of rank 2 of the field \( K \) such that \( v^{(1)}(\pi) = 0 \) and \( v^{(2)} = v \).

We denote by \( \mathfrak{B} \) an arbitrary basis of \( K^{(0)} \) as a vector space over \( \mathbb{F}_p \) and by \( \theta_\ell \) an arbitrary element of \( K^{(0)} \) that generates \( K^{(0)}/\varphi(K^{(0)}) \).

We set

\[
I^+ = \{(i, j, \theta) \mid 0 < i < \frac{pe}{p - 1}, j > 0, p \nmid (i, j), \theta \in \mathfrak{B}\},
\]

\[
I^- = \{(i, j, \theta) \mid 0 < i \leq \frac{pe}{p - 1}, j < 0, p \nmid (i, j), \theta \in \mathfrak{B}\},
\]

\[
I = I^+ \cup I^-.
\]

Any element in \( U(1) \) can be uniquely represented in the form

\[
1 + \sum_{i \geq 0, j \in \mathbb{Z}} [\theta_{i,j}] \pi^i t^j.
\]

Let \( T^+ \) (respectively, \( T^- \)) be the set of elements for which \( \theta_{i,j} = 0 \) for \( j \leq 0 \) (respectively, \( j \geq 0 \)).
Suppose $0 < (j, i) < \frac{p}{p-1} \epsilon$, $j \neq 0$, $p \nmid (i, j)$, $m \in \mathbb{N}_0$, and $\theta \in K^{(0)} \setminus \{0\}$; we define $m_{i,j} \in \mathbb{N}_0$, $f(i, j, m) \in \mathbb{Z}^2$, and $A_{i,j,\theta} \subset U(1)K_2^{\text{top}}K$ in the following way:

$$m_{i,j} = \max \left\{ s \in \mathbb{N}_0 \mid p^s(j, i) \leq \frac{p}{p-1} \epsilon \right\},$$

$$f(i, j, m) = \begin{cases} p^m(j, i) & \text{if } m < m_{i,j}, \\ (m - m_{i,j})\epsilon + p^{m_{i,j}}(j, i) & \text{if } m \geq m_{i,j}, \end{cases}$$

$$A_{i,j,\theta} = U(j + 1, i)K_2^{\text{top}}K + pU(1)K_2^{\text{top}}K + \left\{ \begin{array}{ll} 1 + [\theta]\pi^i t^j, t & \text{if } p \nmid i, \\ 1 + [\theta]\pi^j t^i, \pi & \text{if } p \mid i. \end{array} \right.$$
Lemma 2.3. 1) Any element of $V_k K_2^{\text{top}} k$ can be represented in the form $\{x, \pi\}$, where $x \in V_k$.

2) Any element of $U(1) K_2^{\text{top}} K$ can be represented in the form $\{x, t\} + \{y, \pi\}$, where $x, y \in U(1)$.

3) Let $T^*$ and $I^*$ be equal to $T^+$ and $I^+$ or to $T^-$ and $I^-$. If $a \in V_k$ and $x \in T^*$, then the element $\{x, a\} \in U(1) K_2^{\text{top}} K$ can be presented in the form $\{y, t\} + \{z, \pi\}$, where $y, z \in T^*$.

Moreover, $y$ and $z$ can be chosen so that

$$\begin{align*}
y &= \prod_{(i, j, \theta) \in I^*} (1 + [\theta] \pi^t t^j)^{\alpha_{i, j, \theta}}, \\
z &= \prod_{(i, j, \theta) \in I^*} (1 + [\theta] \pi^t t^j)^{\alpha_{i, j, \theta}}.
\end{align*}$$

Proof. For 1) and 2), see [2, Proposition 2.1].

3) First, we represent $\{x, a\}$ in the form $\{y, t\} + \{z, \pi\}$, where $y$ and $z$ are arbitrary elements in $T^*$.

There is $m' \in \mathbb{N}$ such that for any $u \in U(m')$ we have $\{u, a\} = 0$: it suffices to choose $m'$ satisfying $U(m') \subset U(1)^m$, where $m$ is as in Proposition 2.1.

Thus, it suffices to show that for any $x_i \in U(i) \cap T^*$ there exist elements

$$x_{i+1} \in U(i+1) \cap T^*, \quad y_i, z_i \in T^*,$$

such that

$$\{x_i, a\} = \{y_i, t\} + \{z, \pi\} + \{x_{i+1}, a\}.$$\[\text{Indeed, having constructed the sequences } x_i, y_i, z_i, \text{ starting with } x_1 = x, \text{ we obtain}\]

$$\{x, a\} = \left\{ \prod_{\theta} \prod_{j} (1 + [\theta] \pi^t t^j)^{c_{j, \theta}}, \quad c_{j, \theta} \in \{0, 1, \ldots, p - 1\}, \right.$$\[
\text{the set } J \text{ is equal to } \mathbb{N} \text{ if } T^* = T^+ \text{ and is a finite subset of } -\mathbb{N} \text{ if } T^* = T^- \text{ and } x''_i \in U(i+1). \text{ Since } x_i, x'_i \in T^*, \text{ we have}\]

$$x''_i \in U(i+1) \cap T^*.$$\]

Relation (1) implies that

$$\begin{align*}
\{1 + [\theta] \pi^t t^j\}^{c_{j, \theta}}, a\} &= c_{j, \theta}\left\{ [\theta] \pi^t t^j a, 1 - \frac{[\theta] \pi^t t^j (1 - a)}{1 + [\theta] \pi^t t^j} \right\} \\
&= \left\{ \left(1 - \frac{[\theta] \pi^t t^j (1 - a)}{1 + [\theta] \pi^t t^j}\right)^{-j c_{j, \theta}}, t\right\} + \left\{ \left(1 - \frac{[\theta] \pi^t t^j (1 - a)}{1 + [\theta] \pi^t t^j}\right)^{-j c_{j, \theta}}, \pi\right\} \\
&+ \left\{ \left(1 - \frac{[\theta] \pi^t t^j (1 - a)}{1 + [\theta] \pi^t t^j}\right)^{-j c_{j, \theta}}, a\right\}
\end{align*}$$

for any $j$ and $\theta$. We verify that the products

$$\prod_{\theta} \prod_{j} \left(1 - \frac{[\theta] \pi^t t^j (1 - a)}{1 + [\theta] \pi^t t^j}\right)^{-j c_{j, \theta}}, \prod_{\theta} \prod_{j} \left(1 - \frac{[\theta] \pi^t t^j (1 - a)}{1 + [\theta] \pi^t t^j}\right)^{-j c_{j, \theta}}, \prod_{\theta} \prod_{j} \left(1 - \frac{[\theta] \pi^t t^j (1 - a)}{1 + [\theta] \pi^t t^j}\right)^{-j c_{j, \theta}}, \prod_{\theta} \prod_{j} \left(1 - \frac{[\theta] \pi^t t^j (1 - a)}{1 + [\theta] \pi^t t^j}\right)^{-j c_{j, \theta}}$$
converge; then as \( y_t, z_t, \) and \( x_{t+1} \) we can take the first two products and the third product multiplied by \( x''_t \). If \( T^* = T^- \), these products are finite. For \( T^* = T^+ \), we set

\[
W(s) = \left\{ 1 + \sum_{r \geq i} [\eta_{r,j}] \pi^r \, t^j \mid \eta_{r,j} \in K^{(0)} \right\}.
\]

Then for any neighborhood \( W \) of the unity, we have \( W(s) \subset W \) for \( s \) sufficiently large. Moreover, for any \( s \) the set \( W(s) \) contains factors in the products for which \( j \geq s \).

Now we prove that we can take \( y \) and \( z \) satisfying condition (2). Let \( \{x, a\} = \{y, t\} + \{z, \pi\} \), where

\[
y' = \prod_{(i,j) \in I^*} (1 + [\theta] \pi^i \, t^j)^{\beta_{i,j}}, \quad z' = \prod_{(i,j) \in I^*} (1 + [\theta] \pi^i \, t^j)^{\gamma_{i,j}},
\]

\( \beta_{i,j}, \gamma_{i,j} \in \mathbb{Z}_p \). Then the relation \( i\{1 + [\theta] \pi^i \, t^j, \pi\} + j\{1 + [\theta] \pi^i \, t^j, t\} = 0 \) implies that we can take \( \alpha_{i,j} = \begin{cases} \beta_{i,j} - \frac{j \gamma_{i,j}}{i} & \text{if } p \nmid i, \\ \gamma_{i,j} - \frac{i \beta_{i,j}}{j} & \text{if } p \mid i. \end{cases} \)

The generators in the set \( G \) are not necessarily independent. Example 2.5 describes a dependence between them for the case where the field \( k \) is \( \mathbb{Q}_p(\zeta_{p^n}) \).

**Lemma 2.4.** Let \( n \) be such that \( \zeta_{p^n} \in k \). Then for any \( x \in K^*, x \neq -1 \), we have

\[
p^n \left( 1 + \frac{(\zeta_{p^n} - 1)x}{1 + x} \right) = 0.
\]

**Proof.** We have

\[
\left\{ 1 + \frac{(\zeta_{p^n} - 1)x}{1 + x}, x \right\} = \left\{ 1 + \frac{(\zeta_{p^n} - 1)x}{1 + x}, \zeta_{p^n} \right\} - \left\{ 1 + \frac{(\zeta_{p^n} - 1)x}{1 + x}, \zeta_{p^n} \right\}.
\]

By (1), the first term is equal to \(-\{1 + x, \zeta_{p^n}\}\), whence

\[
p^n \left\{ 1 + \frac{(\zeta_{p^n} - 1)x}{1 + x}, x \right\} = -p^n \left\{ 1 + x, \zeta_{p^n} \right\} - p^n \left\{ 1 + \frac{(\zeta_{p^n} - 1)x}{1 + x}, \zeta_{p^n} \right\} = 0. \quad \Box
\]

**Example 2.5.** Suppose \( n \in \mathbb{N}, k = \mathbb{Q}_p(\zeta_{p^n}), \pi = \zeta_{p^n} - 1, \) and \( a = \{1 + \frac{\pi t}{1 + t}, t\} \). By Lemma 2.4 applied to \( x = t \), we have \( p^n a = 0 \).

On the other hand,

\[
a = \{1 + \pi t - \pi t^2 + \pi t^3 - \pi t^4 + \ldots, t\},
\]

so that \( a \) can be represented as a sum of generators belonging to \( G \) in which the coefficients of \( \{1 + [\theta] \pi^j, t\} \) are congruent to \( \pm 1 \) modulo \( p \). But in Proposition 4.1 it will be proved that, for \( p \mid j \), the orders of such elements are greater than \( p^n \).

## §3. Generators of the group \( V \)

**Lemma 3.1.** Denote by \( \theta_p \) an element of \( K^{(0)} \) such that \( [\theta_p^{-1}] \pi^{-e} p \in V_k \). We set

\[
J = \left\{ (i, j, m) \mid 0 < (j, i) < \frac{p}{p - 1} e, p \nmid (i, j), j \neq 0, m \in \mathbb{N}_0 \right\}
\]

and, for \( (i, j, m) \in J \),

\[
\beta_{i,j,m} = \begin{cases} \theta_p^m & \text{if } m < m_{i,j}, \\ \theta_p^{m - m_{i,j}} \theta_p^{m_{i,j}} & \text{if } m \geq m_{i,j}. \end{cases}
\]
1) Let \((i, j, m) \in J\). We put \((f^{(1)}, f^{(2)}) = f(i, j, m)\) and denote the set \(U(f^{(1)} + 1, f^{(2)})\) by \(W\) if \(p^e i = \frac{e}{p-1}\) for \(s \in \mathbb{N}_0\) and the set \(U(f^{(2)} + 1)\) otherwise. Then

\[
(1 + \lfloor \theta_i \pi^i t^j \rfloor) p^m \in (1 + \lfloor \beta_{i, j, \theta, m} \rfloor \pi^i t^j)^W.
\]

2) For any triple \((i, j, m) \in J\), the set \(\{\beta_{i, j, \theta, m} | \theta \in \mathfrak{M}\}\) is a basis of \(K^{(0)}\) over \(\mathbb{F}_p\).

3) For any \(r > 0\) and \(s \in \mathbb{Z}\), \(s \neq 0\), there exists a unique triple \((i, j, m) \in J\) such that \(f(i, j, m) = (s, r)\).

**Proof.** 1) We have

\[
(1 + \lfloor \theta_i \pi^i t^j \rfloor) p^m = 1 + \sum_{\nu=1}^{p^m} C_{\pi^m}^{\nu} ([\theta_i \pi^i t^j]) \nu, 
\]

and if \(v_p(\nu) = \mu\), then

\[
\bar{v}(C_{\pi^m}^{\nu} ([\theta_i \pi^i t^j]) \nu) = (m - \mu) \bar{e} + \nu(j, i) \geq (m - \mu) \bar{e} + p^\mu(j, i),
\]

with equality only for \(\nu = p^\mu\). Set

\[
g(\mu) = (m - \mu) \bar{e} + p^\mu(j, i).
\]

We have

\[
g(\mu + 1) - g(\mu) = -\bar{e} + p^\mu(j, i) - p^\mu(j, i) = -\bar{e} + (p - 1) p^\mu(j, i) = (p - 1) \left( -\frac{1}{p - 1} \bar{e} + p^\mu(j, i) \right).
\]

The definition of \(m_{i, j}\) and the fact that \(j \neq 0\) show that for \(\mu \geq m_{i, j}\), the right-hand side is greater than \((0, 0)\), and for \(\mu \leq m_{i, j} - 1\) it is less than \((0, 0)\). Moreover, if \(p^\mu i\) is not equal to \(\frac{e}{p-1}\) for any \(\mu\), then the second component of the right-hand side is different from 0 for all \(\mu\).

Consequently,

\[
\min_{1 \leq \mu \leq m, \mu \in \mathbb{N}} g(\mu) = \begin{cases} g(m) & \text{if } m < m_{i, j}, \\ g(m_{i, j}) & \text{if } m \geq m_{i, j} = f(i, j, m), \end{cases}
\]

if \(m < m_{i, j}\), then

\[
(1 + \lfloor \theta_i \pi^i t^j \rfloor) p^m \in (1 + (\lfloor \theta_i \pi^i t^j \rfloor) p^m) W,
\]

and if \(m \geq m_{i, j}\), then

\[
(1 + \lfloor \theta_i \pi^i t^j \rfloor) p^m \in (1 + C_{\pi^m}^{m_{i, j}} ([\theta_i \pi^i t^j]) p^{m_{i, j}}) W.
\]

Moreover, for \(m \geq m_{i, j}\) we have the congruence

\[
C_{\pi^m}^{m_{i, j}} \equiv p^{m_{i, j}} \equiv \lfloor \theta_p^{m_{i, j}} \rfloor \pi^{(m_{i, j})} \mod \pi^{(m_{i, j})+1} O_k.
\]

2) This statement follows from the fact that the maps \(x \rightarrow x^p\) and \(x \rightarrow \theta_p x\) are automorphisms of \(K^{(0)}\) over \(\mathbb{F}_p\).

3) First, we prove existence.

If \((s, r) < \frac{p}{p-1} \bar{e}\), then \((s, r) = f(i, j, m)\), where \(m = v_p(s, r)\) and \((j, i) = \frac{1}{p^m}(s, r)\).

If \((s, r) \geq \frac{p}{p-1} \bar{e}\), then there exists a pair \((j_1, i_1)\) such that

\[
\frac{1}{p - 1} \bar{e} < (j_1, i_1) \leq \frac{p}{p - 1} \bar{e}, \quad (j_1, i_1) = (s, r) - m_1 \bar{e}
\]

for some \(m_1 \in \mathbb{N}_0\). We set \(m_2 = v_p(j_1, i_1)\). Then \((s, r) = f(i, j, m)\), where \((j, i) = \frac{1}{p^{m_2}}(j_1, i_1)\), and \(m = m_1 + m_2\).

Now we prove uniqueness.
We have
\[
\begin{cases}
  f(i, j, m) < \frac{p}{p-1} \bar{e} & \text{if } m < m_{i,j}, \\
  f(i, j, m) \geq \frac{p}{p-1} \bar{e} & \text{if } m \geq m_{i,j}.
\end{cases}
\]
Thus, if \( f(i_1, j_1, m_1) = f(i_2, j_2, m_2) \), then either \( p^{m_1}(j_1, i_1) = p^{m_2}(j_2, i_2) \), or
\[(m_1 - m_{i_1,j_1})\bar{e} + p^{m_{i_1,j_1}}(j_1, i_1) = (m_2 - m_{i_2,j_2})\bar{e} + p^{m_{i_2,j_2}}(j_2, i_2).\]
In the first case, the fact that \( p \nmid (j_1, i_1) \) and \( p \nmid (j_2, i_2) \) implies that \( m_1 = m_2 \), whence \((j_1, i_1) = (j_2, i_2)\). In the second case, we have
\[f(i_s, j_s, m_s) - \frac{p}{p-1} \bar{e} \leq (m_s - m_{i_s,j_s})\bar{e} < f(i_s, j_s, m_s) - \frac{1}{p-1} \bar{e}\]
for \( s = 1, 2 \), so that \( m_1 - m_{i_1,j_1} = m_2 - m_{i_2,j_2} \). Therefore,
\[p^{m_{i_1,j_1}}(j_1, i_1) = p^{m_{i_2,j_2}}(j_2, i_2).\]
This implies that \( m_1 = m_2 \) and \((j_1, i_1) = (j_2, i_2)\). \( \square \)

The first and second statements of Lemma \ref{lemma:independent-generators} show that if
\[f(i, j, m) = (s, r),\]
then the elements \((1 + [\theta] \pi^i t^j)^p m\) for \( \theta \in \mathfrak{B} \) form a basis of \( U(s, r)/U(s + 1, r) \) over \( \mathbb{F}_p \).

**Lemma 3.2.** Let \( x_{i,j,\theta} \in T^+ \) be given for \((i, j, \theta) \in I^+\). Suppose that they satisfy the following conditions:

1) for \( d \geq 0, m > 0, p \nmid m \), there exist \( q_{i,d,\theta,m} \in \mathbb{Z}_p \) such that
\[x_{i, p^d, \theta, m} = \tau_m(x_{i, p^d, \theta})^{q_{i,d,\theta,m}};\]

2) there exists \( d_0 \in \mathbb{N}_0 \) such that \( x_{i, p^d, \theta} = \tau_p^{d-d_0}(x_{i, p^{d_0}, \theta}) \) for \( d \geq d_0 \) and any \( i \) and \( \theta \). Then the product
\[
\prod_{(i,j,\theta) \in I^+} x_{i,j,\theta}^{c_{i,j,\theta}}
\]
converges for any \( c_{i,j,\theta} \in \mathbb{Z}_p \).

**Proof.** Since the set of pairs \((i, \theta)\) is finite, it suffices to prove that, for any \( i_0 \) and \( \theta_0 \), the product
\[a_{i_0,\theta_0} = \prod_j x_{i_0,j,\theta_0}^{c_{i_0,j,\theta_0}}\]
taken over \( j \in \mathbb{N} \) if \( p \nmid i_0 \) and over \( j \in \mathbb{N} \), \( p \nmid j \) if \( p \mid i_0 \), converges. We fix \( i_0, \theta_0 \) and set \( x_j = x_{i_0,j,\theta_0} \). We put
\[W(m) = \left\{ 1 + \sum_{\substack{i \in \mathbb{N} \\ j \geq m}} [\eta_i,j] \pi^i t^j \mid \eta_i,j \in K^{(0)} \right\} \]
For \( p \nmid m \), we have \( x_{p^d, \theta} = \tau_m(x_{p^{d_0}, \theta}^{q_{i,d,\theta,m}}) \in W(m) \) and \( W(m) \to 1 \) as \( m \to \infty \); consequently, the product
\[a_d = \prod_{m > 0} x_{p^d, \theta}^{c_{i_0,p^d,\theta}}\]
converges for any \( d \). For \( d \geq d_0 \), we have \( a_d \in W(p^{d-d_0}) \), whence \( a_d \to 1 \) as \( d \to \infty \), and the product
\[a_{i_0,\theta_0} = \prod_{d \geq 0} a_d\]
converges. \( \square \)
§4. Lower estimates for orders

Proposition 4.1. Let $n$ be such that $\zeta_p^n \in K$. Suppose that, for $a \in U(1) K_2^{top} K$, one of the following conditions is satisfied:

i) $a \equiv \{1 + [\theta] t^n, t\}$ mod $U(j + 1, i) K_2^{top} K + pK_2^{top} K$, $0 < (j, i) < \frac{p}{p - 1} \bar{e}$, $p \nmid i$

ii) $a \equiv \{1 + [\theta] t^n, \pi\}$ mod $U(j + 1, i) K_2^{top} K + pK_2^{top} K$, $0 < (j, i) < \frac{p}{p - 1} \bar{e}$, $p \nmid i$

iii) $a \equiv \alpha \{1 + [\theta] (\zeta_p - 1)^n, t\} + \alpha' \{1 + [\theta] (\zeta_p - 1)^n, \pi\}$ mod $pK_2^{top} K$, $p \nmid (\alpha, \alpha')$

Then:

1) if i) or ii) is valid, then

\[ a \notin U(j + 1, i) K_2^{top} K + pK_2^{top} K, \]

and in the case of iii) we have $a \notin pK_2^{top} K$.

2) we have ord $a \geq p^n$, and if i) is true, then ord $a \geq p^{n+v_p(j)}$; in particular, if $j = 0$, then $a$ is not a torsion element of $U(1) K_2^{top} K$;

3) in the case of i) or ii), if for $s \in \mathbb{N}$ there exists $a' \in M_s$ such that

\[ a' \equiv 1 + [\theta] t^n \mod \pi t^n \theta^{-1} j \mod \pi t^n \theta^{-1} j \mathfrak{M}, \]

then ord $a \geq s$.

This proposition follows from the properties of the reciprocity map, the Vostokov symbol $\Gamma$, and the Miki groups. We state some pertinent facts. Let $\psi$ denote the reciprocity map, $\psi : K_2 K \rightarrow \text{Gal}(K^{ab}/K)$, see [1 §7.2]. The reciprocity map is a homomorphism.

We denote by $\mu_p$ the group of $p$th roots of unity.

From [1] it follows that there exists a map $\Gamma : (K^*)^3 \rightarrow \mu_p$ with the following properties:

a) $\Gamma$ induces a homomorphism $K_2^{top} K \rightarrow \mu_p$;

b) if $u, w \in K^*$, $\theta \in K^{(0)} \setminus \{0\}$, $0 < (j, i) < \frac{p}{p - 1} \bar{e}$,

\[ u \equiv 1 + [\theta] t^n \mod \pi t^n \mathfrak{M}, \text{ and } \]

\[ w \equiv 1 + [\theta] t^n \mod \pi t^n \mathfrak{M}, \]

then $\Gamma(u, t, w) \neq 1$ for $p \nmid i$ and $\Gamma(u, \pi, w) \neq 1$ for $p \nmid j$;

c) $\Gamma(1 + [\theta] (\zeta_p - 1)^n, t, \pi) \neq 1$;

d) $\Gamma$ induces a homomorphism $K_2^{top} K/pK_2^{top} K \times K^*/pK^* \rightarrow \mu_p$, and if $\Gamma(x, y) \neq 1$ for $x \in K_2^{top} K$ and $y \in K^*$, then $\psi(x)$ acts nontrivially on $K(\sqrt{\psi(y)})$.

The map $\Gamma$ is described in the Introduction of [1] by an explicit formula, and the required properties are proved in Theorems 2, 3, 4 and Lemmas 8 and 10.

Lemma 4.2. Suppose $x, y \in U(1)$, $z \in K$, and $\bar{v}(x - 1) + \bar{v}(y - 1) > \frac{p}{p - 1} \bar{e}$. Then $\Gamma(x, y, z) = 1$.

Proof. It suffices to prove that $\{x, y\} \in pK_2^{top} K$.

We set $u = 1 - x$, $w = 1 - y$. Applying relation (1), we obtain

\[ \{x, y\} = \left\{ u(1 - w), 1 + \frac{uw}{1 - u} \right\}. \]

The second component belongs to $U(1, \frac{p e}{p - 1}) \subset U(1)^p$.

Lemma 4.3. Suppose $s \in \mathbb{N}$, $x \in K_2^{top} K$, $y \in M_s$, and $\Gamma(x, y) \neq 1$. Then ord $x \geq p^s$. 

\[ \square \]
Proof. By \[7, \text{Corollary to Proposition 3}\], the fact that \(y \in M_s\) implies that there exists a cyclic extension \(L/K\) such that \([L : K] = p^s, K(\sqrt[2]{y}) \subset L\). By property d), the map \(\psi(x) \in \text{Gal}(L/K)\) acts nontrivially on \(K(\sqrt[2]{y})\). Consequently, \(\psi(x)\) generates \(\text{Gal}(L/K)\), i.e., the order of \(\psi(x)\) in \(\text{Gal}(L/K)\) is equal to \(p^s\). We see that \[
\psi(p^{s-1}x) = (\psi(x))^{p^{s-1}} \neq 1
\] and \(p^{s-1}x \neq 0\). \(\square\)

To find appropriate elements in Miki groups, we need a lemma on the norm mapping for cyclotomic extensions of standard two-dimensional fields. Note that if \(K_1 = k_1 \{t\}\) is such a field and \(K_2 = K_1(\zeta_{p^r})\), then \(K_2 = k_2 \{t\}\), where \(k_2 = k_1(\zeta_{p^r})\). We have \(\text{Gal}(K_2/K_1) = \text{Gal}(k_2/k_1)\), and if \(x \in k_2\), then
\[
N_{K_2/K_1} x = N_{k_2/k_1} x \in k_1, \quad \text{Tr}_{K_2/K_1} x = \text{Tr}_{k_2/k_1} x \in k_1.
\]

Lemma 4.4. Put \(K_s = K(\zeta_{p^r})\). Denote by \(m\) a number such that the extension \(K_m/K\) is unramified and the extensions \(K_s/K_m\) are totally ramified for \(s > m\).

1) For \(r \leq m, i > 0, j \in \mathbb{Z}, \theta \in K_r^{(0)},\) we have
\[
N_{K_r/K}(1 + [\theta]^{i}t^j) \equiv 1 + \text{Tr}_{K_r/K}([\theta]) \pi^j t^j \mod U(i + 1).
\]

2) Suppose \(r > m, i > 0, j \in \mathbb{Z}, j \neq 0,\) and \(\theta \in K_m^{(0)} \setminus \{0\}\). For \(m \leq s \leq r,\) denote by \(\pi_s\) arbitrary uniformizers of the fields \(K_s\) that belong to the constant subfields of these fields and satisfy the condition \(N_{K_{s+1}/K_s} \pi_{s+1} = \pi_s;\) let \(\sigma_s\) be arbitrary generators of the groups \(\text{Gal}(K_{s+1}/K_s)\). Define
\[
\xi, \eta, \theta \in K_m^{(0)}, \quad h_s, i_s \in \mathbb{N}, \quad \alpha \in \mathbb{N}_0,
\]
in such a way that
\[
[\xi] \pi^{-1} \pi_m \in U_{K_m}(1),
\]
\[
s \sigma_s(\pi_{s+1}) \equiv 1 + [\eta_s] \pi_{s+1}^h \mod U_{K_{s+1}}(h_s + 1),
\]
\[
i_m = i, \quad i_{s+1} = \begin{cases} i_s & \text{if } i_s \leq h_s, \\ h_s + (i_s - h_s)p & \text{if } i_s > h_s, \end{cases}
\]
\[
\theta_m = \theta, \quad \theta_{s+1} = \begin{cases} \theta_{s+1}^{1/p} & \text{if } (j, i_s) < (0, h_s), \\ -\theta_s/\eta_{s+1}^{p-1} & \text{if } (j, i_s) > (0, h_s), \end{cases}
\]
\[
a = \#\{s \mid m \leq s \leq r - 1, (j, i_s) < (0, h_s)\}.
\]

Then
\[
N_{K_r/K}(1 + [\theta]^{i}t^j) \equiv 1 + \text{Tr}_{K_m/K}([\theta \xi^{\ell}]) \pi^i j \pi^a \mod \pi^{i+1}O.
\]
Moreover, if \(i_s \neq h_s\) for all \(s,\) then this congruence holds modulo \(\pi^{i+1}O\).

Proof. See \[5, \text{Chapter 3, Propositions 1.2 and 1.5}\].\(\square\)

Corollary 4.5. Suppose \(r, i > 0, j \in \mathbb{Z}, \theta \in K^{(0)} \setminus \{0\},\) and let numbers \(m\) and \(a\) be as in Lemma 4.4.

1) Let \(r \leq m.\) Then there exists \(x \in M_r\) such that
\[
x \equiv 1 + [\theta] \pi^{i} t^j \mod \pi^{i+1}O.
\]

2) Let \(r > m\) and \(v_p(j) \geq a.\) Then there exists \(x \in M_r\) such that
\[
x \equiv 1 + [\theta] \pi^{i} t^j \mod \pi^{i+1}O.
\]
3) Let \( v_p(j) \geq s \) for some \( s \in \mathbb{N}_0 \), and let \( n \) be as in Proposition 4.1. Then there exists \( x \in M_{n+s} \) such that
\[
x \equiv 1 + [\theta] \pi^t t^j \mod \pi^t t^j \mathfrak{M}.
\]

Proof. The map \( \theta \to \theta \xi^i \) and the map \( K_m^{(0)} \to K^{(0)} \) induced by the trace map are surjective. This implies the existence of the required \( x \) for \( r \leq m \), and, for \( r > m \) and any \( l \), the existence of \( x_l \in M_r \) such that
\[
x_l \equiv 1 + [\theta] \pi^t t^j \mod \pi^t t^j \mathfrak{M} + l.
\]
As \( x \) we can take \( x_l \) with \( l = jp^a \).

The third statement follows from the second and the fact that
\[
a \leq r - m \leq r - n.
\]

Proof of Proposition 4.1. We set \( b = 1 + [\theta, \theta^{-1}] = \pi^t \mathfrak{M} \mathfrak{p}^{-i} t^{-j} \) in the cases of i) and ii), \( b = \pi \) in the case of iii) for \( p \nmid \alpha \), and \( b = t \) in the case iii) for \( p \mid \alpha, p \nmid \alpha' \). Then properties b) and c) imply that \( \Gamma(a, b) \neq 1 \).

Lemma 4.2 also ensures that in the cases of i) or ii), for any \( c \) with
\[
c \equiv b \mod \pi^t \mathfrak{M},
\]
we have \( \Gamma(a, c) = \Gamma(a, b) \neq 1 \).

1) Consider the cases where i) or ii) is true. Assume that
\[
a \in U(j + 1, i) K_2^{\text{top}} K + p V K_2^{\text{top}} K.
\]
Then \( a \) can be represented in the form
\[
a = \sum \{x_s, y_s\} + pa',
\]
where \( x_s \in U(j + 1, i), y_s \in K \), and \( a' \in K_2^{\text{top}} K \). By Lemma 4.2 applied to \( x = x_s, y = b, z = y_s \), we have \( \Gamma(x_s, y_s, b) = 1 \) for any \( s \). Moreover, \( \Gamma(pa', b) = 1 \). We obtain \( \Gamma(a, b) = 1 \), a contradiction.

In the case of iii), the claim also follows from the existence of \( b \) such that \( \Gamma(a, b) \neq 1 \).

2) We have \( M_n = K^\ast \); therefore, \( b \in M_n \) and the inequality \( \text{ord} a \geq p^n \) follows from Lemma 4.3. The second statement is a consequence of Lemma 4.3 and Corollary 4.5 applied to \( \frac{p e}{p - 1} = i, -j, \theta_\nu \theta^{-1} \).

3) The third statement is a consequence of Lemma 4.3. \( \square \)

§5. Description of Independent Generators

Our aim in the present section is to prove Proposition 5.1. The elements \( u_{i,j,\theta} \) and \( w_{i,j,\theta} \) that satisfy the assumptions of that proposition will be the required generators.

**Proposition 5.1.** There exists a collection of elements
\[
u_{i,j,\theta}, w_{i,j,\theta}, u_{i,j,\theta}^{(1)}, w_{i,j,\theta}^{(1)}, u_{i,j,\theta}^{(2)}, w_{i,j,\theta}^{(2)} \in U(1),
\]
given for \( 0 < (j, i) < \frac{\nu}{p - 1}, p \nmid (i, j), j \neq 0 \), and \( \theta \in K^{(0)} \setminus \{0\} \), that satisfies the following conditions:

1) \( u_{i,j,\theta}^{(1)} = 1 + [\theta] \pi^t t^j \mod \pi^t t^j \mathfrak{M} \),
\[
w_{i,j,\theta}^{(1)} \in U(j + 1, i) \text{ if } p \nmid i,
\]
\[
u_{i,j,\theta}^{(1)} \in U(j + 1, i), w_{i,j,\theta}^{(1)} = 1 + [\theta] \pi^t t^j \mod \pi^t t^j \mathfrak{M} \text{ if } p \mid i,
\]
\[
u_{i,j,\theta} = u_{i,j,\theta}^{(1)} (w_{i,j,\theta}^{(1)})^p, w_{i,j,\theta} = w_{i,j,\theta}^{(1)} (w_{i,j,\theta}^{(2)})^p \text{ for any } i, j, \theta;
\]
2) \( \text{ord}(\{u_{i,j,\theta}, t\} + \{w_{i,j,\theta}, \pi\}) = p^{\chi_{i,j,\theta}} \) for any \( i, j, \theta \);
3) \( u_{i,j,\theta}, w_{i,j,\theta}, u_{i,j,\theta}^{(1)}, w_{i,j,\theta}^{(2)} \) belong to \( T^+ \) if \( j > 0 \) and to \( T^- \) if \( j < 0 \).
4) there exists \( d_0 \in \mathbb{N} \) such that the following relations are valid for \( d \geq d_0 \), \( s = 1, 2 \), any \( i, \theta \), and \( j = \pm 1 \):

\[
u^{(s)}_{i,j,p^d,\theta} = p^{d-d_0}(u^{(s)}_{i,j,p^{d_0},\theta}), \quad u^{(s)}_{i,j,p^d,\theta} = 1;
\]

5) for \( m > 0 \), \( p \nmid m \), \( s = 1, 2 \), and any \( i, j, \theta \), we have

\[
u^{(s)}_{i,j,m,\theta} = \tau_m(u^{(s)}_{i,j,\theta}), \quad \nu^{(s)}_{i,j,m,\theta} = (\tau_m(u^{(s)}_{i,j,\theta}))^{1/m} \text{ if } p \nmid i,
\]

\[
u^{(s)}_{i,j,m,\theta} = (\tau_m(u^{(s)}_{i,j,\theta}))^m, \quad \nu^{(s)}_{i,j,m,\theta} = \tau_m(u^{(s)}_{i,j,\theta}) \text{ if } p \mid i.
\]

**Lemma 5.2.** We set

\[
U_1 = \left\{ \prod_{(i,j,\theta) \in I^+} (1 + [\theta]p^i t^j)^{\alpha_{i,j,\theta}} \mid \alpha_{i,j,\theta} \in \mathbb{Z}_p \right\},
\]

\[
U_2 = \left\{ \prod_{(i,j,\theta) \in I^+} (1 + [\theta]p^i t^j)^{\alpha_{i,j,\theta}} \mid \alpha_{i,j,\theta} \in \mathbb{Z}_p \right\},
\]

\[
U_3 = \left\{ \prod_{(i,j,\theta) \in I^-} (1 + [\theta]p^i t^j)^{\alpha_{i,j,\theta}} \mid \alpha_{i,j,\theta} \in \mathbb{Z}_p \right\},
\]

\[
U_4 = \left\{ \prod_{(i,j,\theta) \in I^-} (1 + [\theta]p^i t^j)^{\alpha_{i,j,\theta}} \mid \alpha_{i,j,\theta} \in \mathbb{Z}_p \right\},
\]

\[
U_5 = V_k, \quad U_6 = \left\{ (1 + [\theta]p(z - 1)^p)^\alpha \mid \alpha \in \mathbb{Z}_p \right\}.
\]

Suppose \( a \in U(1)K^2_1 \), \( s \in \mathbb{N} \), and \( p^s a = 0 \). Then for \( 1 \leq m \leq 6 \) there exist \( u_m \in U_m \) such that

\[
a = \{u_1 u_3 u_5, t\} + \{u_2 u_4 u_6, \pi\}
\]

and

\[
p^s(\{u_1, t\} + \{u_2, \pi\}) = p^s(\{u_3, t\} + \{u_4, \pi\}) = p^s(\{u_6, \pi\}) = 0, \quad u_5^p = 1.
\]

**Proof.** We argue by induction on \( s \).

First, let \( s = 1 \). By [8, §3], the relation \( p a = 0 \) implies the existence of \( x \in K \) with \( a = \{x, \zeta_p\} \). We represent \( x \) in the form

\[
x = [\theta_0]p^{x_0} t^{x_0} x_+ x_+ x_0,
\]

where \( x_+ \in T^+, \ x_- \in T^- \), and \( x_0 \in V_k \). We have

\[
\{x, \zeta_p\} = \{x_+, \zeta_p\} + \{x_-, \zeta_p\} + \{p^{x_0} x_0, \zeta_p\} + \{\zeta_p^{-x_0}, t\}.
\]

Applying the third assertion of Lemma 2.2.3 to the first two terms and the first assertion of this lemma to the third term, we see that for \( m = 1, 2, 3, 4, 6 \) there exist elements \( u_m \in U_m \) such that

\[
\{x_+, \zeta_p\} = \{u_1, t\} + \{u_2, \pi\}, \quad \{x_-, \zeta_p\} = \{u_3, t\} + \{u_4, \pi\}, \quad \{p^{x_0} x_0, \zeta_p\} = \{u_6, \pi\}.
\]

We set \( u_5 = \zeta_p^{-x_0} \). The collection of elements \( u_m \) constructed in this way satisfies the required condition.

Now we check the induction step from \( s - 1 \) to \( s \). Since the element \( p a \) satisfies \( p^{s-1} p a = 0 \), we can apply the induction hypothesis to it. Let \( u'_m \) be such as in the condition of the lemma, for \( p a \). For \( 1 \leq m \leq 4 \) we represent \( u'_m \) in the form

\[
u'_m = \prod (1 + [\theta]p^i t^j)^{\alpha'_{i,j,\theta}},
\]
where the sets of indices over which the products are taken are the same as in the definition of \( U_m \). Let \( \alpha' \in \mathbb{Z}_p \) be such that

\[
\nu^p_6 = (1 + [\theta_p](\zeta_p - 1)^p)^{\alpha'},
\]

and let \( \beta, \beta_{i, \theta} \in \mathbb{Z}_p \) be such that

\[
\nu^p_5 = (1 + [\theta_p](\zeta_p - 1)^p)^{\beta} \prod_{0 < i < \frac{p}{p^t, \theta \in \mathbb{B}}} (1 + [\theta \pi^i]^{\beta_{i, \theta}}).
\]

Since

\[
\{u_1^i u_3^i t, \pi\} + \{u_2^i u_4^i \nu_6, \pi\} = pa \in pU(1)K_{2^{\top}}K,
\]

Proposition 4.1 shows that \( u^p_m \in U_m \) such that \( (u^p_m)^p = u^p_m \). We have

\[
p^s\{u_1^i u_3^i u_5^i, \pi\} + \{u_2^i u_4^i \nu_6, \pi\} = p^{s-1}\{u_1^i u_3^i, \pi\} = 0 
\quad \text{for } l = 0, 1,
\]

\[
(u^p_m)^{p^p} = (u^p_5)^{p^{s-1}} = 1, \quad p^s\{u^p_m, \pi\} = p^{s-1}\{u^p_5, \pi\} = 0.
\]

We set

\[
b = a - \{u_1^i u_3^i u_5^i, \pi\} = \{u_2^i u_4^i \nu_6, \pi\} = 0.
\]

Then

\[
pb = pa - \{u_1^i u_3^i u_5^i, \pi\} = 0.
\]

Consequently, we can apply the statement of the lemma with \( s = 1 \) to \( b \). Suppose that elements \( u^p_m \in U_m \) satisfy the conditions of the lemma for \( b \). Then

\[
a = \{u_1^i u_3^i u_5^i u_1^i u_3^i u_5^i, \pi\} + \{u_2^i u_4^i u_6^i u_2^i u_4^i u_6^i, \pi\},
\]

and the elements \( u_m = u^p_m u^p_m \) are as required. \( \square \)

**Corollary 5.3.** Suppose \( a \in U(1)K_{2^{\top}}K, c \in \mathbb{Z}_p, p \nmid c \), and

\[
a \equiv c \mod pU(1)K_{2^{\top}}K.
\]

Then \( \text{ord } a \geq p^c \).

**Proof.** Let \( \text{ord } a = p^c \), let \( u_m \) be such as in Lemma 5.2 and let

\[
u_6 = (1 + [\theta_p](\zeta_p - 1)^p)^{\alpha^c}, \quad \alpha^c \in \mathbb{Z}_p.
\]

Then \( p^c \alpha^c \equiv 0 \).

We represent \( u_m, 1 \leq m \leq 4 \), in the form of products as in the definition of \( U_m \) and write \( u_5 \) in the form

\[
u_5 = (1 + [\theta_p](\zeta_p - 1)^p)^{\alpha_1} \prod_{0 < i < \frac{p}{p^t, \theta \in \mathbb{B}}} (1 + [\theta \pi^i]^{\alpha_{i, 0, \theta}}, \alpha_t, \alpha_{i, 0, \theta} \in \mathbb{Z}_p.
\]

We obtain

\[
a = \sum_{0 < \langle j, i \rangle < \frac{p^t \varepsilon}{p^t, \theta \in \mathbb{B}}} \alpha_{i, j, \theta} \{1 + [\theta \pi^j t^i], \pi\} + \sum_{0 < \langle j, i \rangle < \frac{p^t \varepsilon}{p^t, \theta \in \mathbb{B}}} \alpha_{i, j, \theta} \{1 + [\theta \pi^j t^i], \pi\}
\]

\[
+ \alpha_t \{1 + [\theta_p](\zeta_p - 1)^p, \pi\} + \alpha_\pi \{1 + [\theta_p](\zeta_p - 1)^p, \pi\}.
\]

From Proposition 4.1 it follows that \( p | \alpha_{i, j, \theta}, p | \alpha_t, \) and \( p | \alpha_\pi - c \). Consequently, \( p \nmid \alpha_\pi \) and

\[
p^c \geq \text{ord } \alpha^c \varepsilon = \text{ord } \varepsilon = p^\lambda \varepsilon. \quad \square
\]
Lemma 5.4. Suppose $0 < (j, i) < \frac{p}{p-e}$, $p \nmid (i, j)$, $j \neq 0$, and $\theta \in K(0) \setminus \{0\}$. Then there exist elements $u = u_{i,j,\theta}$, $w = w_{i,j,\theta}$, $w^{(1)} = u_{i,j,\theta}^{(1)}$, $w^{(2)} = u_{i,j,\theta}^{(2)}$, and $w^{(2)} = u_{i,j,\theta}^{(2)}$ satisfying conditions 1), 2), and 3) of Proposition 5.1.

Proof. Let $a$ be an arbitrary element of the smallest order in $A_{i,j,\theta}$, and let $u_m$ satisfy the assumptions of Lemma 5.2 for the element $a$. We represent $u_m$ in the form of products $u_m = u_m^{(1)}(u_m^{(2)})^p$ in such a way that $u_m^{(1)}, u_m^{(2)} \in U_m$ and for $1 \leq m \leq 4$ we have

$$u_m^{(1)} = \prod_{\theta \in \mathbb{B}} \left(1 + |\theta| \pi^i t^j\right)^{\alpha_i} \prod_{0 < i < \frac{p}{p-e} \theta \in \mathbb{B}} \left(1 + |\theta| \pi^i t^j\right)^{\alpha_i,0,0}, \quad \alpha_i, \alpha_{i,0,0} \in \{0, 1, \ldots, p-1\},$$

where the products are taken over the sets occurring in the definition of $U_m$, but for $m = 5, 6$ we have

$$u_5^{(1)} = (1 + |\theta| \pi^i t^j)^{\alpha_i} \prod_{\theta \in \mathbb{B}} \left(1 + |\theta| \pi^i t^j\right)^{\alpha_i,0,0}, \quad \alpha_i, \alpha_{i,0,0} \in \{0, 1, \ldots, p-1\},$$

$$u_6^{(1)} = (1 + |\theta| \pi^i t^j)^{\alpha_i}, \quad \alpha_i \in \{0, 1, \ldots, p-1\}.$$

The valuations of the elements $u_m^{(1)} - 1$ for $m = 1, 3, 5$ are different, because so are their first components; therefore,

$$v(u_1^{(1)} u_3^{(1)} u_5^{(1)} - 1) = \min\{v(u_m^{(1)} - 1) \mid m = 1, 3, 5\},$$

and a similar relation is valid for $u_2^{(1)}, u_4^{(1)}, u_6^{(1)}$. We set

$$m_0 = \begin{cases} 1 & \text{if } p \nmid i, j > 0, \\ 2 & \text{if } p \mid i, j > 0, \\ 3 & \text{if } p \mid i, j < 0, \\ 4 & \text{if } p \nmid i, j < 0. \end{cases}$$

Since $a \in A_{i,j,\theta}$, Proposition 4.1 implies

$$u_{m_0}^{(1)} \equiv 1 + |\theta| \pi^i t^j \mod \pi^i t^j \mathfrak{M}, \quad u_m^{(1)} \in U(j + 1, i) \text{ for } m \neq m_0.$$

Thus, for the role of $u^{(s)}, w^{(s)}$ we can take $u_1^{(s)}, u_2^{(s)}$ if $j > 0$ and $u_3^{(s)}, u_4^{(s)}$ if $j < 0$. □

Lemma 5.5. Let $u \in K$ and $s \in \mathbb{N}_0$ be such that $p^s\{u, t\} = 0$. Then for any $d \in \mathbb{N}$ we have $p^{s+d}\{\tau_{p^d} u, t\} = 0$.

Proof. The map $\tau_{p^d}$ induces an isomorphism

$$K_2^{\text{top}} K \to K_2^{\text{top}} k\{\{t^{p^d}\}\};$$

consequently,

$$p^{s+d}\{\tau_{p^d} u, t\} = p^s\{\tau_{p^d} u, t^{p^d}\} = p^s\{\tau_{p^d} u, \tau_{p^d} t\} = 0$$

in $K_2^{\text{top}} k\{\{t^{p^d}\}\}$ and in $K_2^{\text{top}} K$. □

Lemma 5.6. Suppose $0 < i < \frac{p_e}{p-1}$, $p \nmid i$, $j = \pm 1$, $\theta \in K(0) \setminus \{0\}$, and

$$T^* = \begin{cases} T^+ & \text{if } j = 1, \\ T^- & \text{if } j = -1. \end{cases}$$

Then there exist $d_0 \in \mathbb{N}$ and $a \in T^*$ such that for $d \geq d_0$ the element $a_d = \{\tau_{p^d-d_0} u, t\}$ belongs to $A_{i,jp^d,\theta}$ and satisfies $\text{ord} a_d = p^{\lambda_i,jp^d,\theta}$. 

□
Lemma 5.7. There exists a collection of elements $u_{i,j,\theta}, w_{i,j,\theta}, u_{i,j,\theta}^{(1)}, w_{i,j,\theta}^{(1)}, u_{i,j,\theta}^{(2)}, w_{i,j,\theta}^{(2)}$ given for $0 < (j,i) < \frac{p^e}{p-1}, p \nmid (j,i), j = \pm p^e, d \geq 0, \text{ and } \theta \in K^{(0)} \setminus \{0\}$ that satisfies conditions 1), 2), 3), and 4) of Proposition 5.1.

Proof. By Lemma 5.6 for any $i, j_0, \theta$ such that $0 < i < \frac{p^e}{p-1}, p \nmid i, j_0 = \pm 1, \text{ and } \theta \in K^{(0)} \setminus \{0\}$, there exists a number $d_{i,j_0,\theta}$ and a collection of elements $u_{i,j_0,\theta}$ for which the first four conditions of Proposition 5.1 are satisfied. Since the set of triples $i, j_0, \theta$ is finite, for the role of $d_0$ we can take the smallest of the numbers $d_{i,j_0,\theta}$, and for $d < d_0$ we can choose arbitrary $u_{i,j_0,\theta}, w_{i,j_0,\theta}, u_{i,j,\theta}^{(1)}, w_{i,j,\theta}^{(1)}, u_{i,j,\theta}^{(2)}, w_{i,j,\theta}^{(2)}$ that satisfy the first three conditions of Proposition 5.1 such elements exist by Lemma 5.4.

Lemma 5.8. Let $a = \{x, t\} + \{y, \pi\}$, where $x, y \in U(1)$ and $m \in \mathbb{N}$ is such that $p \nmid m$. We set

$$b = \{(\tau_m(x))^m, t\} + \{\tau_m(y), \pi\}, \quad c = \{\tau_m(x, t\} + \{(\tau_m(y))^{1/m}, \pi\}.$$ 

Then $\text{ord } a = \text{ord } b = \text{ord } c$.

Proof. Put $K' = k\{t^m\}$. The homomorphism $\tau_m$ can be regarded as an isomorphism of $K$ to $K'$. It induces an isomorphism of $U(1)K_2^{\text{top}} K$ to $U(1)K_2^{\text{top}} K'$; we shall denote it also by $\tau_m$. Since $p \nmid m = |K' : K|$, the embedding of $U(1)K_2^{\text{top}} K'$ in $U(1)K_2^{\text{top}} K$ is injective by Lemma 3.4 in [5, Chapter 9]. For this reason, the orders of elements in $U(1)K_2^{\text{top}} K'$ do not depend on whether we view them as elements of $U(1)K_2^{\text{top}} K'$ or as elements of $U(1)K_2^{\text{top}} K$. Since

$$b = \{\tau_m(x, t^m\} + \{\tau_m(y), \pi\} = \tau_m(a),$$

we have $\text{ord } b = \text{ord } a$.

The relation $\text{ord } b = \text{ord } c$ follows from the relation $b = mc$. 

Lemma 5.9. Suppose $i, m \in \mathbb{N}, j \in \mathbb{Z}, p \nmid m$, and $K' = k\{t^m\}$. Then

(4) $N_{K/K'}(U(jm+1, i)K_2^{\text{top}} K + pU(1)K_2^{\text{top}} K) \subset U_{K'}(j+1, i)K_2^{\text{top}} K' + pU_{K'}(1)K_2^{\text{top}} K'$. 

Proof.
Proof. From Proposition 4.1 and Lemma 2.3 it follows that the set on the left-hand side of (1) is the set of norms of the elements
\[ a = \{x, t\} + \{y, \pi\} + pb, \]
where \( x, y \in U(jm + 1, i), \ b \in U(1)K_{2}^{top} K \). We have
\[ N_{K/K'} a = \{N_{K/K'} x^{i/m}, t^{m}\} + \{N_{K/K'} y, \pi\} + pN_{K/K'} b. \]
This sum belongs to the right-hand side of (1), because
\[ N_{K/K'} U(jm + 1, i) \subset U_{K'}(j + 1, i) \]
and \((U(jm + 1, i))^{1/m} = U(jm + 1, i). \]
\(\square\)

Proof of Proposition 5.1. For \( j = \pm p^{d} \), we take a collection of elements that satisfy the first four conditions of the proposition; this can be done by Lemma 5.7.
For \( m > 0 \), \( p \nmid m \), \( j = \pm p^{d} \), and \( s = 1, 2 \), we set
\[ u_{i,j,m,\theta}^{(s)} = \tau_{m}(u_{i,j,\theta}^{(s)}), \quad w_{i,j,m,\theta}^{(s)} = \left(\tau_{m}(u_{i,j,\theta}^{(s)})\right)^{1/m} \text{ if } p \nmid i, \]
\[ u_{i,j,m,\theta}^{(s)} = \left(\tau_{m}(u_{i,j,\theta}^{(s)})\right)^{m}, \quad w_{i,j,m,\theta}^{(s)} = \tau_{m}(u_{i,j,\theta}^{(s)}) \text{ if } p \mid i, \]
\[ u_{i,j,\theta} = u_{i,j,\theta}(u_{i,j,\theta})^{p}, \quad w_{i,j,\theta} = w_{i,j,\theta}(w_{i,j,\theta})^{p}, \]
and prove that this collection satisfies all the conditions of the proposition. It suffices to check that the second condition is satisfied.
Fixing \( i, j, \theta, m \), we set
\[ a = \{u_{i,j,\theta}, t\} + \{w_{i,j,\theta}, \pi\}, \quad a' = \{u_{i,j,m,\theta}, t\} + \{w_{i,j,m,\theta}, \pi\}. \]
By Lemma 5.8 we have \( \ord a' = \ord a = p_{i,j,\theta} \). We prove that \( p_{i,j,\theta} \) is the minimal possible order for the elements of \( A_{i,j,m,\theta} \). Let \( b \in A_{i,j,m,\theta} \). Putting \( K' = k\{\{t^{m}\}\} \), we denote by \( A'_{i,j,\theta} \) the subsets of \( U_{K'}(1)K_{2}^{top} K' \) similar to \( A_{i,j,\theta} \). Then \( p_{i,j,\theta} \) is the minimal possible order for the elements of \( A'_{i,j,\theta} \). By Lemma 5.9 for \( p \nmid i \) we have
\[ N_{K/K'} b \equiv N_{K/K'} \{1 + [\theta] \pi^{i}t^{jm}, t\} \]
\[ \equiv \{1 + [\theta] \pi^{i}t^{jm}, t^{m}\} \mod U_{K'}(j + 1, i)K_{2}^{top} K' + pU_{K'}(1)K_{2}^{top} K', \]
and for \( p \mid i \) we have
\[ N_{K/K'} b \equiv N_{K/K'} \{1 + [\theta] \pi^{i}t^{jm}, \pi\} \]
\[ \equiv m\{1 + [\theta] \pi^{i}t^{jm}, \pi\} \mod U_{K'}(j + 1, i)K_{2}^{top} K' + pU_{K'}(1)K_{2}^{top} K'. \]
We see that \( N_{K/K'} b \) or \( \frac{1}{m} N_{K/K'} b \) belongs to \( A'_{i,j,m,\theta} \), whence
\[ \ord b \geq \ord \left( N_{K/K'} b \right) \geq p_{i,j,\theta}. \]
\(\square\)

We prove another property of the sets \( A_{i,j,\theta} \).

Corollary 5.10. The number \( \lambda_{i,j,\theta} \) depends only on \( i, \nu_p(j) \), and \( \sgn(j) \).

Proof. Lemma 5.8 and the fifth property mentioned in Proposition 5.1 imply that \( \lambda_{i,j,\theta} \) depends only on \( i, \nu_p(j), \sgn(j), \) and \( \theta \). Thus, it suffices to prove that \( \lambda_{i,j,\theta} = \lambda_{i,j,\eta} \) for any \( i, j \) and any \( \theta, \eta \in K^{(0)} \setminus \{0\} \), and we may assume that \( j = \pm p^{d} \).
The finite field \( K^{(0)} \) is perfect; consequently, for \( j = \pm p^{d} \) there exists \( \xi \in K^{(0)} \setminus \{0\} \) such that \( \theta \xi^{j} = \eta \). Let \( \sigma \) denote the automorphism of \( K \) over \( k \) for which \( \sigma t = [\xi]t \); its extension \( K_{2}^{top} K \to K_{2}^{top} K \) will also be denoted by \( \sigma \). We have
\[ \sigma \{1 + [\theta] \pi^{i}t^{j}, t\} = \{1 + [\theta \xi^{j}] \pi^{i}t^{j}, [\xi]t\} = \{1 + [\eta] \pi^{i}t^{j}, t\}, \]
and similarly
\[ \sigma \{1 + [\theta] \pi^{i}t^{j}, \pi\} = \{1 + [\eta] \pi^{i}t^{j}, \pi\}. \]
moreover, \(\sigma U(j + 1, i) = U(j + 1, i)\). Therefore, \(\sigma A_{i,j,\theta} = A_{i,j,\theta}\). Since \(\sigma\) is an isomorphism, the minimal orders of the elements in the sets \(A_{i,j,\theta}\) and \(\sigma A_{i,j,\theta}\) coincide. 

\[\boxed{\text{Proof of the main result}}\]

In this section we assume that the elements \(u_{i,j,\theta}\) and \(w_{i,j,\theta}\) are as in Proposition 5.1.

We set

\[\epsilon_{i,j,\theta} = \{u_{i,j,\theta}, t\} + \{w_{i,j,\theta}, \pi\}.\]

We have \(\text{ord} \epsilon_{i,j,\theta} = p^{\lambda_{i,j}}\) for any \(i, j, \theta\).

**Theorem 6.1.** For any \(a \in U(1)K_2^{\text{top}} K\), there exist \(u \in V_k, \alpha \in \mathbb{Z}_p\), and \(\alpha_{i,j,\theta} \in \mathbb{Z}_p\) for \((i, j, \theta) \in I\) such that

\[a = \{u, t\} + \alpha + \sum_{(i,j,\theta) \in I} \alpha_{i,j,\theta} \epsilon_{i,j,\theta}.\]

Moreover, the coefficients \(\alpha\) and \(\alpha_{i,j,\theta}\) are determined uniquely modulo \(p^{\lambda_s}\) and \(p^{\lambda_{i,j}}\), respectively; the element \(u\) is determined uniquely.

Note that the function \(f(i, j, m)\) possesses the following properties.

**Lemma 6.2.** We have

1) \(f(i, j, 0) = (j, i)\) for any \(i, j\);
2) the function \(g(m) = f^{(2)}(i, j, m)\) in monotone increasing for any \(i\) and \(j\);
3) \(\text{sgn}(f^{(1)}(i, j, m)) = \text{sgn}(j)\).

**Lemma 6.3.** Assume that \(r > 0, d \geq 0, x, y \in U(r)^p\), and \(a = \{x, t\} + \{y, \pi\}\). Then there exist \(x', y' \in U(r + 1)^p\), \(x'', y'' \in U(1)^p\), \(c_{j,\theta} \in \mathbb{Z}_p\), such that

\[a = \{x' x''(x''')^p, t\} + \{y' y''(y''')^p, \pi\} + \sum_{j \in \mathbb{Z}, j \neq 0, p(r, j), \theta \in B} c_{j,\theta} \epsilon_{r,j,\theta}.\]

**Proof.** It suffices to prove the lemma in the case where \(d = 0\). Indeed, if \(\bar{x}, \bar{y} \in U(r)\) are such that \(x = \bar{x}^p\), \(y = \bar{y}^p\), and \(\bar{x}', \bar{x}'', \bar{x}''', \bar{y}', \bar{y}'', \bar{y}''', \bar{c}_{j,\theta}\) are elements satisfying the conditions of the lemma applied to \(\bar{x}, \bar{y}\), and \(d = 0\), then the elements \(x' = (\bar{x}')^p\), \(x'' = (\bar{x}'')^p\), \(x''' = (\bar{x}''')^p\), \(y' = (\bar{y}')^p\), \(y'' = (\bar{y}'')^p\), \(y''' = (\bar{y}''')^p\), and \(c_{j,\theta} \in \mathbb{Z}_p\) satisfy the conditions of the lemma for \(x\) and \(y\). In what follows we assume that \(d = 0\).

Let \(u^{(1)}_{i,j,\theta}, u^{(2)}_{i,j,\theta}, w^{(1)}_{i,j,\theta}, w^{(2)}_{i,j,\theta}\) be the same as in Proposition 5.1.

We shall assume that either \(x\) or \(y\) does not belong to \(U(r + 1)\) and denote by \(s_0\) a number satisfying \(x, y \in U(s_0, r)\) and \(s_0 \neq 0\). By Lemma 3.1 for any \(s \neq 0\) there exist unique \(i_s, j_s, m_s\) such that \(f(i_s, j_s, m_s) = (s, r)\), and, by Lemma 6.2 either \((i_s, j_s) = (r, s)\) or \(m_s > 0\). This implies that if \(m_s = 0\) and \(p \mid r\), then \(p \nmid s\). For \(s \geq s_0\) we construct sequences

\[x'_{s}, y'_{s} \in U(s, r)\quad x'', y'' \in U(1),\]

\(c_{j,\theta, s} \in \mathbb{Z}_p, a_{\theta, s}, b_{\theta, s} \in \{0, 1, \ldots, p - 1\}\), and for \(s \geq s_0, s \neq 0, m_s = 0\), sequences \(\gamma_{\theta, s} \in \mathbb{Z}_p\) satisfying the following conditions:

\[a = \{x' x''(x''')^p, t\} + \{y' y''(y''')^p, \pi\} + \sum_{j \in \mathbb{Z}, j \neq 0, p(r, j), \theta \in B} c_{j,\theta, s} \epsilon_{r,j,\theta};\]

\[\gamma_{\theta, s} = \begin{cases} a_{\theta, s} - \frac{z}{\pi} b_{\theta, s} & \text{if } p \nmid r, \\ b_{\theta, s} - \frac{z}{\pi} a_{\theta, s} & \text{if } p \mid r; \end{cases}\]
\[ x_{s+1} = x_s' \prod_{\theta \in \mathfrak{B}} (1 + [\theta] \pi s t^s) - a_{\theta,s} p^m x_s, \text{ where} \]

\[ x_s' = \begin{cases} 
1 & \text{if } m_s > 0, \\
\prod_{\theta \in \mathfrak{B}} (1 + [\theta] \pi r t^s)^{\gamma_{\theta,s}} (w_{r,s,\theta}^{(1)})^{\gamma_{\theta,s}} & \text{if } m_s = 0, \ p \nmid r, \\
\prod_{\theta \in \mathfrak{B}} (u_{r,s,\theta}^{(1)})^{\gamma_{\theta,s}} & \text{if } m_s = 0, \ p \mid r,
\end{cases} \]

\[ y_{s+1} = y_s' \prod_{\theta \in \mathfrak{B}} (1 + [\theta] \pi s t^s) - b_{\theta,s} p^m y_s, \text{ where} \]

\[ y_s' = \begin{cases} 
1 & \text{if } m_s > 0, \\
\prod_{\theta \in \mathfrak{B}} (w_{r,s,\theta}^{(1)})^{\gamma_{\theta,s}} & \text{if } m_s = 0, \ p \nmid r, \\
\prod_{\theta \in \mathfrak{B}} (1 + [\theta] \pi r t^s)^{\gamma_{\theta,s}} (w_{r,s,\theta}^{(1)})^{\gamma_{\theta,s}} & \text{if } m_s = 0, \ p \mid r', \\
\end{cases} \]

\[ x_{s+1}'' = \begin{cases} 
x_s'' \prod_{\theta \in \mathfrak{B}} (1 + [\theta] \pi s t^s) a_{\theta,s} p^{m_s-1} & \text{if } m_s > 0, \\
x_s'' \prod_{\theta \in \mathfrak{B}} (u_{r,s,\theta}^{(2)})^{\gamma_{\theta,s}} & \text{if } m_s = 0;
\end{cases} \]

\[ y_{s+1}'' = \begin{cases} 
y_s'' \prod_{\theta \in \mathfrak{B}} (1 + [\theta] \pi s t^s) b_{\theta,s} p^{m_s-1} & \text{if } m_s > 0, \\
y_s'' \prod_{\theta \in \mathfrak{B}} (w_{r,s,\theta}^{(2)})^{\gamma_{\theta,s}} & \text{if } m_s = 0,
\end{cases} \]

\[ c_{\theta,s} = c_{\theta,s} + \gamma_{\theta,s} \text{ for } s \neq 0, \ m_s = 0; \]
\[ c_{j,\theta,s+1} = c_{j,\theta,s} \text{ for } j \neq s, \text{ for } s = 0, \text{ and for } m_s > 0; \]
\[ x_s'' = y_s'' = 1 \text{ for } s < 0, \text{ and } x_s'' + x_s' = y_s'' \text{ for } s > 0; \]
\[ x_0' = x_1' x_1'', \ y_0' = y_1' y_1'', \ x_1'' = x_0'' + y_1'' \text{ if } s_0 < 0. \]

We set \( x_{s_0}' = x, \ y_{s_0}' = y, \ x_{s_0}'' = x_0'', \ y_{s_0}'' = y_0'', \) and \( c_{j,\theta,s_0} = 0 \) for any \( j \) and \( \theta. \)

Suppose the \( \theta \)th elements of the sequences have already been constructed. We shall construct the \((s + 1)\)st elements.

For \( s = 0 \), the existence of \( x_1', x_1'', y_1', y_1'' \) follows from the fact that \( U(0,r)/U(1,r) \) is generated by elements of \( V_k. \)

Let \( s \neq 0 \). By Lemma \( 3.1 \) there exist \( a_{\theta,s}, b_{\theta,s} \in \{0, 1, \ldots, p - 1\} \) such that

\[ x_s' \equiv \prod_{\theta \in \mathfrak{B}} (1 + [\theta] \pi s t^s) a_{\theta,s} p^m x_s \mod \pi r t^s \mathfrak{M}, \]

\[ y_s' \equiv \prod_{\theta \in \mathfrak{B}} (1 + [\theta] \pi s t^s) b_{\theta,s} p^m y_s \mod \pi r t^s \mathfrak{M}. \]

We define the \((s + 1)\)st elements by formulas \( (7), (8), (9), (10), \) and \( (11) \). Then, these elements satisfy \( (5) \) and \( (6) \); this follows from the corresponding conditions for the \( \theta \)th elements, the fact that \( x_s', y_s' \in U(s + 1, r) \), and the relations

\[ s \{1 + [\theta] \pi r t^s, t\} + r \{1 + [\theta] \pi r t^s, \pi\} = 0, \]

multiplied by \( b_{\theta,s}/r \) if \( m_s = 0, \ p \nmid r \) and multiplied by \( a_{\theta,s}/s \) if \( m_s = 0, \ p \mid r. \)

Thus, sequences that satisfy the conditions described above exist.

The sequences \( x_s'' \) and \( y_s'' \) are constant for \( s > 0 \), and each of the sequences \( c_{j,\theta,s} \) is constant for \( s > j. \) We take the limits of these sequences for the role of \( x'' \), \( y'' \), and \( c_{j,\theta} \).

It remains to verify that the sum

\[ S = \sum_{j \in \mathbb{Z}, \ j \neq 0} c_{j,\theta \in \mathbb{B}}(r, \theta, \theta) \]
converges, and that the sequences $x', y', x'', y''$ also converge. The limits $x', y', x'', y''$ and $x'', y''$, $c_{i,j,\theta}$ of these sequences will satisfy the condition of the lemma.

We have $c_{j,\theta} = 0$ for $j \leq s_0$; therefore,

$$S = \prod_{s \geq s_0} (u_{r,s,\theta})^{\gamma_{s,\theta}, t} + \prod_{s \geq s_0} (w_{r,s,\theta})^{\gamma_{s,\theta}, \pi}.$$

Thus, it suffices to verify the convergence of the products

$$(12) \prod_{s,\theta} u_{r,s,\theta}^{\gamma_{s,\theta}, t}, \prod_{s,\theta} w_{r,s,\theta}^{\gamma_{s,\theta}, \pi}, \prod_{s,\theta} (u_{r,s,\theta})^{\gamma_{s,\theta}, t}, \prod_{s,\theta} (w_{r,s,\theta})^{\gamma_{s,\theta}, \pi}, \text{ where } l = 1, 2,$$

and the products

$$(13) \prod_{s,\theta} (1 + [\theta] \pi^t t^s)^{\gamma_{s,\theta}, t}, \prod_{s,\theta} (1 + [\theta] \pi^t t^s)^{\gamma_{s,\theta}, \pi}$$

for some $\gamma_{s,\theta}, \gamma_{s,\theta} \in \mathbb{Z}_p$; the products are taken over the sets of pairs $(s, \theta)$ in which $s$ ranges over a subset of $\mathbb{Z}$, $\theta$ over $\mathfrak{B}$. By Lemma 6.2, for any pair $(i, j)$ there exists at most one value of $s$ for which $(i, j) = (i_s, j_s)$, and if $s > 0$, then such $s$ exists only for $j > 0$. Consequently, the products in (12) can be written in the form

$$\prod_{(i,j,\theta) \in I} (1 + [\theta] \pi^t t_j)^{\gamma_{i,j,\theta}, \pi}, \gamma_{i,j,\theta} \in \mathbb{Z}_p,$$

and the set of $j < 0$ for which $\gamma_{i,j,\theta}$ are not all equal to 0 is finite. Thus, in each product we can remove finitely many factors, obtaining products over a subset of $I^+$. Such products and the products (12) converge by Lemma 6.2. □

**Lemma 6.4.** Suppose $d \geq 0$, $x, y \in U(1)^d$, and $a = \{x, t\} + \{y, \pi\}$. Then there exist elements $x', y' \in U(1)^{d+1}$, $x'', y'' \in V_k^d$, $c_{i,j,\theta} \in p^d \mathbb{Z}_p$, such that

$$a = \{x', y', t\} + \{y', y'', \pi\} + \sum_{(i,j,\theta) \in I} c_{i,j,\theta} \varepsilon_{i,j,\theta}.$$

**Proof.** We set $x'_0 = x$, $y'_0 = y$. By Lemma 6.3 we can construct sequences

$$x'_s, y'_s \in U(s + 1)^d, \quad x''_s, y''_s \in V_k^d, \quad x''_s, y''_s \in U(1)^p, \quad c_{i,j,\theta, s} \in p^d \mathbb{Z}_p,$$

such that

$$\{x'_s, t\} + \{y'_s, \pi\} = \{x'_{s+1}x''_{s+1}(x''_{s+1})^p, t\} + \{y_{s+1}y''_{s+1}(y''_{s+1})^p, \pi\} + \sum_{(i,j,\theta) \in I} c_{i,j,\theta, s} \varepsilon_{i,j,\theta}.$$

Let $r = \frac{p^d}{p-1}$. Then $U(r + 1) \subset U(1)^p$, and the elements

$$x' = x'_r \prod_{s=1}^r (x''_s)^p, \quad y' = y'_r \prod_{s=1}^r (y''_s)^p, \quad x'' = \prod_{s=1}^r x''_s, \quad y'' = \prod_{s=1}^r y''_s$$

satisfy the conditions of the lemma. □

**Lemma 6.5.** Let $c, c_{i,j,\theta} \in \mathbb{Z}_p$, $x \in V_k$, $a \in U(1)K_{2\text{top}}^d K$, $r \in \mathbb{N}_0$ be such that

$$a = \{x, t\} + c \varepsilon + \sum_{(i,j,\theta) \in I} c_{i,j,\theta} \varepsilon_{i,j,\theta},$$

and $p^r a = 0$. Then $x'^r = 1$.

In particular, if $a = 0$, then $x = 1$. 

Proof. Let $n$ be a number such that $k$ contains a primitive $p^n$th root of unity, but it does not contain a primitive $p^{n+1}$th root of unity.

Assume that $x \neq 1$. Then if for some $m$ we have $p^m \mid c$, $p^m \mid c_{i,j,\theta}$ for $(i,j,\theta) \in I$, and $x = x'^p$ with $x' \in V_k$, then we can replace $c, c_{i,j,\theta}, x, r$ by $p^{-m} c, p^{-m} c_{i,j,\theta}, x', r + m$. Suppose that either $p \nmid c$, or $p \nmid c_{i,j,\theta}$ for some $i, j, \theta$, or $x \notin V_k^p$. Let $c_{1,0,\theta}$ and $c_t$ be such that

$$x = \prod_{0 < i < \frac{m}{p^t}, \theta \in \{0, 1\}} (1 + [\theta] \pi^i)^{c_{1,0,\theta}} (1 + [\theta \pi](\zeta_p - 1))^{c_t}.$$

If $p \nmid c_{i,j,\theta}$ for all $i, j, \theta$, then

$$a \equiv c_t \{1 + [\theta] \pi^i t^i, t\} + c\{1 + [\theta \pi](\zeta_p - 1)^p, \pi\} \mod pK_2^{\top} K,$$

and $p \nmid (c_t, c)$. Otherwise, for

$$(j_0, i_0) = \min \{(j, i) \mid 0 < (j, i) < \frac{p}{p - 1} \epsilon, p \nmid (i, j), p \nmid c_{i,j,\theta} \text{ for some } \theta\}$$

we have $\eta \neq 0$ and

$$a \equiv \begin{cases} \{1 + [\eta] \pi^i t^i, t\} & \text{if } p \nmid i, \\ \{1 + [\eta] \pi^i t^i, t\} & \text{if } p \mid i \text{ mod } U(j + 1, i) K_2^{\top} K + pK_2^{\top} K. \end{cases}$$

In both cases, Proposition [4.11] implies that $p^{n-1} a \neq 0$. Thus, $r \geq n$.

The elements $\epsilon$ and $\xi_{i,j,\theta}$ belong to the torsion subgroup of $U(1)K_2^{\top} K$ by Corollary [2.2]. Therefore, $\{x, t\}$ belongs to the closure of the torsion subgroup in $U(1)K_2^{\top} K$. By [2] Theorem 9.2, this implies that $x$ lies in the torsion subgroup of $V_k$. Hence $x^{p^n} = 1$, and since $r \geq n$, we have $x^{p^n} = 1$. $\square$

Proof of Theorem [5.1] We prove that $a$ can be represented as a sum of the required form. By Lemma [2.3], there exist $x'_0, y'_0 \in U(1)$ such that $a = \{x'_0, t\} + \{y'_0, \pi\}$. Applying Lemma [6.1] we deduce that there exist

$$x'_s, y'_s \in U(1)^p, \quad x''_s, y''_s \in V_k^{p^r}, \quad c_{i,j,\theta,s} \in p^s \mathbb{Z}_p$$

such that, for any $r > 0$,

$$a = \left\{ x'_r \prod_{0 \leq s < r} x''_s, t \right\} + \left\{ y'_r \prod_{0 \leq s < r} y''_s, \pi \right\} + \sum_{0 \leq s < r} \left( \sum_{(i,j,\theta) \in I} c_{i,j,\theta,s} \xi_{i,j,\theta} \right).$$

We have $x'_s, x''_s, y'_s, y''_s \to 1$, $c_{i,j,\theta,s} \to 0$ as $s \to \infty$. Consequently, the products and sums

$$\prod_{s \geq 0} x''_s, \quad \prod_{s \geq 0} y''_s, \quad \sum_{s \geq 0} c_{i,j,\theta,s}$$

converge to certain $u, w \in V_k$ and $\alpha_{i,j,\theta} \in \mathbb{Z}_p$. The sum

$$\alpha_{i,j,\theta} \xi_{i,j,\theta} = \sum_{s \geq 0} \left( \sum_{(i,j,\theta) \in I} c_{i,j,\theta,s} \xi_{i,j,\theta} \right)$$

also converges, because for any $s$ we have

$$\sum_{(i,j,\theta) \in I} c_{i,j,\theta,s} \xi_{i,j,\theta} \in p^s U(1) K_2^{\top} K.$$
Also,

\[ w = (1 + [\theta_p](\zeta_p - 1))^{\alpha} \prod_{p^i, \theta \in \mathbb{G}} (1 + [\theta]^{\pi^j})^{\beta_i,\theta} \]

for some \( \alpha, \beta_i,\theta \in \mathbb{Z}_p \), and \( \{1 + [\theta]^{\pi^j}, \pi\} = 0 \) for \( p \nmid i, \theta \in K^{(0)} \); therefore

\[ \{w, \pi\} = \alpha\{1 + [\theta_p](\zeta_p - 1)^p, \pi\} = \alpha\varepsilon. \]

The elements \( u, \alpha, \alpha_{i,j,\theta} \) satisfy the required relation. Now we prove the independence of the generators, i.e., we verify that if \( a = 0 \), then \( u = 1, p^{\lambda_i} | \alpha, \) and \( p^{\lambda_{i,j,\theta}} | c_{i,j,\theta} \) for any \( i, j, \theta \). The proof is similar to the proof of Proposition 5.1 in [2].

The relation \( u = 1 \) follows from Lemma 6.5.

We assume that either \( p^{\lambda_i} \nmid \alpha \) or \( \alpha = 0 \), and, similarly, for any \( i, j, \theta \), either \( p^{\lambda_{i,j,\theta}} \nmid \alpha_{i,j,\theta} \) or \( \alpha_{i,j,\theta} = 0 \).

Suppose the numbers \( \alpha, \alpha_{i,j,\theta} \) are not all equal to 0. We set

\[ r_1 = \min\{v_p(\alpha_{i,j,\theta}) \mid (i, j, \theta) \in I\}, \quad r = \min\{r_1, v_p(\alpha)\}, \]

\[ \beta = p^{-r}\alpha, \quad \beta_{i,j,\theta} = p^{-r}\alpha_{i,j,\theta}, \quad b = \beta\varepsilon + \sum_{(i,j,\theta) \in I} \beta_{i,j,\theta}\varepsilon_{i,j,\theta}. \]

We have \( p^rb = a = 0 \). Assume that \( r = r_1 \) and put

\[ (j_0, i_0) = \min \{(j, i) \mid v_p(\alpha_{i,j,\theta}) = r \text{ for some } \theta\}, \quad \eta = \sum_{\theta \in \mathbb{G}} \beta_{i_0,j_0,\theta}\theta. \]

Then \( \eta \neq 0 \) and \( b \in A_{i_0,j_0,\eta} \). Consequently, \( \text{ord } b \geq p^{\lambda_{i_0,j_0}} \). Since \( p^rb = 0 \), we conclude that \( r \geq \lambda_{i_0,j_0} \), but this contradicts the fact that either \( p^{\lambda_{i_0,j_0}} \nmid \alpha_{i_0,j_0,\theta} \) or \( \alpha_{i_0,j_0,\theta} = 0 \) for any \( \theta \).

We see that \( r < r_1 \) and, thus,

\[ b \equiv \beta\varepsilon \text{ mod } pU(1)K_{2}^{\text{top}}K, \quad p \nmid \beta. \]

By Corollary 5.3 we have \( \text{ord } b \geq p^{\lambda_i} \), but then \( r \geq \lambda_i \), which contradicts the fact that either \( p^{\lambda_i} \nmid \alpha \) or \( \alpha = 0 \).

\section{Example}

In this section we describe a collection of generators \( \varepsilon_{i,j,\theta} \) for a special type of fields, namely, for fields of the form \( k = Q_p(\sqrt[p^e]{\zeta^{p^n} - 1}) \), where \( p \nmid l \). We assume that \( \pi = \sqrt[p^e]{\zeta^{p^n} - 1} \) and this element is a uniformizer of the field \( k \). Note that in this field we have \( e = p^{n-1}(p-1), \frac{p^e}{p-1} = p^n l. \)

We set \( \text{Tor}(n) = \{x \in U(1)K_{2}^{\text{top}}K \mid p^n x = 0\} \).

For \( 0 < (j, i) < \frac{p}{p-1} \), \( j \neq 0 \), and \( p \nmid (i, j) \), let

\[ \mu_{i,j} = \begin{cases} v_p(j) + n + d & \text{if } \frac{1}{p^e} < i < \frac{1}{p^e e}, \ d \in \mathbb{N}, \\ v_p(j) + n + 1 & \text{if } i = l, \ j < 0, \\ v_p(j) + n & \text{if } (0, l) < (j, i) < (0, \frac{p^e}{p-1}). \end{cases} \]

We prove that the minimal order of the elements in the set \( A_{i,j,\theta} \) is equal to \( p^{\mu_{i,j}} \), and we describe the elements \( u_{i,j,\theta} \) and \( w_{i,j,\theta} \) satisfying the conditions of Proposition 5.1. Note that for this it suffices to check that the order of any element in \( A_{i,j,\theta} \) cannot be less than \( p^{\mu_{i,j}} \), and to find elements that satisfy the conditions of Proposition 5.1 for \( \lambda_{i,j} = \mu_{i,j} \); moreover, Lemmas 5.5 and 5.8 show that it suffices to present these elements only for \( j = \pm 1 \).
The generators to be constructed will satisfy the following additional conditions:

\[(14) \quad u_{i,j,\theta} \in U(i+1), \quad w_{i,j,\theta} \equiv 1 + [\theta] \pi^i t^j \mod \pi^{i+1}O \quad \text{for} \quad p \mid i;\]

\[w_{i,j,\theta} = 1 \quad \text{for} \quad p \nmid i;\]

\[w_{i,j,\theta} \equiv 1 + [\theta] \pi^i t^j \mod \pi^{i+1}O \quad \text{for} \quad p \nmid i, \quad i \neq l \quad \text{and for} \quad i = l, \quad j < 0.\]

A lower estimate for the orders of elements in \(A_{i,j,\theta}\) follows from the second assertion of Proposition 4.4 for \((j, i) > (0, l)\) and from the third assertion of Proposition 4.4 and Lemma 7.1 for \((j, i) < (0, l)\).

**Lemma 7.1.** Suppose \((0, l) < (j, i) < (0, p^n l), \ p \nmid (i, j), \ \text{and} \ \theta \in K(0). \ Then \ there \ exists \ \ u \in M_{\mu, i, j} \ \text{such that} \ \ u \equiv 1 + [\theta] \pi^{p^n l - i} t^{-j} \mod \pi^{p^n l - i} t^{-j} \mathfrak{M}.\]

**Proof.** See [3, Lemmas 4.3 and 4.4] for \(i > l\) and for \(i = l, j > 0\), respectively. \(\square\)

In the remaining part of the section, we shall construct generators of the required order.

We set

\[m_0 = \begin{cases} 2 & \text{if } n = 1, \\ (p+1)p^{n-2} + 1 & \text{if } n > 1. \end{cases}\]

**Lemma 7.2.** Suppose \(x \in K, \ v(x - 1) \geq m_0 l. \ Then \ \{x, \pi\} \in \text{Tor}(n).\]

**Proof.** We have \(\{x, \pi\} = t^{-1} \{x, \zeta_p^n - 1\}\); therefore, it suffices to prove that \(\{x, \zeta_p^n - 1\} \in \text{Tor}(n). \) For \(n = 1\), this follows from [2, Proposition 7.9].

Let \(n > 1\). By [3, Theorem 6.1], for \(K_0 = \mathbb{Q}_p(\zeta_p^n)\{\{t\}\}\) and \(\pi_0 = \zeta_p^n - 1\) we have

\[(15) \quad p^n \{1 + \pi_0^{m_0} t^{-1}, \pi_0\} = 0\]

in \(U_{K_0}(1)K_{2}^{\top}K_0\); therefore, this relation is valid also in \(U(1)K_{2}^{\top}K\). We represent \(x\) in the form

\[x = \prod_{j > 0, \ \theta \in K(0)} (1 + [\theta] \pi_0^{m_0} t^{-j} c_{j,\theta} (1 + \pi_0^{m_0} y),\]

where \(c_{j,\theta} \in \mathbb{Z}_p, \ \tilde{v}(y) \geq 0.\)

Since the homomorphism \(\tau_j\) induces an isomorphism \(K \to k\{\{t'\}\}, \) for any \(j > 0\) we have

\[p^n \{1 + [\theta] \pi_0^{m_0} t^{-j}, \pi_0\} = p^n \{\tau_j (1 + [\theta] \pi_0^{m_0} t^{-1}), \pi_0\} = 0.\]

We prove that \(\{1 + \pi_0^{m_0} y, \pi_0\} \in \text{Tor}(n). \) We set \(t' = \left(\frac{1}{1 + \pi_0^{m_0} y}\right)^{-1}\) and \(\tilde{v}(t') = (1, 0)\) and \(K = k\{\{t'\}\}. \) Consequently,

\[p^n \{1 + \pi_0^{m_0} t'^{-1}, \pi_0\} = 0.\]

Thus,

\[\{1 + \pi_0^{m_0} y, \pi_0\} = \left\{\frac{1 + \pi_0^{m_0} t^{-1}}{1 + \pi_0^{m_0} \cdot \frac{t^{-1} - y}{1 + \pi_0^{m_0} y}}, \pi_0\right\} = \left\{\frac{1 + \pi_0^{m_0} t^{-1}}{1 + \pi_0^{m_0} t'^{-1}}, \pi_0\right\} = \{1 + \pi_0^{m_0} t^{-1}, \pi_0\} - \{1 + \pi_0^{m_0} t'^{-1}, \pi_0\} \in \text{Tor}(n).\] \(\square\)

**Corollary 7.3.** For any \(x \in O\) and \(s \in \mathbb{N}, \) we have

\[p^s \{1 + x, \pi\} \equiv \{1 + x^{p^s}, \pi\} \mod \text{Tor}(n).\]

**Proof.** This follows from the inequality \(e \geq m_0 l.\) \(\square\)
7.1. Description of generators for $p \mid i$, $j = \pm 1$. Throughout this subsection, we assume that $p \mid i$ and $j = \pm 1$.

First, consider the case where $i > l$; then $\mu_{i,j} = n$. We prove that the elements

$$
u_{i,j,\theta} = \left(1 + \frac{[\theta]_i t^j j}{1 + [\theta]_i t^j t^j} \right)^{-j/l} (1 + [\theta]_i t^j t^j)^{j/l},$$

$$w_{i,j,\theta} = \left(1 + \frac{[\theta]_i t^j j}{1 + [\theta]_i t^j t^j} \right)^{(l-i)/l} (1 + [\theta]_i t^j t^j)^{j/l}$$

are as required. Since

$$j\{1 + [\theta]_i t^j t^j, t\} + i\{1 + [\theta]_i t^j t^j, \pi\} = 0,$$

we have

$$\{u_{i,j,\theta}, t\} + \{w_{i,j,\theta}, \pi\} = -\frac{j}{l} \left\{1 + \frac{[\theta]_i t^j j}{1 + [\theta]_i t^j t^j, t}\right\} + \frac{l - i}{l} \left\{1 + \frac{[\theta]_i t^j j}{1 + [\theta]_i t^j t^j, \pi}\right\}$$

$$= -\frac{1}{l} \left\{1 + \frac{[\theta]_i t^j j}{1 + [\theta]_i t^j t^j, \pi^i t^j}\right\}$$

$$= -\frac{1}{l} \left\{1 + \frac{[\theta]_i t^j j}{1 + [\theta]_i t^j t^j, [\theta]_i t^j t^j},\pi\right\},$$

and the right-hand side belongs to Tor($n$) by Lemma 2.4 applied to $x = [\theta]_i t^j t^j$.

Now we consider the case where $i < l$. Let $d$ be such that $\frac{1}{p^d} < i < \frac{1}{p^d-1}$; then $\mu_{i,j} = n + d$ and $d > 0$. We set

$$w_0 = 1 + \frac{[\theta]^{p^d}_i \pi^{p^d_i p^d_j}}{1 + [\theta]^{p^d}_i \pi^{p^d_i - l p^d_j} p^d_j}, \quad u_0 = \left(1 + \frac{[\theta]^{p^d}_i \pi^{p^d_i p^d_j}}{1 + [\theta]^{p^d}_i \pi^{p^d_i - l p^d_j} p^d_j} \right)^{j/(p^d_i - l)}.$$

Then $u_0 \in U(i + 1)$ and

$$\{w_0, \pi\} + p^d \{u_0, t\} \in \text{Tor}(n),$$

because

$$(p^d_i - l)\{w_0, \pi\} + p^d \{u_0, t\} = \left\{1 + \frac{[\theta]^{p^d}_i \pi^{p^d_i p^d_j}}{1 + [\theta]^{p^d}_i \pi^{p^d_i - l p^d_j} p^d_j}, \pi^{p^d_i - l p^d_j}\right\},$$

and the right-hand side belongs to Tor($n$) by Lemma 2.4.

The element $w_0$ can be represented in the form

$$w_0 = w_1 (1 + [\theta]^{p^d}_i \pi^{p^d_i p^d_j}) \prod_{(r,s,n) \in J} (1 + [\eta]^{p^r} \pi^{p^r s})^{c_{r,s,n}},$$

where $J$ is a finite subset of $I^+$ if $j = 1$ and $J$ is a finite subset of $I^-$ if $j = -1$, and

$$(1 + [\eta]^{p^r} \pi^{p^r s})^{c_{r,s,n}} \in U(p^d_i + 1)$$

for any $r, s, \eta$ and $w_1 \in U(m_0l).

By Corollary 7.3 we have

$$(16) \quad \{1 + [\theta]^{p^d}_i \pi^{p^d_i p^d_j}, \pi\} \equiv p^d \{1 + [\theta]_i t^j, \pi\} \mod \text{Tor}(n).$$

For any $r, s, \eta$ we find elements $u'_{r,s,n}, u'_{r,s,n} \in U(i + 1) \cap T^*$, where $T^*$ is equal to $T^+$ or $T^-$ (depending on the sign of $j$) that satisfy the congruences

$$(17) \quad \{1 + [\eta]^{p^r} \pi^{p^r s})^{c_{r,s,n}}, \pi\} \equiv p^d \{w'_{r,s,n}, \pi\} + \{u'_{r,s,n}, t\} \mod \text{Tor}(n).$$

We fix $r, s, \eta$ and set $c = c_{r,s,n}$ and $b = v_p(c)$. We have

$$(18) \quad r p^{b-d} > i$$
because
\[ p^d i < v((1 + [\eta]^{p} t^{d_s})^c - 1) = v((1 + [\eta]^{p} t^{d_s})^{p_b} - 1) \leq v((\eta^{p} t^{d_s})^{p_b}) = r p^b. \]

Case 1: \( p^d \mid cr \). In this case,
\[ \{ (1 + [\eta]^{r} t^{d_s})^c, \pi \} \equiv p^d \{ (1 + [\eta^{p_b} t^{d_s}]^{p_b})^c, \pi \} \mod \text{Tor}(n), \]
where \( c' = p^{-b} c \), and, by (13), we have
\[ 1 + [\eta^{p_b}]^{r} t^{d_s} \subset U(rp^{b - d}) \subset U(i + 1). \]

Thus, we can take
\[ w'_{r, s, n} = (1 + [\eta^{p_b}]^{r} t^{d_s})^c, \quad u'_{r, s, n} = 1. \]

Case 2: \( p^d \nmid cr \). Let \( u \) be such that \( p^u \mid cr \) and \( p^{u+1} \nmid cr \); then \( 0 \leq u \leq d - 1 \). We set
\[ c'' = -\frac{sc}{rp^{b - \mu}}. \]

We have \( v_p(rp^{b - \mu}) = 0, c'' \in \mathbb{Z}_p \), and
\[ \{ (1 + [\eta]^{r} t^{d_s})^c, \pi \} \equiv p^u \{ (1 + [\eta^{p_b}]^{r} t^{d_s})^{p_b}, \pi \} \equiv -p^u \{ [\eta^{p_b}]^{r} t^{d_s}, \pi \} \equiv p^d \{ (1 + [\eta^{p_b}]^{r} t^{d_s})^{c''}, t \} \mod \text{Tor}(n). \]

From (13) it follows that
\[ 1 + [\eta^{p_b}]^{r} t^{d_s} \subset U(rp^{b - \mu}) \subset U(rp^{b - d}) \subset U(i + 1). \]

Consequently, in this case we can take
\[ w'_{r, s, n} = 1, \quad u'_{r, s, n} = (1 + [\eta^{p_b}]^{r} t^{d_s})^{c''}. \]

Summing congruences (16) and (17) and using the fact that \( \{ w_1, \pi \} \in \text{Tor}(n) \) by Lemma 7.2, we obtain
\[ p^d \left( \left\{ (1 + [\theta]^{t^i j}) \prod_{(r, s, n) \in J} w'_{r, s, n}, \prod_{(r, s, n) \in J} u'_{r, s, n}, t \right\} + \left\{ u_0 \prod_{(r, s, n) \in J} u'_{r, s, n}, t \right\} \right) \equiv \{ w_0, \pi \} + p^d \{ u_0, t \} \equiv 0 \mod \text{Tor}(n). \]

Thus, as generators we can take the elements
\[ w_{i, j, \theta} = (1 + [\theta]^{t^i j}) \prod_{(r, s, n) \in J} w'_{r, s, n}, \quad u_{i, j, \theta} = u_0 \prod_{(r, s, n) \in J} u'_{r, s, n}. \]

7.2. Replacement of \( \{ w, \pi \} \) by \( \{ u, t \} \). Suppose \( r \geq l \) and \( w \in U(r) \cap T^s \), where \( T^s \) is either \( T^+ \) or \( T^- \). We construct an element \( u \in U(r) \cap T^s \) such that
\[ \{ w, \pi \} \equiv \{ u, t \} \mod \text{Tor}(n). \]

A) First, we construct \( u' \in U(r) \cap T^s \) and \( w' \in U(r + 1) \cap T^s \) such that
\[ \{ w, \pi \} \equiv \{ u', t \} + \{ w', \pi \} \mod \text{Tor}(n). \]

Let \( u_{i, j, \theta} \) and \( w_{i, j, \theta} \) satisfy the conditions of Proposition 5.1 and condition (14) for \( p \mid i \); such elements were described in Subsection 7.1. The element \( w \) can be represented in the form
\[ w = w'' \prod_{(i, j, \theta) \in I} (1 + [\theta]^{t^i j})^{c_{i, j, \theta}}, \]
where \( w'' \in U(r + 1) \); moreover, Lemma 3.1 and the fact that \( T^+ \) and \( T^- \) are subgroups of \( U(1) \) show that \( c_{i,j,\theta} \) can be chosen in such a way that for \( c_{i,j,\theta} \neq 0 \) we have

\[
f^{(2)}(i, j, v_p(c_{i,j,\theta})) = r,
\]

all the factors belong to \( T^* \), and for \( T^* = T^- \) only finitely many \( c_{i,j,\theta} \) are different from 0. We set

\[
\tilde{w}_{i,j,\theta} = \begin{cases} 
1 + [\theta]^{i}t^{j} & \text{if } p \nmid i, \\
w_{i,j,\theta} & \text{if } p \mid i,
\end{cases}
\]

\[
\tilde{u}_{i,j,\theta} = \begin{cases} 
(1 + [\theta]^{i}t^{j})^{j/i} & \text{if } p \nmid i, \\
u_{i,j,\theta} & \text{if } p \mid i.
\end{cases}
\]

Since \( f^{(2)}(i, j, m) \) is monotone increasing as a function of \( i \) for fixed \( \text{sgn}(j) \) and \( m \), it follows that for \( p \mid i \) and for any \( m \) we have

\[
u_{i,j,\theta}^{p_m} \in U(f^{(2)}(i, j, m) + 1), \quad \tilde{w}_{i,j,\theta}^{p_m} \equiv (1 + [\theta]^{i}t^{j})^{p_m} \mod U(f^{(2)}(i, j, m) + 1),
\]

whence

\[
\tilde{u}_{i,j,\theta}^{c_{i,j,\theta}} \in U(r), \quad (1 + [\theta]^{i}t^{j})^{c_{i,j,\theta}} \equiv \tilde{w}_{i,j,\theta}^{c_{i,j,\theta}} \mod U(r + 1)
\]

for any \( i, j, \theta \). We set

\[
w' = w'' \prod_{(i,j,\theta) \in I} (1 + [\theta]^{i}t^{j})^{c_{i,j,\theta}} \quad u' = \prod_{(i,j,\theta) \in I} \tilde{w}_{i,j,\theta}^{c_{i,j,\theta}}.
\]

The convergence of the above products follows from Lemma 3.2 for \( T^* = T^+ \) and from the fact that only finitely many factors are different from 1 for \( T^* = T^- \). Since

\[
u' \in U(r) \cap T^*, \quad w' \in U(r + 1) \cap T^*
\]

it remains to verify that

\[
c_{i,j,\theta} \left( \{\tilde{u}_{i,j,\theta}, t\} + \{\tilde{w}_{i,j,\theta}, \pi\} \right) \in \text{Tor}(n)
\]

for any \( i, j, \theta \).

For \( p \nmid i \), we have \( \{\tilde{u}_{i,j,\theta}, t\} + \{\tilde{w}_{i,j,\theta}, \pi\} = 0 \).

Let \( p \mid i \). We set \( b_{i,j,\theta} = v_p(c_{i,j,\theta}) \) and \( c_{i,j,\theta}' = c_{i,j,\theta}p^{-b_{i,j,\theta}} \). Since

\[
f^{(2)}(i, j, b_{i,j,\theta}) = r \geq l,
\]

we have \( ip^{b_{i,j,\theta}} > l \). Consequently, \( b_{i,j,\theta} + n \geq \mu_{i,j} \) and

\[
p^{n}c_{i,j,\theta}(\{\tilde{u}_{i,j,\theta}, t\} + \{\tilde{w}_{i,j,\theta}, \pi\}) = c_{i,j,\theta}'p^{b_{i,j,\theta} + n}(\{u_{i,j,\theta}, t\} + \{w_{i,j,\theta}, \pi\}) = 0.
\]

B) With the help of (A), we can find sequences

\[
u_s, w_s \in U(s) \cap T^*
\]

such that \( w_r = w \) and

\[
\{w_s, \pi\} \equiv \{u_s, t\} + \{w_{s+1}, \pi\} \mod \text{Tor}(n)
\]

for any \( s \geq r \). Since \( \{w_{m_0}, \pi\} \in \text{Tor}(n) \) by Lemma 7.2, we conclude that for

\[
u = \prod_{r \leq s < m_0} u_s
\]

the congruence

\[
\{w, \pi\} \equiv \{u, t\} \mod \text{Tor}(n)
\]

is valid, and this element \( u \) satisfies the condition.
Remark 7.4. The condition $w \in T^*$ can be discarded, i.e., if $w \in U(r)$ with $r \geq l$, then there exists $u \in U(r)$ such that

$$\{w, \pi\} \equiv \{u, t\} \mod \text{Tor}(n).$$

Indeed, $w$ can be represented in the form $w = w_+ w_0 w_-$, where

$$w_+ \in U(r) \cap T^+, \quad w_- \in U(r) \cap T^-, \quad w_0 = (1 + [\theta_0](\zeta_p - 1)^p)^c.$$

By Lemma 7.2 we have $\{w_0, \pi\} \in \text{Tor}(n)$. Consequently, we can take $u = u_+ u_-$, where $u_+$ and $u_-$ correspond to $w_+$ and $w_-$. 

7.3. Description of generators for $p \nmid i$, $j = \pm 1$. Now we describe the generators of the form $\{u_{i,j,\theta}, t\}$.

In what follows we assume that $p \nmid i$ and $j = \pm 1$. Denote by $T^*$ the set $T^+$ or $T^-$, depending on the sign of $j$.

If $(i, j) = (l, 1)$, then we can take

$$u_{i,1,\theta} = 1 + \frac{[\theta] \pi^l t}{1 + [\theta] t},$$

because

$$\{1 + \frac{[\theta] \pi^l t}{1 + [\theta] t}, t\} \equiv \{1 + \frac{[\theta] \pi^l t}{1 + [\theta] t}, [\theta] t\} \mod \text{Tor}(n)$$

by Lemma 2.4.

Now consider the case where $i > l$. Let

$$x = \left(1 + \frac{[\theta] \pi^l \pi^j}{1 + [\theta] \pi^{i-1} t^j}\right) \left(1 + [\theta] \pi^i t^j\right)^{-1};$$

then $x \in U(i + 1) \cap T^*$. Using the algorithm described in Subsection 7.2, we construct an element $y \in U(i + 1) \cap T^*$ such that

$$\{x, \pi\} \equiv \{y, t\} \mod \text{Tor}(n).$$

We set

$$u_{i,j,\theta} = (1 + [\theta] \pi^i t^j) x^{i/l} y^{(i-1)/jl}$$

and prove that $u_{i,j,\theta}$ is the required element. We only need to verify that $\{u_{i,j,\theta}, t\} \in \text{Tor}(n)$. We have

$$\frac{j l}{i} \{1 + [\theta] \pi^i t^j, t\} = -\{1 + [\theta] \pi^i t^j, t^{(i-1)/l}\} + \{1 + [\theta] \pi^l, t^j\} \equiv \{1 + [\theta] \pi^i t^j, \pi^{i-1} t^j\} \mod \text{Tor}(n).$$

and

$$\frac{j l}{i} \{x^{i/l} y^{(i-1)/jl}, t\} \equiv j \{x, t\} + (i - l) \{x, \pi\} \equiv \left\{1 + \frac{[\theta] \pi^l \pi^j}{1 + [\theta] \pi^{i-1} t^j}\right\} \left(1 + [\theta] \pi^i t^j\right)^{-1} \pi^{i-1} t^j \mod \text{Tor}(n).$$

Therefore, by Lemma 2.4

$$\frac{j l}{i} \{u_{i,j,\theta}, t\} \equiv \left\{1 + \frac{[\theta] \pi^l t^j}{1 + [\theta] \pi^{i-1} t^j}, \pi^{i-1} t^j\right\} \equiv 0 \mod \text{Tor}(n).$$

It remains to consider the case where $(j, i) \in (0, l)$. Let $d = 1$ if $(i, j) = (l, -1)$, and let $d$ be such that $\frac{l}{p^d} < i < \frac{l}{p^{d-1}}$ if $i < l$. Then $d \geq 1$ and $\mu_{i,j} = n + d$. By Corollary 7.3 we have

$$p^d \{1 + [\theta] \pi^i t^j, \pi\} \equiv -\frac{i p^d}{j} \{1 + [\theta] \pi^i t^j, \pi\} \equiv -\frac{i}{j} \{1 + [\theta^d] \pi^{p^d} t^{p^d j}, \pi\} \mod \text{Tor}(n).$$
Applying the algorithm described in Subsection 7.2 to the field $k\{\{t^p\}\}$, we find an element

$$z \in U(p^d i) \cap T^* \subset U(i + 1) \cap T^*$$

such that

$$\{1 + [θ_i]_p t^i, t\} \equiv \{z, t^p\} \mod \text{Tor}(n).$$

Now we have

$$p^d \{(1 + [θ]_p i)z^{i/j}, t\} \in \text{Tor}(n),$$

and $u_{i,j,θ} = (1 + [θ]_p i)z^{i/j}$ is the required element.

References


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