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REALIZATION AND CHARACTERIZATION OF MODULUS OF SMOOTHNESS IN WEIGHTED LEBESGUE SPACES

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Abstract. A characterization is obtained for the modulus of smoothness in the Lebesgue spaces L^p_ω, $1 < p < \infty$, with weights ω satisfying the Muckenhoupt A_p condition. Also, a realization result and the equivalence between the modulus of smoothness and the Peetre K-functional are proved in L^p_ω for $1 < p < \infty$ and $\omega \in A_p$.

§1. Introduction

One of the main problems of the constructive function theory is finding a relationship between structural characteristics and differential properties of functions. For measuring the structural properties of a function, one of a tool is given by the modulus of smoothness. This concept has various applications in various branches of analysis, such as approximation theory, function spaces, and interpolation theory. The modulus of smoothness of fractional order $r \geq 0$,

$$\omega_r(f,\delta)_p := \sup_{0\leq h\leq\delta} \left\|(T_h - I)^r f\right\|_p, \quad f \in L^p,\ 1 \leq p \leq \infty,$$

was defined by P. L. Butzer, H. Dyckhoff, E. Görlich, R. L. Stens [7] and R. Taberski [30], where I is the identity operator and $T_h f(\,\cdot\,) := f(\,\cdot + h)$ is the translation operator. It is well known that there is a close relationship between the decrease order of the modulus of smoothness of a function f and certain approximation properties of this function. Some of the useful properties of the modulus of smoothness $\omega_r(\,\cdot\,,\delta)_p$ are valid when $\delta \searrow 0$ (near the origin). For $f \in L^p$, the order of decrease $\omega_r(f,\delta)_p \searrow 0$ can be described in terms of the function class Φ_r.

Definition 1. We say that a function φ belongs to the class Φ_r ($r \in \mathbb{R}$) if it satisfies the following conditions: (a) $\varphi(t)$ is nonnegative and bounded on $(0,\infty)$, (b) $\varphi(t) \to 0$ as $t \to 0$, (c) $\varphi(t)$ is nondecreasing, and (d) $\varphi(t)t^{-r}$ is nonincreasing.

The class Φ_r describes completely the class of all majorants for the modulus of smoothness $\omega_r(\,\cdot\,,\delta)_p$ in the space L^p, $1 \leq p \leq \infty$, namely, the following statement is true.

Theorem 1 (see [31]). *Suppose that $r > 0$ and $1 \leq p \leq \infty$.*

(a) *If $f \in L^p$, then there exists a function $\varphi \in \Phi_r$ such that*

$$\varphi(t) \approx \omega_r(f,t)_p$$

for all $t \in (0,\infty)$, where the equivalence constants depend only on r and p.

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(b) *If $\varphi \in \Phi_r$, then there exists $f \in L^p$ and a positive real number t_0 such that*

$$\omega_r(f,\delta)_p \approx \varphi(\delta)$$

for any $\delta \in (0,t_0]$, where the equivalence constants depend only on r and p.

Here and in what follows, $A \lesssim B$ means that there exists a constant c independent of the essential parameters and such that $A \le cB$. If $A \lesssim B$ and $B \lesssim A$ simultaneously, we write $A \approx B$.

For $\omega_r(\,\cdot\,,\delta)_p$, $r \in \mathbb{N}$, some results of the type (a) and (b) in Theorem 1 were obtained in [5, 24, 26]. For more information about the fractional order smoothness moduli, we refer the reader to the book [27] and the papers [6, 30, 31].

An important property of a modulus of smoothness is that $\omega_r(\,\cdot\,,\delta)_p$ is equivalent to the Peetre K-functional

$$K_r(f,t,L^p) := \inf_{g,g^{(r)}\in L^p} \left\{\|f-g\|_p + t^r\|g^{(r)}\|_p\right\}, \qquad t,r>0,\ \ f\in L^p,\ \ 1\le p\le\infty. \tag{1.1}$$

The following equivalence was proved in [7]:

$$\omega_r(f,t)_p \approx K_r(f,t,p), \quad t>0,$$

for $r>0$, $f\in L^p$, $1\le p\le\infty$. In the case where $0<p<1$, the K-functional is equal to 0 identically (see, e.g., [10]). In this case, the realization functional

$$R_r(f,1/n,L^p) := \|f-t_n^*\|_p + \frac{1}{n^r}\|(t_n^*)^{(r)}\|_p, \qquad r>0,\ \ f\in L^p,\ \ 0<p\le\infty,\ \ n\in\mathbb{N},$$

can be used. Here and in what follows, by t_n^* we denote the best (or near-best) trigonometric polynomial approximating f, i.e., $\|f-T\|_{p,\omega} = E_n(f)_{p,\omega} := \inf_{T\in\mathcal{T}_n}\|f-T\|_{p,\omega}$ (respectively, $\|f-T\|_{p,\omega} \lesssim E_n(f)_{p,\omega}$).

The realization result has a lot of applications ([9]). In particular, it is used to get Ul′yanov type inequalities (see, e.g., [11]).

Theorem 2. *If $r>0$, $0<p\le\infty$, $f\in L^p$, then*

$$\omega_r(f,1/n)_p \approx R_r(f,1/n,L^p)$$

for $n=1,2,3,\ldots$

Theorem 2 was proved, in an equivalent form, in [10], for an integral order modulus of smoothness. In the case of the fractional order, Theorem 2 was proved in [28, 7] ($1\le p\le\infty$) and [19] ($0<p\le\infty$).

Our main goal in this paper is to obtain an analog of Theorems 1 and 2 for the weighted Lebesgue spaces L^p_ω, where the weight ω belongs to the Muckenhoupt class A_p, ($1<p<\infty$). A 2π-periodic weight (i.e., a measurable and almost everywhere positive function) ω belongs to the Muckenhoupt class A_p, $1<p<\infty$, (see, e.g., [23]) if

$$\left(\frac{1}{|J|}\int_J \omega(x)\,dx\right)\left(\frac{1}{|J|}\int_J \omega^{\frac{1}{1-p}}(x)\,dx\right)^{p-1} \le c \tag{1.2}$$

with a finite constant c independent of J, where J is any subinterval of $\mathbb{T} := [0,2\pi]$ and $|J|$ denotes the length of J. The smallest constant c satisfying (1.2) is called the A_p constant of ω and is denoted by $A[p] := A[p](\omega)$. The Muckenhoupt weights play a special role in harmonic analysis, because these are precisely the weights for which some singular integrals and maximal operators are bounded in the weighted Lebesgue spaces. We refer to the monograph [13] for a complete account on the theory of Muckenhoupt weights. Throughout this paper, by c we denote positive constants; they may be different at different places.

The weighted Lebesgue space L^p_ω, $1 < p < \infty$, is formed by the measurable functions $f : \mathbb{T} \to \mathbb{R}$ with the norm $\|f\|_{p,\omega} := \left\{ \int_{\mathbb{T}} |f(x)|^p \omega(x)\, dx \right\}^{1/p} < \infty$.

If $\omega \in A_p$, $1 < p < \infty$, then there exists a real number $a > 1$ such that

$$L^\infty \hookrightarrow L^p_\omega \hookrightarrow L^a \tag{1.3}$$

with constants depending only on p and $A[p]$. The left-hand side of (1.3) is seen from (1.2), and the right-hand side of (1.3) was proved in [15, 21].

Since the weighted Lebesgue spaces L^p_ω are, in general, not translation invariant, we need a different definition of the modulus of smoothness in L^p_ω. For the case where $\omega \equiv 1$, there are many different types of moduli of smoothness (see, e.g., [8, 32, 33]). For the Muckenhoupt weights, in 1986 E. A. Gadjieva [12] continued investigations in [33] and defined the modulus of smoothness $\widetilde{\Omega}_r(f, \delta)_{p,\omega} := \sup_{0 \le h_i \le \delta} \left\| \prod_{i=1}^r (I - \sigma_{h_i}) f \right\|_{p,\omega}$ of order $r = 1, 2, 3, \ldots$ constructed with the help of the Steklov operator

$$\sigma_h f(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(t)\, dt, \quad x \in \mathbb{T}, \ \ h > 0. \tag{1.4}$$

For another approach, see [21, 22]. The Steklov mean satisfies the inequality $\sigma_h f(x) \le Mf(x)$ a.e. on $\mathbb{T}$, where M is the Hardy–Littlewood maximal function. It is well known that a necessary and sufficient condition for the boundedness of M in L^p_ω, $1 < p < \infty$ is $\omega \in A_p$. Hence, under the conditions $f \in L^p_\omega$ ($1 < p < \infty$, $\omega \in A_p$), we have

$$\|\sigma_h f\|_{p,\omega} \lesssim \|f\|_{p,\omega}, \tag{1.5}$$

with some constant independent of f and h. We note that for $\omega \equiv 1$ we have (1.5) for the range $1 \le p \le \infty$.

We set $B_h := (I - \sigma_h)$, $[x] := \max\{a \in \mathbb{N} : a \le x\}$, $\{x\} := x - [x]$, and $\prod_{i=k}^l B_{h_i} f := f$ when $l < k$. For $1 < p < \infty$, $\omega \in A_p$, and $f \in L^p_\omega$, we define

$$\Omega_r(f, \delta)_{p,\omega} := \sup_{0 \le h_i, t \le \delta} \left\| \prod_{i=1}^{[r]} B_{h_i} B_t^{\{r\}} f \right\|_{p,\omega}, \quad r \in \mathbb{N}, \tag{1.6}$$

where $\Omega_0(f, \delta)_{p,\omega} := \|f\|_{p,\omega}$. In case $\omega \equiv 1$ and $1 \le p \le \infty$, we write $\Omega_r(\,\cdot\,, \delta)_p := \Omega_r(\,\cdot\,, \delta)_{p,\omega}$.

Using the binomial expansion, for $t > 0$ and $r \in \mathbb{N}$ we get

$$B_t^r f(\,\cdot\,) = \sum_{k=0}^\infty (-1)^k [C]_k^r (\sigma_t^k f)(\,\cdot\,), \quad f \in L^p_\omega. \tag{1.7}$$

Here, as usual, $[C]_0^r := 1$ and $[C]_k^r := \prod_{j=1}^k \frac{r-j+1}{j}$ are the binomial coefficients, $\sigma_t^0 := I$, and $\sigma_t^i := \sigma_t(\sigma_t^{i-1})$ for $i = 1, 2, 3, \ldots$

From [27], we get $|[C]_k^r| \lesssim k^{-r-1}$ ($k \in \mathbb{N}$), whence

$$C_r := \sum_{k=0}^\infty |[C]_k^r| < \infty. \tag{1.8}$$

Then, using (1.7) and (1.8), we have

$$\|B_t^r f\|_{p,\omega} \lesssim \|f\|_{p,\omega}, \quad r \in \mathbb{N}, \ \ t > 0, \tag{1.9}$$

for some constant depending only on r and $A[p]$. Hence, from (1.9), (1.5) and (1.6) we see that if $r \in \mathbb{N}$, $\delta \ge 0$, $1 < p < \infty$, $\omega \in A_p$, and $f \in L^p_\omega$, then

$$\Omega_r(f, \delta)_{p,\omega} \lesssim \|f\|_{p,\omega}, \tag{1.10}$$

with some constant depending only on r and $A[p]$. We note that for $\omega \equiv 1$ we have (1.10) for the range $1 \le p \le \infty$. Our main result is the following.

Theorem 3. *Suppose $r \in \mathbb{N}$, $1 < p < \infty$, and $\omega \in A_p$.*

(a) *If $f \in L^p_\omega$, then there exists a function $\varphi \in \Phi_{2r}$ such that*

$$\varphi(t) \approx \Omega_r(f, t)_{p,\omega}$$

for any $t \in (0, \infty)$, where the equivalence constants depend only on r and $A[p]$.

(b) *If $\varphi \in \Phi_{2r}$, then there exists $f \in L^p_\omega$ and a real number $t_0 > 0$ such that*

$$\Omega_r(f, \delta)_{p,\omega} \approx \varphi(\delta), \quad \delta \in (0, t_0],$$

and

$$\Omega_r(f, \delta)_{p,\omega} \approx \varphi(t_0), \quad \delta \in (t_0, 2\pi),$$

where the equivalence constants depend only on r and $A[p]$.

The following theorem includes the realization result and the equivalence of the modulus of smoothness (1.6) and the Peetre K-functional (1.1).

Theorem 4. *If $r \in \mathbb{N}$, $f \in L^p_\omega$, $1 < p < \infty$, and $\omega \in A_p$, then*

$$\Omega_r(f, 1/n)_{p,\omega} \approx R_{2r}(f, 1/n, L^p_\omega) \tag{1.11}$$

for $n = 1, 2, 3, \dots$, where the equivalence constants depend only on r and $A[p]$. Furthermore, we have

$$\Omega_r(f, \delta)_{p,\omega} \approx K_{2r}(f, \delta, L^p_\omega), \quad \delta \ge 0, \tag{1.12}$$

where the equivalence constants depend only on r and $A[p]$.

The equivalence (1.12) implies the following result.

Corollary 1. (a) *If $r \in \mathbb{N}$, $f \in L^p_\omega$, $1 < p < \infty$, and $\omega \in A_p$, then*

$$\Omega_r(f, \lambda\delta)_{p,\omega} \lesssim (1 + [\lambda])^{2r} \Omega_r(f, \delta)_{p,\omega}, \quad \delta, \lambda > 0,$$

and

$$\Omega_r(f, \delta)_{p,\omega} \delta^{-2r} \lesssim \Omega_r(f, \delta_1)_{p,\omega} \delta_1^{-2r}, \quad 0 < \delta_1 \le \delta.$$

(b) *We have*

$$\Omega_r(f, \cdot)_p \approx \omega_{2r}(f, \cdot)_p \quad \text{for} \quad 1 < p < \infty \quad \text{and} \quad f \in L^p;$$

therefore, Theorems 3 *and* 4 *imply Theorems* 1 *and* 2 (*with* $2r$).

The rest of the paper is organized as follows. In §2, some polynomial inequalities required for the realization theorem are obtained. Also, some estimates are obtained for the characterization theorem. In §3 we give the proof of the main results.

§2. Polynomial inequalities

Before stating the polynomial inequalities, we give some preliminary explanations. Let $\mathcal{T}_n$ denote the class of trigonometrical polynomials of degree at most n. We take a trigonometric polynomial $T \in \mathcal{T}_n$,

$$T(x) = \sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx) = \sum_{k=0}^{n} A_k(x, T), \quad a_k \in \mathbb{R} \ (k = 0, 1, \dots)$$

and we define its conjugate $\widetilde{T}$ by

$$\widetilde{T}(x) = \sum_{k=1}^{n} (a_k \sin kx - b_k \cos kx) =: \sum_{k=1}^{n} A_k(x, \widetilde{T}).$$

We define the *derivative* $T^{(\beta)}$ *of fractional order* $\beta > 0$ *for* the polynomial $T \in \mathcal{T}_n$ as

$$T^{(\beta)}(\,\cdot\,) := \sum_{k=0}^{n} k^{\beta} A_k(\,\cdot\, + \frac{\beta\pi}{2k}, T) = \sum_{k=1}^{n} k^{\beta} A_k(\,\cdot\, + \frac{\beta\pi}{2k}, T) =: \sum_{k=1}^{n} A_k(\,\cdot\,, T^{(\beta)}).$$

We note that, by (1.3), $L^p_\omega \subset L^1$ for $1 < p < \infty$ and $\omega \in A_p$. Hence, given $f \in L^p_\omega$, $1 < p < \infty$, and $\omega \in A_p$, we can define the corresponding Fourier series

$$f(x) \sim \sum_{k=0}^{n} (a_k(f)\cos kx + b_k(f)\sin kx) := \sum_{k=0}^{\infty} A_k(x, f), \tag{2.1}$$

where

$$a_0(f) := (2\pi)^{-1} \int_{\mathbb{T}} f(x)\,dx, \quad a_k(f) := \pi^{-1} \int_{\mathbb{T}} f(x)\cos kx\,dx \quad (k = 1, 2, \dots),$$
$$b_k(f) = \pi^{-1} \int_{\mathbb{T}} f(x)\sin kx\,dx \quad (k = 1, 2, \dots).$$

Since the corresponding Fourier series (2.1) of $f \in L^p_\omega$ converges in the L^p_ω-norm for $1 < p < \infty$ and $\omega \in A_p$, we can write

$$f(x) \overset{p,\omega}{=} \sum_{k=0}^{\infty} A_k(x, f). \tag{2.2}$$

Using (2.2) and (1.4), we find

$$\sigma_h f(x) \overset{p,\omega}{=} \sum_{k=0}^{\infty} \frac{\sin kh}{kh} A_k(x, f) \tag{2.3}$$

with $(\sin 0)/0 := 1$, and hence, consecutively,

$$\sigma_h^2 f(x) \overset{p,\omega}{=} \sum_{k=0}^{\infty} \Big(\frac{\sin kh}{kh}\Big)^2 A_k(x, f), \quad \dots, \quad \sigma_h^k f(x) \overset{p,\omega}{=} \sum_{k=0}^{\infty} \Big(\frac{\sin kh}{kh}\Big)^k A_k(x, f).$$

From (1.7), (2.2), (2.3), (1.8) and the last identities we obtain

$$B_h^r f(x) \overset{p,\omega}{=} \sum_{k=0}^{\infty} \Big(1 - \frac{\sin kh}{kh}\Big)^r A_k(x, f).$$

In particular, if $f(x) = \cos nx$, for all $x \in \mathbb{T}$ and some $n \in \mathbb{N}$, then

$$B^r_{\frac{\pi}{n}} \cos nx \overset{p,\omega}{=} \Big(1 - \frac{\sin \pi}{\pi}\Big)^r \cos nx = \cos nx.$$

The following two lemmas are required for the realization result. We note that all lemmas of this section are new also in the case where $\omega \equiv 1$.

Lemma 1. *Suppose $r \in \mathbb{N}$, $1 < p < \infty$, $\omega \in A_p$, and $T_n \in \mathcal{T}_n$, $n = 1, 2, \dots$ Then*

$$\Omega_r\Big(T_n, \frac{1}{n}\Big)_{p,\omega} \lesssim \frac{1}{n^{2r}} \|T_n^{(2r)}\|_{p,\omega}$$

with some constant depending only on r, p, and $A[p]$.

Proof of Lemma 1. Setting

$$\Big(1 - \frac{\sin x}{x}\Big)_* := \begin{cases} 1 - \frac{\sin x}{x}, & x > 0, \\ 0, & x = 0, \end{cases}$$

we have

$$\Big(1 - \frac{\sin x}{x}\Big)_* \le x^2 \quad \text{for} \quad x \in \mathbb{R}^+ \cup \{0\}.$$

For $0 < t, h_i \leq \frac{1}{n}$, we have

$$\begin{aligned}
&\left\| \prod_{i=1}^{[r]} B_{h_i} B_t^{\{r\}} T_n \right\|_{p,\omega} \\
&= \left\| \sum_{k=0}^{n} \Big(1 - \frac{\sin kh_1}{kh_1}\Big)_* \dots \Big(1 - \frac{\sin kh_{[r]}}{kh_{[r]}}\Big)_* \Big(1 - \frac{\sin kt}{kt}\Big)_*^{\{r\}} A_k(x, T_n) \right\|_{p,\omega} \\
&= \left\| \sum_{k=1}^{n} \frac{\big(1 - \frac{\sin kh_1}{kh_1}\big)(kh_1)^2}{(kh_1)^2} \dots \frac{\big(1 - \frac{\sin kh_{[r]}}{kh_{[r]}}\big)(kh_{[r]})^2}{(kh_{[r]})^2} \left(\frac{\big(1 - \frac{\sin kt}{kt}\big)}{(kt)^2} \right)^{\{r\}} (kt)^{2\{r\}} A_k(x, T_n) \right\|_{p,\omega} \\
&\leq n^{-2r} \left\| \sum_{k=1}^{n} \frac{\big(1 - \frac{\sin kh_1}{kh_1}\big)}{(kh_1)^2} k^2 \dots \frac{\big(1 - \frac{\sin kh_{[r]}}{kh_{[r]}}\big)}{(kh_{[r]})^2} k^2 \left(\frac{\big(1 - \frac{\sin kt}{kt}\big)}{(kt)^2} \right)^{\{r\}} k^{2\{r\}} A_k(x, T_n) \right\|_{p,\omega} \\
&\leq n^{-2r} \left\| \sum_{k=1}^{n} k^{2r} \frac{\big(1 - \frac{\sin kh_1}{kh_1}\big)}{(kh_1)^2} \dots \frac{\big(1 - \frac{\sin kh_{[r]}}{kh_{[r]}}\big)}{(kh_{[r]})^2} \left(\frac{\big(1 - \frac{\sin kt}{kt}\big)}{(kt)^2} \right)^{\{r\}} A_k(x, T_n) \right\|_{p,\omega}.
\end{aligned}$$

Using the Marcinkiewicz multiplier theorem for Lebesgue spaces with Muckenhoupt weight (see, e.g,, [20]), we get

$$\left\| \prod_{i=1}^{[r]} B_{h_i} B_t^{\{r\}} T_n \right\|_{p,\omega} \lesssim n^{-2r} \left\| \sum_{k=1}^{n} k^{2r} A_k(x, T_n) \right\|_{p,\omega}.$$

For $k = 1, 2, 3, \dots$ we observe that

$$\begin{aligned}
A_k(x, T_n) &= a_k \cos k\Big(x + \frac{r\pi}{k} - \frac{r\pi}{k}\Big) + b_k \sin k\Big(x + \frac{r\pi}{k} - \frac{r\pi}{k}\Big) \\
&= \cos r\pi \Big[a_k \cos k\Big(x + \frac{r\pi}{k}\Big) + b_k \sin k\Big(x + \frac{r\pi}{k}\Big)\Big] \\
&\qquad + \sin r\pi \Big[a_k \sin k\Big(x + \frac{r\pi}{k}\Big) - b_k \cos k\Big(x + \frac{r\pi}{k}\Big)\Big] \\
&= A_k\Big(x + \frac{r\pi}{k}, T_n\Big) \cos r\pi + A_k\Big(x + \frac{r\pi}{k}, \widetilde{T_n}\Big) \sin r\pi,
\end{aligned}$$

whence

$$\begin{aligned}
\left\| \prod_{i=1}^{[r]} B_{h_i} B_t^{\{r\}} T_n \right\|_{p,\omega} &\lesssim \frac{1}{n^{2r}} \left\| \sum_{k=1}^{n} k^{2r} \Big[A_k\Big(x + \frac{r\pi}{k}, T_n\Big) \cos r\pi + A_k\Big(x + \frac{r\pi}{k}, \widetilde{T_n}\Big) \sin r\pi\Big] \right\|_{p,\omega} \\
&\lesssim n^{-2r} \left(\left\| \sum_{k=0}^{n} k^{2r} A_k\Big(x + \frac{r\pi}{k}, T_n\Big) \right\|_{p,\omega} + \left\| \sum_{k=1}^{n} k^{2r} A_k\Big(x + \frac{r\pi}{k}, \widetilde{T_n}\Big) \right\|_{p,\omega} \right).
\end{aligned}$$

Since

$$A_k(x, T_n^{(2r)}) = k^{2r} A_k(x + \frac{r\pi}{k}, T_n), \quad k = 1, 2, 3, \dots,$$

we obtain

$$\begin{aligned}
\Omega_r\Big(T_n, \frac{1}{n}\Big)_{p,\omega} &= \sup_{0 \leq t, h_i \leq 1/n} \left\| \prod_{i=1}^{[r]} B_{h_i} B_t^{\{r\}} T_n \right\|_{p,\omega} \\
&\lesssim n^{-2r} \left(\left\| \sum_{k=1}^{n} k^{2r} A_k\Big(x + \frac{r\pi}{k}, T_n\Big) \right\|_{p,\omega} + \left\| \sum_{k=1}^{n} k^{2r} A_k\Big(x + \frac{r\pi}{k}, \widetilde{T_n}\Big) \right\|_{p,\omega} \right) \\
&\lesssim n^{-2r} \big(\|T_n^{(2r)}\|_{p,\omega} + \|\widetilde{T_n}^{(2r)}\|_{p,\omega}\big) = n^{-2r} \big(\|T_n^{(2r)}\|_{p,\omega} + \|\widetilde{T_n^{(2r)}}\|_{p,\omega}\big) \\
&\lesssim n^{-2r} \|T_n^{(2r)}\|_{p,\omega},
\end{aligned}$$

* completing the proof. □

The following lemma is an improvement of the Bernstein inequality.

Lemma 2. *If $r \in \mathbb{N}$, $1 < p < \infty$, $\omega \in A_p$, and $T_n \in \mathcal{T}_n$, $n = 1, 2, \dots,$ then*

$$\frac{1}{n^{2r}} \|T_n^{(2r)}\|_{p,\omega} \lesssim \Omega_r\Big(T_n, \frac{1}{n}\Big)_{p,\omega}$$

with some constant depending only on r, p, and $A[p]$.

Proof of Lemma 2. Let

$$T(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx).$$

Then

$$\begin{aligned}
n^{-2r}\|T_n^{(2r)}\|_{p,\omega} &= n^{-2r}\Big\| \sum_{k=1}^{n} k^{2r} A_k\Big(x + \frac{r\pi}{k}, T_n\Big)\Big\|_{p,\omega} \\
&= n^{-2r}\Big\| \sum_{k=1}^{n} k^{2r} (\cos r\pi A_k(x, T_n) - \sin r\pi A_k(x, \widetilde{T_n}))\Big\|_{p,\omega} \\
&\le n^{-2r}\Big\| \sum_{k=1}^{n} k^{2r} \cos r\pi A_k(x, T_n)\Big\|_{p,\omega} + n^{-2r}\Big\| \sum_{k=1}^{n} k^{2r} \sin r\pi A_k(x, \widetilde{T_n})\Big\|_{p,\omega} \\
&= \Big\| \sum_{k=1}^{n} \cos r\pi \Bigg(\frac{\left(\frac{k}{n}\right)^2}{\Big(1 - \frac{\sin\frac{k}{n}}{\frac{k}{n}}\Big)}\Bigg)^r \Big(1 - \frac{\sin\frac{k}{n}}{\frac{k}{n}}\Big)^r A_k(x, T_n)\Big\|_{p,\omega} \\
&\qquad + \Big\| \sum_{k=1}^{n} \sin r\pi \Bigg(\frac{\left(\frac{k}{n}\right)^2}{\Big(1 - \frac{\sin\frac{k}{n}}{\frac{k}{n}}\Big)}\Bigg)^r \Big(1 - \frac{\sin\frac{k}{n}}{\frac{k}{n}}\Big)^r A_k(x, \widetilde{T_n})\Big\|_{p,\omega}.
\end{aligned}$$

Using the Marcinkiewicz multiplier theorem [20] for Lebesgue spaces with Muckenhoupt weight, we get

$$\begin{aligned}
n^{-2r}\|T_n^{(2r)}\|_{p,\omega} &\lesssim \Big\| \sum_{k=1}^{n} \Big(1 - \frac{\sin\frac{k}{n}}{\frac{k}{n}}\Big)^r A_k(x, T_n)\Big\|_{p,\omega} + \Big\| \sum_{k=1}^{n} \Big(1 - \frac{\sin\frac{k}{n}}{\frac{k}{n}}\Big)^r A_k(x, \widetilde{T_n})\Big\|_{p,\omega} \\
&= \Big\| \sum_{k=1}^{n} \Big(1 - \frac{\sin\frac{k}{n}}{\frac{k}{n}}\Big)^r A_k(x, T_n)\Big\|_{p,\omega} + \Big\| \Big(\sum_{k=1}^{n} \Big(1 - \frac{\sin\frac{k}{n}}{\frac{k}{n}}\Big)^r A_k(x, T_n)\Big)^{\sim}\Big\|_{p,\omega}.
\end{aligned}$$

At the last step, we have used the linearity of the conjugation operator. Thus, since the conjugation operator is bounded (see, e.g., [14]), we see that

$$\begin{aligned}
n^{-2r}\|T_n^{(2r)}\|_{p,\omega} &\lesssim \Big\| \sum_{k=1}^{n} \Big(1 - \frac{\sin\frac{k}{n}}{\frac{k}{n}}\Big)^r A_k(x, T_n)\Big\|_{p,\omega} + \Big\| \sum_{k=1}^{n} \Big(1 - \frac{\sin\frac{k}{n}}{\frac{k}{n}}\Big)^r A_k(x, T_n)\Big\|_{p,\omega} \\
&\lesssim \big\|B_{1/n}^{r} T_n\big\|_{p,\omega} = \big\|B_{1/n}^{[r]+\{r\}} T_n\big\|_{p,\omega} \\
&\lesssim \sup_{\substack{i=1,2,\dots,r \\ 0<h_i,u\le 1/n}} \Big\| \prod_{i=1}^{[r]} B_{h_i} B_u^{\{r\}} T_n\Big\|_{p,\omega} \lesssim \Omega_r(T_n, 1/n)_{p,\omega}.
\end{aligned}$$

Then

$$\frac{1}{n^{2r}} \big\|T_n^{(2r)}\big\|_{p,\omega} \lesssim \Omega_r(T_n, 1/n)_{p,\omega}$$

as required. □

For our characterization theorem we need the following lemma.

Lemma 3. *Suppose $r \in \mathbb{N}$ and $n \in \mathbb{N}$.*

(a) *If $f(x) = \cos x$, $1 < p < \infty$, and $\omega \in A_p$, then there exists $t_1 > 0$ such that*

$$\Omega_r(f, \delta)_{p,\omega} \approx \delta^{2r}$$

for all $\delta \in (0, t_1]$, where the constants depend only r and $A[p]$.

(b) *If $f(x) = \cos nx$ and $1 \leq p \leq \infty$, then for any $\delta \in (0, \pi/n)$ we have*

$$\Omega_r(f, \delta)_p \leq (2\pi)^{1/p} 6^{-2r} (n\delta)^{2r},$$

with $(1/\infty) := 0$.

(c) *If $f(x) = \cos nx$, then*

$$\|B^r_{\pi/n} f\|_1 = 4.$$

(d) *If $f(x) = \cos nx$, then, for any $\delta \in (0, \pi/n)$,*

$$\|B^r_\delta f\|_1 \geq 2^{r+2} 3^{-r} \pi^{-2r} (n\delta)^{2r}.$$

Proof of Lemma 3. (a) Let $f(x) = \cos x$ and $\delta \geq 0$. Then

$$\begin{aligned}
\Omega_r(f, \delta)_{p,\omega} &= \sup_{0 \leq u, h_i \leq \delta} \left\| \prod_{i=1}^{[r]} B_{h_i} B_u^{\{r\}} (\cos x) \right\|_{p,\omega} \\
&= \sup_{0 \leq u, h_i \leq \delta} \left\| \left(1 - \frac{\sin h_1}{h_1}\right) \dots \left(1 - \frac{\sin h_{[r]}}{h_{[r]}}\right) \left(1 - \frac{\sin u}{u}\right)^{\{r\}} \cos x \right\|_{p,\omega}.
\end{aligned}$$

Since

$$\left(1 - \frac{\sin x}{x}\right) \leq 6^{-1} x^2 \text{ for } x \in \mathbb{R}^+,$$

we have

$$\Omega_r(f, \delta)_{p,\omega} \leq \sup_{0 \leq u, h_i \leq \delta} 6^{-2r} h_1^2 \dots h_{[r]}^2 u^{2\{r\}} \|\cos x\|_{p,\omega} \lesssim 6^{-2r} \delta^{2r} \|\cos x\|_\infty \lesssim 6^{-2r} \delta^{2r}.$$

On the other hand, using the inequalities

$$\left(1 - \frac{\sin x}{x}\right) \geq (2/3) \sin^2(x/2), \quad x > 0,$$

and

$$\sin x \geq (2/\pi) x, \quad 0 \leq x \leq \pi/2,$$

we conclude that, for $0 \leq \delta \leq \pi/2$,

$$\begin{aligned}
&\sup_{0 \leq u, h_i \leq \delta} \left\| \left(1 - \frac{\sin h_1}{h_1}\right) \dots \left(1 - \frac{\sin h_{[r]}}{h_{[r]}}\right) \left(1 - \frac{\sin u}{u}\right)^{\{r\}} \cos x \right\|_{p,\omega} \\
&\qquad \geq \sup_{0 \leq h \leq \delta} \left\| \left(1 - \frac{\sin h}{h}\right)^r \cos x \right\|_{p,\omega} \geq \left\| \left(1 - \frac{\sin \delta}{\delta}\right)^r \cos x \right\|_{p,\omega} \\
&\qquad \geq (2/3)^r (2/\pi)^{2r} \delta^{2r} 2^{-2r} \|\cos x\|_{p,\omega} \gtrsim (1/6)^r (2/\pi)^{2r} \delta^{2r} \|\cos x\|_1 \\
&\qquad \gtrsim 4 \cdot (1/6)^r (2/\pi)^{2r} \delta^{2r}.
\end{aligned}$$

Now, taking $t_1 = \pi/2$, we complete the proof of (a).

(b) Suppose $f(x)=\cos nx$, $1\le p\le\infty$, $\delta\in(0,\pi/n)$ and $h_1,\dots,h_{[r]}$, $u\in[0,\delta]$. Then

$$\begin{aligned}\Omega_r(f,\delta)_p &= \sup_{0\le u,h_i\le\delta}\left\|\prod_{i=1}^{[r]} B_{h_i}B_u^{\{r\}}\cos nx\right\|_p\\ &= \sup_{0\le u,h_i\le\delta}\left\|\Big(1-\frac{\sin nh_1}{nh_1}\Big)\dots\Big(1-\frac{\sin nh_{[r]}}{nh_{[r]}}\Big)\Big(1-\frac{\sin nu}{nu}\Big)^{\{r\}}\cos nx\right\|_p\\ &\le \sup_{0\le u,h_i\le\delta} 6^{-2r}n^{2r}h_1^2\dots h_{[r]}^2u^{2\{r\}}\|\cos nx\|_p\\ &\le 6^{-2r}(n\delta)^{2r}\|1\|_p\le(2\pi)^{1/p}6^{-2r}\delta^{2r}.\end{aligned}$$

(c) Let $f(x)=\cos nx$. Then

$$\|B^r_{\pi/n}\cos nx\|_1=\left\|\Big(1-\frac{\sin\pi}{\pi}\Big)^r\cos nx\right\|_1=\|\cos nx\|_1=4.$$

(d) Let $f(x)=\cos nx$ and $\delta\in(0,\pi/n)$. Then

$$\begin{aligned}\|B^r_\delta f\|_1 &= \left\|\Big(1-\frac{\sin n\delta}{n\delta}\Big)^r\cos nx\right\|_1\\ &\ge(2/3)^r(2/\pi)^{2r}(n\delta)^{2r}2^{-2r}\|\cos nx\|_1=2^{r+2}3^{-r}\pi^{-2r}(n\delta)^{2r}.\end{aligned}$$ □

§3. Proof of Theorems 3 and 4

Proof of Theorem 3. Part (a). We define

$$\varphi(t):=t^{2r}\inf_{0<\zeta\le t}\frac{\Omega_r(f,\zeta)_{p,\omega}}{\zeta^{2r}},\quad t>0.$$

It is easily seen that the function φ has the following properties [29]: (i) $\varphi(t)$ is nonnegative and bounded on $(0,\infty)$, (ii) $\varphi(t)\to0$ as $t\to0$, (iii) $\varphi(t)$ is monotone nondecreasing, (iv) $\varphi(t)t^{-2r}$ is monotone nonincreasing. Hence, $\varphi\in\Phi_{2r}$. From the definition we have

$$\varphi(t)=t^{2r}\inf_{0<\zeta\le t}\frac{\Omega_r(f,\zeta)_{p,\omega}}{\zeta^{2r}}\le t^{2r}\frac{\Omega_r(f,t)_{p,\omega}}{t^{2r}}=\Omega_r(f,t)_{p,\omega}.$$

Also,

$$\Omega_r(f,t)_{p,\omega}=t^{2r}\frac{\Omega_r(f,t)_{p,\omega}}{t^{2r}}$$

and using the second inequality of Corollary 1 (a) and then taking the infimum, we get

$$\Omega_r(f,t)_{p,\omega}\lesssim t^{2r}\inf_{0<\zeta\le t}\frac{\Omega_r(f,\zeta)_{p,\omega}}{\zeta^{2r}}=c\varphi(t).$$

Part (b). Employing some ideas from [31], we consider the following two possibilities. The first one is (A) $\lim_{t\to0}(\varphi(t)/t^{2r})=C\in[0,\infty)$ and the second one is (B) $\lim_{t\to0}(\varphi(t)/t^{2r})=\infty$.

Consider case (A). Then, the definition of Φ_{2r} and the relation $\lim_{t\to0}(\varphi(t)/t^{2r})=C$ show that

$$\varphi(t)\lesssim t^{2r}\quad\text{for}\quad t\in(0,\pi), \tag{3.1}$$

so that there exists $t_0>0$ such that

$$\varphi(t)\ge Ct^{2r}\quad\text{for}\quad t\in(0,t_0]. \tag{3.2}$$

Now we put $f(x)=\cos x$. Then $f\in L^p_\omega$. Using, (a) of Lemma 3, we see that there is $t_1>0$ and constants depending only on r and $A[p]$ such that

$$\delta^{2r}\lesssim\Omega_r(f,\delta)_{p,\omega}\lesssim\delta^{2r}$$

for all $\delta \in (0, t_1]$. Hence, by (3.1) and (3.2),

$$\varphi(\delta) \lesssim \Omega_r(f, \delta)_{p,\omega} \lesssim \varphi(\delta)$$

for $\delta \in (0, t_2]$, where $t_2 := \min\{\pi, t_0, t_1\}$. This completes the analysis of case (A).

We consider case (B). From $\lim_{t\to 0}(\varphi(t)/t^{2r}) = \infty$ we obtain $\lim_{t\to 0}(t^{2r}/\varphi(t)) = 0$. We take $a \geq 2$ and fix it. Later we shall state the exact conditions on a when necessary. Following the construction given in [25] and [31], it is possible to find a sequence $\{n_v\}_{v=1}^{\infty}$ such that $n_v = 2^{m_v}$, $m_1 = 2$ and

$$m_{v+1} := \min\left\{m \in \mathbb{N} : \ \max\left\{\frac{\varphi(2^{-m})}{\varphi(2^{-m_v})}, \frac{2^{2m_v r}\varphi(2^{-m_v})}{2^{2mr}\varphi(2^{-m})}\right\} \leq \frac{1}{a}\right\}.$$

From this construction we have $m_{v+1} > m_v$, $n_{v+1} \geq 2n_v$ $(v = 1, 2, \dots)$, and

$$\varphi\Big(\frac{1}{n_{v+1}}\Big) \leq \frac{1}{a}\varphi\Big(\frac{1}{n_v}\Big), \tag{3.3}$$

$$n_v^{2r}\varphi\Big(\frac{1}{n_v}\Big) \leq \frac{1}{a}n_{v+1}^{2r}\varphi\Big(\frac{1}{n_{v+1}}\Big). \tag{3.4}$$

We take $\chi = 2^l$, $l \in \mathbb{N}$, $\chi > 2\pi$, and fix χ. Hence,

$$\sum_{v=1}^{\infty}\varphi\Big(\frac{1}{n_v}\Big) \leq \varphi\Big(\frac{1}{n_1}\Big)\sum_{v=1}^{\infty}\Big(\frac{1}{a}\Big)^{v-1} < +\infty.$$

Define

$$f(x) := \sum_{v=1}^{\infty}\varphi\Big(\frac{1}{n_v}\Big)\cos(\chi n_v x).$$

Then $f \in L^p_\omega$ (in fact, $f \in L^\infty$). From (1.3) we get

$$\|f\|_{p,\omega} \lesssim \|f\|_\infty,$$

whence $\Omega_r(f, \delta)_{p,\omega} \lesssim \Omega_r(f, \delta)_\infty$. If we prove that $\Omega_r(f, \delta)_\infty \lesssim \varphi(\delta)$, we shall have the lower estimate of Theorem 3 (b). Let $\delta \in (0, 1/n_1]$. For any $h_i \in (0, 1/n_1]$ with $i = 1, 2, 3, \dots, [r] + 1$, there are natural numbers N_i such that $n_{N_i+1}^{-1} < h_i \leq n_{N_i}^{-1}$. We set $h_{\max} := \max\{h_i : i = 1, 2, 3, \dots, [r] + 1\}$. Then there is $i_0 \in \{1, 2, 3, \dots, [r] + 1\}$ such that $h_{\max} = h_{i_0}$. Also we set $N_{\max} := N_{i_0}$. Then

$$\begin{aligned}
&\|(I - \sigma_{h_1})\dots(I - \sigma_{h_{[r]}})(I - \sigma_{h_{[r]+1}})^{\{r\}}f\|_\infty \\
&\quad\leq \Big\|\sum_{v=1}^{N_{\max}}\varphi\Big(\frac{1}{n_v}\Big)\big((I - \sigma_{h_1})\dots(I - \sigma_{h_{[r]}})(I - \sigma_{h_{[r]+1}})^{\{r\}}\big)\cos(\chi n_v x)\Big\|_\infty \\
&\qquad+ \Big\|\sum_{v=N_{\max}+1}^{\infty}\varphi\Big(\frac{1}{n_v}\Big)\big((I - \sigma_{h_1})\dots(I - \sigma_{h_{[r]}})(I - \sigma_{h_{[r]+1}})^{\{r\}}\big)\cos(\chi n_v x)\Big\|_\infty \\
&\quad=: I_1 + I_2.
\end{aligned}$$

We estimate I_1. Using (b) of Lemma 3, (3.4), and (d) of the definition of Φ_r, we get

$$\begin{aligned}
I_1 &\leq \sum_{v=1}^{N_{\max}}\varphi\Big(\frac{1}{n_v}\Big)\big\|\big((I - \sigma_{h_1})\dots(I - \sigma_{h_{[r]}})(I - \sigma_{h_{[r]+1}})^{\{r\}}\big)\cos(\chi n_v x)\big\|_\infty \\
&\lesssim \sum_{v=1}^{N_{\max}}\varphi\Big(\frac{1}{n_v}\Big)(\chi n_v)^{2r}h_1^2\dots h_{[r]}^2 h_{[r]+1}^{2\{r\}} \lesssim (\chi h_{\max})^{2r}\sum_{v=1}^{N_{\max}}n_v^{2r}\varphi(\frac{1}{n_v}) \\
&\lesssim (\chi h_{\max})^{2r}n_{N_{\max}}^{2r}\varphi\Big(\frac{1}{n_{N_{\max}}}\Big)\sum_{v=1}^{N_{\max}}\Big(\frac{1}{a}\Big)^{N_{\max}-v} \lesssim (h_{\max}n_{N_{\max}})^{2r}\varphi\Big(\frac{1}{n_{N_{\max}}}\Big) \lesssim \varphi(h_{\max}).
\end{aligned}$$

We estimate I_2. Relations (1.10) and (3.3) yield

$$\begin{aligned} I_2 &\le \sum_{v=N_{\max}+1}^{\infty} \varphi\Big(\frac{1}{n_v}\Big) \Big\| \big((I-\sigma_{h_1})\dots(I-\sigma_{h_{[r]}})(I-\sigma_{h_{[r]+1}})^{\{r\}}\big)\cos(\chi n_v x)\Big\|_\infty \\ &\lesssim \sum_{v=N_{\max}+1}^{\infty} \varphi\Big(\frac{1}{n_v}\Big) \lesssim \varphi\Big(\frac{1}{n_{N_{\max}+1}}\Big) \sum_{v=N_{\max}+1}^{\infty} \Big(\frac{1}{a}\Big)^{v-N_{\max}-1} \\ &\lesssim \varphi\Big(\frac{1}{n_{N_{\max}+1}}\Big) \lesssim \varphi(h_{\max}). \end{aligned}$$

Now, for $i = 1, 2, 3, \dots, [r]+1$, we have $n_{N_i+1}^{-1} < h_i \le n_{N_i}^{-1}$ and

$$\Big\|(I-\sigma_{h_1})\dots(I-\sigma_{h_{[r]}})(I-\sigma_{h_{[r]+1}})^{\{r\}} f\Big\|_\infty \lesssim \varphi(h_{\max}).$$

Thus,

$$\Omega_r(f,\delta)_\infty = \sup_{\substack{i=1,\dots,[r]+1\\ 0\le h_i\le\delta}} \Big\|(I-\sigma_{h_1})\dots(I-\sigma_{h_{[r]}})(I-\sigma_{h_{[r]+1}})^{\{r\}} f\Big\|_\infty \lesssim \varphi(\delta).$$

From this we obtain $\Omega_r(f,\delta)_{p,\omega} \lesssim \varphi(\delta)$. To reverse the last inequality, we shall use the estimate $\|f\|_1 \lesssim \|f\|_{p,\omega}$ (see, e.g., (1.3)). We shall prove that

$$\Omega_r(f,\delta)_1 \gtrsim \varphi(\delta)$$

and this will imply the inequality $\Omega_r(f,\delta)_{p,\omega} \gtrsim \varphi(\delta)$. We choose an integer i such that $n_{i+1}^{-1} = 2^{-m_{i+1}} < \delta \le 2^{-m_i} = n_i^{-1}$. The definition of m_i shows that at least one of the following conditions is fulfilled:

$$2^{2r(m_{i+1}-1)}\varphi\Big(\frac{1}{2^{m_{i+1}-1}}\Big) < a2^{2rm_i}\varphi\Big(\frac{1}{2^{m_i}}\Big), \tag{3.5}$$

$$\varphi\Big(\frac{1}{2^{m_{i+1}-1}}\Big) > \frac{1}{a}\varphi\Big(\frac{1}{2^{m_i}}\Big). \tag{3.6}$$

For the case of (3.5), we decompose

$$f(x) = \sum_{v=1}^{i-1} \varphi\Big(\frac{1}{n_v}\Big)\cos(\chi n_v x) + \varphi\Big(\frac{1}{n_i}\Big)\cos(\chi n_i x) + \sum_{v=i+1}^{\infty} \varphi\Big(\frac{1}{n_v}\Big)\cos(\chi n_v x) =: f_1+f_2+f_3.$$

We estimate f_2. For $\frac{1}{\chi n_i} < \delta < \frac{\pi}{\chi n_i}$, we use statement (d) of Lemma 3 to get

$$\|(I-\sigma_\delta)^r f_2\|_1 = \varphi\Big(\frac{1}{n_i}\Big)\|(I-\sigma_\delta)^r\cos(\chi n_i x)\|_1 \ge \frac{2^{r+2}}{3^r\pi^{2r}}(\chi\delta)^{2r} n_i^{2r}\varphi\Big(\frac{1}{n_i}\Big).$$

We estimate f_1. For $\frac{1}{\chi n_i} < \delta < \frac{\pi}{\chi n_i}$, using (b) of Lemma 3 and (3.4), we obtain

$$\begin{aligned} \|(I-\sigma_\delta)^r f_1\|_1 &\le \sum_{v=1}^{i-1} \varphi\Big(\frac{1}{n_v}\Big)\|(I-\sigma_\delta)^r\cos(\chi n_v x)\|_1 \\ &\le \frac{4}{6^{2r}}(\chi\delta)^{2r}\sum_{v=1}^{i-1} n_v^{2r}\varphi\Big(\frac{1}{n_v}\Big) \le \frac{4}{6^{2r}}(\chi\delta)^{2r} n_{i-1}^{2r}\varphi\Big(\frac{1}{n_{i-1}}\Big)\sum_{v=1}^{i-1}\Big(\frac{1}{a}\Big)^{i-1-v} \\ &\le \frac{4\cdot 2}{6^{2r}}(\chi\delta)^{2r} n_{i-1}^{2r}\varphi\Big(\frac{1}{n_{i-1}}\Big) \le \frac{4\cdot 2}{6^{2r}a}(\chi\delta)^{2r} n_i^{2r}\varphi\Big(\frac{1}{n_i}\Big). \end{aligned}$$

We estimate f_3. For $\frac{1}{\chi n_i} < \delta < \frac{\pi}{\chi n_i}$, using the inequalities (1.10), $C_r \le 2^{[r]+1}$, (3.3), $1 < \delta \chi n_i$ and the identity $\|\cos(\chi n_v x)\|_1 = 4$, we obtain

$$\begin{aligned}\|(I-\sigma_\delta)^r f_3\|_1 &\le \sum_{v=i+1}^{\infty} \varphi\Big(\frac{1}{n_v}\Big)\|(I-\sigma_\delta)^r \cos(\chi n_v x)\|_1 \\ &\le 4\cdot 2^{[r]+1}\sum_{v=i+1}^{\infty}\varphi\Big(\frac{1}{n_v}\Big) \le 4\cdot 2^{[r]+1}\varphi\Big(\frac{1}{n_{i+1}}\Big)\sum_{v=i+1}^{\infty}\Big(\frac{1}{a}\Big)^{v-i-1} \\ &\le 2\cdot 4\cdot 2^{[r]+1}\varphi\Big(\frac{1}{n_{i+1}}\Big) \le \frac{2\cdot 4\cdot 2^{[r]+1}}{a}\varphi\Big(\frac{1}{n_i}\Big) \\ &\le \frac{2\cdot 4\cdot 2^{[r]+1}}{a}(\chi n_i\delta)^{2r}\varphi\Big(\frac{1}{n_i}\Big).\end{aligned}$$

Hence, for $\frac{1}{\chi n_i} < \delta < \frac{\pi}{\chi n_i}$ we have

$$\begin{aligned}\|(I-\sigma_\delta)^r f\|_1 &\ge \|(I-\sigma_\delta)^r f_2\|_1 - \|(I-\sigma_\delta)^r f_1\|_1 - \|(I-\sigma_\delta)^r f_3\|_1 \\ &\ge \varphi\Big(\frac{1}{n_i}\Big)(\chi n_i\delta)^{2r}\Big(\frac{2^{r+2}}{3^r\pi^{2r}} - \frac{4\cdot 2}{6^{2r}a} - \frac{2\cdot 4\cdot 2^{[r]+1}}{a}\Big) \\ &= \varphi\Big(\frac{1}{n_i}\Big)(\chi n_i\delta)^{2r}4\Big(\frac{2^r}{3^r\pi^{2r}} - \frac{1}{6^{2r}a} - \frac{2^{[r]+1}}{a}\Big).\end{aligned}$$

We choose a in such a way that $a \ge 2$ and

$$\frac{2^r}{3^r\pi^{2r}} - \frac{1}{a}\Big(\frac{1}{6^{2r}} + 2^{[r]+1}\Big) > 0, \quad 1 - \frac{1}{a}\Big(\frac{\pi^{2r+1}}{6^{2r}} + 2\cdot 2^{[r]+1}\Big) > 0.$$

Since $\frac{\pi}{\chi n_i} < \frac{1}{n_{i+1}} < \frac{1}{n_i}$, for $\frac{1}{\chi n_i} < \delta < \frac{\pi}{\chi n_i}$ we get

$$\|(I-\sigma_\delta)^r f\|_1 \gtrsim (\chi n_i\delta)^{2r}\varphi\Big(\frac{1}{n_i}\Big) \gtrsim \varphi(\delta).$$

Thus, for $0 < \delta < \frac{\pi}{\chi n_i}$,

$$\Omega_r(f,\delta)_1 \ge \sup_{0<h\le\delta}\|(I-\sigma_h)^r f\|_1 \ge \sup_{\chi^{-1}n_i^{-1}<h\le\delta}\|(I-\sigma_h)^r f\|_1 \ge \|(I-\sigma_\delta)^r f\|_1 \gtrsim \varphi(\delta).$$

If $\frac{\pi}{\chi n_i} \le \delta \le \frac{1}{n_i}$, then

$$\Omega_r(f,\delta)_1 \ge \Omega_r(f,\frac{\pi}{\chi n_i})_1.$$

If we prove that

$$\Omega_r\Big(f,\frac{\pi}{\chi n_i}\Big)_1 \gtrsim \varphi\Big(\frac{1}{n_i}\Big), \tag{3.7}$$

then this will complete the proof in the case where (3.5) is true:

$$\Omega_r(f,\delta)_1 \ge \Omega_r\Big(f,\frac{\pi}{\chi n_i}\Big)_1 \gtrsim \varphi\Big(\frac{1}{n_i}\Big) \gtrsim \varphi(\delta).$$

Now we prove (3.7). Since

$$f = \sum_{v=1}^{i-1}\varphi\Big(\frac{1}{n_v}\Big)\cos(\chi n_v x) + \varphi\Big(\frac{1}{n_i}\Big)\cos(\chi n_i x) + \sum_{v=i+1}^{\infty}\varphi\Big(\frac{1}{n_v}\Big)\cos(\chi n_v x) = f_1 + f_2 + f_3$$

we have

$$\big\|\big(I-\sigma_{\frac{\pi}{\chi n_i}}\big)^r f\big\|_1 \ge \big\|\big(I-\sigma_{\frac{\pi}{\chi n_i}}\big)^r f_2\big\|_1 - \big\|\big(I-\sigma_{\frac{\pi}{\chi n_i}}\big)^r f_1\big\|_1 - \big\|\big(I-\sigma_{\frac{\pi}{\chi n_i}}\big)^r f_3\big\|_1.$$

Using (c) of Lemma 3, we see that

$$\big\|\big(I-\sigma_{\frac{\pi}{\chi n_i}}\big)^r f_2\big\|_1=\varphi\Big(\frac{1}{n_i}\Big)\big\|\big(I-\sigma_{\frac{\pi}{\chi n_i}}\big)^r\cos(\chi n_i x)\big\|_1=4\varphi\Big(\frac{1}{n_i}\Big).$$

Using (b) of Lemma 3 and the inequalities $\sum_{v=1}^{i-1}(\frac{1}{a})^{i-1-v}\le 2$ and (3.4), we get

$$\begin{aligned}\big\|\big(I-\sigma_{\frac{\pi}{\chi n_i}}\big)^r f_1\big\|_1&\le\sum_{v=1}^{i-1}\varphi\Big(\frac{1}{n_v}\Big)\big\|\big(I-\sigma_{\frac{\pi}{\chi n_i}}\big)^r\cos(\chi n_v x)\big\|_1\\&\le\frac{2\pi}{6^{2r}}\Big(\chi\frac{\pi}{\chi n_i}\Big)^{2r}\sum_{v=1}^{i-1}n_v^{2r}\varphi\Big(\frac{1}{n_v}\Big)=\frac{2\pi}{6^{2r}}\Big(\frac{\pi}{n_i}\Big)^{2r}\sum_{v=1}^{i-1}n_v^{2r}\varphi\Big(\frac{1}{n_v}\Big)\\&\le\frac{2\pi}{6^{2r}}\Big(\frac{\pi}{n_i}\Big)^{2r}n_{i-1}^{2r}\varphi\Big(\frac{1}{n_{i-1}}\Big)\sum_{v=1}^{i-1}\Big(\frac{1}{a}\Big)^{i-1-v}\\&\le\frac{4\pi}{6^{2r}}\Big(\frac{\pi}{n_i}\Big)^{2r}n_{i-1}^{2r}\varphi\Big(\frac{1}{n_{i-1}}\Big)\le\frac{4\pi^{2r+1}}{6^{2r}a}\varphi\Big(\frac{1}{n_i}\Big).\end{aligned}$$

Taking into account (1.10), the inequalities $C_r\le 2^{[r]+1}$ and (3.3)), and the identity $\|\cos(\chi n_v x)\|_1=4$, we arrive at the estimate

$$\begin{aligned}\big\|\big(I-\sigma_{\frac{\pi}{\chi n_i}}\big)^r f_3\big\|_1&\le\sum_{v=i+1}^{\infty}\varphi\Big(\frac{1}{n_v}\Big)\big\|\big(I-\sigma_{\frac{\pi}{\chi n_i}}\big)^r\cos(\chi n_v x)\big\|_1\\&\le 4\cdot 2^{[r]+1}\sum_{v=i+1}^{\infty}\varphi\Big(\frac{1}{n_v}\Big)\\&\le 4\cdot 2^{[r]+1}\varphi\Big(\frac{1}{n_{i+1}}\Big)\sum_{v=i+1}^{\infty}\Big(\frac{1}{a}\Big)^{v-i-1}\\&\le 2\cdot 4\cdot 2^{[r]+1}\varphi\Big(\frac{1}{n_{i+1}}\Big)\le\frac{2\cdot 4\cdot 2^{[r]+1}}{a}\varphi\Big(\frac{1}{n_i}\Big).\end{aligned}$$

Hence,

$$\begin{aligned}\big\|\big(I-\sigma_{\frac{\pi}{\chi n_i}}\big)^r f\big\|_1&\ge\big\|\big(I-\sigma_{\frac{\pi}{\chi n_i}}\big)^r f_2\big\|_1-\big\|\big(I-\sigma_{\frac{\pi}{\chi n_i}}\big)^r f_1\big\|_1-\big\|\big(I-\sigma_{\frac{\pi}{\chi n_i}}\big)^r f_3\big\|_1\\&\ge\varphi\Big(\frac{1}{n_i}\Big)\Big(4-\frac{4\pi^{2r+1}}{6^{2r}a}-\frac{2\cdot 4\cdot 2^{[r]+1}}{a}\Big)\\&=4\varphi\Big(\frac{1}{n_i}\Big)\Big(1-\frac{1}{a}\Big(\frac{\pi^{2r+1}}{6^{2r}}+2^{[r]+2}\Big)\Big)=c(r)\varphi\Big(\frac{1}{n_i}\Big)\end{aligned}$$

and we have proved (3.7). Now we analyze the remaining case (3.6). Since $\varphi(t)t^{-2r}\downarrow$, we have

$$\varphi\Big(\frac{1}{2^{m_{i+1}-1}}\Big)\le 2^{-2r}\varphi\Big(\frac{1}{2^{m_{i+1}}}\Big),$$

whence

$$\varphi\Big(\frac{1}{n_{i+1}}\Big)>\frac{2^{2r}}{a}\varphi(\frac{1}{n_i}).$$

Using the last inequality and (3.7) yields

$$\Omega_r(f,\delta)_1\ge\Omega_r\Big(f,\frac{1}{n_{i+1}}\Big)_1\ge\Omega_r\Big(f,\frac{\pi}{\chi n_{i+1}}\Big)_1\gtrsim\varphi\Big(\frac{1}{n_{i+1}}\Big)\gtrsim\varphi\Big(\frac{1}{n_i}\Big)\gtrsim\varphi(\delta).$$

The proof of Theorem 3 is complete. □

It is well known that the class of trigonometric polynomials is a dense (see, e.g., [18, Lemma 3]) subset of L^p_ω, $1<p<\infty$, for any weight $\omega\in A_p$. Hence, approximation problems make sense in L^p_ω for $1<p<\infty$ and $\omega\in A_p$.

The following Jackson type theorem relates the best approximation error $E_n(f)_{p,\omega}$ with the modulus of smoothness $\Omega_r(f,n^{-1})_{p,\omega}$.

Theorem 5 (see [12, 2])**.** *If $r\in\mathbb{N}$, $f\in L^p_\omega$, $1<p<\infty$, and $\omega\in A_p$, then*

$$E_n(f)_{p,\omega}\lesssim \Omega_r\Big(f,\frac{1}{n}\Big)_{p,\omega}$$

for $n=1,2,3,\dots$ with some constant depending only on r and p.

This theorem was proved in [12] for the integral order case, and in [1, 2] for the fractional case. We note that the integral order case was considered in a more general setting in the case of weighted Orlicz spaces [4, 16, 17].

Proof of Theorem 4. We prove (1.11). Let T_n be the near-best trigonometric polynomial approximating f. From Theorem 5 we know that

$$\|f-T_n\|_{p,\omega}\lesssim E_n(f)_{p,\omega}\lesssim \Omega_r\Big(f,\frac{1}{n}\Big)_{p,\omega}.$$

Thus, by Lemma 2,

$$\begin{aligned}\frac{1}{n^{2r}}\|T_n^{(2r)}\|_{p,\omega}&\lesssim \Omega_r(T_n,1/n)_{p,\omega}\lesssim \Omega_r(T_n-f,1/n)_{p,\omega}+\Omega_r(f,1/n)_{p,\omega}\\ &\lesssim \|f-T_n\|_{p,\omega}+\Omega_r(f,1/n)_{p,\omega}\lesssim \Omega_r(f,1/n)_{p,\omega}\end{aligned}$$

and

$$\|f-T_n\|_{p,\omega}+\frac{1}{n^{2r}}\|T_n^{(2r)}\|_{p,\omega}\lesssim \Omega_r(f,1/n)_{p,\omega}.$$

On the other hand, by Lemma 1,

$$\begin{aligned}\Omega_r(f,1/n)_{p,\omega}&\le \Omega_r(f-T_n,1/n)_{p,\omega}+\Omega_r(T_n,1/n)_{p,\omega}\\ &\lesssim \|f-T_n\|_{p,\omega}+\frac{1}{n^{2r}}\|T_n^{(2r)}\|_{p,\omega}=R_{2r}(f,1/n).\end{aligned}$$

Thus, (1.11) is proved. The proof of the equivalence (1.12) follows from the properties of the modulus of smoothness and the K-functional and from the following lemma. □

Lemma 4. *Suppose $1<p<\infty$, $\omega\in A_p$, $f\in L^p_\omega$, and let $\beta\in\mathbb{N}$. Then for any $0<t<2$,*

$$\Omega_\beta(f,t)_{p,\omega}\lesssim t^{2\beta}\|f^{(2\beta)}\|_{p,\omega}$$

with some constant depending only on β and $A[p]$.

Proof of Lemma 4. Since $0<t<2$, there exists $n=1,2,3,\dots$ such that $(1/n)<t\le(2/n)$. By Lemma 1, we have

$$\Omega_\beta(f,t)_{p,\omega}\le \Omega_\beta(f-T_n,t)_{p,\omega}+\Omega_\beta(T_n,t)_{p,\omega}\lesssim E_n(f)_{p,\omega}+t^{2\beta}\|T_n^{(2\beta)}\|_{p,\omega}. \tag{3.8}$$

On the other hand, [4, Theorem 1] and Theorem 5 show that

$$E_n(f)_{p,\omega}\lesssim \frac{1}{n^{2\beta}}E_n(f^{(2\beta)})_{p,\omega}\lesssim \frac{1}{n^{2\beta}}\Omega_\beta(f^{(2\beta)},1/n)_{p,\omega}\lesssim t^{2\beta}\|f^{(2\beta)}\|_{p,\omega} \tag{3.9}$$

and the result follows from (3.8), (3.9), Theorem 1 of [3] and Theorem 5. □

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