Martingale Transforms of the Rademacher Sequence in Rearrangement Invariant Spaces

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Abstract. Let $v_k = c_k X_{(\tau \geq k)}$, where $\tau$ is a stopping time with respect to the Rademacher system $\{r_k\}$ and $c_k \in \mathbb{R}, k = 1, 2, \ldots$. Then $\| \sum_{k=1}^{n} v_k r_k \|_X \leq \| \sum_{k=1}^{n} v_k^2 \|_X^{1/2}$ if and only if the rearrangement invariant Banach function space $X$ has nontrivial Boyd indices. If the $v_k$ are the vectors $\sum_{i=0}^{k-1} a_k^i r_i, k = 1, 2, \ldots$, then the same relation is fulfilled if and only if $X$ contains the closure of $L_\infty$ in the Orlicz space $\exp L_1$. In the second part of the paper, a new unconditionality criterion for the Haar system in a rearrangement invariant space is obtained in terms of a decoupling version of the transforms $f_n = \sum_{k=1}^{n} v_k r_k$.

§1. Introduction

In 1966, Burkholder proved that for every $1 < p < \infty$ there exists a constant $C_p > 0$ such that

\begin{equation}
C_p^{-1} \| f_\infty \|_p \leq \| P(f) \|_p \leq C_p \| f_\infty \|_p
\end{equation}

for all uniformly integrable martingales $f = \{f_n\}_{n=0}^{\infty}$, where $f_\infty = \lim_{n \to \infty} f_n$ a.e. and $P(f) := (f_0^2 + \sum_{n=1}^{\infty} (f_n - f_{n-1})^2)^{1/2}$, see [11]. At the same time, [11] fails if $p = 1$ or $p = \infty$. Considering a similar question for an arbitrary rearrangement invariant (r.i.) space $X$, Kikuchi proved in [17] that

\begin{equation}
C^{-1} \| f_\infty \|_X \leq \| P(f) \|_X \leq C \| f_\infty \|_X
\end{equation}

for all uniformly integrable martingales $f = \{f_n\}_{n=0}^{\infty} \subset X$ if and only if the Boyd indices of $X$ are nontrivial, i.e.,

\begin{equation}
0 < \alpha_X \leq \beta_X < 1.
\end{equation}

The present paper is devoted to the study of martingales of the form

\begin{equation}
f_n := \sum_{k=1}^{n} v_k r_k, \quad n = 1, 2, \ldots,
\end{equation}

where the $r_k$ are the Rademacher functions, i.e., $r_k(t) = \text{sgn} \left[ \sin(2^k \pi t) \right], 0 \leq t \leq 1,$ and $\{v_k\}_{k=1}^{\infty}$ is an arbitrary predictable sequence with respect to the filtration $\{\Sigma_k\}_{k=0}^{\infty}$ of $\sigma$-algebras generated by the system $\{r_k\}_{k=0}^{\infty}$, where $r_0 = 1$. In the simplest case, when the $v_k$ are constants, the classical Rodin–Semenov theorem (see [25]), the inequalities

\begin{equation}
C^{-1} \left( \sum_{k=1}^{\infty} c_k^2 \right)^{1/2} \leq \left\| \sum_{k=1}^{\infty} c_k r_k \right\|_X \leq C \left( \sum_{k=1}^{\infty} c_k^2 \right)^{1/2}
\end{equation}

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are fulfilled for arbitrary \( c_k \in \mathbb{R} \) if and only if the r.i. space \( X \) contains \( G \), where \( G \) is the closure of \( L_\infty \) in the Orlicz space \( \exp L_2 \). If we talk about sequences of independent functions (that are also independent from the Rademacher system), the result is different: the inequalities

\[
C^{-1} \left\| \left( \sum_{k=1}^{\infty} \frac{v_k}{r_k} \right)^2 \right\|_X^{1/2} \leq \left\| \sum_{k=1}^{\infty} \frac{v_k(s)r_k(t)}{\| X([0,1] \times [0,1])} \right\|_X \leq C \left\| \left( \sum_{k=1}^{\infty} \frac{v_k^2}{r_k^2} \right)^{1/2} \right\|_X
\]

are valid for an arbitrary sequence \( \left\{ v_k \right\}_{k=1}^{\infty} \subset X \) of independent functions if and only if \( X \) possesses the so-called Kruglov property (in symbols: \( X \in \mathbb{K} \)), see [2, Theorem 2]. Here \( X([0,1] \times [0,1]) \) is the r.i. space of functions on the square \( [0,1] \times [0,1] \) with the norm \( \| f \|_X([0,1] \times [0,1]) = \| f^* \|_X \), where \( f^* \) is the nonincreasing rearrangement of \( |f| \) (see the next section for the definitions). We refer to [10, 8, 9] for more information concerning the Kruglov property. Here we only note that the condition \( \alpha_X > 0 \) implies the Kruglov property and that the Orlicz space \( \exp L_p \) belongs to \( \mathbb{K} \) if and only if \( p \geq 1 \).

In the first part of the paper, the behavior of the martingale transforms (4) is studied for other classes of \( \{ \Sigma_k \} \)-predictable sequences. Refining the result of Kikuchi [17] cited above, we prove that inequalities (3) must be true already in the case where (2) is fulfilled for other classes of \( \{ \Sigma_k \} \)-predictable sequences. Refining the result of Kikuchi [17] cited above, we prove that inequalities (3) must be true already in the case where (2) is fulfilled.

In the second part of the paper, we prove a new criterion for the unconditionality of the Haar system in an r.i. space in terms of the transforms (4). Moreover, it is shown that the following two stopping times:

\[
\tau_1 := \inf \{ k = 2, 3, \ldots : r_k \neq r_{k-1} \} \quad \text{and} \quad \tau_2 := \inf \{ k = 2, 3, \ldots : r_k = r_{k-1} \}
\]

are “critical”: if

\[
C^{-1} \left\| \sum_{i=1}^{\infty} c_i \chi_{\{ \tau_j \geq i \}} r_i \right\|_X \leq \left\| \left( \sum_{i=1}^{\infty} c_i^2 \chi_{\{ \tau_j \geq i \}} \right)^{1/2} \right\|_X \leq C \left\| \sum_{i=1}^{\infty} c_i \chi_{\{ \tau_j \geq i \}} r_i \right\|_X
\]

for some \( C > 0 \), \( j = 1, 2 \), and all \( c_i \in \mathbb{R} \), then the Boyd indices of \( X \) satisfy (5).

At the same time, if the \( v_k \) are arbitrary linear combinations \( \sum_{i=0}^{k-1} a_i r_i \), \( k = 1, 2, \ldots \), then inequalities (2) are true if and only if \( X \) contains the closure of \( L_\infty \) in the Orlicz space \( \exp L_1 \) (Theorem 2).

A natural decoupling version of the transforms (4) is given by the martingales

\[
g_n(s,t) := \sum_{k=1}^{n} v_k(s)r_k(t), \quad 0 \leq s, \ t \leq 1.
\]

On the basis of Hitczenko’s results [13] about comparison of these martingales with the transforms (4) in \( L_p \)-spaces and also on Talagrand’s example exposed in the same paper, in the second part of this article we discuss a similar question for general r.i. spaces. Also in the second part of the paper, we prove a new criterion for the unconditionality of the Haar system in an r.i. space in terms of the transforms \( g_n, n = 1, 2, \ldots \) (see Theorem 8).

§2. Definitions and preliminaries

(a) Rearrangement invariant spaces. A Banach space \( X \) of measurable functions on \([0,1]\) is said to be rearrangement invariant (or symmetric) if 1) it is an ideal space, i.e., the conditions \( |x(t)| \leq |y(t)| \) for a.e. \( t \in [0,1] \) and \( y \in X \) imply that \( x \in X \) and \( \| x \|_X \leq \| y \|_X \); 2) whenever \( y \in X \) and \( x \) is a function such that \( x \) and \( y \) are equimeasurable, i.e.,

\[
m\{ t \in [0,1] : |y(t)| > u \} = m\{ t \in [0,1] : |x(t)| > u \} \quad (u > 0),
\]

where \( m(a) \) is the Lebesgue measure of a set \( a \subset \mathbb{R} \), we have \( x \in X \) and \( \| x \|_X = \| y \|_X \). In particular, an arbitrary measurable function \( x(t) \) on \([0,1]\) is equimeasurable with its
nonincreasing rearrangement continuous from the left:

\[ x^*(t) := \inf \{ u \geq 0 : m\{ s \in [0,1] : |x(s)| > u \} < t \}, \quad 0 \leq t \leq 1. \]

If \( X \) is an r.i. space on \([0,1]\), its dual \( X' \) is defined to consist of all \( y \) such that
\[
\|y\|_{X'} = \sup \left\{ \int_0^1 x(t) y(t) \, dt : \|x\|_X \leq 1 \right\} < \infty.
\]
The space \( X' \) is also rearrangement invariant; it embeds isometrically to the conjugate space \( X^* \), moreover, \( X' = X^* \) if and only if \( X \) is separable. The space \( X \) is said to be maximal if \( X = X'' \). This property is equivalent to the following one: given \( x_n \in X \) \((n = 1, 2, \ldots)\) with \( \sup_{n=1,2,\ldots} \|x_n\|_X < \infty \) and \( x_n \to x \) a.e. on \([0,1]\), we have \( x \in X \) and \( \|x\|_X \leq \liminf_{n \to \infty} \|x_n\|_X \). Throughout the paper, we shall assume that an r.i. space is either separable or maximal. Recall that any such space has the following property: if \( x \in X \) and \[
\int_0^t y^*(s) \, ds \leq \int_0^t x^*(s) \, ds, \quad 0 \leq t \leq 1,
\]
then \( y \in X \) and \( \|y\|_X \leq \|x\|_X \). Furthermore, the norm in any such r.i. space is order semicontinuous, that is, \( X \) is isometrically embedded in \( X'' \). We also assume the normalization condition \( \|\chi_{[0,1]}\|_X = 1 \), where \( \chi_e(t) \) is the characteristic function of a measurable set \( e \subset [0,1] \). Then \( L_\infty \subset X \subset L_1 \) and \( \|x\|_{L_1} \leq \|x\|_X \leq \|x\|_{L_\infty} \) for every \( x \in L_\infty \). If \( X \neq L_\infty \), then the closure of \( L_\infty \) in \( X \) is called the separable part of \( X \) and is denoted by \( X_0 \); this is a separable r.i. space. In particular, the norm of an arbitrary function in \( X_0 \) is absolutely continuous. This means that \( \lim_{k \to \infty} \|x_{e_k}\|_X = 0 \) whenever \( \{e_k\}_{k=1}^\infty \) is a monotone decreasing sequence of subsets of \([0,1]\) with \( \lim_{k \to \infty} m(e_k) = 0 \).

The dilation operator \( \sigma_x(t) = x(t/\tau) \chi_{[0,1]}(t/\tau) \) \((0 \leq t \leq 1)\), where \( \tau > 0 \), acts boundedly in every r.i. space \( X \). Moreover, \( \|\sigma_x\|_{X \to X} \leq \max(1,\tau) \) (see [13 Theorems 2.4.4 and 2.4.5]). The quantities
\[
\alpha_X = \lim_{\tau \to 0} \frac{\ln \|\sigma_{\tau}\|_X}{\ln \tau} \quad \text{and} \quad \beta_X = \lim_{\tau \to \infty} \frac{\ln \|\sigma_{\tau}\|_X}{\ln \tau}
\]
are called the Boyd indices of the r.i. space \( X \). We always have \( 0 \leq \alpha_X \leq \beta_X \leq 1 \). For example, \( \alpha_{L_p} = \beta_{L_p} = 1/p \) for all \( p \in [1,\infty] \).

The Orlicz spaces (a natural generalization of \( L_p \)) constitute an important class of r.i. spaces. Let \( M(u) \) be an Orlicz function, i.e., a strictly monotone increasing convex function on \([0,\infty)\), \( M(0) = 0 \). The Orlicz space \( L_M \) consists of all measurable functions \( x(t) \) on \([0,1]\) such that
\[
\int_0^1 M(|x(t)|/\lambda) \, dt \leq 1
\]
for sufficiently large \( \lambda > 0 \). The norm on \( L_M \) is defined as follows: \( \|x\|_{L_M} = \inf \lambda \), where the infimum is taken over all \( \lambda > 0 \) for which the preceding inequality is true. The classical Orlicz space built on the basis of an Orlicz function equivalent to \( e^{u^p} - 1 \) is usually denoted by \( \exp L_p \). We quote a well-known description of the separable part of this space (see [21 or [3 Lemma 3.2]): \( x \in (\exp L_p)_0 \) if and only if
\[
\lim_{t \to +0} \frac{x^*(t)}{\log^{1/p}(2/t)} = 0.
\]

(b) MARTINGALE TRANSFORMS AND THE HAAR SYSTEM. Let \( \{\mathcal{A}_k\}_{k=0}^\infty \) be a filtration of \( \sigma \)-algebras on a probability space \((\Omega, \mathcal{A}, P)\), i.e., \( \mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_k \subset \cdots \subset \mathcal{A} \). A sequence of random variables \( \{v_k\}_{k=1}^\infty \) is said to be predictable with respect to this filtration (or \( \{\mathcal{A}_k\}-\text{predictable} \) if \( v_k \) is \( \mathcal{A}_{k-1} \)-measurable for all \( k = 0, 1, \ldots \).
Suppose we are given a sequence \( \{\xi_k\}_{k=0}^{\infty} \) of independent random variables on \((\Omega, \mathcal{A}, \mathbb{P})\) with \( \int_{\Omega} \xi_k \, d\mathbb{P} = 0 \). Let \( \mathcal{A}_k \) be the \( \sigma \)-algebra generated by the collection \( \{\xi_0, \xi_1, \ldots, \xi_k\} \), \( k = 0, 1, \ldots \). Then for every \( \{\mathcal{A}_k\} \)-predictable sequence \( \{v_k\}_{k=1}^{\infty} \), the random variables

\[
\eta_n := \sum_{k=1}^{n} v_k \xi_k, \quad n = 1, 2, \ldots,
\]

form a martingale (see [23, 26] or [28] for the definition and properties of martingales). For this reason, the sequence \( \{\eta_n\}_{n=1}^{\infty} \) is called a martingale transform of the sequence \( \{\xi_k\}_{k=0}^{\infty} \). We remind the reader that a random variable \( \tau = \tau(\omega) \) with values in \( \mathbb{N} \cup \{0\} \) is called a stopping time relative to the filtration \( \{\mathcal{A}_k\}_{k=0}^{\infty} \) of \( \sigma \)-algebras if \( \{\omega \in \Omega : \tau(\omega) \leq k\} \in \mathcal{A}_k \) \( (k = 0, 1, \ldots) \).

In this paper, we study martingale transforms of the simplest sequence of independent functions, namely, the Rademacher system \( \{r_k\}_{k=0}^{\infty} \), where \( r_0 \equiv 1 \). In this case, \( \mathcal{A}_k \) is the \( \sigma \)-algebra \( \Sigma_k \) generated by the dyadic intervals of rank \( k \), i.e., by the intervals \( \Delta^k_i = ((i-1)2^{-k}, i2^{-k}), i = 1, \ldots, 2^k, k = 0, 1, \ldots \). By the Rademacher–Kolmogorov theorem, the series \( \sum_{k=0}^{\infty} a_k r_k(t) \) converges a.e. if and only if \( a = (a_k)_{k=1}^{\infty} \in \ell_2 \) (see [16, §4.5] or [3, Theorems 1.1 and 1.2]).

Recall that the Haar system is formed by the functions \( h_{0,0}(t) = 1 \),

\[
h_{n,k}(t) = \begin{cases} 
1, & t \in \Delta^{2k-1}_{n+1}, \\
-1, & t \in \Delta^{2k}_{n+1}, \\
0 & \text{for all other } t \in [0,1],
\end{cases}
\]

where \( n = 0, 1, \ldots, k = 1, \ldots, 2^n \). See the monographs [15, 12, 24] and [22, Chapter 1] for more details. The complete orthonormal system \( \{2^{n/2} h_n^k\} \) is a monotone basis in every separable r.i. space. The Haar system is an unconditional basis in a separable r.i. space \( X \) if and only if \( 0 < \alpha_X \leq \beta_X < 1 \) (see [20, 2.c.6] or [18, 2.9.6]). Another equivalent condition is \( \|x\|_X \approx \|P x\|_X \), where

\[
(5) \quad x = \sum_{k=1}^{\infty} \sum_{i=1}^{2^{k-1}} c_{k,i} h_{k-1,i} \quad \text{and} \quad P x = \left( \sum_{k=1}^{\infty} \sum_{i=1}^{2^{k-1}} c_{k,i}^2 h_{k-1,i}^2 \right)^{1/2},
\]

see [15, Theorem 1.11]. Throughout, when we write \( f \asymp g \) we mean that \( f \) and \( g \) are nonnegative functions (in particular, they may be quasinorms) satisfying \( C^{-1} \leq g/f \leq C \) for some constant \( C \) uniformly on the domain of \( f \) and \( g \). The function \( P x(t) \) is called the Paley function. The Fourier–Haar coefficients \( c_{n,k} = c_{n,k}(x) \) of a function \( x \in L_1 = L_1[0,1] \) are defined by the formula

\[
c_{n,k} = 2^n \int_0^1 x(s) h_{n,k}(s) \, ds, \quad n = 0, 1, \ldots, \quad k = 1, \ldots, 2^n.
\]

Let \( \{v_k\}_{k=1}^{\infty} \) be a \( \{\Sigma_k\} \)-predictable sequence. By Doob’s classical theorem (see [28, Theorem 3.14] or [20, Theorem 7.4.1]), the condition

\[
\sup_{n=1,2,\ldots} \int_0^1 \left| \sum_{k=1}^{n} v_k(t) r_k(t) \right| \, dt < \infty
\]

implies the a.e. convergence of the series

\[
(6) \quad R v := \sum_{k=1}^{\infty} v_k r_k
\]
on $[0, 1]$ to an integrable function. Observe that the terms $v_k$, $k = 1, 2, \ldots$, of any $\{\Sigma_k\}$-predictable sequence can be represented in the form

$$v_k := \sum_{i=1}^{2^{k-1}} c_{k,i} \chi_{\Delta_k^{i-1}}, \quad k = 1, 2, \ldots,$$

with some $c_{k,i} \in \mathbb{R}$. Therefore, since $r_{n+1}(t) = \sum_{k=1}^{2^n} h_{n,k}(t)$, $n = 0, 1, 2, \ldots$, we have

$$Rv = \sum_{k=1}^{\infty} \sum_{i=1}^{2^{k-1}} c_{k,i} h_{k-1,i}.$$  

This formula shows that the $c_{k,i}$ are the Fourier–Haar coefficients of the function $Rv$. In the same terms, the Paley function acquires the following form:

$$Pv = \left( \sum_{k=1}^{\infty} v_k^2 \right)^{1/2}.$$  

The last formulas, when combined with Theorem 2 in [7] about Haar series in r.i. spaces, yield the following result.

\textbf{Proposition 1.} Let $X$ be an r.i. space on $[0, 1]$.

1. The relation

$$\|Rv\|_X \leq C\|Pv\|_X$$

is fulfilled for every $\{\Sigma_k\}$-predictable sequence $\{v_k\}$ with $C > 0$ independent of the sequence if and only if $\alpha_X > 0$.

2. The following statements are equivalent.

(a) There exists $C > 0$ such that for every $\{\Sigma_k\}$-predictable sequence $\{v_k\}$ we have

$$\|Pv\|_X \leq C\|Rv\|_X;$$

(b) $0 < \alpha_X \leq \beta_X < 1$.

In the sequel, we shall also consider a decoupling version of the series $Rv$, specifically, the (formal) series

$$R'v(s, t) := \sum_{k=1}^{\infty} v_k(s) r_k(t), \quad 0 \leq s, \ t \leq 1.$$  

If the coefficients $v_k$ are defined by (7), we have

$$R'v(s, t) = \sum_{k=1}^{\infty} \sum_{i=1}^{2^{k-1}} c_{k,i} \chi_{\Delta_k^{i-1}}(s) r_k(t).$$

In distinction with $\{v_k r_k\}_{k=1}^{\infty}$, the sequence $\{v_k(s) r_k(t)\}_{k=1}^{\infty}$ is unconditional with constant 1 in every r.i. space on $[0, 1] \times [0, 1]$. Moreover, the following slightly stronger statement holds true (we shall prove it for completeness).

\textbf{Lemma 1.} The family of functions

$$\{ \chi_{\Delta_k^{i-1}}(s) r_k(t) : k = 1, 2, \ldots; \ i = 1, 2, \ldots, 2^{k-1} \},$$

where the $\Delta_k^i$ are the dyadic subintervals of $[0, 1]$, is 1-unconditional in every r.i. space $X$ on $([0, 1] \times [0, 1])$.  

Proof. For arbitrary \( c_{k,i} \in \mathbb{R}, \varepsilon_{k,i} = \pm 1 \), and \( \tau > 0 \) we have

\[
m \left\{ (s,t) \in [0,1] \times [0,1] : \left| \sum_{k=1}^{\infty} \sum_{i=1}^{2^{k-1}} \varepsilon_{k,i} c_{k,i} \chi_{\Delta_{k-1}^i}(s)r_k(t) \right| > \tau \right\} = \int_0^1 m \left\{ t \in [0,1] : \left| \sum_{k=1}^{\infty} \sum_{i=1}^{2^{k-1}} \varepsilon_{k,i} c_{k,i} \chi_{\Delta_{k-1}^i}(s)r_k(t) \right| > \tau \right\} ds
\]

\[
= \int_0^1 m \left\{ t \in [0,1] : \left| \sum_{k=1}^{\infty} \sum_{i=1}^{2^{k-1}} \varepsilon_{k,i} c_{k,i} \chi_{\Delta_{k-1}^i}(s)r_k(t) \right| > \tau \right\} \exp \left( \sum_{k=1}^{\infty} \sum_{i=1}^{2^{k-1}} \varepsilon_{k,i} c_{k,i} \chi_{\Delta_{k-1}^i}(s)r_k(t) \right) ds,
\]

where the numbers \( i_k(s) = 1, \ldots, 2^{k-1} \) are chosen so that \( s \in \Delta_{k-1}^{i_k(s)} \) for arbitrary \( s \in [0,1] \) and \( k = 1, 2, \ldots \) Elementary properties of the Rademacher system allow us to rewrite the last quantity as follows:

\[
\int_0^1 m \left\{ t \in [0,1] : \left| \sum_{k=1}^{\infty} c_{k,i_k(s)} r_k(t) \right| > \tau \right\} ds = m \left\{ (s,t) \in [0,1] \times [0,1] : \left| \sum_{k=1}^{\infty} \sum_{i=1}^{2^{k-1}} c_{k,i} \chi_{\Delta_{k-1}^i}(s)r_k(t) \right| > \tau \right\}.
\]

By the definition of r.i. spaces, it follows that

\[
\left( \sum_{k=1}^{\infty} \sum_{i=1}^{2^{k-1}} \varepsilon_{k,i} c_{k,i} \chi_{\Delta_{k-1}^i}(s)r_k(t) \right)_{X([0,1] \times [0,1])} = \left( \sum_{k=1}^{\infty} \sum_{i=1}^{2^{k-1}} c_{k,i} \chi_{\Delta_{k-1}^i}(s)r_k(t) \right)_{X([0,1] \times [0,1])}
\]

for every \( \varepsilon_{k,i} = \pm 1 \). This completes the proof by [15, Theorem 1.10].

§3. Martingale transforms generated by a stopping time

In this section, we consider the martingale transforms [1] in the case where \( v_k = c_k \chi_{\{ \tau \geq k \}} \), where \( \tau \) is a stopping time with respect to the filtration \( \{ \Sigma_k \} \) of \( \sigma \)-algebras and \( c_k \in \mathbb{R} \) \( (k = 1, 2, \ldots) \). We refine Proposition [1] by showing that the Boyd indices must be nontrivial provided relations (2) are fulfilled for this special class of \( \{ \Sigma_k \} \)-predictable sequences.

Theorem 1. For an arbitrary r.i. space \( X \) on \([0,1]\), the following properties are equivalent.

(a) We have

\[
\left| \left( \sum_{i=1}^{\infty} c_i \chi_{\{ \tau \geq i \}} r_i \right) \right|_X \times \left( \sum_{i=1}^{\infty} c_i^2 \chi_{\{ \tau \geq i \}} \right)^{1/2} \right|_X
\]

with constants independent of the stopping time \( \tau \) and \( c_i \in \mathbb{R} \).

(b) \( 0 < \alpha_X \leq \beta_X < 1 \).

Proof. By Proposition [1] it suffices to show that \( (a) \Rightarrow (b) \). Consider the stopping time

\[
\tau_1 := \inf \{ k = 2, 3, \cdots : r_k \neq r_{k-1} \}.
\]

Putting

\[
R_1 = R_1(c) := \sum_{i=1}^{\infty} c_i \chi_{\{ \tau_1 \geq i \}} r_i,
\]
where \( c_i \geq 0, \ c_{2i} = c_{2i-1}, \ i = 1, 2, \ldots, \) we obtain

\[
|R_1| = \sum_{k=2}^{\infty} \left| \sum_{i=1}^{k} c_i r_i \right| \chi_{\{\tau_1 = k\}} = |c_1 + c_2 - c_3|\chi_{\{\tau_1 = 3\}} + (c_1 + c_2)\chi_{\{\tau_1 = 4\}} + \ldots + \left( \sum_{i=1}^{2k-2} c_i - c_{2k-1} \right)\chi_{\{\tau_1 = 2k-1\}} + \left( \sum_{i=1}^{2k-2} c_i \right)\chi_{\{\tau_1 = 2k\}} + \ldots
\]

\[
\geq \sum_{k=1}^{\infty} \left( \sum_{i=1}^{k} c_{2i} \right)\chi_{\{\tau_1 = 2k+2\}}.
\]

The sets \( \{\tau_1 = 2k + 2\}, \ k = 1, 2, \ldots, \) are mutually disjoint and \( m\{\tau_1 = 2k + 2\} = 2^{-2k-1}. \)

Thus, the last relation implies

\[
R_1(t) \geq \sum_{k=1}^{\infty} \left( \sum_{i=1}^{k} d_i \right)\chi_{\{2^{-2k-1}, 2^{-2k}\}}(t), \quad 0 < t \leq 1,
\]

where \( d_k = c_{2k}. \) Denoting by \( x_a \) the right-hand side of this inequality, we observe that

\[
x_a + \sigma_{1/2} x_a = \sum_{k=1}^{\infty} \left( \sum_{i=1}^{k} a_i \right)\chi_{\{2^{-2k-1}, 2^{-2k}\}}.
\]

Since \( \|\sigma_{1/2}\|_{X \rightarrow X} \leq 1, \) it follows that

\[
(13) \quad \left\| \sum_{k=1}^{\infty} \left( \sum_{i=1}^{k} a_i \right)\chi_{\{4^{-k-1}, 4^{-k}\}} \right\|_X \leq 2 \|R_1\|_X.
\]

On the other hand, putting

\[
\tau_2 := \inf \{ k = 2, 3, \ldots : r_k = r_{k-1} \}, \quad R_2 = R_2(c) := \sum_{i=1}^{\infty} c_i \chi_{\{\tau_2 \geq i\}} r_i
\]

and assuming as before that \( c_i \geq 0, \ c_{2i} = c_{2i-1}, \ i = 1, 2, \ldots, \) we obtain

\[
|R_2| = \sum_{k=2}^{\infty} \left| \sum_{i=1}^{k} c_i r_i \right| \chi_{\{\tau_2 = k\}} = (c_1 + c_2)\chi_{\{\tau_2 = 2\}} + c_3 \chi_{\{\tau_2 = 3\}} + \ldots
\]

\[
= 2 \sum_{k=1}^{\infty} c_{2k} \chi_{\{\tau_2 = 2k\}} + \sum_{k=1}^{\infty} c_{2k} \chi_{\{\tau_2 = 2k-1\}}.
\]

Since the sets \( \{\tau_2 = j\}, \ j = 2, 3, \ldots, \) are mutually disjoint and \( m\{\tau_2 = j\} = 2^{-j+1}, \) we see that

\[
\left( \sum_{k=2}^{\infty} c_{2k} \chi_{\{\tau_2 = 2k-1\}} \right)^*(t) = \sigma_2 \left( \sum_{k=2}^{\infty} c_{2k} \chi_{\{\tau_2 = 2k\}} \right)^*(t), \quad 0 \leq t \leq 1.
\]

Since \( \|\sigma_2\|_{X \rightarrow X} \leq 2, \) it follows that

\[
\left\| 2 \sum_{k=1}^{\infty} c_{2k} \chi_{\{\tau_2 = 2k\}} + \sum_{k=2}^{\infty} c_{2k} \chi_{\{\tau_2 = 2k-1\}} \right\|_X \leq 4 \left\| \sum_{k=1}^{\infty} c_{2k} \chi_{\{4^{-k-1}, 4^{-k}\}} \right\|_X
\]

\[
= 4 \left\| \sigma_2 \left( \sum_{k=1}^{\infty} c_{2k} \chi_{\{4^{-k-1}, 4^{-k}\}} \right) \right\|_X \leq 8 \left\| \sum_{k=1}^{\infty} c_{2k} \chi_{\{4^{-k-1}, 4^{-k}\}} \right\|_X.
\]
Together with (14), this implies the inequality

\[ \|R_2\|_X \leq 8 \left\| \sum_{k=1}^{\infty} a_k \chi_{[4^{-k-1}, 4^{-k})} \right\|_X, \tag{15} \]

where \( a_k = c_{2k} \).

Next, since \( m\{\tau_1 = k\} = m\{\tau_2 = k\}, \) \( k = 1, 2, \ldots, \) our assumptions imply

\[ \|R_1(c)\|_X \preceq \left\| \left( \sum_{i=1}^{\infty} c_i^2 \chi_{\{\tau_i \geq 0\}} \right)^{1/2} \right\|_X = \left\| \sum_{k=1}^{\infty} \left( \sum_{i=1}^{k} c_i^2 \chi_{\{\tau_i = k\}} \right)^{1/2} \right\|_X, \tag{16} \]

As a result, by (13) and (15) it follows that

\[ \left\| \sum_{k=1}^{\infty} \left( \sum_{i=1}^{k} a_i \right) \chi_{[4^{-k-1}, 4^{-k})} \right\|_X \leq K \left\| \sum_{k=1}^{\infty} a_k \chi_{[4^{-k-1}, 4^{-k})} \right\|_X, \tag{17} \]

for some \( K > 0 \) and an arbitrary sequence \((a_k)_{k=1}^{\infty}\) of nonnegative numbers.

We show that for an arbitrary function \( y \in L_1[0, 1] \) with \( y = y^* \) and for every \( m \in \mathbb{N} \) we have

\[ \int_0^t \sigma_{4^{-m}} y(s) \, ds \leq \frac{1}{m} \int_0^t \sum_{k=0}^{m} \sigma_{4^{-k}} y(s) \, ds, \quad 0 \leq t \leq 1. \tag{17} \]

Indeed, there is no loss of generality in assuming that \( 0 \leq t \leq 4^{-m} \). Then, since the function \( \frac{1}{t} \int_0^t y(s) \, ds \) is monotone decreasing, we see that

\[ \int_0^t \sum_{k=0}^{m} \sigma_{4^{-k}} y(s) \, ds = \sum_{k=0}^{m} \int_0^t \sigma_{4^{-k}} y(s) \, ds = \sum_{k=0}^{m} 4^{-k} \int_0^{4^k t} y(s) \, ds \]

\[ \geq m 4^{-m} \int_0^{4^m t} y(s) \, ds = m \int_0^t \sigma_{4^{-m}} y(s) \, ds, \]

and (17) follows. By the assumption (see §2) imposed on the r.i. space \( X \), we obtain

\[ \|\sigma_{4^{-m}} y\|_X \leq \frac{1}{m} \left\| \sum_{k=0}^{m} \sigma_{4^{-k}} y \right\|_X, \tag{18} \]

if the function \( y \) is nonnegative and monotone nonincreasing.

Put

\[ v_a := \sum_{k=1}^{\infty} a_k \chi_{[4^{-k-1}, 4^{-k})}, \tag{19} \]

where \( a_k \geq 0, \) \( k = 1, 2, \ldots, \) and the sequence \((a_k)_{k=1}^{\infty}\) is monotone nondecreasing. Since

\[ \sum_{k=1}^{\infty} \left( \sum_{i=1}^{k} a_i \right) \chi_{[4^{-k-1}, 4^{-k})} = \sum_{k=0}^{\infty} \sigma_{4^{-k}} v_a, \]

we can rewrite (16) as follows:

\[ \left\| \sum_{k=0}^{m} \sigma_{4^{-k}} v_a \right\|_X \leq K \| v_a \|_X, \quad m \in \mathbb{N}. \]
Since the function \( v_a \) is monotone nonincreasing, this and formula (18) imply

\[
\| \sigma_{4^{-m}} v_a \|_X \leq \frac{K}{m} \| v_a \|_X, \quad m \in \mathbb{N}.
\]

Now, assume that \( x \in X \) is arbitrary. Putting

\[
v := \sum_{k=1}^{\infty} x^*(4^{-k}) \chi_{[4^{-k-1}, 4^{-k})},
\]

we see that \( \sigma_{1/4} x^*(t) \leq v(t) \leq x^*(t), \quad 0 < t \leq 1. \)

Observe that \( v = v_a \) if \( a_k = x^*(4^{-k}) \), \( k = 1, 2, \ldots \). Thus, combining the last inequality with (20), we arrive at

\[
\| \sigma_{4^{-m}} (\sigma_{1/4} x^*) \|_X \leq \| \sigma_{4^{-m}} v \|_X \leq \frac{K}{m} \| v \|_X \leq \frac{K}{m} \| x \|_X, \quad m \in \mathbb{N}.
\]

Consequently,

\[
\| \sigma_{4^{-m+1}} x \|_X \leq \frac{K}{m} \| x \|_X,
\]

and therefore, \( \lim_{s \to +0} \| \sigma_s \|_{X \to X} = 0 \), i.e., \( \alpha_X > 0 \).

To show that \( \beta_X < 1 \), it suffices to prove that the adjoint \( Q \),

\[
Qx(t) := \int_{t}^{1} \frac{x(s)}{s} ds, \quad 0 < t \leq 1,
\]

to the Hardy operator is bounded on \( X \) (see [18, Theorem 2.6.8]).

Take \( x \in X \) with \( x = x^* \) and define \( v \) by (21). Since \( \text{supp} v \subset [0, 1/4] \), we have

\[
Q(\sigma_4 v)(t) = \int_{t}^{1} \frac{v(s/4)}{s} ds = \int_{t/4}^{1} \frac{v(s)}{s} ds = \sigma_4(Qv)(t).
\]

Since

\[
v(t) \leq x(t) \leq \sigma_4 v(t), \quad 0 < t \leq 1,
\]

it follows that

\[
\| Qx \|_X \leq \| Q(\sigma_4 v) \|_X = \| \sigma_4(Qv) \|_X \leq 4 \| Qv \|_X.
\]

It can easily be shown that

\[
\| Q \|_{X \to X} = \sup_{x=x^*, \| x \|_X \leq 1} \| Q_x \|_X
\]

(see also [19, Proof of Theorem 1]). Thus, the above relations imply that

\[
\| Q \|_{X \to X} \leq 4 \sup \| Qv_a \|_X,
\]

where the supremum is taken over all functions \( v_a, \| v_a \|_X \leq 1 \), of the form (19), where the \( a_k \geq 0 \) are nonnegative and monotone nondecreasing. To estimate the supremum in (22), we observe that

\[
Qv_a(4^{-k-1}) = \ln 4 \cdot \sum_{i=1}^{k} a_i,
\]

for all \( k \in \mathbb{N} \); thus, since \( Qv_a \) is monotone decreasing, we have

\[
Qv_a(t) \leq \ln 4 \cdot \sum_{k=1}^{\infty} \left( \sum_{i=1}^{k} a_i \right) \chi_{[4^{-k-1}, 4^{-k})}(t).
\]

By (17), this implies the inequality \( \| Qv_a \|_X \leq K \ln 4 \| v_a \|_X \leq K \ln 4 \). So, \( Q \) is bounded on \( X \) by (22). This completes the proof of the theorem. \( \square \)
§4. Martingale transforms generated by linear combinations of Rademacher functions

In the case where the \( v_k \) are linear combinations of Rademacher functions, the martingale transforms of the Rademacher sequence behave not quite so as in Theorem 1.

We remind the reader that the symbol \( X_0 \) stands for the closure of \( L_\infty \) in an r.i. space \( X \).

**Theorem 2.** Suppose \( X \) is an r.i. space on \([0,1]\). Let \( v_k = \sum_{i=0}^{k-1} a_k r_i \), \( k = 1, 2, \ldots \), where \( r_0(t) = 1 \). Then the inequalities

\[
C^{-1} \|Pv\|_X \leq \|Rv\|_X \leq C \|Pv\|_X
\]

are fulfilled with some \( C \) independent of the \( a_k^i \in \mathbb{R} \) if and only if \( X \supset (\exp L_1)_0 \).

**Proof.** First, we recall that every r.i. space \( X \) on \([0,1]\) embeds in \( L_1 \) with constant 1 (see §2). Thus, applying the Khinchin and Minkowski inequalities, we obtain

\[
\|Pv\|_X = \left\| \left( \sum_{k=1}^{\infty} \left( \sum_{i=0}^{k-1} a_k^i r_i \right)^2 \right)^{1/2} \right\|_X \geq \left\| \sum_{k=1}^{\infty} \left( \sum_{i=0}^{k-1} a_k^i r_i \right)^2 \right\|_{L_1}^{1/2}.
\]

We show that the reverse inequality is true whenever \( X \supset (\exp L_1)_0 \). After squaring, we have

\[
\|Pv\|_X = \left\| \left( \sum_{k=1}^{\infty} \left( \sum_{i=0}^{k-1} a_k^i r_i \right)^2 \right)^{1/2} \right\|_X = \left\| \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} a_k^i r_i \right\|_{L_1}^{1/2}
\]

So, putting

\[
A^2 := \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} (a_k^i)^2 \quad \text{and} \quad U := \sum_{k=1}^{\infty} \sum_{0 \leq i < j < k} a_k^i a_k^j r_i r_j,
\]

we arrive at

\[
\|Pv\|_X = \left\| (A^2 + 2U)^{1/2} \right\|_X \leq \|A^2 + 2U\|^{1/2}_X \leq (A^2 + 2\|U\|_X)^{1/2}.
\]

In [1] it was proved that the system \( \{r_i r_k\}_{1 \leq i < k < \infty} \) is equivalent in an r.i. space \( X \) to the standard basis of \( \ell_2 \) if and only if \( X \supset (\exp L_1)_0 \). So, if this embedding does occur, we use the identity

\[
U = \sum_{0 \leq i < j < \infty} \left( \sum_{k=j+1}^{\infty} a_k^i a_k^j \right) r_i r_j
\]

and the Minkowski inequality to obtain

\[
\|U\|_X \leq \left( \sum_{0 \leq i < j < \infty} \left( \sum_{k=j+1}^{\infty} (a_k^i a_k^j)^2 \right) \right)^{1/2} \leq \sum_{k=1}^{\infty} \left( \sum_{0 \leq i < j < k} (a_k^i a_k^j)^2 \right)^{1/2} \leq \sum_{k=1}^{\infty} \left( \sum_{j=1}^{k-1} \sum_{i=0}^{j-1} (a_k^i)^2 \right)^{1/2} \leq A^2.
\]

By (25), it follows that

\[
\|Pv\|_X \leq \sqrt{3} A = \sqrt{3} \left( \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} (a_k^i)^2 \right)^{1/2}.
\]
On the other hand, since \( X \ni (\exp L_1)_0 \) and
\[
Rv = \sum_{k=1}^{\infty} v_k r_k = \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} a_k^i r_i r_k,
\]
we can apply the result from \([1]\) already used to conclude that
\[
\|Rv\|_X \asymp \left( \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} (a_k^i)^2 \right)^{1/2}.
\]
Together with (24) and (26), this implies (23).

Now, conversely, let (23) be fulfilled. Then, in particular,
\[
\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_X \asymp \left( \sum_{k=1}^{\infty} a_k^2 \right)^{1/2},
\]
with constants independent of \( a_k \in \mathbb{R} \), \( k = 1, 2, \ldots \). Therefore, \( X \ni G \), where \( G \) is the closure of \( L_\infty \) in the Orlicz space \( \exp L_2 \) (see \([25]\) and also \( \S 1 \)). Since \( G \) is 2-convex (see, e.g., \([14]\)), we obtain
\[
\|Pv\|_X \leq \|Pv\|_G \leq \left( \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} (a_k^i)^2 \right)^{1/2} \leq C \left( \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} (a_k^i)^2 \right)^{1/2}.
\]
Together with (24) and (23), this implies (27), i.e., the system \( \{r_i r_k\}_{1 \leq i < k < \infty} \) in the r.i. space \( X \) is equivalent to the standard basis of \( \ell_2 \). But then, again, \( X \ni (\exp L_1)_0 \) by the results of \([1]\), and the theorem follows.

\[\square\]

\textbf{§ 5. The Unconditionality of the Haar System in an r.i. Space}

It has already been mentioned that the Haar system is an unconditional basis in a separable r.i. space \( X \) if and only if \( \|x\|_X \asymp \|Px\|_X \), where \( Px \) is the Paley function defined by (5) (see \([15, \text{Theorem 1.11}]\)). We prove a similar unconditionality criterion in terms of the decoupling sum (10).

We start with a statement about comparison of \( Pv = \left( \sum_{k=1}^{\infty} v_k^2 \right)^{1/2} \) and the sum
\[
R'v(s, t) := \sum_{k=1}^{\infty} v_k(s) r_k(t),
\]
which is of independent interest.

\textbf{Proposition 2.} For every r.i. space \( X \) on \( I = [0, 1] \), the following conditions are equivalent.

(a) There exists \( M > 0 \) such that for every \( \{\Sigma_k\} \)-predictable sequence \( \{v_k\} \) we have
\[
\|R'v\|_{X(I \times I)} \leq M \|Pv\|_X.
\]
(b) The lower Boyd index \( \alpha_X \) is strictly positive.

To prove this statement, we need the following lemma (see \([1]\) or \([3, \text{Lemma 7.2(a)}]\)) about the boundedness of the tensor product operator in an r.i. space.

\textbf{Lemma 2.} Let \( \varphi \) be an unbounded measurable function on \( I \). If the operator \( T_\varphi x(s, t) = x(s) \varphi(t) \) is bounded from \( X \) to \( X(I \times I) \), then \( \alpha_X > 0 \).
Proof of Proposition 2. The implication \((b) \Rightarrow (a)\) is well known (see, e.g., [20, Proposition 2.4.1]). So, we only need to prove that \((b)\) is a consequence of \((a)\).

So, let \((28)\) be fulfilled in \(X\). Let \(\varphi\) be the mapping inverse to the function

\[
\Psi(u) = \frac{2}{\sqrt{2\pi}} \int_u^\infty e^{-\frac{v^2}{2}} dv, \quad 0 \leq u < \infty.
\]

Since \(\Psi(0) = 1\), we see that \(\varphi\) is a monotone decreasing function defined on \([0, 1]\) with \(\lim_{t \to +0} \varphi(t) = +\infty\). Let \(z_n := n^{-\frac{1}{2}} \sum_{i=1}^n r_i\). The central limit theorem (see [26, Theorem 3.3.3]) shows that

\[
\lim_{n \to \infty} m\{t \in [0, 1] : |z_n(t)| > u\} = \Psi(u), \quad u \geq 0,
\]

whence \(\lim_{n \to \infty} z_n(t) = \varphi(t)\) \((0 < t \leq 1)\) by [3, Lemma 3.1].

For fixed \(k, l \in \mathbb{N}\) with \(l > k\) and an arbitrary \(\Sigma_k\)-measurable function \(v\), we put \(v_i = 0\) if \(1 \leq i \leq k\) or \(i > l\) and \(v_i = \frac{1}{\sqrt{l-k}} v\) if \(i = k+1, \ldots, l\). Clearly, the sequence \(\{v_i\}\) is \(\{\Sigma_i\}\)-predictable and, moreover,

\[
\sum_{i=1}^\infty v_i(s)r_i(t) = v(s)z_{k,l}(t), \quad \text{where } z_{k,l} := \frac{1}{\sqrt{l-k}} \sum_{i=k+1}^l r_i.
\]

By \((28)\), we obtain

\[
(29) \quad \|v(s)z_{k,l}(t)\|_{X(I \times I)} \leq M\|v\|_X,
\]

where \(M\) is independent of \(v, k,\) and \(l\). Next, since \(z_{k,l}^* = z_{l-k}^\ast\), the above discussion ensures the relation \(\lim_{t \to +0} z_{k,l}^\ast(t) = \varphi(t)\) \((0 < t \leq 1)\) for every \(k \in \mathbb{N}\). Since the norm of \(X\) is order semicontinuous, the properties of \(X''\) and \((29)\) show that

\[
(30) \quad \|T\varphi v\|_{X''(I \times I)} = \|v(s)\varphi(t)\|_{X''(I \times I)} \leq M\|v\|_{X''}.
\]

Clearly, any \(\Sigma_k\)-measurable function (with some \(k \in \mathbb{N}\)) can play the role of \(v\) here. Thus, in particular, \(X \neq L_\infty\). Therefore, the norm of \(X''\) is absolutely continuous on \(L_\infty\).

Let \(x \in L_\infty\). It is easily seen that there exists a sequence \(\{v_k\}\), \(v_k \geq 0\), of \(\Sigma_k\)-measurable functions with \(v_k \leq \|x\|_\infty\) and \(v_k \to |x|\) a.e. on \([0, 1]\). Then \(\|v_k\|_{X''} \to \|x\|_{X''}\) and, by \((30)\),

\[
\|v_k(s)\varphi(t)\|_{X''(I \times I)} \leq M\|v_k\|_{X''}, \quad k = 1, 2, \ldots.
\]

Consequently, by the maximality of \(X''\), the limit passage as \(k \to \infty\) in the last inequality yields

\[
\|T\varphi x\|_{X''(I \times I)} = \|x(s)\varphi(t)\|_{X''(I \times I)} \leq M\|x\|_{X''}
\]

for all \(x \in L_\infty\). Now, if \(x \in X''\) is arbitrary, there exists a monotone increasing sequence of functions \(\{x_k\} \subset L_\infty, x_k \geq 0,\) with \(x_k \to |x|\) a.e. on \([0, 1]\). As before, this allows us to extend the preceding inequality to the entire space \(X''\). As a result, we obtain

\[
\|T\varphi x\|_{X''(I \times I)} \leq M\|x\|_{X''},
\]

whence we see that \(T\varphi\) is bounded from \(X''\) to \(X''(I \times I)\). Then \(\alpha_{X''} > 0\) by Lemma 2. Since \(\alpha_X = \alpha_{X''}\) (see [18, Theorem 2.4.11]), the proposition is proved. \(\square\)

Remark 1. An inequality reverse to \((28)\), more precisely, the estimate

\[
\|Pv\|_X \leq \sqrt{2}\|R'v\|_{X(I \times I)},
\]

holds true in an arbitrary r.i. space \(X\) and for arbitrary \(v_k \in X, k = 1, 2, \ldots\) (see, e.g., [20, Proposition 2.d.1]).
Theorem 3. For an arbitrary r.i. space \( X \), the following conditions are equivalent.

(i) The Haar system is unconditional in \( X \).

(ii) We have

\[
\begin{align*}
\left\| \sum_{k=1}^{\infty} \sum_{i=1}^{2^{k-1}} c_{k,i} h_{k-1,i} \right\|_X \leq C \left\| \sum_{k=1}^{\infty} \sum_{i=1}^{2^{k-1}} c_{k,i}^2 \chi_{\Delta_{k-1}^i}(s) r_k(t) \right\|_{X(I \times I)}
\end{align*}
\]

with constants independent of \( c_{k,i} \in \mathbb{R} \).

Proof. If (i) is fulfilled, then \( 0 < \alpha_X \leq \beta_X < 1 \) by \([20, 2.c.6]\) or \([18, 2.9.6]\). But then, by \([15, \text{Theorem 1.11}]\) and Proposition 2 of the present paper, each of the two sides of (31) is equivalent to the \( X \)-norm of the Paley function, i.e., to the quantity

\[
\left\| \left( \sum_{k=1}^{\infty} \sum_{i=1}^{2^{k-1}} (c_{k,i}^t h_{k-1,i}^t) \right)^{1/2} \right\|_X.
\]

So, we obtain (31).

The reverse implication is a direct consequence of the fact that, by Lemma 1, the family of functions \( \{ \chi_{\Delta_{k-1}^i}(s) r_k(t) : k = 1, 2, \ldots, i = 1, 2, \ldots, 2^{k-1} \} \) is unconditional in an arbitrary r.i. space. \( \square \)

§6. Decoupling-inequality for martingale transforms in r.i. spaces

Now, we want to discuss the r.i. spaces \( X \) in which the inequality

\[
\begin{align*}
\left\| \sum_{k=1}^{\infty} \sum_{i=1}^{2^{k-1}} c_{k,i} h_{k-1,i} \right\|_X \leq C \left\| \sum_{k=1}^{\infty} \sum_{i=1}^{2^{k-1}} c_{k,i} \chi_{\Delta_{k-1}^i}(s) r_k(t) \right\|_{X(I \times I)}
\end{align*}
\]

is fulfilled with some constant \( C > 0 \) depending only on \( X \) or, equivalently, for every \( \{\Sigma_k\} \)-predictable sequence \( \{v_k\} \) we have

\[
\begin{align*}
\left\| \sum_{k=1}^{\infty} v_k r_k \right\|_X \leq C \left\| \sum_{k=1}^{\infty} v_k(s) r_k(t) \right\|_{X(I \times I)}.
\end{align*}
\]

By the first statement of Proposition 1, Proposition 2 and Remark 1 these inequalities are true if \( \alpha_X > 0 \). However, (33) is also true in many r.i. spaces with zero lower Boyd index. Indeed, the main result of [13] shows that this inequality is fulfilled for \( X = L_p \), \( 1 \leq p < \infty \), with a constant \( C \) independent of \( p \). Thus, (33) is fulfilled in an arbitrary r.i. space that is extrapolatory with respect to the scale \( L_p \) (see [5, 6]), in particular, in all exponential Orlicz spaces \( \exp L_p \), \( p > 0 \). At the same time, an example of Talagrand (see [13]) shows that the corresponding estimate for the distributions of the sums in (33) fails. More precisely, there is no constant \( C \) such that the inequality

\[
m \left\{ u \in I : \left| \sum_{k=1}^{\infty} v_k(u) r_k(u) \right| > C \tau \right\} \leq C m \left\{ (s, t) \in I \times I : \left| \sum_{k=1}^{\infty} v_k(s) r_k(t) \right| > \tau \right\}
\]

is true for every \( \{\Sigma_k\} \)-predictable sequence \( \{v_k\} \) and all \( \tau > 0 \). By using that example, we can prove an even stronger statement.

Proposition 3. There is no constant \( C > 0 \) such that for every \( \{\Sigma_k\} \)-predictable sequence \( \{v_k\} \) and all \( 0 < t \leq 1 \) we have

\[
\int_0^t \left( \sum_{k=1}^{\infty} v_k r_k \right)^*(u) \, du \leq C \int_0^t \left( \sum_{k=1}^{\infty} v_k(s) r_k(t) \right)^*(u) \, du.
\]
Proof. As in Talagrand’s example (see [13, Theorem 6.1]), for every $k \in \mathbb{N}$, $k \geq 4$, we put $N_1 = 2^{2k}$ and

$$N_i - N_{i-1} = 2^{1-i}N_1, \quad i = 2, \ldots, k.$$ 

Next, we define the sets

$$A_1 = \{ t \in [0, 1] : r_1(t) = \cdots = r_{N_1}(t) = 1 \},$$

$$A_i = A_{i-1} \cap \{ t \in [0, 1] : r_{N_i-1+1}(t) = \cdots = r_{N_i}(t) \}, \quad i = 2, \ldots, k,$$

and introduce a $\{ \Sigma_k \}$-predictable sequence $\{ v_k \}$ by setting

$$v_1 = \cdots = v_{N_1} = 1,$$

$$v_{N_1+1} = \cdots = v_{N_2} = 2\chi_{A_1},$$

$$v_{N_k-1+1} = \cdots = v_{N_k} = 2^{k-1}\chi_{A_{k-1}}.$$ 

By the definitions, we obtain

$$\left| \sum_{j=1}^{N_k} v_j(u)r_j(u) \right| \leq \sum_{i=1}^{k} \sum_{j=N_{i-1}+1}^{N_i} |v_j(u)r_j(u)| \leq \sum_{i=1}^{k} 2^{i-1}2^{1-i}N_1 = kN_1, \quad 0 \leq u \leq 1,$$

and, similarly,

$$\left(34\right) \left| \sum_{j=1}^{N_k} v_j(s)r_j(t) \right| \leq kN_1, \quad 0 \leq s, \ t \leq 1.$$ 

Consequently,

$$m \left\{ u \in I : \left| \sum_{j=1}^{N_k} v_j(u)r_j(u) \right| \geq kN_1 \right\} = m \left\{ u \in I : \left| \sum_{j=1}^{N_k} v_j(u)r_j(u) \right| = kN_1 \right\}$$

$$= 2m \{ r_j = 1, \ j = 1, 2, \ldots, N_k \} = 2 \cdot 2^{-N_1(1+2+\cdots+2^{1-k})} = 2^{-2N_1(1-2^{-k})},$$

whence for $\alpha_k := 2^{-2N_1(1-2^{-k})}$ we obtain

$$\left(35\right) \int_0^{\alpha_k} \left( \sum_{j=1}^{N_k} v_j r_j \right)^*(u) \, du = kN_1 \alpha_k.$$ 

On the other hand, as was shown in the proof of Theorem 6.1 in [13], we have

$$m \left\{ (s, t) \in I \times I : \left| \sum_{j=1}^{N_k} v_j(s)r_j(t) \right| \geq 4N_1 \right\} \leq k \cdot 2^{-N_12^{2-k}} \cdot \alpha_k,$$

and, therefore, from (34) and (35) it follows that

$$\int_0^{\alpha_k} \left( \sum_{j=1}^{N_k} v_j(s)r_j(t) \right)^*(u) \, du \leq kN_1 \cdot k \cdot 2^{-N_12^{2-k}} \cdot \alpha_k + 4N_1 \cdot \alpha_k$$

$$= kN_1 \alpha_k \cdot \left( k2^{-2k+2} + 4 \right) = \left( k2^{-2k+2} + 4 \right) \int_0^{\alpha_k} \left( \sum_{j=1}^{N_k} v_j r_j \right)^*(u) \, du.$$

Since $k \in \mathbb{N}$ may be taken arbitrarily large, the statement follows. \qed
References


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