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OSCILLATION METHOD IN THE SPECTRAL PROBLEM FOR A FOURTH ORDER DIFFERENTIAL OPERATOR WITH A SELF-SIMILAR WEIGHT

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Abstract. Selfadjoint boundary problems are considered for the differential equation $y^{(4)} - \lambda\rho y = 0$, where the weight $\rho \in W_2^{-1}[0,1]$ is the generalized derivative of a self-similar function of the Kantor type. On the basis of the study of oscillation properties of eigenfunctions, the characteristics of the known spectral asymptotics of such problems are refined.

§1. Introduction

1.1. The present paper is aimed at the application of the oscillation method for exploring the spectral asymptotics of the Sturm–Liouville type problems with self-similar weight, as elaborated in [1], to the case of the selfadjoint problem

$$y^{(4)} - \lambda\rho y = 0, \tag{1.1}$$

$$(U-1)y^{\vee} + i(U+1)y^{\wedge} = 0, \tag{1.2}$$

where $\rho \in W_2^{-1}[0,1]$ is a nonnegative generalized weight function, $U \in \mathbb{C}^{4\times 4}$ is a unitary matrix of boundary conditions, and $y^{\wedge}$ and $y^{\vee}$ are numerical vectors,

$$y^{\wedge} \rightleftharpoons \begin{pmatrix} y(0) & y'(0) & y(1) & y'(1) \end{pmatrix}^T,$$
$$y^{\vee} \rightleftharpoons \begin{pmatrix} -y'''(0) & y''(0) & y'''(1) & -y''(1) \end{pmatrix}^T$$

(cf. [2, (7.50)]). The investigation of the oscillation properties of eigenfunctions for fourth order problems requires application of methods other than those employed usually in the theory of Sturm–Liouville problems (see, e.g., [3, 4]). Therefore, the transfer of the results of the paper [1], based on such investigations, to the case of problem (1.1), (1.2) is not immediate.

A principal tool of the construction developed in [1], which will also be used below in the present paper, is the following criterion for the singularity of a monotone function.

Proposition 1 (see [1, §4.1.2, §4.1.3]). *A bounded and monotone nondecreasing function $f \in L_2[0,1]$ is singular if and only if there is a sequence $\{f_n\}_{n=0}^{\infty}$ of monotone nondecreasing step functions such that*

$$(\#\mathfrak{A}_n + 2)\,\|f - f_n\|_{L_2[0,1]} = o(1),$$

where $\mathfrak{A}_n$ denotes the set of discontinuity points for f_n.

The form of the central result of [1] coincides, up to a somewhat different definition of the coefficient ν, with Proposition 10 stated below. The restrictions used there on the

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character of the weight function are the same as in the next subsection. Some results on the spectrum of higher order problems, which are rougher than those in the present paper, but are valid under less restrictive assumptions about the weight function, can be found in [5, §3].

In the papers [1, 5], some historical information was presented about the study of ordinary differential equations with self-similar weights.

1.2. In what follows, the boundary problems (1.1), (1.2) will not be treated in the maximal generality. Namely, we shall assume[1] that relations (1.2) can be written in the form

$$\begin{aligned} y''(0) + \alpha y(0) - \beta y'(0) &= -\beta y'''(0) + \alpha y''(0) \\ &= y''(1) + \alpha y(1) + \beta y'(1) = -\beta y'''(1) - \alpha y''(1) = 0, \end{aligned} \tag{1.3}$$

with $\alpha \geq 0$, $\beta > 0$. Also, we usually impose the assumption that the function ρ the generalized derivative of a monotone nondecreasing function $P \in C[0,1]$ with self-similarity of Kantor type.

Definition 1 (see [1, §2])**.** A function $P \in C[0,1]$ is called a self-similar function of Kantor type if $P(0) = 0$, $P(1) = 1$, and there exists a natural number $\varkappa \geq 2$ and real numbers $a \in (0, 1/\varkappa)$ and $b \rightleftharpoons (1 - \varkappa a)/(\varkappa - 1)$ with the following properties.

(1) For any index $k = 0, \dots, \varkappa - 1$, the function $P_k \in C[0,1]$ defined by

$$P_k(x) \rightleftharpoons \varkappa P(k[a+b] + ax)$$

coincides with P up to an additive constant.

(2) For any index $k = 1, \dots, \varkappa - 1$, the function P is constant on the interval

$$(k[a+b] - b, k[a+b]).$$

In particular, the well-known Kantor function is a self-similar Kantor-like function with the parameters $\varkappa = 2$, $a = b = 1/3$. It is easily seen that every self-similar function P of Kantor type satisfies the identity $P(x) \equiv 1 - P(1-x)$.

1.3. In a standard way (see [6]), the formal problem (1.1), (1.3) gives rise to the linear pencil $T\colon \mathbb{C} \to \mathcal{B}(W_2^2[0,1], W_2^{-2}[0,1])$ of operators of the form

$$\langle T(\lambda)y, y\rangle \equiv \int_0^1 |y''|^2\,dx + \frac{|\alpha y(0) - \beta y'(0)|^2 + |\alpha y(1) + \beta y'(1)|^2}{\beta} - \lambda\langle \rho, |y|^2\rangle. \tag{1.4}$$

Integrating by parts (see [6, Lemma 2]), it is easy to check that a pair $\{\lambda, y\}$ consisting of a number $\lambda \in \mathbb{C}$ and a nontrivial function $y \in W_2^2[0,1]$ is an eigenpair of the pencil T if and only if the functions y'' and $y''' - \lambda P y$ are continuously differentiable and satisfy the equation

$$[y''' - \lambda P y]' + \lambda P y' = 0 \tag{1.5}$$

together with the boundary conditions (1.3). This observation will be used repeatedly in what follows.

1.4. We outline the layout of the paper. In §2 we present some information about the oscillation of eigenfunctions of the problems under consideration. This material is fairly standard [3, 4] and cannot be fully qualified as new. In §3 we consider the spectral periodicity phenomenon and the properties of spectral asymptotics implied by it. Also, here we present some results of numerical experiments that illustrate the theoretical results obtained.

[1]This restriction is technical and does not affect the applicability (implied by the variational principles, see Proposition 10) of the final result to the case of arbitrary boundary conditions (1.2).

§2. Oscillation of eigenfunctions

2.1. We start with two statements.

Proposition 2. *Let $\lambda > 0$, and let $y \in C^3[0,1]$ be a nontrivial solution of equation* (1.5) *that satisfies the inequalities*

$$y(a) \geq 0, \quad y'(a) \geq 0, \quad y''(a) \geq 0, \quad y'''(a) \geq 0$$

for some $a \in [0,1)$. Then we also have

$$y(1) > 0, \quad y'(1) > 0, \quad y''(1) > 0, \quad y'''(1) > 0.$$

Proof. Let $\{P_n\}_{n=0}^{\infty}$ be a sequence of functions of class $C^1[0,1]$ that converge uniformly to P, satisfy $P_n(a) = P(a)$, and have positive derivatives uniformly on $[0,1]$. Also, let $\{y_n\}_{n=0}^{\infty}$ be the sequence of solutions of the initial problems

$$[y_n''' - \lambda P_n y_n]' + \lambda P_n y_n' = 0, \quad y_n^{(k)}(a) = y^{(k)}(a), \quad k \in \overline{0,3}. \tag{2.1}$$

The standard methods of the theory of linear differential equations (see [7, §16] show that the sequences $\{y_n\}_{n=0}^{\infty}$, $\{y_n'\}_{n=0}^{\infty}$, $\{y_n''\}_{n=0}^{\infty}$ and $\{y_n''' - \lambda P_n y_n\}_{n=0}^{\infty}$ converge uniformly on $[0,1]$ to the functions y, y', y'' and $y''' - \lambda P y$, respectively.

By [4, Lemma 2.1], each of the functions y_n, y_n', y_n'', and y_n''' is strictly positive and monotone increasing on $(a,1]$. Therefore, for any $x \in (a,1)$ we have

$$\begin{aligned} y_n'''(1) &\geq \int_x^1 y_n^{(4)}(t)\,dt = \int_x^1 \lambda P_n'(t) y_n(t)\,dt \\ &\geq \int_x^1 \lambda P_n'(t)\,dt \cdot y_n(x) = \lambda \cdot [P_n(1) - P_n(x)]\, y_n(x). \end{aligned}$$

Letting $n \to \infty$, we immediately see that the functions y', y'', and y''' are nonnegative and monotone nondecreasing on $(a,1]$ and that

$$(\forall x \in (a,1)) \quad y'''(1) \geq \lambda \cdot [P(1) - P(x)] y(x). \tag{2.2}$$

Since the function y is nontrivial, we can find $\gamma > 0$ such that

$$(\forall x \in (a,1]) \quad y(x) \geq \gamma \cdot (x-a)^3. \tag{2.3}$$

Definition 1 implies that the function P is nonconstant in any left neighborhood of the point 1; therefore, using (2.2) and (2.3), we conclude that $y'''(1) > 0$. Now the remaining required inequalities follow from the Lagrange mean value theorem. □

Proposition 3. *Let $\lambda > 0$, and let $y \in C^3[0,1]$ be a nontrivial solution of equation* (1.5) *that satisfies the inequalities*

$$y(a) \geq 0, \quad y'(a) \leq 0, \quad y''(a) \geq 0, \quad y'''(a) \leq 0$$

for some $a \in (0,1]$. Then we also have

$$y(0) > 0, \quad y'(0) < 0, \quad y''(0) > 0, \quad y'''(0) < 0.$$

The proof is completely similar to that of Proposition 2.

2.2. We present two more facts.

Proposition 4. *The spectrum of the pencil T is formed by a sequence $\{\lambda_n\}_{n=0}^{\infty}$ of nonnegative (and even strictly positive if $\alpha > 0$) simple eigenvalues. For any index n, the eigenfunction y_n corresponding to λ_n has only simple zeros and satisfies $y_n(0) \neq 0$, $y_n(1) \neq 0$.*

Proof. Formula (1.4) for the quadratic form of the operator $T(0)$ shows that the kernel of this operator consists of the linear functions that satisfy

$$\alpha y(0) - \beta y'(0) = \alpha y(1) + \beta y'(1) = 0.$$

If $\alpha > 0$, then such a function is necessarily equal to zero identically. Accordingly, in this case all eigenvalues of the pencil T are strictly positive. For $\alpha = 0$, such functions form the 1-dimensional subspace of constant functions.

Suppose that an eigenvalue $\lambda > 0$ of T has an eigenfunction y with $y(0) = 0$. Then there is no loss of generality in assuming that the function y is real-valued and that the signs of the quantities $y'(0)$, $y''(0)$, and $y'''(0)$ coincide, see [(1.3)]. But then, by Proposition 3, the quantities $y''(1) \neq 0$ and $y'''(1) \neq 0$ have equal signs, which contradicts (1.3).

Suppose that an eigenvalue $\lambda > 0$ of the pencil T is multiple; then there is an eigenfunction y corresponding to λ and such that $y(1) = 0$, which contradicts the said above.

Finally, suppose that, for some eigenvalue $\lambda > 0$ of T, the corresponding eigenfunction y has a multiple zero $a \in (0, 1)$.

If the signs of $y''(a)$ and $y'''(a)$ coincide, then, by Proposition 2, so do the signs of $y''(1) \neq 0$ and $y'''(1)$, which contradicts (1.3). Otherwise, if the signs of $y''(a)$ and $y'''(a)$ are different, then the signs of $y''(0) \neq 0$ and $-y'''(0) \neq 0$ coincide by Proposition 3, again contradicting (1.3). □

Proposition 5. *If $\alpha > 0$, then the operator $[T(0)]^{-1}T'(0)\colon W_2^2[0,1] \to W_2^2[0,1]$ cannot enlarge the number of sign changes of any real-valued function.*

Proof. Since the operator $[T(0)]^{-1}T'(0)$ depends on the choice of the weight function $\rho \in W_2^{-1}[0,1]$ continuously in the uniform operator topology, it suffices to handle the case where ρ is continuous and uniformly positive. In other words, it suffices to check that, for any natural[2] $n > 0$ and m, if a real-valued function $y \in C^4[0,1]$ satisfying (1.3) has at least $n + (4-m)^+$ sign changes, then $y^{(4)}$ has at least n sign changes. For $m = 0$, this follows immediately from the Lagrange mean value theorem. The general case will be treated below with the help of induction on m.

So, suppose that, for some $n > 0$, if $y \in C^4[0,1]$ is an arbitrary real-valued function that satisfies (1.3) and has at least $n + 1 + (4-m)^+$ sign changes, then $y^{(4)}$ has at least n sign changes. Now, let $y \in C^4[0,1]$ be a real-valued function satisfying (1.3) and such that there are $n + 1 + (4-m)^+$ points

$$0 < \xi_{0,1} < \cdots < \xi_{0,n+1+(4-m)^+} < 1$$

with $y(\xi_{0,k}) \cdot y(\xi_{0,k+1}) < 0$, where $k = 1, \ldots, n+1+(4-m)^+$. By the Lagrange theorem, there are $n + (4-m)^+$ points $\xi_{1,k} \in (\xi_{0,k}, \xi_{0,k+1})$ such that $y'(\xi_{1,k}) \cdot y(\xi_{0,k}) < 0$. Observe that either we can find $\xi \in (0, \xi_{0,1})$ with $y(\xi) \cdot y(\xi_{0,1}) < 0$, or $y'(0) \cdot y'(\xi_{1,1}) < |y'(\xi_{1,1})|^2$, or $y''(0) \cdot y'(\xi_{1,1}) > 0$; this follows from the expression of $y''(0)$ in terms of $y(0)$ and $y'(0)$ that occurs among the boundary conditions (1.3).

In the first case, y has at least $n+1+(4-m)^+$ sign changes, whence $y^{(4)}$ has at least n sign changes by the inductive assumption. In the second and third cases, there exists a point $\xi_{2,1} \in (0, \xi_{1,1})$ such that either $y''(\xi_{2,1}) \cdot y'(\xi_{1,1}) > 0$, or $y(\xi) \cdot y(\xi_{0,n+1+(4-m)^+}) < 0$, or $y'(1) \cdot y'(\xi_{1,n+(4-m)^+}) < |y'(\xi_{1,n+(4-m)^+})|^2$, or $y''(1) \cdot y'(\xi_{1,n+(4-m)^+}) < 0$.

In the first case, y has at least $n+1+(4-m)^+$ sign changes. In the second and third cases, there exists a point $\xi_{2,n+1+(4-m)^+} \in (\xi_{1,n+(4-m)^+}, 1)$ such that

$$y''(\xi_{2,n+1+(4-m)^+}) \cdot y'(\xi_{1,n+(4-m)^+}) < 0.$$

[2]We always assume that 0 is the smallest natural number.

Summarizing, we see that either $y^{(4)}$ has at least n sign changes, or there are $n+1+(4-m)^+$ points

$$0 < \xi_{2,1} < \cdots < \xi_{2,n+1+(4-m)^+} < 1$$

such that $y''(\xi_{2,k}) \cdot y''(\xi_{2,k+1}) < 0$. In the second case, by the boundary conditions (1.3), we have either $y''(0) \cdot y''(\xi_{2,1}) < |y''(\xi_{2,1})|^2$, or $y'''(0) \cdot y''(\xi_{2,1}) > 0$. Similarly, we have either $y''(1) \cdot y''(\xi_{2,n+1+(4-m)^+}) < |y''(\xi_{2,n+1+(4-m)^+})|^2$, or $y'''(1) \cdot y''(\xi_{2,n+1+(4-m)^+}) < 0$. In either subcase we can find a point $\xi_{3,n+1+(4-m)^+} \in (\xi_{2,n+1+(4-m)^+}, 1)$ such that $y'''(\xi_{3,n+1+(4-m)^+}) \cdot y''(\xi_{2,n+1+(4-m)^+}) < 0$. Accordingly, the function y''' has at least $n+1+(4-m)^+$ sign changes, whence $y^{(4)}$ has at least $n+(4-m)^+ \geq n$ sign changes by the Lagrange theorem. This completes the inductive step. □

2.3.

Proposition 6. *Suppose that a real-valued function $f \in W_2^2[0,1]$ satisfies $f(0) \neq 0$ and $f(1) \neq 0$ and has exactly n simple zeros on $(0,1)$. Then there exists $\varepsilon > 0$ such that any real-valued function $y \in W_2^2[0,1]$ with $\|y - f\|_{W_2^2[0,1]} < \varepsilon$ also has exactly n simple zeros on $(0,1)$.*

This claim follows trivially from the continuity of the natural embedding $W_2^2[0,1] \hookrightarrow C^1[0,1]$.

Proposition 7. *Let $\{\lambda_n\}_{n=0}^{\infty}$ be the sequence of eigenvalues of the pencil T, enumerated in the increasing order. Then, for any $n \in \mathbb{N}$, the eigenfunction y_n corresponding to λ_n has exactly n zeros on $(0,1)$.*

Proof. First, let $\alpha > 0$. Observe that Proposition 4 allows us, starting with any real-valued function of the form

$$f = \sum_{k=0}^{n} c_k y_k, \tag{2.4}$$

to build the following sequence of functions:

$$f_m \rightleftharpoons \sum_{k=0}^{n} c_k \lambda_k^m \lambda_n^{-m} y_k.$$

Obviously, this sequence satisfies $f_m = \lambda_n \cdot [T(0)]^{-1} T'(0) f_{m+1}$ and converges in $W_2^2[0,1]$ to $c_n y_n$, see Proposition 4. Propositions 5, 4, and 6 show that if $c_n \neq 0$, then the number of sign changes of f is at most the number of zeros of y_n. However, since the eigenfunctions of the pencil T are linearly independent, we can find a function of the form (2.4) such that $c_n \neq 0$ and having at least n sign changes on $(0,1)$. Thus, the function y_n has at least n zeros.

Next, we fix a nontrivial real-valued polynomial Q having degree at most n and belonging to the invariant subspace of the operator $[T(0)]^{-1}T'(0)$ that corresponds to the part of the spectrum complementary to $\{\lambda_k\}_{k=0}^{n-1}$. Since the support of the weight function ρ is infinite, the polynomial Q cannot belong to the kernel of the operator $[T(0)]^{-1}T'(0)$. Therefore, there is an index $N \geq n$ such that the sequence $\{Q_m\}_{m=0}^{\infty}$ defined by

$$Q_m \rightleftharpoons \lambda_N^m \{[T(0)]^{-1} T'(0)\}^m Q$$

converges in $W_2^2[0,1]$ to a nontrivial multiple of the eigenfunction y_N. By Propositions 5, 4, and 6, the number of zeros of y_N cannot exceed the number of the sign changes of Q, and, thus, is at most n. Summarizing, we conclude that $N = n$ and that y_n has precisely n zeros on $(0,1)$.

The extension of the results obtained to the general case of $\alpha \geq 0$ is done by the limit passage, with the help of Propositions 4 and 6. □

§3. Spectral periodicity and asymptotics of eigenvalues

3.1. We formulate two statements.

Proposition 8. *Let $\{\lambda_n\}_{n=0}^{\infty}$ be the sequence of eigenvalues of the boundary problem* (1.1), (1.3) *with $\alpha = 0$ and $\beta = 2/b$, enumerated in the increasing order, and let $\{\mu_n\}_{n=0}^{\infty}$ be a similar sequence in the case where $\alpha = 0$, $\beta = 2a/b$. Then for any $n \in \mathbb{N}$ we have*

$$\lambda_{\varkappa n} = (\varkappa/a^3)\,\mu_n.$$

Proof. Since the identity in question is obvious for $\lambda_0 = \mu_0 = 0$, we assume that $n > 0$. We fix an eigenfunction y that corresponds to the eigenvalue $\mu_n > 0$, has precisely n distinct zeros on $(0,1)$, and does not vanish at 0 or 1, see Propositions 7 and 4. Since the eigenvalue μ_n is simple, the identity $P(x) \equiv 1 - P(1-x)$ (see Subsection 1.2) shows that the eigenfunction y, which satisfies the equation

$$[y''' - \mu_n P y]' + \mu_n P y' = 0$$

and the boundary conditions (1.3), is either even, or odd relative to the point 1/2. This observation allows us to construct a function $z \in C^3[0,1]$ with the following properties.

(1) For any $k = 0, \dots, \varkappa - 1$, the function

$$z_k(x) \rightleftharpoons z(k[a+b] + ax)$$

coincides with y up to the sign.

(2) for any $k = 1, \dots, \varkappa - 1$, on the interval $(k[a+b] - b, k[a+b])$ we have

$$|z(x)| \equiv \left| y(0) + \frac{y''(0)}{2a^2} \cdot (x - k[a+b] + b) \cdot (x - k[a+b]) \right|.$$

Since the function P is self-similar, direct verification shows that z satisfies the equation

$$[z''' - (\varkappa/a^3)\,\mu_n P z]' + (\varkappa/a^3)\,\mu_n P z = 0.$$

Moreover, the inequality $y(0) \cdot y''(0) < 0$ (see (1.3) and Proposition 2) implies that z has precisely $\varkappa n$ zeros on $(0,1)$. Thus, the required claim is true by Proposition 7. □

Proposition 9. *Let $\{\lambda_n\}_{n=0}^{\infty}$ be the sequence of eigenvalues of problem* (1.1), (1.3) *with $\alpha = 12/b^2$, $\beta = 6/b$, enumerated in the increasing order, and let $\{\mu_n\}_{n=0}^{\infty}$ be a similar sequence in the case where $\alpha = 12a^2/b^2$, $\beta = 6a/b$. Then for any $n \in \mathbb{N}$ we have*

$$\lambda_{\varkappa(n+1)-1} = (\varkappa/a^3)\,\mu_n.$$

Proof. The proof is similar to that of Proposition 8, with the main difference that for the "gluing" of copies of the initial eigenfunction we now use cubic (rather than quadratic) parabolas of the form

$$\zeta_k \cdot \left[\frac{y''(0)}{3a^2 b} \cdot \left(\zeta_k^2 - \frac{b^2}{4} \right) + \frac{2y(0)}{b} \right], \tag{3.1}$$

where $\zeta_k \rightleftharpoons x - k[a+b] + b/2$. In view of the fact that the signs of $y(0)$ and $y''(0)$ are different (see (1.3) and Proposition 2), each of the functions (3.1) has a single zero on the interval $(k[a+b] - b, k[a+b])$ corresponding to it. This means that the function z has precisely $\varkappa(n+1) - 1$ zeros on $(0,1)$. □

3.2. We finish with yet another fact.

Proposition 10. *Let $N\colon (0,+\infty) \to \mathbb{N}$ be the counting function for the eigenvalues of the pencil T. Then, as $\lambda \to +\infty$, we have the following asymptotic relation*:

$$N(\lambda) = \lambda^D \cdot [s(\ln\lambda) + o(1)], \tag{3.2}$$

where $D \rightleftharpoons \nu^{-1}\ln\varkappa$, $\nu \rightleftharpoons \ln\varkappa - 3\ln a$, and s is a ν-periodic function given on $[0,\nu]$ by the formula

$$s(t) \equiv e^{-Dt}\sigma(t), \tag{3.3}$$

in which σ is some purely singular and monotone decreasing function.

Proof. Relation (1.4) shows that changing the values of the parameters α and β results in perturbing the operators of the pencil T by some operator whose rank is at most 4. Therefore, the leading term of the asymptotics of the counting function N does not depend on the values indicated. In the remaining part of the proof we mainly keep in mind the problem for which $\alpha = 0$, $\beta = 2a/b$, with the eigenvalues $\{\mu_n\}_{n=0}^{\infty}$. The sequence of eigenvalues of the problem with $\alpha = 0$, $\beta = 2/b$ will be denoted by $\{\lambda_n\}_{n=0}^{\infty}$.

We introduce the functions $\sigma_k(t) \rightleftharpoons \varkappa^{-k}N(e^{k\nu+t})$ defined on the segment $[0,\nu]$. Note that, for any k, $n \in \mathbb{N}$ and $t \in [0,\nu]$, if

$$\mu_n < e^{k\nu+t} \le \mu_{n+1},$$

then

$$\mu_{\varkappa n} \le \lambda_{\varkappa n} < e^{(k+1)\nu+t} \le \lambda_{\varkappa(n+1)} \le \mu_{\varkappa(n+1)+2}.$$

see Proposition 8 and (1.4).

Thus, for any $k \in \mathbb{N}$ and $t \in [0,\nu]$ we have

$$|\sigma_{k+1}(t) - \sigma_k(t)| \le \varkappa^{-k}, \tag{3.4}$$

which implies that the sequence $\{\sigma_k\}_{k=0}^{\infty}$ converges uniformly to some function σ satisfying (3.2) and (3.3).

Next, for any k, $n \in \mathbb{N}$ and $t \in [0,\nu]$, if

$$\sup(\mu_{\varkappa(n+1)-1}, \lambda_{\varkappa n}) < e^{(k+1)\nu+t} \le \mu_{\varkappa(n+1)}$$

then $\sigma_{k+1}(t) = \sigma_k(t)$. Repeating the arguments from the proofs in [1, §5.1.1] and [1, §5.2.1] almost word for word, we conclude (see Propositions 8 and 9) that the partial sums of the series

$$\sum_{n=1}^{\infty} |\ln\mu_{\varkappa(n+1)-1} - \ln\mu_{\varkappa n}|, \quad \sum_{n=1}^{\infty} |\ln\lambda_{\varkappa n} - \ln\mu_{\varkappa n}|.$$

are bounded. Therefore, the sequence of the measures of the sets

$$\{t \in [0,\nu]\colon \sigma_{k+1}(t) \ne \sigma_k(t)\}$$

is asymptotically $o(1)$ as $k \to \infty$, which implies (see (3.4)) that

$$\|\sigma_{k+1} - \sigma_k\|_{L_2[0,1]} = o(\varkappa^{-k}), \quad \|\sigma_k - \sigma\|_{L_2[0,1]} = o(\varkappa^{-k}).$$

As was already noted, see (3.4), the functions σ_k are uniformly bounded; thus, the number of discontinuity points of these functions is asymptotically $O(\varkappa^k)$ as $k \to \infty$. Now, to complete the proof it remains to refer to the criterion of singularity (Proposition 1). □

3.3. The tables contain data pertaining to the case where the role of the weight function is played by the generalized derivative of the Kantor staircase function. Table 1 illustrates Proposition 8. Table 2 illustrates Proposition 9.

TABLE 1. Estimates of first eigenvalues for problems with $\alpha = 0$, $\beta = 2$ and $\alpha = 0$, $\beta = 6$ in the case where $\varkappa = 2$, $a = b = 1/3$.

n	μ_n	$54\mu_n$	λ_n
1	$2,2131 \cdot 10^1 \pm 10^{-3}$	$1,1951 \cdot 10^3 \pm 10^{-1}$	$4,0965 \cdot 10^1 \pm 10^{-3}$
2	$8,1717 \cdot 10^2 \pm 10^{-2}$	$4,4127 \cdot 10^4 \pm 10^0$	$1,1951 \cdot 10^3 \pm 10^{-1}$
3	$3,175 \cdot 10^3 \pm 10^0$	$1,714 \cdot 10^5 \pm 10^2$	$3,867 \cdot 10^3 \pm 10^0$
4	$3,849 \cdot 10^4 \pm 10^1$	$2,078 \cdot 10^6 \pm 10^3$	$4,412 \cdot 10^4 \pm 10^1$

TABLE 2. Estimates of first eigenvalues for problems with $\alpha = 12$, $\beta = 6$ and $\alpha = 108$, $\beta = 18$ in the case where $\varkappa = 2$, $a = b = 1/3$.

n	μ_n	$54\mu_n$	λ_n
0	$8,2987 \cdot 10^0 \pm 10^{-4}$	$4,4813 \cdot 10^2 \pm 10^{-2}$	$4,0965 \cdot 10^1 \pm 10^{-3}$
1	$1,3784 \cdot 10^2 \pm 10^{-2}$	$7,443 \cdot 10^3 \pm 10^0$	$4,4813 \cdot 10^2 \pm 10^{-2}$
2	$1,6311 \cdot 10^3 \pm 10^{-1}$	$8,808 \cdot 10^4 \pm 10^1$	$3,867 \cdot 10^3 \pm 10^0$
3	$4,380 \cdot 10^3 \pm 10^0$	$2,365 \cdot 10^5 \pm 10^2$	$7,443 \cdot 10^3 \pm 10^0$
4	$4,586 \cdot 10^4 \pm 10^1$	$2,476 \cdot 10^6 \pm 10^3$	$6,251 \cdot 10^4 \pm 10^1$
5	$6,465 \cdot 10^4 \pm 10^1$	$3,491 \cdot 10^6 \pm 10^3$	$8,808 \cdot 10^4 \pm 10^1$

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