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DISCRETE SPECTRUM OF A PERIODIC SCHRÖDINGER OPERATOR WITH VARIABLE METRIC PERTURBED BY A NONNEGATIVE RAPIDLY DECAYING POTENTIAL

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To the memory of M. Sh. Birman

Abstract. The discrete spectrum is investigated that emerges in spectral gaps of the elliptic periodic operator $A = -\mathrm{div} a(x)\mathrm{grad} + b(x)$, $x \in \mathbb{R}^d$, perturbed by a nonnegative, "rapidly" decaying potential

$$0 \le V(x) \sim v(x/|x|)|x|^{-\varrho}, \quad |x| \to +\infty, \quad \varrho \ge d.$$

The asymptotics of the number of eigenvalues for the perturbed operator $B(t) = A + tV$, $t > 0$, that have crossed a fixed point of the gap, is established with respect to the large coupling constant t.

§0. Introduction

In the present paper, we study the discrete spectrum that emerges in spectral gaps of an elliptic periodic selfadjoint second order differential operator perturbed by a nonnegative decaying potential. It is assumed that the unperturbed operator A acts in $L_2(\mathbb{R}^d)$ and has the form

$$A = -\operatorname{div} a(x) \operatorname{grad} + b(x). \tag{0.1}$$

Here the real-valued symmetric $(d \times d)$-matrix-valued function $a(x)$, $x \in \mathbb{R}^d$, is bounded, $\mathbb{Z}^d$-periodic, and uniformly elliptic; the function $b(x)$, $x \in \mathbb{R}^d$, is real-valued, bounded, and $\mathbb{Z}^d$-periodic. The operator A is perturbed by a *nonnegative* bounded potential $V(x)$, $x \in \mathbb{R}^d$, with power-like decay at infinity,

$$V(x) \sim v(x/|x|)|x|^{-\varrho}, \quad |x| \to +\infty, \quad \varrho > 0. \tag{0.2}$$

It is assumed that the spectrum of A possesses an *internal* gap (α, β). The spectrum of the perturbed operator $B(t) = A + tV$, $t > 0$ in the gap (α, β) is discrete. The eigenvalues of $B(t)$ inside the gap (α, β) move from left to right as t increases. The principal object of investigation is the *counting function* $N(\lambda, \tau, V)$ equal to the number of eigenvalues of the operator $B(t)$ that have passed through a fixed "observation point" $\lambda \in (\alpha, \beta)$ as t grew from 0 to τ. At the left endpoint of the gap the counting function is defined by $N(\alpha, \tau, V) := \lim_{\lambda \to \alpha+0} N(\lambda, \tau, V)$; the quantity $N(\alpha, \tau, V)$ can be interpreted as the number of eigenvalues that have been "born" at the left endpoint of the gap as t increased from 0 to τ.

In a nutshell, the main result of the paper (see Theorems 1.2 and 1.3) is the following statement: *if the perturbation admits a power-like asymptotics* (0.2)*, then for the counting*

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function we have

$$N(\lambda,\tau,V)\sim\Gamma_\varrho(\lambda)\tau^{d/\varrho},\quad \tau\to+\infty,\quad \lambda\in(\alpha,\beta). \tag{0.3}$$

The coefficient $\Gamma_\varrho(\lambda)$ can be determined in terms of *band functions* of the operator A and the perturbation V. Under certain conditions, the asymptotics (0.3) remains valid also in the case where $\lambda=\alpha$.

The sign of the perturbation affects significantly both the result and the technique utilized. The negative perturbations are in a certain sense better studied than the nonnegative ones (see, e.g., [1]). The present paper deals with the case of nonnegative perturbations.

For the first time, the asymptotics (0.3) was obtained in the paper [2] for the case $A=-\Delta+b(x)$. There, the potential $b(x)$ was not assumed to be periodic; instead, the restriction that the operator A possesses an integrated density of states was imposed. In [2], the asymptotics (0.3) was only established inside the gap. The validity of (0.3) at the left endpoint of the gap was discussed in [3] for the case of $A=-\Delta+b(x)$, where $b(x)$ is a periodic function. In the survey [1], it was conjectured that the asymptotics (0.3) occurs in the case where A admits the form (0.1); the conditions for the asymptotics (0.3) to be valid with $\lambda=\alpha$ were also discussed. This conjecture was proved in [4] in the case of a "slow" decay of the potential V (in the case where $\varrho\in(0,d)$). In the present paper, we continue the investigation started in [4]. Namely, the asymptotics (0.3) is ascertained for the operator (0.1) perturbed by a potential (0.2) in the case where $\varrho\geq d$. Moreover, conditions are given under which the asymptotics (0.3) holds true for $\lambda=\alpha$. No smoothness conditions on the coefficients of the operator A or the potential V are imposed.

The case of a "fast" decay of the potential V (with $\varrho\geq d$) is technically more involved than the "slow" case. The required results pertaining to the estimates and asymptotics for singular values of compact operators admitting the form $\varphi(A)W(x)$ were obtained in [5, 6]; the results of [5, 6] allows us to reduce the case of a "fast" decay to that of a "slow" decay and then to utilize the relevant results of [4] (cf. Proposition 2.6).

Let us agree on the notation. Let $(\mathcal{X},d\rho)$ be a measurable space; for a measurable function $f\colon\mathcal{X}\to\mathbb{C}$ the symbol $[f(x)]$ denotes the operator of multiplication by f in the space $L_2(\mathcal{X},d\rho)$. The norm in a (quasi)normed space X is denoted by $\|\cdot\|_X$; if the spaces X, Y are (quasi)normed, the standard norm of a bounded linear operator $T\colon X\to Y$ is denoted either by $\|T\|_{X\to Y}$ or by $\|T\|$ (with no subscript), provided that the latter does not lead to a confusion. For a separable measurable space with a σ-finite measure $(\mathcal{Z},d\nu)$, the notation $L_p(\mathcal{Z},d\nu)$ means the standard class L_p; if $\mathcal{Z}$ is a countable set, we employ the notation $L_p(\mathcal{Z},d\nu)=:l_p(\mathcal{Z},d\nu)$. Moreover, $\mathbf{1}$ is a $(d\times d)$-matrix identity; $\langle\,\cdot\,,\cdot\,\rangle$ is the standard inner product in $\mathbb{C}^d$; $\mathbb{Z}^d$ is the lattice of integers in $\mathbb{R}^d$; $H^1(\Omega)$ is the standard Sobolev class in the domain $\Omega\subset\mathbb{R}^d$; $H^1_{\mathrm{loc}}(\Omega)$ is the class of functions $u(x)$, $x\in\Omega$, such that $u\varphi\in H^1(\Omega)$ for any function $\varphi\in C_0^\infty(\Omega)$; Δ is the Laplacian in $\mathbb{R}^d$. For a densely defined closed operator A in a Hilbert space, its spectrum is denoted by $\sigma(A)$ and its adjoint by A^*. If A is selfadjoint, $E_A(\,\cdot\,)$ is its spectral measure. For a Borel set $\delta\subset\mathbb{R}$ we put $\pi_A(\delta):=\operatorname{rank}E_A(\delta)$. Next, $n_\pm(s,A):=\pi_{\pm A}(s,+\infty)$, $s>0$, are the distribution functions of the positive and negative spectra of the operator A. The symbols $\mathfrak{B}$ and $\mathfrak{S}_\infty$ stand for the classes of all bounded and all compact linear operators, respectively.

The paper consists of the Introduction and three sections. In §1 we pose the problem and describe the main results. In §2 we collect of essential information pertaining to estimates and asymptotics for singular values of compact operators admitting the form $\varphi(A)[W(x)]$. The proofs of the main results are contained in §3.

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§1. The problem and main result

1. The unperturbed operator. Let a be a real-valued symmetric $(d \times d)$-matrix-valued function, and let b be a real-valued function such that

$$a, b \in L_\infty(\mathbb{R}^d), \quad a(x) \geq c_0 \mathbf{1}, \quad c_0 > 0, \tag{1.1}$$

$$a(x+n) = a(x), \quad b(x+n) = b(x), \quad x \in \mathbb{R}^d, \quad n \in \mathbb{Z}^d. \tag{1.2}$$

The operator (0.1) acts in $L_2(\mathbb{R}^d)$, $d \geq 1$, and is generated by the closed semibounded quadratic form

$$\mathfrak{a}[u,u] := \int_{\mathbb{R}^d} \left(\langle a(x)\nabla u, \nabla u\rangle + b(x)|u|^2 \right) dx, \quad u \in H^1(\mathbb{R}^d).$$

The operator A is $\mathbb{Z}^d$-periodic. In order to describe the spectrum of A, we introduce the standard objects that are linked with the spectral resolution of a periodic elliptic operator (see, e.g., [7, XIII.16]; [8]). Let $\Omega := [0,1)^d$ be the periodicity cell of $\mathbb{Z}^d$, and let $\widetilde{\Omega} := [-\pi,\pi)^d$ be the cell of the dual lattice $(2\pi\mathbb{Z})^d$. In $L_2(\Omega)$, consider the family of selfadjoint operators $A(k)$, $k \in \widetilde{\Omega}$, generated by the quadratic forms

$$\mathfrak{a}(k)[u,u] := \int_{\Omega} \left(\langle a(x)\nabla u, \nabla u\rangle + b(x)|u|^2 \right) dx, \quad u \in \operatorname{dom}\mathfrak{a}(k), \ k \in \widetilde{\Omega}.$$

The domain $\operatorname{dom}\mathfrak{a}(k)$, $k \in \widetilde{\Omega}$, consists of functions $u \in H^1(\Omega)$ such that the product $e^{-ikx}u(x)$ is periodically extendible to a function of class $H^1_{\mathrm{loc}}(\mathbb{R}^d)$. In other words, these are functions that belong to $H^1(\Omega)$ and satisfy k-quasiperiodic boundary conditions. The spectra of operators $A(k)$ are discrete. Denote by

$$E_1(k) \leq E_2(k) \leq \cdots \leq E_n(k) \leq \dots \tag{1.3}$$

the consecutive eigenvalues of the operators $A(k)$, counting multiplicities. The maps $E_s \colon \widetilde{\Omega} \to \mathbb{R}$ (*band* functions) can be periodically (with respect to the lattice $(2\pi\mathbb{Z})^d$) extended up to continuous functions in $\mathbb{R}^d$. Let $\{\psi_s(k,\cdot)\}_{s=1}^\infty$ denote orthonormal in $L_2(\Omega)$ eigenfunctions of the operators $A(k)$, $k \in \widetilde{\Omega}$, corresponding to the eigenvalues (1.3). We extend the definition of $\psi_s(k,x)$ to all $x \in \mathbb{R}^d$ so that the products $e^{-ikx}\psi_s(k,x)$ be $\mathbb{Z}^d$-periodic in $x \in \mathbb{R}^d$ for all $k \in \widetilde{\Omega}$, $s \in \mathbb{N}$. We single out the property (see, e.g., [9])

$$\psi_s(\cdot\,,\cdot) \in L_\infty(\widetilde{\Omega} \times \mathbb{R}^d). \tag{1.4}$$

We introduce the operators Φ_s that partially diagonalize the operator A:

$$(\Phi_s u)(k) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \overline{\psi_s(k,x)}u(x)\,dx, \quad u \in \mathcal{S}(\mathbb{R}^d), \quad s \in \mathbb{N}. \tag{1.5}$$

Here $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz class. The operators (1.5) extend by continuity up to partial isometries of $L_2(\mathbb{R}^d)$ onto $L_2(\widetilde{\Omega})$. The products $\Phi_s^*\Phi_s =: P_s$, $s \in \mathbb{N}$, are pairwise orthogonal projections in $L_2(\mathbb{R}^d)$, and moreover, $\sum_{s\in\mathbb{N}} P_s = I$. The operator A is decomposable into the sum (the Floquet decomposition)

$$A = \sum_{s\in\mathbb{N}} \Phi_s^*[E_s(k)]\Phi_s.$$

The spectrum of the operator A is the union of *bands*, which are the ranges of the continuous mappings E_s:

$$\sigma(A) = \bigcup_{s=1}^{\infty} \operatorname{Im} E_s.$$

The spectrum $\sigma(A)$ may contain gaps. Henceforth it is assumed that (α, β) is a gap in $\sigma(A)$ such that

$$\alpha = \max_{k\in\widetilde{\Omega}} E_{l-1}(k) < \beta = \min_{k\in\widetilde{\Omega}} E_l(k) \tag{1.6}$$

for some $l \geq 2$.

2. Perturbation. The operator (0.1) is perturbed by a real-valued potential $V(x)$. Assume that the potential $V(x)$ is such that

$$V(x) \geq 0, \quad V \in L_\infty(\mathbb{R}^d), \tag{1.7}$$

$$V(x) = v(x/|x|)|x|^{-\rho} + o(|x|^{-\rho}), \quad |x| \to +\infty, \quad \rho > 0. \tag{1.8}$$

Define the *perturbed* operator $B(t) := A + tV$, $t > 0$. It is easily seen (see, e.g., [4]) that, under conditions (1.1), (1.7), and (1.8), the perturbation is nonnegative and relatively compact:

$$B(t) \geq A; \quad (B(t) - iI)^{-1} - (A - iI)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H}), \quad t > 0.$$

Therefore, the spectrum of $B(t)$, $t > 0$, within the gap (α, β) is discrete and does not accumulate to the right endpoint of the gap. Consider the *counting function*

$$N(\lambda, \tau, V) := \sum_{t\in(0,\tau)} \dim \operatorname{Ker}(B(t) - \lambda I), \quad \tau > 0, \quad \lambda \in (\alpha, \beta). \tag{1.9}$$

It is natural to assume that the right-hand side in (1.9) is infinite if there exist infinitely many points $t \in (0, \tau)$ for which $\dim \operatorname{Ker}(B(t)-\lambda I) > 0$. Variational considerations yield the following statement (the Birman–Schwinger principle; see, e.g., [10]).

Proposition 1.1. *Assume that conditions* (1.1), (1.7), (1.8) *and* $\sigma(A) \cap (\alpha, \beta) = \varnothing$ *are fulfilled. Then*

$$\begin{gathered} Z(\lambda) := V^{1/2}(\lambda I - A)^{-1}V^{1/2} \in \mathfrak{S}_\infty, \quad \lambda \in (\alpha, \beta), \\ N(\lambda, \tau, V) = n_+(\tau^{-1}, Z(\lambda)), \quad \tau > 0, \quad \lambda \in (\alpha, \beta). \end{gathered} \tag{1.10}$$

Proposition 1.1 implies that the quantity $N(\lambda, \tau, V)$ is finite for all $\tau > 0$, $\lambda \in (\alpha, \beta)$, and is monotone nonincreasing as λ grows (for a fixed $\tau > 0$). Therefore, the (either finite or infinite) value of the counting function at the left endpoint of the gap can be defined as

$$N(\alpha, \tau, V) := \lim_{\lambda\to\alpha+0} N(\lambda, \tau, V), \quad \tau > 0.$$

3. The main result. We describe the asymptotics of the function $N(\lambda, \tau, V)$ as $\tau \to +\infty$. Consider the quantity

$$\Gamma_\rho(\lambda) := (2\pi)^{-d}d^{-1}\sum_{s=1}^{l-1}\int_{\widetilde{\Omega}}(\lambda - E_s(k))^{-d/\rho}\,dk\int_{\mathbb{S}^{d-1}} v^{d/\rho}(\theta)\,dS(\theta), \quad \lambda \in [\alpha, \beta].$$

Obviously, $\Gamma_\rho(\lambda)$ is finite for $\lambda \in (\alpha, \beta)$; $\Gamma_\rho(\alpha)$ is finite provided that

$$(\alpha - E_s(\,\cdot\,))^{-1} \in L_{d/\rho}(\widetilde{\Omega}), \quad s = 1, \dots, l-1. \tag{1.11}$$

The following result was proved in [4].

Theorem 1.2. *Assume the conditions* (1.1), (1.2), (1.6), (1.7), *and* (1.8), *and let* $\rho \in (0, d)$, $d \geq 1$. *Then for all* $\lambda \in (\alpha, \beta)$ *we have*

$$\lim_{\tau\to+\infty} \tau^{-d/\rho}N(\lambda, \tau, V) = \Gamma_\rho(\lambda). \tag{1.12}$$

If, moreover, relations (1.11) *are valid, then the asymptotics* (1.12) *holds true for* $\lambda = \alpha$ *as well.*

Our goal in the present paper is to investigate the case where $\rho \geq d \geq 1$, $\lambda \in [\alpha, \beta)$. In doing this, we replace condition (1.11) by a more restrictive one:

$$(1.13)\qquad (\alpha - E_s(\,\cdot\,))^{-1} \in L_\sigma(\widetilde{\Omega}), \quad s = 1, \dots, l-1, \quad \text{where} \quad \begin{cases} \sigma = 1, & \rho > d, \\ \sigma > 1, & \rho = d. \end{cases}$$

In the sequel (see §3) we shall establish the following.

Theorem 1.3. *Under conditions* (1.1), (1.2), (1.6), (1.7), *and* (1.8), *suppose that* $\rho \geq d \geq 1$. *Then for all* $\lambda \in (\alpha, \beta)$ *the asymptotics* (1.12) *is valid. If, moreover,* (1.13) *is true, then* (1.12) *is fulfilled for* $\lambda = \alpha$ *as well.*

The left endpoint of the gap (α, β) is said to be *regular* if $\max_{k \in \widetilde{\Omega}} E_{l-2}(k) < \alpha$, and the identity $E_{l-1}(k) = \alpha$ is attained at finitely many points $k_j \in \widetilde{\Omega}$, $j = 1, \dots, M$, where each of the points k_j is a nondegenerate maximum for E_{l-1}, i.e.,

$$\alpha - E_{l-1}(k) = q_j(k - k_j) + O(|k - k_j|^3), \quad k \to k_j, \quad j = 1, \dots, M,$$

where q_j is a positive definite quadratic form.

If the left endpoint of a gap is regular, then

$$(\alpha - E_s(\,\cdot\,))^{-1} \in L_\sigma(\widetilde{\Omega}), \quad s = 1, \dots, l-1, \quad \forall \sigma < d/2.$$

Thus, Theorem 1.3 implies the following claim.

Corollary 1.4. *Let* $\rho \geq d \geq 3$, *and assume conditions* (1.1), (1.2), (1.6), (1.7), *and* (1.8). *Next, suppose that the left endpoint of the gap is regular. Then the asymptotics* (1.12) *holds true for* $\lambda = \alpha$.

Commentary. The order of the asymptotics (1.12) differs from the "standard" $\tau^{d/2}$. The asymptotic coefficient depends on the observation point λ. The "Weyl nature" of the asymptotics (1.12) becomes clear if one swaps the rôles of the coordinates and quasiimpulses. A statement similar to Corollary 1.4 is also valid in the case where $\varrho \in (0, d)$ (see [4, Corollary 1.4]). If $d = 2$ and the left endpoint of the gap is regular, Theorems 1.2 and 1.3 yield nothing, because conditions (1.11) and (1.13) are violated *a priori.*

§2. Preliminaries

1. Compact operators: necessary information. For an arbitrary compact operator $\mathbb{T}$ acting from a Hilbert space $\mathfrak{H}_1$ to a Hilbert space $\mathfrak{H}_2$, denote by $s_m(\mathbb{T})$, $m \in \mathbb{N}$, the singular values of $\mathbb{T}$ (i.e., consecutive eigenvalues of the operator $(\mathbb{T}^*\mathbb{T})^{1/2}$); let $n(s, \mathbb{T}) := \#\{m \in \mathbb{N} : s_m(\mathbb{T}) > s\}$ be the distribution function of the singular values. A number of evident relations are true: $n(s, \mathbb{T}^*\mathbb{T}) = n(\sqrt{s}, \mathbb{T}) = n(\sqrt{s}, \mathbb{T}^*) = n(s, \mathbb{T}\mathbb{T}^*)$, $s > 0$; $n(s, \mathbb{T}) = n_+(s, \mathbb{T}) + n_-(s, \mathbb{T})$, $s > 0$, $\mathbb{T} = \mathbb{T}^*$; $n(s, \mathbb{T}) = n_+(s, \mathbb{T})$, $s > 0$, $\mathbb{T} \geq 0$. We single out the "variational" property of the distribution function $n_+(s, \mathbb{T})$ of a selfadjoint compact operator $\mathbb{T}$ in the Hilbert space $\mathfrak{H}$, which will be used below (see, e.g., [11], §10.2, Subsection 2):

$$(2.1)\qquad n_+(s, \mathbb{T}) = \sup\{\dim \mathcal{F},\ \mathcal{F} \subset \mathfrak{H} : (\mathbb{T}u, u) > s\|u\|^2,\ \forall u \in \mathcal{F} \setminus \{0\}\}.$$

The class $\mathfrak{S}_{p,\infty}(\mathfrak{H}_1, \mathfrak{H}_2)$ (see, e.g., [12]) is singled out by the condition that the functional $\|\mathbb{T}\|_{\mathfrak{S}_{p,\infty}} := \sup_{s>0} s n^{1/p}(s, \mathbb{T})$ is finite. The space $\mathfrak{S}_{p,\infty}$ is complete with respect to the quasinorm $\|\cdot\|_{\mathfrak{S}_{p,\infty}}$; it is, in general, nonseparable and contains the separable subspace $\mathfrak{S}^0_{p,\infty} := \{\mathbb{T} \in \mathfrak{S}_{p,\infty} : n(s, \mathbb{T}) = o(s^{-p}),\ s \to +0\}$, in which the set of finite rank operators is dense. The continuous (see, e.g., [13]) functionals $\Delta_p(\mathbb{T}) := \limsup_{s \to 0} s^p n(s, \mathbb{T})$; $\delta_p(\mathbb{T}) := \liminf_{s \to 0} s^p n(s, \mathbb{T})$ are defined on the space $\mathfrak{S}_{p,\infty}$. The functionals $\Delta_p^{\pm}(\mathbb{T}) := \limsup_{s\to+0} s^p n_{\pm}(s,\mathbb{T})$, $\delta_p^{\pm}(\mathbb{T}) := \liminf_{s\to+0} s^p n_{\pm}(s,\mathbb{T})$ are continuous on the set of selfadjoint operators in $\mathfrak{S}_{p,\infty}$. In certain formulations, the following notation will be used: $D_p(\mathbb{T})$ for any one of the functionals $\Delta_p(\mathbb{T})$, $\Delta_p^{\pm}(\mathbb{T})$, $\delta_p(\mathbb{T})$, $\delta_p^{\pm}(\mathbb{T})$. Here the case where $D_p(\mathbb{T}) = \Delta_p^{\pm}(\mathbb{T}), \delta_p^{\pm}(\mathbb{T})$ automatically implies that $\mathbb{T} = \mathbb{T}^*$. It is easily seen that, for an operator $\mathbb{T} \in \mathfrak{S}_{p,\infty}$, the fact that $\mathbb{T} \in \mathfrak{S}^0_{p,\infty}$ is equivalent to $\Delta_p(\mathbb{T}) = 0$. We single out the following property (see, e.g., [13]), to be used in the sequel:

$$D_p(\mathbb{T}+\mathbb{K}) = D_p(\mathbb{T}), \quad D = \Delta, \Delta^{\pm}, \delta, \delta^{\pm}, \quad \mathbb{T} \in \mathfrak{S}_{p,\infty}, \quad \mathbb{K} \in \mathfrak{S}^0_{p,\infty}. \tag{2.2}$$

We shall also make use of the following claim (see, e.g., [11, §11.6, Subsection 3]).

Proposition 2.1. *If $\mathbb{T}_1 \in \mathfrak{S}_{p,\infty}$ and $\mathbb{T}_2 \in \mathfrak{S}_{q,\infty}$, then $\mathbb{T}_1\mathbb{T}_2 \in \mathfrak{S}_{r,\infty}$, where $r^{-1} = p^{-1} + q^{-1}$; moreover, we have*

$$\|\mathbb{T}_1\mathbb{T}_2\|_{\mathfrak{S}_{r,\infty}} \le C(p,q)\|\mathbb{T}_1\|_{\mathfrak{S}_{p,\infty}}\|\mathbb{T}_2\|_{\mathfrak{S}_{q,\infty}}, \quad r^{-1} = p^{-1} + q^{-1}.$$

If, moreover, either $\mathbb{T}_1 \in \mathfrak{S}^0_{p,\infty}$ or $\mathbb{T}_2 \in \mathfrak{S}^0_{q,\infty}$, then $\mathbb{T}_1\mathbb{T}_2 \in \mathfrak{S}^0_{r,\infty}$.

2. Required estimates for the singular values of compact operators. Suppose that a real-valued $(d\times d)$-matrix-valued function $a(x)$ and a real-valued function $b(x)$ satisfy condition (1.1), let the operator A in $L_2(\mathbb{R}^d)$ be defined by (0.1), let $\varphi(\lambda)$, $\lambda \in \mathbb{R}$, be a bounded Borel function, and let $W \in L_\infty(\mathbb{R}^d)$. Assume that for some $p > 0$ and $\varepsilon > 0$ the following conditions are met:

$$\varphi(\lambda) = O(\lambda^{-\frac{d}{2p}-\varepsilon}), \quad \lambda \to +\infty; \quad W(x) = O(|x|^{-d/p}), \quad |x| \to +\infty. \tag{2.3}$$

The following statement is true (see [5, Proposition 3.2]).

Proposition 2.2. *If for some $p > 0$ and $\varepsilon > 0$ conditions* (2.3) *are satisfied, then $\varphi(A)[W(x)] \in \mathfrak{S}_{p,\infty}$. If, moreover, $W(x) = o(|x|^{-d/p})$, $|x| \to +\infty$, then $\varphi(A)[W(x)] \in \mathfrak{S}^0_{p,\infty}$.*

Assume that for some $p \in (0,+\infty)$ the function $W \in L_\infty(\mathbb{R}^d)$ admits the asymptotics

$$W(x) = \omega(x/|x|)|x|^{-d/p} + o(|x|^{-d/p}), \quad |x| \to +\infty. \tag{2.4}$$

Let $\varphi(\lambda)$, $\lambda \in \mathbb{R}$, be a continuous function with compact support. By Proposition 2.2, we have $\varphi(A)W \in \mathfrak{S}_{p,\infty}$. By [6, Theorems 0.4 and 0.5], the following is true.

Proposition 2.3. *Suppose that, for some $p \in (0,+\infty)$, the function $W \in L_\infty(\mathbb{R}^d)$ admits the asymptotics* (2.4)*; let $\varphi(\lambda)$, $\lambda \in \mathbb{R}$, be a continuous compactly supported function. Then*

$$\varphi(A)W - W\varphi(A) \in \mathfrak{S}^0_{p,\infty}$$

and

$$D_p(\varphi(A)W) = D_{p/n}(\varphi^n(A)W^n), \quad n \in \mathbb{N}, \quad D = \Delta, \delta.$$

Let $(\mathcal{K}, d\mu)$ be a separable measurable space with a σ-finite measure, and suppose that $T\colon L_2(\mathbb{R}^d) \to L_2(\mathcal{K}, d\mu)$ is a bounded integral linear operator with a bounded kernel $t(\,\cdot\,,\cdot\,) \in L_\infty(\mathcal{K}\times\mathbb{R}^d, d\mu dx)$ (dx is Lebesque measure in $\mathbb{R}^d$). Assume that there exists a measurable function $h\colon \mathcal{K} \to \mathbb{C}$ such that, for all $\sigma > 0$, the inclusion $u \in \operatorname{dom} e^{\sigma A}$ implies that $Tu \in \operatorname{Dom}[e^{\sigma h(k)}]$ and the following condition is fulfilled:

$$Te^{\sigma A}u = \big[e^{\sigma h(k)}\big]Tu, \quad u \in \operatorname{dom} e^{\sigma A}, \quad \sigma > 0. \tag{2.5}$$

For brevity, we denote $a_0 := \min\{0, \inf\sigma(A)\}$, $a_j := 2^j$, $j \in \mathbb{N}$. Put $\mathcal{K}_j := \{k \in \mathcal{K} : |h(k)| \in [a_j, a_{j+1})\}$, $j \in \mathbb{Z}_+$. With every measurable function $f\colon \mathcal{K} \to \mathbb{C}$ we associate the sequence $u(f) := \{u_j(f)\}_{j=0}^\infty$, $u_j(f) := a_{j+1}^{-d/4}\|f\|_{L_2(\mathcal{K}_j)}$, $j \in \mathbb{Z}_+$. Define the measure $d\tau(j) = 2^{\frac{d}{2}(j+1)}dj$ (dj is the counting measure on $\mathbb{Z}_+$) on the set $\mathbb{Z}_+$.

The next statement is taken from [5], see Proposition 3.3 therein.

Proposition 2.4. *Let a function $W \in L_\infty(\mathbb{R}^d)$ admit the estimate $W(x) = O(|x|^{-d/p})$, $|x| \to +\infty$, $p > 0$, and let $f\colon \mathcal{K} \to \mathbb{C}$ be a measurable function satisfying any of the following three conditions:*

a) *if $p > 2$, then $f \in L_p(\mathcal{K}, d\mu)$;*
b) *if $p \in (0, 2)$, then $u(f) \in l_p(\mathbb{Z}_+, d\tau)$;*
c) *if $p = 2$, then for some $\alpha \in (1, 2)$ we have*

$$u(f) \in l_\alpha(\mathbb{Z}_+, d\tau), \quad f \in L_\beta(\mathcal{K}, d\mu), \quad 1/\alpha + 1/\beta = 1.$$

Then $[f(k)]T[W(x)] \in \mathfrak{S}_{p,\infty}$ and $\|[f(k)]T[W(x)]\|_{\mathfrak{S}_{p,\infty}} \le C\|f\|$. Here it is implied that

$$\begin{aligned} &\|f\| := \|f\|_{L_p}, && C := C(A, d, p, T, W), \quad p > 2;\\ &\|f\| := \|u(f)\|_{l_p}, && C := C(A, d, p, T, W), \quad p \in (0, 2);\\ &\|f\| := \|f\|_{L_\beta}^{1/2} \|u(f)\|_{l_\alpha}^{1/2}, && C := C(A, d, \alpha, T, W), \quad p = 2. \end{aligned}$$

Under the additional condition $W(x) = o(|x|^{-d/p})$ as $|x| \to +\infty$ we have $fTW \in \mathfrak{S}^0_{p,\infty}$.

Commentary. Strictly speaking, Propositions 2.2–2.4 were proved in the papers [5, 6] under the condition that the semigroup $e^{-\sigma A}$ is an integral operator in $L_2(\mathbb{R}^d)$ with the kernel $K(\sigma, x, y)$, $x, y \in \mathbb{R}^d$, $\sigma > 0$, and

$$|K(\sigma, x, y)| \le M_1 \sigma^{-d/2} e^{M_2\sigma - \frac{|x-y|^2}{M_3\sigma}}, \quad x, y \in \mathbb{R}^d, \quad \sigma > 0, \tag{2.6}$$

where the $M_i > 0$, $i = 1, 2, 3$, are independent of $x, y \in \mathbb{R}^d$, $\sigma > 0$. Estimate (2.6) holds true automatically as long as condition (1.1) is satisfied (cf. [5, Theorem 0.1]).

Now, suppose that a real $(d\times d)$-matrix-valued function $a(x)$ and a real-valued function $b(x)$ satisfy (1.2) along with condition (1.1), i.e., the operator A is $\mathbb{Z}^d$-periodic. Moreover, the operator $\Phi_s\colon L_2(\mathbb{R}^d) \to L_2(\widetilde{\Omega})$, $s \in \mathbb{N}$, is a bounded linear operator with a bounded kernel (see (1.4)). The operator Φ_s, $s \in \mathbb{N}$, satisfies (2.5); here $h(k) = E_s(k)$, $s \in \mathbb{N}$, is the corresponding band function. Under the choice $h(k) = E_s(k)$, $k \in \widetilde{\Omega}$, for an arbitrary measurable function $f\colon \widetilde{\Omega} \to \mathbb{C}$ we have $\|u(f)\|_{l_p} \le C(s, p)\|f\|_{L_2}$. Proposition 2.4 yields the following claim.

Corollary 2.5. *Suppose that the function $W \in L_\infty(\mathbb{R}^d)$ admit the estimate $W(x) = O(|x|^{-d/p})$, $|x| \to +\infty$, $p > 0$, and let $f\colon \widetilde{\Omega} \to \mathbb{C}$ be a measurable function satisfying any of the following three conditions*:

a) *if $p > 2$, then $f \in L_p(\widetilde{\Omega})$;*
b) *if $p \in (0, 2)$, then $f \in L_2(\widetilde{\Omega})$;*
c) *if $p = 2$, then $f \in L_\beta(\widetilde{\Omega})$ for some $\beta > 2$.*

Then $[f(k)]\Phi_s[W(x)] \in \mathfrak{S}_{p,\infty}$ and $\|[f(k)]\Phi_s[W(x)]\|_{\mathfrak{S}_{p,\infty}} \le C\|f\|$. Here it is implied that

$$\begin{aligned} &\|f\| := \|f\|_{L_p}, && C := C(A, d, p, s, W), \quad p > 2;\\ &\|f\| := \|f\|_{L_2}, && C := C(A, d, p, s, W), \quad p \in (0, 2);\\ &\|f\| := \|f\|_{L_\beta}, && C := C(A, d, \beta, s, W), \quad p = 2. \end{aligned}$$

Under the additional condition $W(x) = o(|x|^{-d/p})$ as $|x| \to +\infty$, we have $f\Phi_s W \in \mathfrak{S}^0_{p,\infty}$.

3. In the sequel, we shall need the singular values asymptotics for some specially chosen integral operators, which was obtained in [4]. As above, we assume that a is a real symmetric $(d \times d)$-matrix-valued function, b is a real-valued function, and conditions (1.1), (1.2) are satisfied. Suppose that, for some $p \in (1, +\infty)$, the function $W \in L_\infty(\mathbb{R}^d)$

admits the asymptotics (2.4). Let $f_s, h_s \in L_{2p}(\widetilde{\Omega})$, $s = 1, \dots, l-1$. Denote for brevity

$$Y := \sum_{s,t=1}^{l-1} \Phi_s^*[f_s(k)]\Phi_s[W(x)]\Phi_t^*[h_t(k)]\Phi_t.$$

The following statement was proved in [4, Corollary 2.4].

Proposition 2.6. *Suppose that, for some $p \in (1, +\infty)$, the function $W \in L_\infty(\mathbb{R}^d)$ admits the asymptotics (2.4), and let $f_s, h_s \in L_{2p}(\widetilde{\Omega})$, $s = 1, \dots, l-1$. Then $Y \in \mathfrak{S}_{p,\infty}$, and*

$$\delta_p(Y) = \Delta_p(Y) = d^{-1}(2\pi)^{-d} \sum_{s=1}^{l-1} \int_{\widetilde{\Omega}} |f_s(k)h_s(k)|^p \, dk \int_{\mathbb{S}^{d-1}} |\omega(\theta)|^p \, dS(\theta).$$

§3. Proof of Theorem 1.3

1. Proof of Theorem 1.3 "inside the gap". By Proposition 1.1, in order to establish the asymptotics (1.12) for $\lambda \in (\alpha, \beta)$, we need to prove that

$$\lim_{s \to +0} s^q n_+(s, V^{1/2}(\lambda I - A)^{-1}V^{1/2}) = \Gamma_\rho(\lambda), \quad \lambda \in (\alpha, \beta), \quad q\rho = d. \tag{3.1}$$

For brevity, denote $E_1 := E_A((-\infty, \alpha])$, $E_2 := E_A([\beta, +\infty))$. The variational property (2.1) implies the two-sided estimate

$$\begin{aligned} n_+(s, E_1V^{1/2}(\lambda I - A)^{-1}V^{1/2}E_1) &\le n_+(s, V^{1/2}(\lambda I - A)^{-1}V^{1/2}) \\ &\le n_+(s, V^{1/2}E_1(\lambda I - A)^{-1}E_1V^{1/2}). \end{aligned} \tag{3.2}$$

Since the operator A is lower bounded and condition (1.6) is fulfilled, there exists a continuous compactly supported function $\varphi(\lambda)$, $\lambda \in \mathbb{R}$, such that $E_1 = \varphi(A)$. By Proposition 2.2, we have $V^{1/2}E_1 \in \mathfrak{S}_{2q,\infty}$, $q\varrho = d$. Therefore, to obtain the asymptotics (1.12) for $\lambda \in (\alpha, \beta)$ it suffices to check that

$$\delta_q^+(E_1V^{1/2}(\lambda I - A)^{-1}V^{1/2}E_1) = \Delta_q^+(V^{1/2}E_1(\lambda I - A)^{-1}E_1V^{1/2}) = \Gamma_\varrho(\lambda), \quad q\varrho = d. \tag{3.3}$$

Proposition 2.3 shows that $V^{1/2}E_1 - E_1V^{1/2} \in \mathfrak{S}^0_{2q,\infty}$, $q\varrho = d$, whence

$$E_1V^{1/2}(\lambda I - A)^{-1}V^{1/2}E_1 - V^{1/2}E_1(\lambda I - A)^{-1}E_1V^{1/2} \in \mathfrak{S}^0_{q,\infty}, \quad q\varrho = d.$$

Therefore, by (2.2), identities (3.3) are equivalent to

$$\delta_q(V^{1/2}E_1(\lambda I - A)^{-1}E_1V^{1/2}) = \Delta_q(V^{1/2}E_1(\lambda I - A)^{-1}E_1V^{1/2}) = \Gamma_\varrho(\lambda), \quad q\varrho = d. \tag{3.4}$$

Since $[\inf \sigma(A), \alpha]$ is an isolated part of the spectrum of the operator A, we can find a continuous compactly supported function $\varphi_1(\lambda)$, $\lambda \in \mathbb{R}$, such that $E_1(\lambda I - A)^{-1/2}E_1 = \varphi_1(A)$. By Proposition 2.3, for an arbitrary $n \in \mathbb{N}$ and all $\lambda \in (\alpha, \beta)$ we have

$$\begin{aligned} D_q(V^{1/2}E_1(\lambda I - A)^{-1}E_1V^{1/2}) &= D_{2q}(E_1(\lambda I - A)^{-1/2}E_1V^{1/2}) \\ &= D_{2qn}(E_1(\lambda I - A)^{-1/2n}E_1V^{1/2n}) \\ &= D_{qn}(E_1(\lambda I - A)^{-1/2n}E_1V^{1/n}E_1(\lambda I - A)^{-1/2n}E_1), \\ D = \delta, \Delta, \quad q\varrho = d. \end{aligned}$$

Thus, for any $n \in \mathbb{N}$, identities (3.4) are equivalent to the fact that

$$D_{qn}(E_1(\lambda I - A)^{-1/2n}E_1V^{1/n}E_1(\lambda I - A)^{-1/2n}E_1) = \Gamma_\varrho(\lambda), \quad D = \delta, \Delta, \quad q\varrho = d. \tag{3.5}$$

We pick $n \in \mathbb{N}$ in (3.5) sufficiently large to ensure that $\varrho/n \in (0, d)$. Now, (3.5) is true by Proposition 2.6 and the identity

$$E_1(\lambda I - A)^{-1/2n} E_1 V^{1/n} E_1 (\lambda I - A)^{-1/2n} E_1 \\ = \sum_{s,t=1}^{l-1} \Phi_s^*[(\lambda - E_s(k))^{-1/2n}]\Phi_s V^{1/n} \Phi_t^*[(\lambda - E_t(k))^{-1/2n}]\Phi_t.$$

□

2. Proof of Theorem 1.3 "at the edge of the gap".

Lemma 3.1. *Under conditions* (1.1), (1.2), (1.6), (1.7), *and* (1.8), *let* $\rho \geq d \geq 1$, *and let condition* (1.13) *be satisfied. Then for all* $s = 1, \dots, l-1$ *the operator* $[(\lambda - E_s(k))^{-1/2}]\Phi_s V^{1/2}$ *converges to the operator* $[(\alpha - E_s(k))^{-1/2}]\Phi_s V^{1/2}$ *in* $\mathfrak{S}_{2q,\infty}$, $q\varrho = d$, *as* $\lambda \to \alpha+0$, *and the quasinorm* $\|[(\lambda - E_s(k))^{-1/2}]\Phi_s V^{1/2}\|_{\mathfrak{S}_{2q,\infty}}$, $\lambda \in (\alpha, \beta)$, *is uniformly bounded.*

Proof of Lemma 3.1. Since $0 < (\lambda - E_s(k))^{-1} \leq (\alpha - E_s(k))^{-1}$, $\lambda \in (\alpha, \beta)$, $k \in \widetilde{\Omega}$, $s = 1, \dots, l-1$, condition (1.13) implies that the relations

$$\begin{gathered} \|(\lambda - E_s(\,\cdot\,))^{-1/2}\|_{L_{2\sigma}} \leq \|(\alpha - E_s(\,\cdot\,))^{-1/2}\|_{L_{2\sigma}}, \quad \lambda \in (\alpha, \beta), \\ \|(\lambda - E_s(\,\cdot\,))^{-1/2} - (\alpha - E_s(\,\cdot\,))^{-1/2}\|_{L_{2\sigma}} \to 0, \quad \lambda \to \alpha + 0, \\ \text{where} \quad \begin{cases} 2\sigma = 2, & 2q \in (0,2), \\ 2\sigma > 2, & 2q = 2, \end{cases} \qquad q\rho = d, \quad s = 1, \dots, l-1. \end{gathered} \tag{3.6}$$

By Corollary 2.5, relation (3.6) and the estimate $V^{1/2}(x) = O(|x|^{-d/2q})$, $|x| \to \infty$, $q\varrho = d$ (see conditions (1.7) (1.8)) yield all the required assertions:

$$\begin{aligned} &\|[(\lambda - E_s(k))^{-1/2}]\Phi_s V^{1/2}\|_{\mathfrak{S}_{2q,\infty}} \leq C \|(\alpha - E_s(\,\cdot\,))^{-1/2}\|_{L_{2\sigma}}, \quad \lambda \in (\alpha, \beta); \\ &\|[(\lambda - E_s(k))^{-1/2} - (\alpha - E_s(k))^{-1/2}]\Phi_s V^{1/2}\|_{\mathfrak{S}_{2q,\infty}} \\ &\qquad \leq C \|(\lambda - E_s(\,\cdot\,))^{-1/2} - (\alpha - E_s(\,\cdot\,))^{-1/2}\|_{L_{2\sigma}} \to 0, \quad \lambda \to \alpha + 0, \\ &\text{where} \quad C = C(A, d, q, \sigma, s, V), \quad q\varrho = d, \quad s = 1, \dots, l-1. \end{aligned}$$

□

Proof of formula (1.12) *with* $\lambda = \alpha$. By condition (1.13), we have $\Gamma_\varrho(\lambda) \to \Gamma_\varrho(\alpha)$, $\lambda \to \alpha + 0$. Moreover, by Lemma 3.1, the operator

$$\Upsilon(\lambda) := E_1(\lambda I - A)^{-1/2} E_1 V^{1/2} = \sum_{s=1}^{l-1} \Phi_s^*[(\lambda - E_s(k))^{-1/2}]\Phi_s V^{1/2}$$

converges in $\mathfrak{S}_{2q,\infty}$, as $\lambda \to \alpha + 0$, $q\varrho = d$, to the operator

$$\Upsilon(\alpha) := \sum_{s=1}^{l-1} \Phi_s^*[(\alpha - E_s(k))^{-1/2}]\Phi_s V^{1/2},$$

the quasinorm $\|\Upsilon(\lambda)\|_{\mathfrak{S}_{2q,\infty}}$, $\lambda \in (\alpha, \beta)$, is uniformly bounded. Therefore (see Proposition 2.1), the operator $Z_1(\lambda) := V^{1/2} E_1 (\lambda I - A)^{-1} E_1 V^{1/2} = \Upsilon^*(\lambda)\Upsilon(\lambda)$ converges in $\mathfrak{S}_{q,\infty}$, $q\varrho = d$, as $\lambda \to \alpha + 0$, to the operator $Z_1(\alpha) := \Upsilon^*(\alpha)\Upsilon(\alpha)$, whence $\delta_q(Z_1(\alpha)) = \Delta_q(Z_1(\alpha)) = \Gamma_\varrho(\alpha)$.

By monotonicity, we have $Z_1(\lambda) \leq Z_1(\alpha)$, $\lambda \in (\alpha, \beta)$. Therefore (cf. (1.10), (3.2)), the following relation is true:

$$N(\lambda, \tau, V) \leq n_+(\tau^{-1}, Z_1(\lambda)) \leq n(\tau^{-1}, Z_1(\alpha)), \quad \lambda \in (\alpha, \beta).$$

Passing to the limit as $\lambda \to \alpha + 0$ on the left-hand side, we obtain $N(\alpha, \tau, V) \leq n(\tau^{-1}, Z_1(\alpha))$. By the monotonicity of $N(\,\cdot\,, \tau, V))$, we obtain

$$N(\lambda, \tau, V) \leq N(\alpha, \tau, V) \leq n(\tau^{-1}, Z_1(\alpha)), \quad \lambda \in (\alpha, \beta) \tag{3.7}$$

for each $\lambda \in (\alpha, \beta)$. Now, (3.7) yields the relations

$$\Gamma_\varrho(\lambda) \leq \liminf_{\tau \to +\infty} \tau^{-d/\varrho} N(\alpha, \tau, V) \leq \limsup_{\tau \to +\infty} \tau^{-d/\varrho} N(\alpha, \tau, V) \leq \Gamma_\varrho(\alpha), \quad \lambda \in (\alpha, \beta).$$

Passing to the limit as $\lambda \to \alpha + 0$, we finally see that

$$\lim_{\tau \to +\infty} \tau^{-d/\varrho} N(\alpha, \tau, V) = \Gamma_\varrho(\alpha). \qquad \square$$

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