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A SIMPLE EMBEDDING THEOREM FOR KERNELS OF TRACE CLASS INTEGRAL OPERATORS IN $L^2(\mathbb{R}^m)$. APPLICATION TO THE FREDHOLM TRACE FORMULA

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Easy reading for professionals

Abstract. The present paper by Mikhail Shlemovich Birman was written in 1989 and circulated among specialists as a preprint published in English by Linköping University (the original manuscript of M. Sh. Birman was translated by A. A. Laptev). In this paper a transparent approach to the proof of the Fredholm formula for the traces of integral operators of trace class was found. By communication with D. R. Yafaev we knew that M. Sh. Birman did not publish this paper because he discovered that a similar construction was used in the book by M. A. Shubin on pseudodifferential operators. This is so, but the presentation in the present text is much more general, clear, and detailed. In this connection, and also in connection with the renewed interest to integral formulas for traces of integral operators, the editorial board decided to publish this paper under the heading "Easy Reading for Professionals".

The paper is based on a talk given by the author at a seminar organized by Prof. L. Hedberg in Linköping University, 5 June 1989.

1. Our purpose in this paper is mainly pedagogical. We would like to eliminate one common misunderstanding related to the application of Fredholm's classical formula

$$\sum_n \lambda_n(K) = \int \mathcal{K}(x,x)\,dx \tag{1}$$

to arbitrary operators of trace class S_1 in $L^2(\mathbb{R}^m)$. Here the $\lambda_n(K)$ are the eigenvalues of an operator K. If $K \in S_1$, then, by the well-known Lidskiĭ theorem (see [1, 2]), the left-hand side in (1) coincides with the functional $\operatorname{Tr} K$, i.e., with the "matrix trace" of the operator K. Thus, it suffices to check the identity

$$\operatorname{Tr} K = \int \mathcal{K}(x,x)\,dx, \quad K \in S_1, \tag{2}$$

where $\mathcal{K}$ is the kernel of the operator K. This can be written as an integral operator: for a.e. $x \in \mathbb{R}^m$ we have

$$(Ku)(x) = \int \mathcal{K}(x,y)u(y)\,dy. \tag{3}$$

However, in formula (3), one only needs to know the value of the kernel $\mathcal{K}$ almost everywhere in $\mathbb{R}^{2m}$ with respect to Lebesgue measure. The "diagonal" $x = y$ is of zero measure

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in $\mathbb{R}^{2m}$. This fact forces us to interpret formula (2) more carefully if the kernel is not continuous. It turns out (see Subsection 4) that if $K \in S_1$, then after the change of variables $y = x + a$ the kernel $\mathcal{K}$ becomes a continuous function of a, with values in $L^1(\mathbb{R}^m)$ with respect to x. This "embedding theorem" removes the difficulty in understanding formula (2).

2. Let $\mathcal{H}$ be a separable Hilbert space, let $S_p = S_p(\mathcal{H})$, $p = 1, 2$, be (respectively) the trace class and the class of Hilbert–Schmidt operators in $\mathcal{H}$, and let $\|\cdot\|_p$ be the corresponding norm in S_p. If $K \in S_1$, then there exists a (nonunique) representation

$$K = LM, \quad L, M \in S_2. \tag{4}$$

If (4) is satisfied, then

$$\|K\|_1 \le \|L\|_2 \|M\|_2, \tag{5}$$

$$\|K\|_1 = \min_{L,M} \|L\|_2 \|M\|_2 \quad (K = LM). \tag{6}$$

We mention the following elementary result.

Proposition. *Let $\widetilde{S}_2$ $(\subset S_2)$ be set dense in S_2. Then the set $\widetilde{S}_1 = \{K : K = LM, L \in \widetilde{S}_2, M \in \widetilde{S}_2\}$ is dense in S_1.*

Proof. Indeed, inequality (5) reduces to the estimate

$$\begin{aligned} \|K - \widetilde{K}\|_1 &= \|(L - \widetilde{L})M + \widetilde{L}(M - \widetilde{M})\|_1 \\ &\le \|M\|_2 \|L - \widetilde{L}\|_2 + \|\widetilde{L}\|_2 \|M - \widetilde{M}\|_2. \end{aligned}$$ □

3. First, we consider a situation more general than in Subsection 1. Let $\mathcal{H} = L^2(X, \mu)$, where X is a separable space with measure μ. We shall also need the tensor square of the space $\mathcal{H}$, i.e., space $L^2(\Omega, \nu)$, where $\Omega = X^2$, $\nu = \mu \times \mu$. It is well known that every operator $K \in S_2$ can be written as an integral operator:

$$(Ku)(x) = \int_X \mathcal{K}(x, y) u(y)\, d\mu(y), \tag{7}$$

where $\mathcal{K} \in L^2(\Omega, \nu)$. Moreover, formula (7) gives an isometric isomorphism of the space $S_2(\mathcal{H})$ onto $L^2(\Omega, \nu)$:

$$\|K\|_2^2 = \iint_{X^2} |\mathcal{K}(x, y)|^2\, d\mu(x) d\mu(y). \tag{8}$$

In terms of kernels, formula (4) means that

$$\mathcal{K}(x, y) = \int_X \mathcal{L}(x, z) \mathcal{M}(z, y)\, d\mu(z). \tag{9}$$

In order to define the operator K, it suffices to know the kernel $\mathcal{K} \in L^2(\Omega, \nu)$ (ν)-a.e. on Ω. At the same time, if $K \in S_1$, then the kernel $\mathcal{K}$ is defined by formula (9) for (μ)-a.e. $x \in X$ and for (μ)-a.e. $y \in X$, and the exceptional sets of μ-measure zero are independent of each other. Such a kernel (a "representative" of the equivalence class $\mathcal{K} \in L^2(\Omega, \nu)$) will be called a *regular kernel* of the operator $K \in S_1$; we denote it by $\mathcal{K}_0$. (For the relationship of the regular kernel with scattering theory, see [3].) Note that the regular kernel $\mathcal{K}_0(x, y)$ is already defined (μ)-a.e. on the diagonal $x = y$. Moreover, the formula

$$\operatorname{Tr} K = \int_X \mathcal{K}_0(x, x)\, d\mu(x) \tag{10}$$

is valid. Indeed, denoting $\mathcal{M}^*(x,y) = \overline{\mathcal{M}(y,x)}$, we have

$$\operatorname{Tr} K = \operatorname{Tr} LM = (\mathcal{L}, \mathcal{M}^*)_{L^2(\Omega,\nu)} = \iint_{X^2} \mathcal{L}(x,z)\mathcal{M}(z,x)\, d\mu(x)\, d\mu(z),$$

which coincides with (10).

It seems that our goal is reached. However, under this approach, the definition of the canonical (regular) representative for $\mathcal{K} \in L^2(\Omega,\nu)$ needs the factorization (4). It would be good to have a method for constructing $\mathcal{K}_0$ directly in terms of the function $\mathcal{K}$. We are able to do this for $\mathcal{H} = L^2(\mathbb{R}^m)$.

4. Now we assume that $X = \mathbb{R}^m$, and that μ is the Lebesgue measure on $\mathbb{R}^m$, i.e., $\mathcal{H} = L^2(\mathbb{R}^m)$. Put

$$\widehat{\mathcal{K}}(x,a) = \mathcal{K}(x, x+a), \quad x, a \in \mathbb{R}^m. \tag{11}$$

The kernels $\widehat{\mathcal{K}}$ and $\mathcal{K}$ determine each other uniquely a.e. on $\mathbb{R}^{2m}$. If $\mathcal{K}$ is regularized as in (9), then

$$\Big(\int |\widehat{\mathcal{K}}(x,a)|\, dx\Big)^2 \le \Big(\iint |\mathcal{L}(x,z)|^2\, dx\, dz\Big)\Big(\iint |\mathcal{M}(z,x+a)|^2\, dx\, dz\Big) = \|L\|_2^2 \|M\|_2^2.$$

Combining this inequality with (4) and (6), for $K \in S_1(L^2(\mathbb{R}^m))$ we see that (using obvious notation)

$$\widehat{\mathcal{K}} \in L^\infty(\mathbb{R}^m_a; L^1(\mathbb{R}^m_x)), \tag{12}$$

$$\|\widehat{\mathcal{K}}\|_{L^\infty(\mathbb{R}^m_a; L^1(\mathbb{R}^m_x))} \le \|K\|_1. \tag{13}$$

From Subsection 2 it is clear that the set of operators with kernels of class $C_0^\infty(\mathbb{R}^{2m})$ is dense in $S_1(L^2(\mathbb{R}^m))$. Together with estimate (13), it allows us to replace (12) by the stronger property

$$\widehat{\mathcal{K}} \in C_0(\mathbb{R}^m_a; L^1(\mathbb{R}^m_x)), \tag{14}$$

which implies that the $L^1(\mathbb{R}^m)$-valued function $\widehat{\mathcal{K}}(\cdot, a)$ is continuous with respect to the variable a and converges to zero as $|a| \to \infty$. Thus, we obtain the following *embedding theorem.*

Theorem. *Let $K \in S_1(L^2(\mathbb{R}^m))$, let $\mathcal{K}$ be the corresponding kernel from* (7)*, and let $\widehat{\mathcal{K}}$ be defined as in* (11)*. Then relation* (14) *and estimate* (13) *hold true.*

The natural identification of an operator of class S_1 and its kernel (11) allows us to write (13) and (14) in the form of an embedding:

$$S_1(L^2(\mathbb{R}^m)) \hookrightarrow C_0(\mathbb{R}^m_a; L^1(\mathbb{R}^m_x)). \tag{15}$$

As usual, we understand this embedding as follows: for $K \in S_1$ the kernel $\widehat{\mathcal{K}}$ modulo the equivalence class with respect to Lebesgue measure on $\mathbb{R}^{2m}$ coincides with some element of the space $C_0(\mathbb{R}^m_a; L^1(\mathbb{R}^m_x))$.

Now we shall give an estimate for the modulus of continuity $\omega_h\big(\widehat{\mathcal{K}}; C_0(\mathbb{R}^m_a; L^1(\mathbb{R}^m_x))\big)$ of the function $\widehat{\mathcal{K}}$ with respect to the variable a. Obviously,

$$\begin{aligned}\Big(\int |\widehat{\mathcal{K}}(x,b) - \widehat{\mathcal{K}}(x,a)|\, dx\Big)^2 &= \Big(\int \Big|\int \mathcal{L}(x,z)\,(\mathcal{M}(z,x+b) - \mathcal{M}(z,x+a))\, dz\Big|\, dx\Big)^2 \\ &\le \|L\|_2^2 \iint |\mathcal{M}(z,x+b) - \mathcal{M}(z,x+a)|^2\, dx\, dz,\end{aligned}$$

i.e.,

$$\omega_h\big(\widehat{\mathcal{K}}; C_0(\mathbb{R}^m_a; L^1(\mathbb{R}^m_x))\big) \le \|L\|_2\, \omega_h(\mathcal{M}; L^2(\mathbb{R}^{2m})). \tag{16}$$

The proof of inequality (16) shows that no uniform estimate is available for $\omega_h(\widehat{\mathcal{K}})$ on the unit ball in the space $S_1(L^2(\mathbb{R}^m))$.

5. To get (13) and (14), we used the regular value $\mathcal{K}_0$ for the kernel $\mathcal{K}$. So, if the conditions of the theorem are fulfilled, we obtain formula (10), which now can be written in the form

$$\operatorname{Tr} K = \int_{\mathbb{R}^m} \mathcal{K}_0(x,x)\,dx = \int_{\mathbb{R}^m} \widehat{\mathcal{K}}(x,0)\,dx. \tag{17}$$

By continuity, the value $\widehat{\mathcal{K}}(x,0)$ is defined for a.e. x in terms of the kernel $\mathcal{K}$.

Now, let the operator Γ_a be a "shift by a", i.e., $\Gamma_a : u(y) \to u(y-a)$. Then $\mathcal{K}(x,y+a)$ is the kernel of the operator $K\Gamma_a$. If $K \in S_1$, then from (17) we obtain

$$\operatorname{Tr} K\Gamma_a = \int_{\mathbb{R}^m} \mathcal{K}_0(x,x+a)\,dx = \int_{\mathbb{R}^m} \widehat{\mathcal{K}}(x,a)\,dx. \tag{18}$$

Now, if $a \to 0$, then formula (18) becomes the same as (17). To calculate $\operatorname{Tr} K$, we can use any averaging process, which, after taking the limit, gives us the value of a continuous function at some fixed point. In particular, taking the usual mean value over the ball $|a| \le h$ immediately leads to a natural and efficient expression:

$$\begin{aligned} \operatorname{Tr} K &= \lim_{h\to 0} \frac{1}{\kappa_m h^m} \int_{|a|\le h} da \int_{\mathbb{R}^m} \widehat{\mathcal{K}}(x,a)\,dx \\ &= \lim_{h\to 0} \frac{1}{\kappa_m h^m} \iint_{|y-x|\le h} \mathcal{K}(x,y)\,dx\,dy; \end{aligned} \tag{19}$$

here κ_m is the volume of the unit ball in $\mathbb{R}^m$. It is easy to write other versions of efficient expressions for $\operatorname{Tr} K$.

6. The considerations of Sections 4 and 5 are automatically extended to the case where

$$\mathcal{H} = L^2(\mathbb{R}^m; H),$$

H being an auxiliary Hilbert space, $\dim H \le \infty$. Then in formula (3) the kernel $\mathcal{K}$ is already an operator-valued function: $\mathcal{K}(x,y) \in S_2(H)$ for a.e. $(x,y) \in \mathbb{R}^{2m}$. Identity (8) becomes the formula

$$\|K\|^2_{2,\mathcal{H}} = \iint_{\mathbb{R}^{2m}} \|\mathcal{K}(x,y)\|^2_{2,H}\,dx\,dy.$$

If $K \in S_1(\mathcal{H})$ and (4) is satisfied, then formula (9) (for $d\mu(z) = dz$) retains its meaning if the kernels $\mathcal{L}$, $\mathcal{M}$ take their values in $S_2(H)$. It follows that $\mathcal{K}(x,y) \in S_1(H)$ for a.e. $x \in \mathbb{R}^m$ and a.e. $y \in \mathbb{R}^m$. The kernel $\widehat{\mathcal{K}}$ can still be defined by (11). Relation (14) and estimate (13) can be replaced (respectively) by

$$\widehat{\mathcal{K}} \in C_0\left(\mathbb{R}^m_a; L^1(\mathbb{R}^m_x; S_1(H))\right)$$

and

$$\sup_a \int_{\mathbb{R}^m} \|\widehat{\mathcal{K}}(x,a)\|_{1,H}\,dx \le \|K\|_{1,\mathcal{H}}.$$

Now formula (17) looks like this:

$$\operatorname{Tr}_{\mathcal{H}}\, K = \int (\operatorname{Tr}_H\, \mathcal{K}_0(x,x))\,dx = \int \Big(\operatorname{Tr}_H\, \widehat{\mathcal{K}}(x,0)\Big)\,dx.$$

An obvious modification of formula (19) can also be written.

Note that representation of an operator in this integral form with an operator-valued kernel is often used in mathematical scattering theory (see [3, 4]).

7. The results of Sections 4–6 can be generalized to the case of the space $\mathcal{H} = L^2(X,\mu;H)$, where X is a group and μ is a shift invariant measure.

8. Great attention has previously been paid to embedding theorems of the form

$$W \hookrightarrow S_1(L^2(\mathbb{R}^m)),$$

where W is some suitable class of functions on $\mathbb{R}^{2m}$ (class of kernels). See, e.g., [1, 5]. At the same time, it seems that the simple embedding (15) has not been observed before. Concerning formulas like (19), we mention that in [1] (for $m = 1$ and for a finite interval $[\alpha, \beta]$), a more complicated expression was presented:

$$\operatorname{Tr} K = \lim_{h\to 0} \int_\alpha^\beta \mathcal{K}_h(x,x)\,dx,$$

$$\mathcal{K}_h(x,y) = \frac{1}{4h^2} \int_{x-h}^{x+h} \int_{y-h}^{y+h} \mathcal{K}(x',y')\,dx'dy'.$$

The kernel $\mathcal{K}_h$ is continuous on $[\alpha,\beta]^2$.

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