Abstract. A modification of the total variation image inpainting method is investigated. By using DeGiorgi type arguments, the partial regularity results established previously are improved to the $C^{1,\alpha}$ interior differentiability of solutions of this new variational problem.

Suppose we are given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ (e.g., a rectangle), a subset $D$ of $\Omega$ that is assumed to be measurable with ($L^2$ denoting Lebesgue’s measure on $\mathbb{R}^2$)

$$0 < L^2(D) < L^2(\Omega),$$

as well as an observed black and white image described through a measurable function $f: \Omega - D \rightarrow [0,1]$, where $f(x)$ is the intensity of the grey level at $x \in \Omega - D$.

Roughly speaking, the region $D$, which is also called an “inpainting domain” (see [12]), represents a certain part of this image for which image data are missing or inaccessible.

Our goal is to restore this missing part from the part that is known. In the image processing community, this kind of image interpolation is called “inpainting”, respectively, “image inpainting” (compare [12, 21, 22]).

At this point, we would like to add some general comments concerning inpainting. We are concerned with the attempt to recover the original image in terms of a function $u: \Omega \rightarrow [0,1]$ that measures the intensity of the grey level at $x \in \Omega$ on the whole domain $\Omega$ based on the partial observation $f: \Omega - D \rightarrow [0,1]$, which is usually corrupted by noise stemming from transmission or measuring errors.

Due to [21], there are essentially four different methods to handle the inpainting problem, depending on being variational or nonvariational and local or nonlocal.

Local inpainting methods take the information that is needed to fill in the inpainting domain $D$ only from neighboring points of the boundary of $D$ (compare [21]). In the case where the inpainting domain is quite small, these methods seem to be more desirable (compare [6, 13, 14, 15, 16, 21]).

Using nonlocal inpainting methods means that all the information of the known part of the image is taken into account and the information is weighted by its distance to the point that is to be filled in (compare [21]). Although these methods are suitable to fill in structures and textures, there are also several disadvantages as for instance the high computational costs arising in their numerical solution (compare [2, 21]).

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In our paper we concentrate on a TV-like variational nonlocal type approach, which leads to the minimization of a functional of the form

\[ I[u] := \int_{\Omega} \psi(|\nabla u|) \, dx + \frac{\lambda}{2} \int_{\Omega-D} (u - f)^2 \, dx. \]

Here, \( \lambda \) is a positive regularization parameter and \( \psi \) is assumed to be a convex and monotone increasing function with nonnegative values.

The second term on the right-hand side of (2) measures the quality of data fitting, i.e., the deviation of the original image \( u \) from the given data on \( \Omega - D \), while the first term produces a kind of mollification and allows one to incorporate some kind of \textit{a priori} information of the generated image on the entire domain \( \Omega \) into the minimization process.

In this setting, a common choice of \( \psi \) is \( \psi(|\nabla u|) := |\nabla u| \). This leads to the total variation inpainting model (compare [3, 21]). To discuss this variational problem, one has to work with functions \( \Omega \to \mathbb{R} \) of bounded variation, i.e., in the space \( \text{BV}(\Omega) \). In this situation, \( \nabla u \) denotes the distributional gradient, which is represented by a vector-valued Radon measure on \( \Omega \) with finite total variation \( \int_{\Omega} |\nabla u| \) (for the details, we refer the reader to [18]).

Due to the lack of ellipticity, in general, variational problems involving the total variation admit only irregular solutions, and in the papers [9, 10] and also in the related work [8] the basic idea was to replace the TV-density \( \psi(|\nabla u|) = |\nabla u| \) by a family of densities that are still of linear growth but with better ellipticity properties leading to appropriate regularity results for the corresponding minimizers. At first glance, our investigations seem to be of rather theoretical interest. However, first numerical experiments (using the densities introduced in formulas (7) and (8)) carried out in collaboration with J. Weickert (compare the forthcoming joint paper [11]) clearly indicate that our \textit{Ansatz} is numerically comparable to the standard TV-model with the clear advantage of a complete analysis of the regularity properties of minimizers, which is not available in the TV-case.

Now we pass to the details: as in the papers [9] and [10], we introduce the energy

\[ I[w] := \int_{\Omega} F(\nabla w) \, dx + \frac{\lambda}{2} \int_{\Omega-D} (w - f)^2 \, dx \]

for functions \( w \) in the Sobolev space \( W^{1,1}(\Omega) \) (see [1] for the details concerning this space), where \( F: \mathbb{R}^2 \to [0, \infty) \) is a \( \mu \)-elliptic density of linear growth satisfying \( F \in C^2 \), \( F(0) = 0 \), and \( DF(0) = 0 \).

More precisely, we impose the following conditions on \( F \): there exist positive constants \( \nu_1, \nu_2, \nu_3 \) and a real number \( \mu > 1 \) such that for any \( Y, Z \in \mathbb{R}^2 \) we have

\[ |DF(Z)| \leq \nu_1 \]

and

\[ \frac{\nu_2}{(1 + |Z|)^\mu} |Y|^2 \leq D^2 F(Z)(Y, Y) \leq \nu_3 \frac{1}{1 + |Z|} |Y|^2. \]

Based on these hypotheses, we can state some useful properties of \( F \), which were established, e.g., in [12] p. 97, 98.

**Lemma 1.** Suppose that \( F \) satisfies (4) and (5) for some number \( \mu > 1 \). Then \( F \) is strictly convex on \( \mathbb{R}^2 \), and:

i) there are real constants \( \nu_1 > 0 \), \( \nu_2 \in \mathbb{R} \) such that for all \( Z \in \mathbb{R}^2 \) we have

\[ DF(Z) \cdot Z \geq \nu_1 |Z| - \nu_2; \]
ii) $F$ is of linear growth in the sense that for real numbers $\nu_3$, $\nu_4 > 0$, $\nu_5$, $\nu_6 \in \mathbb{R}$ and for all $Z \in \mathbb{R}^2$ we have
\[
\nu_3 |Z| - \nu_5 \leq F(Z) \leq \nu_4 |Z| + \nu_6;
\]
iii) the integrand satisfies a balancing condition: there exists a real constant $\nu_7 > 0$ such that
\[
|D^2F(Z)| |Z|^2 \leq \nu_7 (1 + F(Z))
\]
for all $Z \in \mathbb{R}^2$.

In this context, the minimal surface integrand given by $F(Z) := \sqrt{1 + |Z|^2}$ serves as the most prominent example for which we have (4) and (5). It remains to note that (5) holds here for the optimal choice $\mu = 3$.

Furthermore, as was outlined in [8], another explicit example of a density $F$ satisfying the above hypothesis exactly with a given value $\mu > 1$ is generated by the function
\[
\Phi_\mu(t) := \int_0^t \int_0^s (1 + r)^{-\mu} dr ds, \quad t \geq 0,
\]
if we define
\[
F_\mu(Z) := \Phi_\mu(|Z|), \quad Z \in \mathbb{R}^2.
\]
Note that $(\mu - 1)F_\mu(Z) \to |Z|$ as $\mu \to \infty$, which follows from the formula
\[
\Phi_\mu(t) = \frac{t}{\mu - 1} + \frac{1}{(\mu - 1)(\mu - 2)}(t + 1)^{-\mu + 2} - \frac{1}{(\mu - 1)(\mu - 2)}, \quad \mu \neq 2,
\]
wheras
\[
\Phi_2(t) = t - \ln(1 + t).
\]
With respect to the explicit formulas (7) and (8), the density $F_\mu(\nabla u)$ serves as a good candidate for an approximation of $|\nabla u|$ by more regular integrands of linear growth and, as mentioned before, working with the expression $F_\mu(\nabla u)$ leads to adequate computational results.

The following theorem on the existence, uniqueness, and regularity properties of minimizers was shown in [9].

**Theorem 1.** Suppose (1) is true and define the energy $I$ by (3) with $F$ satisfying (4) and (5) for some $\mu \in (1, 2)$. Then:

i) the problem $I \to \min$ admits a unique solution $u$ in the space $W^{1,1}(\Omega)$;
ii) the solution satisfies $0 \leq u \leq 1$ a.e. on $\Omega$;
iii) we have $u \in W^{1,p}_{\text{loc}}(\Omega)$ for any finite $p$, whence $u$ is Hölder continuous in the interior of $\Omega$ for any exponent strictly smaller than 1;
iv) there is an open subset $\Omega_0$ of $\Omega$ such that $\dim_H(\Omega - \Omega_0) = 0$ and $u \in C^{1,\beta}(\Omega_0)$ for any $\beta < 1$;
v) if $D$ is an open set, then $D \subset \Omega_0$, i.e., $u \in C^{1,\alpha}(D)$ for any $\alpha \in (0, 1)$. For arbitrary sets $D$ we have $\text{Int}(D) \subset \Omega_0$, where $\text{Int}(D)$ is the set of interior points of $D$.

**Remark 1.** Recall, by definition, that $\dim_H(\Omega - \Omega_0) = 0$ means $\mathcal{H}^\varepsilon(\Omega - \Omega_0) = 0$ for any $\varepsilon > 0$ ($\mathcal{H}^\varepsilon$ denoting the Hausdorff measure of dimension $\varepsilon$).

**Remark 2.** Let us give a comment on the almost everywhere regularity result stated in iv) of Theorem 1. In [9] Theorem 1.4 we applied a technique of Frehse and Seregin [17] to show that $\sigma := DF(\nabla u)$ is a continuous vector-field on $\Omega$, however, due to the linear growth of $F$, only the continuity of $\nabla u$ up to a small closed exceptional set follows via “almost everywhere inversion” using $(DF)^{-1}$. For proving this result, the inpainting
term, i.e., the quantity $\int_{\Omega-D} (f-w)^2 \, dx$ occurring in (3) causes no essential technical problem, when going through the arguments of Frehse and Seregin in the situation at hand (compare Remark 3).

In this note we are going to show that, actually, interior singularities can be excluded, more precisely we have the following substantial improvement of Theorem 1 iv).

**Theorem 2.** Suppose that (11), (14), and (15) are fulfilled together with $\mu \in (1,2)$. Then $u \in C^{1,\alpha}(\Omega)$ for any $0 < \alpha < 1$, where $u$ is the solution from Theorem 1.

**Remark 3.** In contrast to the situation described in the previous remark, the occurrence of the inpainting quantity $\int_{\Omega-D} (f-w)^2 \, dx$ causes some severe problems if one likes to prove interior $C^{1,\alpha}$-regularity of minimizers via DeGiorgi-type arguments, which means that we cannot directly refer to, e.g., [7] and add some obvious modifications. Moreover, during our proof we also want to investigate in detail (see Lemma 4) what initial integrability of $\nabla u$ is actually needed to obtain its local boundedness.

**Remark 4.** We emphasize that it is easy to check that Theorem 1 and Theorem 2 also hold true in the case where $D = \emptyset$ (“pure denoising of $f$”).

The proof of Theorem 2 is split into four steps: regularization, Caccioppoli-type inequality, DeGiorgi-type iteration, and the conclusion.

**Step 1. Regularization**

**Lemma 2.** Suppose that we have (11), (14), and (15) for some $\mu > 1$. For $\delta > 0$, we let

$$I_\delta[w] := \int_{\Omega} F_\delta(\nabla w) \, dx + \frac{\lambda}{2} \int_{\Omega-D} (w-f)^2 \, dx,$$

$$F_\delta(Z) := \frac{\delta}{2}|Z|^2 + F(Z), \quad Z \in \mathbb{R}^2.$$

Then the problem $I_\delta \to \min$ in $W^{1,2}(\Omega)$ admits a unique solution $u_\delta$ and we have $0 \leq u_\delta \leq 1$ a.e. in $\Omega$ as well as $u_\delta \in W^{2,2}_{\text{loc}}(\Omega)$.

Moreover, if $\mu \in (1,2)$, then

i) $I_\delta[u_\delta] \to I[u]$, $\delta \int_{\Omega} |\nabla u_\delta|^2 \, dx \to 0$, $u_\delta \to u$ in $L^1_{\text{loc}}(\Omega)$ as $\delta \to 0$;

ii) $\nabla u_\delta \in L^p_{\text{loc}}(\Omega, \mathbb{R}^2)$ uniformly in $\delta$ for any finite $p$, i.e.,

$$\sup_\delta \int_{\Omega'} |\nabla u_\delta|^p \, dx = c(p, \Omega') < \infty,$$

where $\Omega' \subset \subset \Omega$. As a consequence, we also have $\nabla u_\delta \to \nabla u$ in $L^p_{\text{loc}}(\Omega, \mathbb{R}^2)$ as $\delta \to 0$ for all $p < \infty$.

**Proof.** The lemma was established in [9, §3].

**Step 2. Caccioppoli-type inequality**

First, we introduce some notation: we fix a point $x_0 \in \Omega$ and consider radii $0 < r < R < R_0$ with $B_{R_0}(x_0) \Subset \Omega$. Moreover, we let

$$\Gamma_\delta := 1 + |\nabla u_\delta|^2, \quad A_\delta(k, R) := \{x \in B_R(x_0) : \Gamma_\delta > k\}, \quad k > 0,$$

and consider $\eta \in C^\infty_0(B_R(x_0))$ with $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_r(x_0)$ and $|\nabla \eta| \leq c/(R-r)$.

Finally, for functions $v : \Omega \to \mathbb{R}$ we denote $\max\{v, 0\}$ by $v^+$.

In the following Lemma 4 we establish a Caccioppoli-type inequality which, in fact, is valid for any $\mu > 1$. 

Lemma 3. With the previous notation and under the general assumption of Lemma 2, i.e., with (1), (3), and (5) for some $\mu > 1$, we have the following variant of Caccioppoli’s inequality:

$$
\int_{A_\delta(k,R)} \frac{\delta}{R - r} |\nabla \Gamma_\delta| \eta^2 \, dx \leq c \int_{A_\delta(k,R)} |D^2 F_\delta(\nabla u_\delta)| |\nabla \eta|^2 (\Gamma_\delta - k)^2 \, dx
$$

(10)

$$
+ c \int_{A_\delta(k,R)} \eta^2 |\nabla u_\delta|^2 + \mu \, dx + \int_{A_\delta(k,R)} \eta |\nabla \eta| |\nabla u_\delta|^3 \, dx
$$

$$
\leq \frac{c}{(R - r)^2} \int_{A_\delta(k,R)} \frac{\delta}{\lambda} \, dx
$$

with $\nu := \max\{4, 2 + \mu\}$ and for a positive constant $c$ independent of $\delta, r$ and $R$.

Proof. The second inequality follows from the first because from now we assume without loss of generality that $R_0 \leq 1$, i.e., $1 < 1/(R - r)$, and $k \geq 2$, i.e., $\Gamma_\delta - k \leq \Gamma_\delta \leq c |\nabla u_\delta|^2 \leq 2 \Gamma_\delta$ on $A_\delta(k,R)$. We also recall that $\mu > 1$, i.e., $3 < 2 + \mu$. Finally, on $A_\delta(k,R)$ we use the inequalities

$$
|D^2 F(\nabla u_\delta)| (\Gamma_\delta - k)^2 \leq c |\nabla u_\delta|^3,
$$

$$
(\delta(\Gamma_\delta - k))^2 \leq c |\nabla u_\delta|^4.
$$

Remark 5. Of course, the choice of $\nu$ in Lemma 3 is not optimal. Regularizing the energy density $F$ with

$$
F_{\delta,q}(Z) := \frac{\delta}{q} |Z|^q + F(Z),
$$

where $q > 1$ is sufficiently close to 1, we could choose any $\nu > \max\{3, 2 + \mu\} = 2 + \mu$. Details are left to the reader.

It remains to prove the first inequality in (10). Observe that $u_\delta$ solves the Euler equation

$$
\int_{\Omega} DF_\delta(\nabla u_\delta) \cdot \nabla \varphi \, dx = - \int_{\Omega - D} \lambda (u_\delta - f) \varphi \, dx
$$

for all $\varphi \in C_0^\infty(\Omega)$. Setting $\varphi = \partial_\alpha \psi, \alpha \in \{1, 2\}$, with $\psi \in C_0^\infty(\Omega)$, we have

$$
\int_{\Omega} DF_\delta(\nabla u_\delta) \partial_\alpha \nabla \psi \, dx = - \int_{\Omega - D} \lambda (u_\delta - f) \partial_\alpha \psi \, dx
$$

for all $\psi \in C_0^\infty(\Omega)$. Since $u_\delta \in W^{2,2}_{\text{loc}}(\Omega)$ (compare Lemma 2) and thereby $DF_\delta(\nabla u_\delta) \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^2)$, integration by parts leads to

$$
\int_{\Omega} D^2 F_\delta(\nabla u_\delta)(\partial_\alpha \nabla u_\delta, \nabla \psi) \, dx = \int_{\Omega - D} \lambda (u_\delta - f) \partial_\alpha \psi \, dx
$$

for all $\psi \in W^{1,2}_{\text{loc}}(\Omega)$ with compact support in $\Omega$. Observing that $\psi = \eta^2 \partial_\alpha u_\delta (\Gamma_\delta - k)^+$ is admissible, we get (from now on summation is with respect to $\alpha \in \{1, 2\}$)

$$
\int_{A_\delta(k,R)} D^2 F_\delta(\nabla u_\delta)(\partial_\alpha \nabla u_\delta, \partial_\alpha \nabla u_\delta)(\Gamma_\delta - k) \eta^2 \, dx
$$

(11)

$$
+ \int_{A_\delta(k,R)} D^2 F_\delta(\nabla u_\delta)(\partial_\alpha \nabla u_\delta, \partial_\alpha u_\delta \nabla \Gamma_\delta) \eta^2 \, dx
$$

$$
+ 2 \int_{A_\delta(k,R)} D^2 F_\delta(\nabla u_\delta)(\partial_\alpha \nabla u_\delta, \partial_\alpha u_\delta \nabla \eta)(\Gamma_\delta - k) \, dx
$$

$$
= \int_{B_R(x_0) - D} \lambda (u_\delta - f) \partial_\alpha [\eta^2 \partial_\alpha u_\delta (\Gamma_\delta - k)^+] \, dx.$$
For the second integral on the left-hand side we have
\[
\int_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\partial_\alpha \nabla u_\delta, \partial_\alpha u_\delta \nabla \eta) \eta^2 \, dx = \frac{1}{2} \int_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\nabla \Gamma_\delta, \nabla \eta) \eta^2 \, dx.
\]

In accordance with \((12)\), for the third integral on the left-hand side of \((11)\) we also have
\[
\int_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\partial_\alpha \nabla u_\delta, \partial_\alpha u_\delta \nabla \eta) \eta(\Gamma_\delta - k) \, dx
= \frac{1}{2} \int_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\nabla \eta, \nabla \Gamma_\delta) \eta(\Gamma_\delta - k) \, dx.
\]

Summarizing, relations \((11)-(13)\), the Cauchy–Schwarz inequality applied to the bilinear form \(D^2F_\delta(\nabla u_\delta)\), and Young’s inequality \((\varepsilon > 0)\) yield
\[
\int_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\partial_\alpha \nabla u_\delta, \partial_\alpha u_\delta)(\Gamma_\delta - k) \eta^2 \, dx
= \frac{1}{2} \int_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\nabla \Gamma_\delta, \nabla \eta) \eta^2 \, dx
\]
\[
\leq \varepsilon \int_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\nabla \eta, \nabla \Gamma_\delta) \eta^2 \, dx
+ \frac{1}{\varepsilon} \int_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\nabla \eta, \eta)(\Gamma_\delta - k)^2 \, dx
+ \int_{B_R(x_0) - D} \lambda(u_\delta - f) \partial_\alpha [\eta^2 \partial_\alpha u_\delta(\Gamma_\delta - k)^+] \, dx.
\]

In what follows next, we concentrate on the last integral on the right in \((14)\), which is denoted by \(I\). Recalling that \(0 \leq u_\delta \leq 1\), \(0 \leq f \leq 1\) a.e., we get
\[
I \leq c \int_{A_\delta(k,R)} |\nabla \eta| |\nabla u_\delta|(\Gamma_\delta - k) \, dx + c \int_{A_\delta(k,R)} \eta^2 |\nabla^2 u_\delta|(\Gamma_\delta - k) \, dx
\]
\[
+ c \int_{A_\delta(k,R)} \eta^2 |\nabla u_\delta| |\nabla \Gamma_\delta| \, dx.
\]

Another application of Young’s inequality \((\varepsilon > 0)\) gives
\[
\int_{A_\delta(k,R)} \eta^2 |\nabla^2 u_\delta|(\Gamma_\delta - k) \, dx
\]
\[
\leq \varepsilon \int_{A_\delta(k,R)} \eta^2 |\nabla^2 u_\delta|^2 (\Gamma_\delta - k) \Gamma_\delta^{-\mu} \, dx + \frac{1}{\varepsilon} \int_{A_\delta(k,R)} \eta^2 (\Gamma_\delta - k) \Gamma_\delta^\mu \, dx.
\]
as well as
\[
\int_{A_\delta(k,R)} \eta^2 |\nabla u_\delta| |\nabla \Gamma_\delta| \, dx \leq \varepsilon \int_{A_\delta(k,R)} \eta^2 |\nabla \Gamma_\delta|^2 \Gamma_\delta^{-\mu} \, dx + \frac{1}{\varepsilon} \int_{A_\delta(k,R)} \eta^2 |\nabla u_\delta|^2 \Gamma_\delta^\mu \, dx.
\]

Recalling that \(k \geq 2\), we have \(|\nabla u_\delta| \geq 1\) on \(A_\delta(k,R)\), and therefore \(\Gamma_\delta \leq c |\nabla u_\delta|^2\) on \(A_\delta(k,R)\). Incorporating \((16)\) and \((17)\) in \((15)\), we see that
\[
I \leq c \int_{A_\delta(k,R)} |\nabla \eta| |\nabla u_\delta|^3 \, dx + c \varepsilon \int_{A_\delta(k,R)} \eta^2 |\nabla^2 u_\delta|^2 (\Gamma_\delta - k) \Gamma_\delta^{-\mu} \, dx
\]
\[
+ c \varepsilon \int_{A_\delta(k,R)} \eta^2 |\nabla \Gamma_\delta|^2 \Gamma_\delta^\mu \, dx + \frac{c}{\varepsilon} \int_{A_\delta(k,R)} \eta^2 |\nabla u_\delta|^{2+\mu} \, dx.
\]
Connecting (18) with (14), we get
\[
\int_{A_{\delta}(k,R)} D^2 F_\delta(\nabla u_{\delta})(\partial_\alpha \nabla u_{\delta}, \partial_\alpha \nabla u_{\delta})(\Gamma_\delta - k) \eta^2 \, dx \\
+ \frac{1}{2} \int_{A_{\delta}(k,R)} D^2 F_\delta(\nabla u_{\delta})(\nabla \Gamma_\delta, \nabla \Gamma_\delta) \eta^2 \, dx \\
\leq \varepsilon \int_{A_{\delta}(k,R)} D^2 F_\delta(\nabla u_{\delta})(\nabla \eta, \nabla \eta)(\Gamma_\delta - k)^2 \, dx \\
+ \frac{1}{\varepsilon} \int_{A_{\delta}(k,R)} D^2 F_\delta(\nabla u_{\delta})(\nabla \eta, \nabla \eta)(\Gamma_\delta - k)^2 \, dx \\
+ c \int_{A_{\delta}(k,R)} |\nabla \eta|^2 |\nabla u_{\delta}|^3 \, dx + c \varepsilon \int_{A_{\delta}(k,R)} \eta^2 |\nabla^2 u_{\delta}|^2 (\Gamma_\delta - k) \Gamma_\delta^{\frac{p}{2}} \, dx \\
+ c \varepsilon \int_{A_{\delta}(k,R)} \eta^2 |\nabla \Gamma_\delta|^2 \Gamma_\delta^{\frac{p}{2}} \, dx + \frac{c}{\varepsilon} \int_{A_{\delta}(k,R)} \eta^2 |\nabla u_{\delta}|^{2+\mu} \, dx.
\]
Choosing \( \varepsilon > 0 \) sufficiently small and using (5), we obtain
\[
\int_{A_{\delta}(k,R)} \eta^2 |\nabla^2 u_{\delta}|^2 (\Gamma_\delta - k) \Gamma_\delta^{\frac{p}{2}} \, dx + \int_{A_{\delta}(k,R)} \eta^2 |\nabla \Gamma_\delta|^2 \Gamma_\delta^{\frac{p}{2}} \, dx \\
\leq c \int_{A_{\delta}(k,R)} D^2 F_\delta(\nabla u_{\delta})(\nabla \eta, \nabla \eta)(\Gamma_\delta - k)^2 \, dx \\
+ c \int_{A_{\delta}(k,R)} |\nabla \eta|^2 |\nabla u_{\delta}|^3 \, dx + c \int_{A_{\delta}(k,R)} \eta^2 |\nabla u_{\delta}|^{2+\mu} \, dx.
\]
Finally, the nonnegative first integral on the left-hand side of (19) may be neglected, which immediately proves the first inequality in (10). \( \square \)

**Step 3. DeGiorgi-type iteration**

Now we come to a DeGiorgi-type lemma that, with the help of Caccioppoli’s inequality, gives a sufficient condition to close the gap between uniform local higher \( \bar{p} \)-integrability of the gradients and uniform local \textit{a priori} gradient bounds.

Of course, in the inpainting model at hand, we even have (9) for any finite \( p \). However, replacing (21) by (9) gives no essential simplification of the proof.

**Lemma 4.** Suppose that \( \{u_{\delta}\} \) is a sequence of class \( W^{2,2}_{\text{loc}}(\Omega) \). Moreover, suppose that we are given real numbers \( \bar{p}, \nu > 3, \mu > 1 \) such that
\[
\mu + \nu < \bar{p}.
\]
If with a uniform constant \( c > 0 \) (and with \( \Gamma_\delta, A_{\delta}(k,R), r, R, R_0, \eta \) as above) we have
\[
\int_{A_{\delta}(k,R)} \Gamma_\delta^{\frac{p}{2}} |\nabla \Gamma_\delta|^2 \eta^2 \, dx \leq \frac{c}{(R-r)^2} \int_{A_{\delta}(k,R)} \Gamma_\delta^{\frac{\nu}{2}} \, dx,
\]
and if we know in addition that
\[
\sup_{\delta} \int_{\Omega'} |\nabla u_{\delta}|^p \, dx = c(\bar{p}, \Omega') < \infty,
\]
where \( \Omega' \Subset \Omega \), then \( \nabla u_{\delta} \in L^\infty_{\text{loc}}(\Omega, \mathbb{R}^2) \) uniformly in \( \delta \).

As has already been mentioned, we immediately obtain the following.

**Corollary 1.** Suppose that \( \mu \in (1,2) \). If \( \{u_{\delta}\} \) denotes the approximating sequence of Lemma 2 to the inpainting model under consideration, then we have uniform (in \( \delta \)) local \textit{a priori} gradient bounds for \( u_{\delta} \).
For proving Lemma 4 we need a technical proposition which is of pure algebraic nature.

**Proposition 1.** Consider real numbers \( \bar{p}, \nu > 3 \), and \( \mu > 1 \) with \( \mu + \nu < \bar{p} \).

Then there exist real numbers \( s_1, s_2, s_3 \) such that

\[
\begin{align*}
i) & \quad 2 \frac{s_1}{s_1 - 1} < \bar{p}, \quad ii) & \quad 2 \frac{s_1}{s_1} > 1, \quad iii) & \quad \mu \frac{s_2}{s_2 - 1} < \bar{p}, \quad iv) & \quad \nu \frac{s_3}{s_3 - 1} < \bar{p}, \quad v) & \quad \frac{1}{s_3} + \frac{1}{s_2} > 1.
\end{align*}
\]

**Proof of Proposition 1.** First we choose \( \bar{p} < \bar{p} \) so that we still have \( \mu + \nu < \bar{p} \); in particular, \( \bar{p} > 4 \) and \( \bar{p} > \nu \). Then i) and iv) are obvious by setting

\[
s_1 := \frac{\bar{p}}{\bar{p} - 2} > 1, \quad s_3 := \frac{\bar{p}}{\bar{p} - \nu} > 1.
\]

Since \( \bar{p} > 4 \), we also get ii). To prove v) we observe that

\[
1 - \frac{1}{s_3} = \frac{\nu}{\bar{p}} < 1
\]

and we may choose \( s_2 > 1 \) in such a way that

\[
1 - \frac{1}{s_3} < \frac{1}{s_2} < 1.
\]

In particular, the left-hand side of (22) implies v). It remains to show iii), which will follow from

\[
\frac{1}{s_2} < 1 - \frac{\mu}{\bar{p}}.
\]

Here we note that \( \mu + \nu < \bar{p} \) implies \( \nu/\bar{p} < 1 - \mu/\bar{p} \); thus, we may choose \( s_2 > 1 \) in addition to (22) in such a way that

\[
1 - \frac{1}{s_3} < \frac{1}{s_2} < 1 - \frac{\mu}{\bar{p}}
\]

which proves iii) and altogether Proposition 1.

**Proof of Lemma 4.** With the previous notation and using Sobolev’s inequality, we have

\[
\int_{A_{\delta}(k,r)} (\Gamma_{\delta} - k)^2 \, dx \leq \int_{B_{\delta}(x_0)} (\eta(\Gamma_{\delta} - k))^2 \, dx \leq c \left[ \int_{B_{\delta}(x_0)} |\nabla [\eta(\Gamma_{\delta} - k)]| \, dx \right]^2.
\]

Moreover,

\[
\left[ \int_{B_{\delta}(x_0)} |\nabla [\eta(\Gamma_{\delta} - k)]| \, dx \right]^2 = \left[ \int_{A_{\delta}(k,R)} |\nabla [\eta(\Gamma_{\delta} - k)]| \, dx \right]^2 \leq c \left[ \int_{A_{\delta}(k,R)} |\nabla \eta(\Gamma_{\delta} - k) \eta| \, dx \right]^2 + c \left[ \int_{A_{\delta}(k,R)} \eta |\nabla \Gamma_{\delta} | \, dx \right]^2 =: c[I_1^2 + I_2^2].
\]

Thus, we get

\[
\int_{A_{\delta}(k,r)} (\Gamma_{\delta} - k)^2 \, dx \leq c[I_1^2 + I_2^2].
\]

In the following we apply the algebraic Proposition 1 with the same parameters as given in Lemma 4. In particular, we have \( \mu + \nu < \bar{p} \) and the claims i)–v) of Proposition 1 are valid.
Thus (21) and i) of Proposition 1 give \( \Gamma_\delta - k \in L^{\frac{s_1}{s_1-t}}(B_R(x_0)) \) uniformly in \( \delta \), and Hölder’s inequality implies

\[
I_1^2 = \left[ \int_{A_\delta(k,R)} |\nabla \eta| (\Gamma_\delta - k) \, dx \right]^2 \\
\leq \frac{c}{(R-r)^2} \left[ \mathcal{L}^2(A_\delta(k,R)) \right]^{\frac{2}{s_1}} \left[ \int_{A_\delta(k,R)} (\Gamma_\delta - k)^{\frac{s_1}{s_1-t}} \, dx \right]^{\frac{2(s_1-1)}{s_1}} \\
\leq \frac{c}{(R-r)^2} \left[ \mathcal{L}^2(A_\delta(k,R)) \right]^{\frac{2}{s_1}},
\]

i.e., by ii) of Proposition 1 there exists a real number \( \bar{\beta} = 2/s_1 > 1 \) such that

\[
I_1^2 \leq \frac{c}{(R-r)^2} \left[ \mathcal{L}^2(A_\delta(k,R)) \right]^{\bar{\beta}}.
\]

Next we discuss \( I_2 \). We apply Caccioppoli’s inequality of assumption (20), which, once again after an application of Hölder’s inequality, gives

\[
I_2^2 \leq \left[ \int_{A_\delta(k,R)} \eta^2 |\nabla \Gamma_\delta|^2 \Gamma_\delta^{\bar{\beta}} \, dx \right] \cdot \left[ \int_{A_\delta(k,R)} \Gamma_\delta^{\bar{\beta}} \, dx \right] \\
\leq \frac{c}{(R-r)^2} \left[ \int_{A_\delta(k,R)} \Gamma_\delta^{\bar{\beta}} \, dx \right] \cdot \left[ \int_{A_\delta(k,R)} \Gamma_\delta^{\bar{\beta}} \, dx \right].
\]

With iii) of Proposition 1 we estimate

\[
\int_{A_\delta(k,R)} \Gamma_\delta^{\bar{\beta}} \, dx \leq \left[ \int_{A_\delta(k,R)} \Gamma_\delta^{\frac{2(s_1-1)}{s_1}} \, dx \right]^{\frac{2}{s_1}} \left[ \mathcal{L}^2(A_\delta(k,R)) \right]^{\frac{1}{s_1}} \leq c \left[ \mathcal{L}^2(A_\delta(k,R)) \right]^{\frac{1}{s_1}}.
\]

Moreover, item iv) of Proposition 1 gives

\[
\int_{A_\delta(k,R)} \Gamma_\delta^{\bar{\beta}} \, dx \leq \left[ \int_{A_\delta(k,R)} \Gamma_\delta^{\frac{2(s_1-1)}{s_1}} \, dx \right]^{\frac{2}{s_1}} \left[ \mathcal{L}^2(A_\delta(k,R)) \right]^{\frac{1}{s_1}} \leq c \left[ \mathcal{L}^2(A_\delta(k,R)) \right]^{\frac{1}{s_1}}.
\]

Combining (26)–(28) with v) of Proposition 1 we obtain a real number \( \tilde{\beta} = 1/s_2 + 1/s_3 > 1 \) such that

\[
I_2^2 \leq \frac{c}{(R-r)^2} \left[ \mathcal{L}^2(A_\delta(k,R)) \right]^{\tilde{\beta}}.
\]

Now, without loss of generality we have \( \mathcal{L}^2(A_\delta(k,R)) < 1 \), so that (21), (25), and (29) show the existence of a real number \( \beta > 1 \) such that

\[
\int_{A_\delta(k,R)} (\Gamma_\delta - k)^2 \, dx \leq \frac{c}{(R-r)^2} \left[ \mathcal{L}^2(A_\delta(k,R)) \right]^{\beta}.
\]

Next we define the following quantities for \( k \geq 2 \) and \( r < R \):

\[
\tau_\delta(k,r) := \int_{A_\delta(k,R)} (\Gamma_\delta - k)^2 \, dx, \quad a_\delta(k,r) := \mathcal{L}^2(A_\delta(k,r)).
\]

Moreover, suppose that two real numbers \( h, k \) with \( h > k > 2 \) are given, whence \( \frac{\Gamma_\delta - k}{h - k} \geq 1 \) on \( A_\delta(h,R) \). We get

\[
a_\delta(h,R) = \int_{A_\delta(h,R)} 1 \, dx \leq \int_{A_\delta(h,R)} (\Gamma_\delta - k)^2 (h - k)^{-2} \, dx.
\]

Thus,

\[
a_\delta(h, R) \leq \frac{1}{(h - k)^2} \tau_\delta(k, R),
\]
and \((30), (31)\) give
\[
\tau_\delta(h, r) \leq \frac{c}{(R - r)\gamma(h - k)^\alpha [\tau_\delta(k, R)]^\beta}
\]
with exponents
\[
\gamma := 2, \quad \alpha := 2\beta > 0, \quad \beta > 1.
\]
With \((32)\) and \((33)\), the assumptions of Stampacchia’s well-known lemma (see [23, Lemma 5.1, p. 219] or [20, Lemma B.1, p. 63]) are fulfilled, and we see that, for all \(\sigma \in (0, 1)\),
\[
\tau_\delta(d_\delta + k_0, R_0 - \sigma R_0) = 0
\]
with
\[
d_\delta = \frac{2(\alpha + \beta)\beta}{\sigma \gamma R_0^\gamma [\tau_\delta(k_0, R_0)]^{\beta - 1}} \leq d^\alpha,
\]
where \(d\) is a constant not depending on \(\delta\) (recall that \(p > 4\)). Choosing \(k_0 = 2\) and \(\sigma = \frac{1}{2}\), we arrive at
\[
0 = \tau_\delta(d_\delta + 2, R_0/2) \geq \tau_\delta(d + 2, R_0/2) \geq 0,
\]
i.e., we have shown that
\[
\tau_\delta(d + 2, R_0/2) = 0.
\]
As a consequence, we obtain the uniform estimate
\[
|\nabla u_\delta| \leq c
\]
a.e. on \(B_{R_0/2}(x_0)\). Using an immediate covering argument we then get
\[
\|\nabla u_\delta\|_{L^\infty(\omega, \mathbb{R}^2)} \leq c(\omega)
\]
for all \(\omega \in \Omega\) and \(\delta \in (0, 1)\), i.e., \(u_\delta\) is locally uniformly Lipschitz continuous with Lipschitz constant \(c(\omega) > 0\). This completes the proof of Lemma 4.

\[\square\]

**Step 4. Conclusion**

By assumption we have \(\mu \in (1, 2)\) and Corollary 1 gives \(\nabla u_\delta \in L^\infty_{\text{loc}}(\Omega, \mathbb{R}^2)\) uniformly in \(\delta\). Moreover, \(u_\delta \to u\) in \(L^1_{\text{loc}}(\Omega)\) as \(\delta \to 0\) (see Lemma 2 i)) and since \(u_\delta\) is locally uniformly Lipschitz continuous (with respect to \(\delta\)) we may apply Arzelà–Ascoli’s theorem to show that \(u \in C^{0,1}(\Omega)\).

To check that \(u\) has locally Hölder continuous first partial derivatives in \(\Omega\), we let \(\omega \in \Omega\) be arbitrary.

Moreover, we observe that \(u\) is a solution of the Euler equation
\[
\int_{\Omega} DF(\nabla u) \cdot \nabla \varphi \, dx = - \int_{\Omega} g \varphi \, dx
\]
for all \(\varphi \in C^\infty_0(\Omega)\), where \(g := \lambda \chi_{\Omega - D}(u - f)\). Setting \(\varphi = \partial_\alpha \psi, \alpha \in \{1, 2\}\), with \(\psi \in C^\infty_0(\Omega)\), we arrive at
\[
\int_{\Omega} DF(\nabla u) \cdot \partial_\alpha \nabla \psi \, dx = - \int_{\Omega} g \partial_\alpha \psi \, dx
\]
for all \(\psi \in C^\infty_0(\Omega)\). Since \(u\) is Lipschitz continuous, we may argue with the standard difference quotient technique to get \(u \in W^{2,2}_{\text{loc}}(\Omega)\). Moreover, we have \(DF(\nabla u) \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^2)\) by applying the chain rule for Sobolev functions. By means of these results, integration by parts leads to
\[
- \int_{\omega} D^2 F(\nabla u)(\partial_\alpha \nabla u, \nabla \psi) \, dx = - \int_{\omega} g \partial_\alpha \psi \, dx.
\]
Setting \( v := \partial_\alpha u \), we get
\[
\int_\Omega D^2F(\nabla u)(\nabla v, \nabla \psi) \, dx = \int_\Omega g \partial_\alpha \psi \, dx.
\]
The coefficients \( a_{\alpha \beta}(x) := \frac{\partial^2 F}{\partial p_\alpha \partial p_\beta}(\nabla u) \) are strictly elliptic and bounded on \( \omega \), which follows from (5) and from the local Lipschitz continuity of \( u \). Finally, [19, Theorem 8.22, p. 200] ensures the interior Hölder continuity of \( v \) and therefore of \( \partial_\alpha u \) for all \( \alpha \in \{1, 2\} \), i.e., \( u \) has locally Hölder continuous first partial derivatives in \( \Omega \). This completes the proof of Theorem 2.

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Department of Mathematics, Saarland University, P.O. Box 151150, 66041 Saarbrücken, Germany

E-mail address: bibi@math.uni-sb.de

Department of Mathematics, Saarland University, P.O. Box 151150, 66041 Saarbrücken, Germany

E-mail address: fuchs@math.uni-sb.de

Department of Mathematics, Saarland University, P.O. Box 151150, 66041 Saarbrücken, Germany

E-mail address: tietz@math.uni-sb.de

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