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CONTACT OF A THIN FREE BOUNDARY WITH A FIXED ONE IN THE SIGNORINI PROBLEM

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Dedicated to N. N. Ural'tseva on the occasion of her 80th birthday

Abstract. The Signorini problem is studied near a fixed boundary where the solution is "clamped down" or "glued". It is shown that, in general, the solutions are at least $C^{1/2}$ regular and that this regularity is sharp. Near the actual points of contact of the free boundary with the fixed one, the blowup solutions are shown to have homogeneity $\kappa \geq 3/2$, while at the noncontact points the homogeneity must take one of the values: $1/2, 3/2, \dots, m-1/2, \dots$.

§1. Introduction and main results

1.1. The Signorini problem. Our purpose in this paper is to study the behavior of the thin free boundary as it approaches the fixed boundary in the so-called (scalar) *Signorini problem* (also known as the *thin obstacle problem*).

The Signorini problem consists in minimizing the Dirichlet energy functional

$$J(v) := \int_{B_1^+} |\nabla v|^2 \tag{1.1}$$

on the closed convex set

$$\mathcal{K} = \mathcal{K}(g) := \{v \in W^{1,2}(B_1^+) \,:\, v = g \ \text{ on } \ (\partial B_1)^+,\ v \geq 0 \ \text{ on } \ B_1'\}, \tag{1.2}$$

for a given function $g \in L^2((\partial B_1)^+)$. Here and throughout, we use the following notation:

$$B_r(x) := \{y \in \mathbb{R}^n \,:\, |x-y| < r\}, \quad B_r := B_r(0),$$
$$E^+ := E \cap \{x_n > 0\}, \quad E' := E \cap \{x_n = 0\},$$

for a subset $E \subset \mathbb{R}^n$. We assume that $n \geq 2$. Using direct methods of calculus of variations, one can verify that a minimizer $u \in \mathcal{K}$ exists and satisfies the following variational inequality:

$$\int_{B_1^+} \nabla u \nabla (v-u) \geq 0 \ \text{ for any } \ v \in \mathcal{K}. \tag{1.3}$$

The problem above goes back to the fundamental paper [LS] on variational inequalities. It has been known for quite some time that the minimizers are in the class $C^{1,\alpha}(B_1^+ \cup B_1')$

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for some $\alpha > 0$ (see [Ca1] and also [U1]) and even $C^{1,1/2}(B_1^+ \cup B_1')$ in the dimension $n = 2$, see [Ri]. Moreover, the minimizers satisfy

$$\begin{gathered}\Delta u = 0 \text{ in } B_1^+,\\ u \geq 0, \quad -\partial_{x_n} u \geq 0, \quad u\,\partial_{x_n} u = 0 \text{ on } B_1'.\end{gathered}$$

These conditions are known as the *Signorini* or *complementarity boundary conditions.*

The problem features the following *a priori* unknown subsets of B_1':

$$\begin{aligned}&\Lambda(u) := \{x \in B_1' \,:\, u = 0\} && \textit{the coincidence set,}\\ &\Omega(u) := \{x \in B_1' \,:\, u > 0\} && \textit{the noncoincidence set,}\\ &\Gamma(u) := \partial_{B_1'}\Omega(u) && \textit{the free boundary.}\end{aligned}$$

The study of the geometric and analytic properties of the free boundary is one of the objectives of the Signorini problem. Sometimes it is said that the free boundary $\Gamma(u) \subset B_1'$ is *thin*, to indicate that it is (expected to be) of dimension $n - 2$.

Recent years have seen some interesting new developments in the problem, starting with the proof in [AC] that the minimizers u are in the class $C^{1,1/2}(B_1^+ \cup B_1')$, in any dimension $n \geq 2$, which is the optimal regularity. This opened up the possibility of studying the free boundary $\Gamma(u)$, which has been done in [ACS, CSS, GP], see also [PSU, Chapter 9]. An effective tool in the study of the free boundary is *Almgren's frequency formula*

$$N^x(r,u) := \frac{r\int_{B_r^+(x)} |\nabla u|^2}{\int_{(\partial B_r)^+} u^2}.$$

It originated in the work of Almgren on multivalued harmonic functions [Alm] and has an important property of being monotone in r, even for solutions of the Signorini problem. Then the free boundary points can be classified in accordance with

$$\kappa := N^x(0+,u).$$

It is known that $\kappa \geq 3/2$ for $x \in \Gamma(u)$ in the Signorini problem and that, more precisely, $\kappa = 3/2$ or $\kappa \geq 2$ [ACS]. This results in the decomposition

$$\Gamma(u) = \Gamma_{3/2}(u) \cup \bigcup_{\kappa \geq 2} \Gamma_\kappa(u), \quad \text{where } \Gamma_\kappa(u) := \{x \in \Gamma(u) \,:\, N^x(0+,u) = \kappa\}.$$

The set $\Gamma_{3/2}(u)$ is known as the *regular set.* Recently, it was shown that $\Gamma_{3/2}(u)$ is real analytic [KPS] by using a partial hodograph-Legendre transform, with the help of the $C^{1,\alpha}$ regularity proved in [ACS]. See also [DS1] for a different proof of C^∞ regularity, based on a generalization of the boundary Harnack principle. The only other free boundary points studied in the literature are those in $\Gamma_{2m}(u)$, $m \in \mathbb{N}$, which correspond to the points where the coincidence set $\Lambda(u)$ has zero H^{n-1} density, see [GP]. Such points are known as *singular points.* It was proved in [GP] that $\Gamma_{2m}(u)$ is contained in a countable union of C^1 manifolds.

An interesting question is finding all possible values for $\kappa = N^x(0+,u)$. In dimension $n = 2$ the answer is known (the proof is a simple exercise): κ must take one of the following values:

$$3/2,\ 2,\ 7/2,\ 4,\ \ldots,\ 2m - 1/2,\ 2m,\ \ldots\ .$$

However, this is still an open problem in dimensions $n \geq 3$.

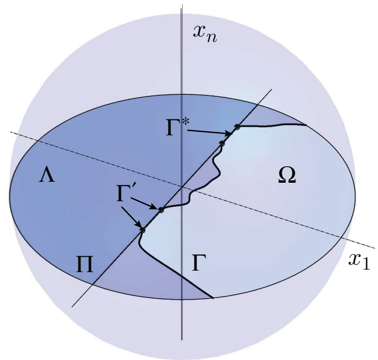


FIGURE 1. Free boundary Γ near the contact points Γ' with the fixed boundary Π considered in the hyperplane $\{x_n = 0\}$.

1.2. Contact of the free and fixed boundaries. Our objective in this paper is the study of the behavior of the free boundary $\Gamma(u)$ in the Signorini problem as it approaches a set where u is forced to be zero. More precisely, consider a closed subset $\mathcal{K}_0$ of the set $\mathcal{K}$ in (1.2), defined by

$$\mathcal{K}_0 = \mathcal{K}_0(g) := \{v \in \mathcal{K}(g) : v = 0 \text{ on } B_1' \cap \{x_1 \leq 0\}\}, \tag{1.4}$$

and minimize the Dirichlet energy J in (1.1) over $\mathcal{K}_0$. That is, compared to the Signorini problem, we have an additional constraint that the functions must vanish on $B_1' \cap \{x_1 \leq 0\}$. If we think of the solution of the Signorini problem as an elastic membrane that is forced to stay above zero in B_1', the new constraint in $\mathcal{K}_0$ can be thought of as "clamping down" or "gluing" the membrane on $B_1' \cap \{x_1 \leq 0\}$. The boundary of the last mentioned set in B_1' is

$$\Pi := \{x_1 = 0, x_n = 0\},$$

which we call the *fixed boundary.* Note that the coincidence set $\Lambda(u)$ will now contain $B_1' \cap \{x_1 \leq 0\}$ and the truly free part of $\Gamma(u)$ is $\Gamma(u) \cap \{x_1 > 0\}$. The points in

$$\Gamma'(u) := \overline{\Gamma(u) \cap \{x_1 > 0\}} \cap \Pi$$

are categorized as *contact points*, and those in

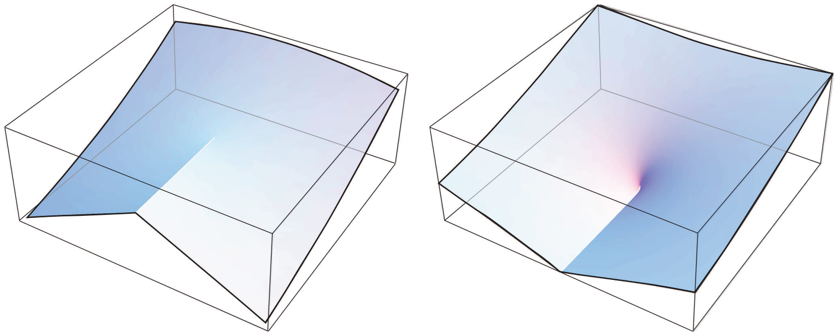
$$\Gamma^*(u) := (\Gamma(u) \cap \Pi) \setminus \Gamma'(u)$$

are *noncontact points*, see Figure 1. We note that the minimizers in $\mathcal{K}_0$ still solve the Signorini problem in small halfballs $B_r^+(x_0)$ with $x_0 \in B_1' \cap \{x_1 > 0\}$, and therefore we will have $u \in C^{1,1/2}_{\text{loc}}(B_1^+ \cup (B_1' \cap \{x_1 > 0\}))$ and

$$\begin{aligned} \Delta u = 0 &\text{ in } B_1^+, \\ u = 0 &\text{ on } B_1' \cap \{x_1 \leq 0\}, \\ u \geq 0, \quad -\partial_{x_n} u \geq 0, \quad u\,\partial_{x_n} u = 0 &\text{ on } B_1' \cap \{x_1 > 0\}. \end{aligned}$$

In the literature, there are many papers dealing with the contact of the free and fixed boundaries in various free boundary problems. The case of the classical obstacle problem, for instance, was studied by [U2, AU]. We also refer to [PSU, Chapter 8] and references therein for some of these results, including also extensions to other obstacle-type problems.

In contrast to the case of the classical obstacle problem, where the presence of the fixed boundary actually helps — for instance, to avoid a geometric "thickness" condition



$\widehat{u}_{3/2}(x) = \operatorname{Re}(x_1 + i|x_n|)^{3/2}$ $\qquad$ $\widehat{u}_{1/2}(x) = \operatorname{Re}(x_1 + i|x_n|)^{1/2}$

FIGURE 2. Examples of solutions limiting the optimal regularity: $\widehat{u}_{3/2}(x)$ is an explicit solution of the Signorini problem and $\widehat{u}_{1/2}(x)$ is a minimizer over $\mathcal{K}_0$ with worst possible regularity.

on the coincidence set, needed for the regularity of the free boundary — in the Signorini problem the presence of the fixed boundary introduces a serious handicap. Indeed, as we have mentioned earlier, the optimal regularity of the Signorini problem is $C^{1,1/2}$. This regularity is exhibited by the following explicit solution:

$$\widehat{u}_{3/2}(x) := \operatorname{Re}(x_1 + i|x_n|)^{3/2}. \tag{1.5}$$

On the other hand, it is easy to check that

$$\widehat{u}_{1/2}(x) := \operatorname{Re}(x_1 + i|x_n|)^{1/2} \tag{1.6}$$

is a minimizer of J over $\mathcal{K}_0$ (simply because it is harmonic in $B_1 \setminus (B_1' \cap \{x_1 \le 0\})$), thus limiting the generally expected regularity of minimizers of J to at most $C^{1/2}$. (See Figure 2 for the illustration of these solutions.)

This lower regularity of minimizers undercuts many techniques used for the Signorini problem, calling for caution even when we deal with the first derivatives of the solution. Luckily, however, one of the most important tools in our analysis, Almgren's frequency formula, still works: a step in the proof is based on a Rellich-type identity, which in our case becomes an inequality in the correct direction and allows the proof to go through.

1.3. Main results. The first main result in this paper establishes the optimal regularity of the minimizers.

Theorem 1.1 (Optimal regularity)**.** *If u is a minimizer of the functional J in* (1.1) *over $\mathcal{K}_0$ in* (1.4)*, then $u \in C^{1/2}_{\mathrm{loc}}(B_1^+ \cup B_1')$ with*

$$\|u\|_{C^{1/2}(B^+_{1/2} \cup B'_{1/2})} \le C_n \|u\|_{L^2(B_1^+)}.$$

The regularity above implies that for any $x \in \Gamma(u)$ we have

$$\kappa = N^x(0+, u) \ge 1/2.$$

The knowledge of the possible values of κ is important for the classification of free boundary points (as we discussed at the end of Subsection 1.1). Concerning these values we have the following results.

Theorem 1.2 (Minimal Almgren's frequency at contact points)**.** *If u is a minimizer of J over $\mathcal{K}_0$, then for a contact point $\bar{x} \in \Gamma'(u)$ we have*

$$\kappa = N^{\bar{x}}(0+, u) \ge 3/2.$$

At noncontact points we give a more complete picture.

Theorem 1.3 (Almgren's frequency at noncontact points)**.** *If u is a minimizer of J over $\mathcal{K}_0$, then for a noncontact point $\bar{x} \in \Gamma^*(u)$ the quantity*

$$\kappa = N^{\bar{x}}(0+, u)$$

can take only the following values:

$$1/2,\ 3/2,\ 5/2,\ \ldots,\ m - 1/2,\ \ldots$$

§2. Optimal regularity

2.1. Symmetrization. It will be convenient for our considerations to extend every function $v \in \mathcal{K}_0$ by even symmetry in the x_n-variable to the entire ball B_1:

$$v(x', -x_n) := v(x', x_n) \quad \text{for} \quad (x', x_n) \in B_1^+.$$

With such extension in mind, the energy J in (1.1) can be replaced with

$$J(v) := \frac{1}{2} \int_{B_1} |\nabla v|^2. \tag{2.1}$$

2.2. Hölder continuity. As the first result towards the optimal regularity, we show that the minimizers are C^α regular for some $\alpha > 0$.

Proposition 2.1 (Hölder continuity)**.** *If u is a minimizer of J over $\mathcal{K}_0$, then $u \in C^\alpha(B_{1/2})$, with a dimensional constant $\alpha > 0$, and*

$$\|u\|_{C^\alpha(B_{1/2})} \leq C_n \|u\|_{L^2(B_1)}.$$

We start with showing that the positive and negative parts of the minimizer u are subharmonic. Note that at this stage we have not yet established the continuity of u, so we will resort to the energy methods.

Lemma 2.2. *The functions $u_\pm = \max\{\pm u, 0\}$ are subharmonic in B_1.*

Proof. Proving the lemma is equivalent to showing that for any nonnegative test function $\eta \in C_0^\infty(B_1)$ we have

$$\int_{B_1} \nabla u_\pm \nabla \eta \leq 0. \tag{2.2}$$

Let $\psi_\varepsilon \in C^\infty(\mathbb{R})$ be a monotone nondecreasing function such that

$$\psi_\varepsilon = 0 \;\text{ in }\; (-\infty, \varepsilon), \quad 0 \leq \psi_\varepsilon \leq 1 \;\text{ in }\; (\varepsilon, 2\varepsilon), \quad \psi_\varepsilon = 1 \;\text{ in }\; (2\varepsilon, \infty).$$

Then for a fixed $\varepsilon > 0$ and sufficiently small $|t|$ we have

$$\{u > 0\} = \{u + t\eta\psi_\varepsilon(u_\pm) > 0\},$$
$$\{u < 0\} = \{u + t\eta\psi_\varepsilon(u_\pm) < 0\}$$

and thus $u + t\eta\psi_\varepsilon(u_\pm)$ are admissible functions belonging to $\mathcal{K}_0$. Since u is a minimizer, we have $J(u + t\eta\psi_\varepsilon(u_\pm)) \geq J(u)$, implying

$$0 = \int_{B_1} \nabla u \nabla \bigl(\eta \psi_\varepsilon(u_\pm)\bigr) = \int_{B_1} \nabla u \nabla \eta \psi_\varepsilon(u_\pm) \pm \int_{B_1} |\nabla u|^2 \psi_\varepsilon'(u_\pm)\eta.$$

Since the second integral is nonnegative, sending ε to 0 yields (2.2). □

Once we know that the $u_\pm$ are subharmonic in B_1, we immediately see that u is locally bounded.

Lemma 2.3 (Local boundedness)**.** *If u is a minimizer of J over $\mathcal{K}_0$, then $u \in L^\infty(B_{3/4})$ and, more precisely,*
$$\sup_{B_{3/4}} |u| \le C_n \|u\|_{L^2(B_1)}.$$

Now we can proceed to the proof of Hölder continuity.

Proof of Proposition 2.1. Using the local boundedness, the fact that the $u_\pm$ vanish on $B'_1 \cap \{x_1 \le 0\}$, and the comparison principle, we can write
$$|u| \le Mv \quad \text{in} \quad B_{3/4}, \tag{2.3}$$
where $M = C_n \|u\|_{L^2(B_1)}$ and v solves the problem
$$\begin{aligned} \Delta v &= 0 \quad \text{in} \quad B_{3/4} \setminus (B'_{3/4} \cap \{x_1 \le 0\}),\\ v &= 0 \quad \text{on} \quad B'_{5/8} \cap \{x_1 \le 0\},\\ v &= 1 \quad \text{on} \quad \partial B_{3/4} \end{aligned} \tag{2.4}$$
with boundary values changing continuously from 0 to 1 in $(B'_{3/4} \setminus B'_{5/8}) \cap \{x_1 \le 0\}$. Next we claim that the barrier function v above is in $C^\alpha(B_{1/2})$. Indeed, we can use a bi-Lipschitz transformation to map $B_{3/4} \setminus (B'_{3/4} \cap \{x_1 \le 0\})$ to $B^+_{3/4}$ preserving the distance from the origin. Then v will transform into w, which would be a solution of a uniformly elliptic equation in divergence form with measurable coefficients:
$$\begin{aligned} \operatorname{div}(a_{ij} w_j) &= 0 \quad \text{in} \quad B^+_{3/4},\\ w &= 0 \quad \text{on} \quad B'_{5/8}. \end{aligned} \tag{2.5}$$
By the De Giorgi–Nash–Moser theorem, we have $w \in C^\alpha(B^+_{1/2} \cup B'_{1/2})$, and since the transformation is bi-Lipschitz we also get $v \in C^\alpha(B_{1/2})$, which provides
$$|v(x)| \le C \operatorname{dist}(x, B'_1 \cap \{x_1 \le 0\})^\alpha. \tag{2.6}$$
This formula together with (2.3), gives
$$|u(x)| \le CM \operatorname{dist}(x, B'_1 \cap \{x_1 \le 0\})^\alpha. \tag{2.7}$$
Combined with the next lemma, this implies $u \in C^\alpha(B_{1/2})$. □

Lemma 2.4. *Let u be a minimizer of J over $\mathcal{K}_0$. If*
$$|u(x)| \le C_0 \operatorname{dist}\left(x, B'_1 \cap \{x_1 \le 0\}\right)^\beta \tag{2.8}$$
for some $0 < \beta \le 1$ and all $x, y \in B_{1/2}$, then $u \in C^\beta(B_{1/2})$ with $\|u\|_{C^\beta(B_{1/2})}$ depending only on C_0, n, β.

Proof. Denote $d_x := \operatorname{dist}(x, B'_1 \cap \{x_1 \le 0\})$. Take any $x, y \in B_{1/2}$. Without loss of generality we may assume $x \in B^+_1$ and $d_y \le d_x$. We consider three cases:

1) $|x - y| > d_x/8$. Using (2.8), we get
$$|u(x) - u(y)| \le C_0(d_x^\beta + d_y^\beta) \le 2C_0 8^\beta |x - y|^\beta.$$

2) $|x - y| \le d_x/8$ and for the nth coordinate of x we have $x_n > d_x/4$. In this case we observe that $B_{d_x/4}(x) \subset B^+_1$ and thus u is harmonic there, $x, y \in B_{d_x/8}(x)$, and the interior gradient estimates for harmonic functions imply
$$\begin{aligned} |u(x) - u(y)| &\le C_n \|u\|_{L^\infty(B_{d_x/4}(x))} \frac{|x - y|}{d_x}\\ &\le C_n C_0 (5/4)^\beta d_x^\beta \frac{|x - y|^\beta (d_x/8)^{1-\beta}}{d_x} = C|x - y|^\beta. \end{aligned}$$

3) $|x-y| \le d_x/8$ and $x_n \le d_x/4$. In this case $B_{(3/4)d_x}(x',0) \subset B_{d_x}(x)$. Thus, u solves the Signorini problem in $B_{(3/4)d_x}(x',0)$ and $x, y \in B_{(3/8)d_x}(x',0)$. Using the interior Lipschitz regularity for the solutions of the Signorini problem, see [AC, Theorem 1], we have

$$|u(x)-u(y)| \le C_n \|u\|_{L^\infty(B_{(3/4)d_x}(x))} \frac{|x-y|}{d_x}$$

and the proof can be completed as in the previous case. □

2.3. Monotonicity formula in the halfball. As we observed in the Introduction, we know that the function $\widehat{u}_{1/2}$ restricts the regularity of our solutions to $C^{1/2}$. In order to rigorously show that $C^{1/2}$ is also the minimal expected (and thus optimal) regularity, we need the following monotonicity formula for the halfball, first introduced in [AC].

Lemma 2.5 (Monotonicity formula, [AC, Lemma 4])**.** *If $w \in C(\overline{B_1^+})$ satisfies*

$$\begin{aligned} \Delta w = 0 \quad &\text{in} \quad B_1^+, \\ w = 0 \quad &\text{on} \quad B_1' \cap \{x_1 \le 0\}, \\ w \ge 0, \quad w\,\partial_{x_n} w = 0 \quad &\text{on} \quad B_1', \end{aligned}$$

then the function

$$\varphi(r) := \frac{1}{r}\int_{B_r^+} \frac{|\nabla w|^2}{|x|^{n-2}}\,\mathrm{d}x$$

is monotone nondecreasing for $r \in (0,1)$.

Proof. The proof is a *verbatim* repetition of that of [AC, Lemma 4], despite of the slight difference in the assumptions. Namely, instead of requiring the convexity of the set $\{x' \in B_1' : w(x',0) > 0\}$, we note that it is only used to show that the complement of the support of w contains the lower dimensional halfball $B_1' \cap \{x_1 \le 0\}$. In our setting, this is satisfied automatically. □

2.4. Optimal $C^{1/2}$ regularity of minimizers. Now we are ready to prove our first main result.

Proof of Theorem 1.1. We apply the monotonicity formula in Lemma 2.5 to the minimizer u of J to obtain

$$\varphi(r) \le \varphi(3/4) \le C\|u\|^2_{L^2(B_1)}. \tag{2.9}$$

Here, the last inequality is standard for nonnegative subharmonic functions (for the proof see, e.g., [Ca2]). Applying this to $u_\pm$, we obtain the corresponding inequality for u.

Now, using the fact that u vanishes on $B_1' \cap \{x_1 \le 0\}$, we also have the Poincaré inequality for the halfball

$$\int_{B_r^+} u^2 \le C_n r^2 \int_{B_r^+} |\nabla u|^2. \tag{2.10}$$

Then by scaling in Lemma 2.3 we obtain

$$\begin{aligned} \sup_{B_{r/2}} |u| &\le C_n r^{-\frac{n}{2}} \|u\|_{L^2(B_r)} \le C_n r^{1-\frac{n}{2}} \|\nabla u\|_{L^2(B_r^+)} \\ &\le C_n r^{\frac12}\left(\frac{1}{r}\int_{B_r^+} \frac{|\nabla u|^2}{|x|^{n-2}}\,\mathrm{d}x\right)^{1/2} \le C_n r^{\frac12}\varphi(r)^{1/2} \le C_n r^{\frac12}\|u\|_{L^2(B_1)}. \end{aligned} \tag{2.11}$$

Observe that the above estimate is also true for any ball $B_{r/2}(x)$ with center $x \in B'_{1/2} \cap \{x_1 \le 0\}$ and $r \le 1/4$:

$$\sup_{B_{r/2}(x)} |u| \le C_n r^{\frac12}\|u\|_{L^2(B_1)} \le C_n r^{\frac12}\|u\|_{L^\infty(B_1)}, \tag{2.12}$$

yielding

$$|u| \le C\operatorname{dist}(x, B_1' \cap \{x_1 \le 0\})^{1/2}. \tag{2.13}$$

Using Lemma 2.4, we see that $u \in C^{1/2}(B_{1/4})$. □

Remark 2.6. Without loss of generality we will further assume that $u \in C^{1/2}(B_1)$.

§3. Monotonicity of the frequency

3.1. Almgren's frequency formula. As we mentioned in the Introduction, Almgren's frequency formula plays an important role in the Signorini problem. Since we have an additional constraint for functions in $\mathcal{K}_0$, it is not automatic that the frequency will still be monotone. However, fortunately, it is the case.

Theorem 3.1 (Monotonicity of the frequency). *If u is a minimizer of J over $\mathcal{K}_0$, then*

$$N(r) = N^{x_0}(r,u) := \frac{r\int_{B_r(x_0)} |\nabla u|^2}{\int_{\partial B_r(x_0)} u^2} \tag{3.1}$$

is monotone nondecreasing in r for $r \in (0,R)$ and $x_0 \in B_1' \cap \{x_1 \ge 0\}$ such that $B_R(x_0) \subset B_1$. Moreover, $N^{x_0}(r,u) \equiv \kappa$ for all $0 < r \le R$ if and only if u is homogeneous of degree κ in $B_R(x_0)$, with respect to the center x_0.

The following notation will be used in the proof:

$$D(r) := \int_{B_r(x_0)} |\nabla u|^2 \quad \text{and} \quad H(r) := \int_{\partial B_r(x_0)} u^2.$$

Now if we consider the logarithm of $N(r)$ and formally differentiate it, we obtain

$$\frac{N'(r)}{N(r)} = (\log N(r))' = \frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)}.$$

In order to prove the theorem, we need to show that the right-hand side is nonnegative. We accomplish this by proving differentiation formulas/inequalities in Lemmas 3.2, 3.3, and 3.4, following similar proofs in [GP] or [ACS].

We start with the following alternative formula for $D(r)$.

Lemma 3.2 (First identity). *For the minimizers u of J over $\mathcal{K}_0$, the following identity is true for $B_r(x_0) \Subset B_1$ with $x_0 \in B_1'$:*

$$D(r) = \int_{B_r(x_0)} |\nabla u|^2 = \int_{\partial B_r(x_0)} u u_\nu. \tag{3.2}$$

Proof. To prove the lemma we note that, for any test function $\eta \in W^{1,2}(B_r(x_0))$ that vanishes in a neighborhood of $B_1' \cap \{x_1 \le 0\}$, we have

$$\int_{B_r^+(x_0)} \nabla u \nabla \eta = \int_{B_r'(x_0)} u_\nu \eta + \int_{(\partial B_r(x_0))^+} u_\nu \eta. \tag{3.3}$$

For a small $\varepsilon > 0$, choose $\eta_\varepsilon(x) = u\psi(d(x)/\varepsilon)$, where

$$d(x) = \operatorname{dist}\big(x, B_1' \cap \big\{x_1 \le 0\big\}\big)$$

and $\psi \in C^\infty(\mathbb{R})$ is such that

$$\begin{aligned} &\psi = 0 \ \text{ in } \ (-\infty, 1), \quad 0 \le \psi \le 1 \ \text{ in } \ (1,2), \quad \psi = 1 \ \text{ in } \ (2,\infty),\\ &0 \le \psi' \le M \ \text{ in } \ (-\infty,\infty). \end{aligned}$$

We want to plug $\eta = \eta_\varepsilon$ into (3.3) and let $\varepsilon \to 0$. First, we claim that

$$\lim_{\varepsilon\to 0} \int_{B_r^+(x_0)} \nabla u \nabla \eta_\varepsilon = \int_{B_r^+(x_0)} |\nabla u|^2, \tag{3.4}$$

which is the same as

$$\lim_{\varepsilon\to 0} \int_{B_r^+(x_0)} \nabla u (\nabla \eta_\varepsilon - \nabla u) = 0. \tag{3.5}$$

Indeed,

$$\nabla \eta_\varepsilon = \psi\Big(\frac{d}{\varepsilon}\Big)\nabla u + u\psi'\Big(\frac{d}{\varepsilon}\Big)\frac{\nabla d}{\varepsilon},$$

$$\nabla \eta_\varepsilon - \nabla u = \Big(\psi\Big(\frac{d}{\varepsilon}\Big) - 1\Big)\nabla u + \frac{u}{\varepsilon}\psi'\Big(\frac{d}{\varepsilon}\Big)\nabla d.$$

Multiplying both sides of the above by ∇u and integrating over B_1, we obtain

$$\Big| \int_{B_r^+(x_0)} \nabla u (\nabla \eta_\varepsilon - \nabla u)\Big| \le \int_{\{d\le 2\varepsilon\}} |\nabla u|^2 + \frac{M}{\varepsilon}\int_{\{d\le 2\varepsilon\}} u|\nabla u|, \tag{3.6}$$

using the inequalities $|\psi'| \le M$ and $|\nabla d| \le 1$. Since the first integral on the right-hand side goes to 0 as $\varepsilon \to 0$, it only remains to estimate the second integral. We have

$$\frac{M}{\varepsilon}\int_{\{d\le 2\varepsilon\}} u|\nabla u| \le \Big(\int_{\{d\le 2\varepsilon\}} |\nabla u|^2\Big)^{1/2} \frac{M}{\varepsilon}\Big(\int_{\{d\le 2\varepsilon\}} u^2\Big)^{1/2}. \tag{3.7}$$

Again the first integral goes to 0, and to estimate the second we use the $C^{1/2}$ regularity of u to show that

$$u^2 \le C\varepsilon \quad \text{in} \quad \{d \le 2\varepsilon\}.$$

Moreover, we also have $|\{d \le 2\varepsilon\}| \le C\varepsilon$, which gives

$$\frac{1}{\varepsilon}\Big(\int_{\{d\le 2\varepsilon\}} u^2\Big)^{1/2} \le C$$

and establishes (3.4). Now, to complete the proof of the lemma, we let $\eta = \eta_\varepsilon$ in (3.3) and pass to the limit as $\varepsilon \to 0$. Since

$$u_\nu \eta_\varepsilon = u_\nu u \psi = 0 \quad \text{on} \quad B_1',$$

we obtain (3.2). □

Lemma 3.3 (Second identity)**.** *For the minimizer u of J over $\mathcal{K}_0$, the following identity holds true for $B_r(x_0) \Subset B_1$ with $x_0 \in B_1'$:*

$$H'(r) = \frac{n-1}{r}H(r) + 2\int_{\partial B_r(x_0)} u u_\nu. \tag{3.8}$$

The differentiation formula should be understood in the sense that $H(r)$ is an absolutely continuous function of r and identity is fulfilled for a.e. r.

Proof. We have

$$\begin{aligned} H(r) &= 2\int_{(\partial B_r(x_0))^+} u^2 = 2\int_{(\partial B_r(x_0))^+} \Big(\frac{x-x_0}{r}\nu\, u^2\Big) \\ &= \frac{2}{r}\int_{B_r^+(x_0)} \operatorname{div}((x-x_0)u^2) \\ &= \frac{1}{r}\int_{B_r(x_0)} \operatorname{div}(x-x_0)u^2 + \frac{2}{r}\int_{B_r(x_0)} (x-x_0)(\nabla u)u. \end{aligned}$$

Hence,

$$H'(r) = \frac{n}{r}\int_{\partial B_r(x_0)} u^2 + \frac{2}{r}\int_{\partial B_r(x_0)} (x-x_0)(\nabla u)u - \frac{1}{r}H(r),$$

which yields the desired identity. □

While the above two identifies were the same as in the Signorini problem, the third one becomes actually an inequality, which suffices for our purposes.

Lemma 3.4 (Third (Rellich-type) inequality)**.** *For the minimizer u of J over $\mathcal{K}_0$, the following inequality holds true for $B_r(x_0) \Subset B_1$ with $x_0 \in B_1' \cap \{x_1 \geq 0\}$:*

$$D'(r) \geq \frac{n-2}{r}D(r) + 2\int_{\partial B_r(x_0)} u_\nu^2 \tag{3.9}$$

or, equivalently,

$$r\int_{\partial B_r(x_0)} |\nabla u|^2 \geq \int_{B_r(x_0)} (n-2)|\nabla u|^2 + 2r\int_{\partial B_r(x_0)} u_\nu^2. \tag{3.10}$$

We emphasize that the center x_0 of the ball $B_r(x_0)$ must be in the upper thin halfball $B_1' \cap \{x_1 \geq 0\}$ for the inequality to be valid.

Proof. The proof of this lemma involves domain variation in the radial direction, similar to that in [We, p. 444]. The main difference is that our constraints allow us to make perturbations that increase the distance from the origin, thus yielding an inequality (with the correct sign) instead of an identity in the nonconstrained case. We consider the function

$$\eta_k(y) := \max\Big\{0, \min\Big\{1, \ \frac{r-|y|}{k}\Big\}\Big\}.$$

Then for $\varepsilon > 0$, we have

$$u_\varepsilon(x) = u(x + \varepsilon\eta_k(x-x_0)(x-x_0)) \in \mathcal{K}_0.$$

Note that the same will not be true for negative ε (which is why we only have an inequality), that variation will transfer the zero values of u from $B_1' \cup \{x_1 \leq 0\}$ into $B_1' \cup \{x_1 > 0\}$, making the result to fail to be an admissible function. Once the admissibility of u_ε is established, we can translate x_0 to the origin and continue the rest of the proof for balls centered at the origin.

Using the minimality of u, we have

$$0 \geq \frac{J(u) - J(u_\varepsilon)}{\varepsilon} = \frac{J(u(x)) - J(u(x+\varepsilon\eta_k(x)x))}{\varepsilon}.$$

Letting $\varepsilon \to 0$ yields

$$\begin{aligned} 0 &\geq \int_{B_r} \big(|\nabla u|^2 \operatorname{div}(\eta_k(x)x) - 2\nabla u D(\eta_k(x)x)\nabla u\big) \\ &= \int_{B_r} \big((n-2)|\nabla u|^2\eta_k(x) + |\nabla u|^2 x\nabla\eta_k(x) - 2(x\nabla u)(\nabla u\nabla\eta_k(x))\big). \end{aligned}$$

This time we let $k \to \infty$ to obtain

$$0 \geq \int_{B_r} (n-2)|\nabla u|^2 - \int_{\partial B_r} \big(|\nabla u|^2 x\nu + 2(x\nabla u)(\nu\nabla u)\big),$$

which is equivalent to (3.10). □

Now we can prove the monotonicity of Almgren's frequency.

Proof of Theorem 3.1. The three lemmas proved above imply

$$\frac{N'(r)}{N(r)} \geq \frac{1}{r} + \frac{n-2}{r} - \frac{n-1}{r} + 2\left(\frac{\int_{\partial B_r(x)} u_\nu^2}{\int_{\partial B_r(x)} uu_\nu} - \frac{\int_{\partial B_r(x)} uu_\nu}{\int_{\partial B_r(x)} u^2}\right) \geq 0.$$

The last inequality follows from the Cauchy–Schwartz inequality, the equality case of which is satisfied if $N'(r) = 0$, showing that u is homogeneous (see [GP] or [ACS]). From the scaling properties of $N(r, u)$ we can also see that it is constant when the function u is homogeneous; thus, the theorem is proved. □

§4. Blowups and possible homogeneities

4.1. Blowups. An important tool for us will be the following rescaling of the minimizers at some points $x_0 \in \Gamma(u)$:

$$u_r(x) = u_{x_0,r}(x) := \frac{u(rx + x_0)}{\left(\frac{1}{r^{n-1}} \int_{\partial B_r(x_0)} u^2\right)^{1/2}}. \tag{4.1}$$

The limits of the rescaled functions $\{u_r\}$ as $r = r_j \to 0+$ will be called the *blowups* of u at the point x_0. The above definition makes the $L^2(\partial B_1)$ norm of the rescaled functions to be one:

$$\int_{\partial B_1} u_r^2 = 1. \tag{4.2}$$

Another useful property is the identity

$$N(\rho, u_r) = N^{x_0}(\rho r, u). \tag{4.3}$$

Next we want to let $r = r_j \to 0$ and study the convergence of the rescaled functions u_{r_j}. We start with showing that this convergence will be strong in $W^{1,2}$.

Lemma 4.1 (Strong convergence)**.** *Let u_j be a minimizer of J over $\mathcal{K}_0(g_j)$ with some $g_j \in L^2((\partial B_1)^+)$. Suppose that $\|u_j\|_{W^{1,2}(B_1)} \leq C$, and that $u_j \rightharpoonup u_0$ weakly in $W^{1,2}(B_1)$ and $u_j \to u_0$ in $C^\alpha_{\mathrm{loc}}(B_1)$. Then $u_j \to u_0$ strongly in $W^{1,2}_{\mathrm{loc}}(B_1)$:*

$$\int_{B_\rho} |\nabla u_j|^2 \to \int_{B_\rho} |\nabla u_0|^2 \quad \textit{for all} \quad 0 < \rho < 1. \tag{4.4}$$

Moreover, u_0 minimizes J over $\mathcal{K}_0(g_0)$ with boundary values $g_0 = \lim_{j\to\infty} g_j$.

Proof. 1) We first prove that for any two solutions u_1 and u_2, the functions $(u_2 - u_1)_\pm$ are subharmonic:

$$\int_{B_1} \nabla(u_2 - u_1)_\pm \nabla\eta \leq 0 \tag{4.5}$$

for all nonnegative test functions $\eta \in C_0^\infty(B_1)$. We show only the subharmonicity of $(u_2 - u_1)_+$, the other argument being similar. Now, since the only complications can occur on $B_1' \cap \{x_1 > 0\}$, without loss of generality we may assume that $E = \{u_2 > u_1\} \cap B_1' \subset B_1' \cap \{x_1 > 0\}$ is nonempty. Then the Signorini conditions on B_1' show that

$$\partial_{x_n} u_2 = 0 \ \text{ on } \ E, \quad \partial_{x_n} u_1 \leq 0 \ \text{ on } \ E, \quad \partial_{x_n}(u_2 - u_1) \geq 0 \ \text{ on } \ E.$$

For any point $x_0 \in E$, let $\delta > 0$ be such that $B_\delta'(x_0) \subset E$. Then by the harmonicity of $u_2 - u_1$ in $B_1^\pm$, for any test function $\eta \geq 0$, $\eta \in C_0^\infty(B_\delta(x_0))$, we have

$$\int_{B_\delta(x_0)} \nabla(u_2 - u_1)\nabla\eta = 2\int_{B_\delta^+(x_0)} \nabla(u_2 - u_1)\nabla\eta = -\int_{B_\delta'(x_0)} \partial_{x_n}(u_2 - u_1)\eta \leq 0.$$

This implies the subharmonicity of $(u_2 - u_1)_+$ in a neighborhood of any point $x_0 \in E$, implying subharmonicity in B_1.

2) Take the sequence $\{u_j\}$ and u_0 as in the statement of the lemma. The previous step shows that $(u_j - u_k)_\pm$ are subharmonic. Letting $k \to \infty$, we see that $(u_j - u_0)_\pm$ is also subharmonic. Now the energy inequality yields

$$\int_{B_\rho} |\nabla(u_j - u_0)_\pm|^2 \leq C(\rho) \int_{B_1} (u_j - u_0)_\pm^2 \to 0 \text{ as } j \to \infty, \tag{4.6}$$

which implies the strong convergence in B_ρ.

3) Recall now that u_j minimizes J over $\mathcal{K}_0(g_j)$. Since the u_j are bounded in $W^{1,2}(B_1)$ and the trace mapping is compact, we can take a subsequence such that $g_j \to g_0$ in $L^2(\partial B_1)$ as $j \to \infty$. Taking the minimizer $\widehat{u}_0$ of J on $\mathcal{K}_0(g_0)$, letting u_0 be the strong limit of u_j obtained at the previous step, and using the fact that $(u_j - \widehat{u}_0)_\pm$ is subharmonic, we obtain

$$\sup_{B_\rho} |(u_j - \widehat{u}_0)_\pm| \leq C(\rho) \int_{\partial B_1^+} (g_j - g_0)_\pm^2. \tag{4.7}$$

Thus u_j converges uniformly to $\widehat{u}_0$ on B_ρ for any $0 < \rho < 1$, which means that $\widehat{u}_0 \equiv u_0$ in B_1. □

4.2. Homogeneity of blowups. Next we show that the blowups are homogeneous.

Lemma 4.2 (Homogeneity of blowups)**.** *Let u be a minimizer of J over $\mathcal{K}_0$, and let and $u_0 = \lim_{r_j \to 0} u_{r_j}$ be a blowup of u at $x_0 \in \Gamma(u)$. Then u_0 is homogeneous of degree $\kappa = N^{x_0}(0+, u)$.*

Proof. Indeed, using (4.3) and Theorem 3.1 for $0 < r < 1/2$, we obtain

$$N(1, u_r) = N(r, u) \leq N(1/2, u) =: M.$$

Using (4.2) and the above estimate, we arrive at the estimate

$$\int_{B_1} |\nabla u_r|^2 = N(1, u_r) \leq M,$$

which shows that the sequence $\{u_r\}$ is bounded in $W^{1,2}(B_1)$. Thus, we can choose a weakly converging subsequence $u_{r_j} \rightharpoonup u_0$ in $W^{1,2}(B_1)$. By Lemma 4.1, we also have the strong convergence $u_{r_j} \to u_0$ in $W^{1,2}_{\mathrm{loc}}(B_1)$, which means in particular that

$$\lim_{r_j \to 0} N(\rho, u_{r_j}) = N(\rho, u_0), \tag{4.8}$$

provided $\int_{\partial B_\rho} u_0^2 \neq 0$. Now suppose that $\int_{\partial B_\rho} u_0^2 = 0$. Then, by the maximum principle, the subharmonic functions $(u_0)_\pm$ would vanish in B_ρ, and since u_0 is harmonic in B_1^+, we would have $u_0 \equiv 0$ in B_1. But, due to the compactness of the trace mapping,

$$\int_{\partial B_1} u_0^2 = \lim_{r_j \to 0} \int_{\partial B_1} u_{r_j}^2 \, d\sigma = 1,$$

which contradicts the vanishing of u_0 in B_1. Thus (4.8) holds for any $0 < \rho < 1$. Moreover, we can write

$$N(\rho, u_0) = \lim_{r_j \to 0} N(\rho, u_{r_j}) = \lim_{r_j \to 0} N^{x_0}(\rho r_j, u) = N^{x_0}(0+, u) =: \kappa,$$

yielding

$$N(\rho, u_0) \equiv \kappa \text{ for any } 0 < \rho < 1. \tag{4.9}$$

Then we use the last part of Theorem 3.1 to complete the proof of the lemma. □

We can now proceed to the proof of Theorems 1.2 and 1.3.

4.3. Minimal homogeneity at contact points.

Proof of Theorem 1.2. For a fixed $r > 0$, consider the functional

$$\Gamma(u) \ni x \mapsto N^x(r,u) = \frac{r\int_{B_r(x)} |\nabla u|^2}{\int_{\partial B_r(x)} u^2}. \tag{4.10}$$

Then, since for a fixed $r > 0$ this functional is continuous and $N^x(r,u)$ is monotone nondecreasing in r, we obtain the upper semicontinuity of the functional $x \mapsto N^x(0+,u)$ on $\Gamma(u)$. More precisely, we have

$$N^{x_0}(0+,u) \geq \limsup_{\substack{x\to x_0\\ x\in\Gamma(u)}} N^x(0+,u). \tag{4.11}$$

Now, for a contact point $\bar{x} \in \Gamma'(u)$ we have a sequence of free boundary points $x_j \in \Gamma(u) \cap \{x_1 > 0\}$ converging to $\bar{x}$. Now, near x_j, the minimizer u solves the Signorini problem, and therefore

$$N^{x_j}(0+,u) \geq 3/2 \quad \text{for} \quad x_j \in \Gamma(u)\cap\{x_1 > 0\}.$$

Thus, using the upper semicontinuity, we conclude that

$$N^{\bar{x}}(0+,u) \geq 3/2. \qquad \square$$

4.4. Possible homogeneities at noncontact points.

Proof of Theorem 1.3. Since $\bar{x}$ is not a contact point, we know that there exists a positive number δ such that u is harmonic in $B_\delta(\bar{x}) \setminus (B'_\delta(\bar{x}) \cap \{x_1 \leq 0\})$. Let u_0 be a blowup of u at $\bar{x}$:

$$u_0 = \lim_{r_j\to 0} u_{\bar{x},r_j} = \lim_{r_j\to 0} \frac{u(\bar{x}+r_j x)}{\left(\frac{1}{r_j^{n-1}}\int_{\partial B_{r_j}(\bar{x})} u^2 d\sigma\right)^{1/2}}.$$

We know that u_0 is homogeneous of degree $\kappa = \kappa(\bar{x}) := N^{\bar{x}}(0+,u)$, meaning that $u_0(r\theta) = r^\kappa u_0(\theta)$ for $r > 0$ and $\theta \in \partial B_1$. We also know that u_0 is harmonic in $\mathbb{R}^n \setminus (\mathbb{R}^{n-1}\cap\{x_1 \leq 0\})$ and u_0 is nonnegative in $\mathbb{R}^{n-1}\cap\{x_1 > 0\}$. Next, for $m \in \mathbb{N}$, define

$$\widehat{u}_{m-1/2}(x) := \operatorname{Re}(x_1 + i|x_n|)^{m-1/2}. \tag{4.12}$$

It is easily seen that $\widehat{u}_{m-1/2}$ is homogeneous of degree $(m-1/2)$ and

$$\Delta\widehat{u}_{m-1/2} = 0 \text{ in } \mathbb{R}^n \setminus (\mathbb{R}^{n-1}\cap\{x_1\leq 0\}),$$
$$\widehat{u}_{m-1/2} = 0 \text{ on } \mathbb{R}^{n-1}\cap\{x_1\leq 0\}.$$

Thus, the set of possible values of κ includes $\{m - 1/2 : m \in \mathbb{N}\}$. We want to show that those are the only possible values of κ. This fact will follow from the expansion of harmonic functions in slit domains, recently established in [DS0, Theorem 3.1]. That theorem implies that for any $k \geq 0$ there exists a polynomial $P_0(x,r)$ of degree $k+1$ such that

$$u_0(x) = \widehat{u}_{1/2}(x)\left(P_0(x',r) + o(|x|^{k+1})\right), \quad r = \sqrt{x_1^2 + x_n^2},$$

solely under the assumption that u_0 is harmonic in $B_1 \setminus B'_1 \cap \{x_1 \leq 0\}$, vanishes continuously on $B'_1 \cap \{x_1 \leq 0\}$ and is even in x_n. Taking $k > \kappa$ and using the fact that u_0 is homogeneous of degree κ, we see that

$$u_0(x) = \widehat{u}_{1/2}(x)P_0(x',r)$$

for a homogeneous polynomial $P_0(x',r)$ of degree $\kappa - 1/2$. Thus, $\kappa = m - 1/2$ for some $m \in \mathbb{N}$. The proof is complete. $\square$

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