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GEOMETRIC PROPERTIES OF SYSTEMS OF VECTOR STATES AND EXPANSION OF STATES IN PETTIS INTEGRALS

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ABSTRACT. The relationship is studied between the geometry ofsystems unit vectors in Hilbert space and the state on the algebra of bounded operators that is obtained by integration of the vector states determined by the system in question with respect to a finitely additive measure on the set of natural numbers.

§1. INTRODUCTION

A positive and continuous linear functional ω on the algebra of all bounded operators on a Hilbert space H is called a *state* if $\omega(I) = 1$ (see [1]). The set of all states will be denoted by $\Sigma(H)$. A state ω is said to be *normal* if $x_n \uparrow x$ implies that $\omega(x_n) \uparrow \omega(x)$ for $x_n, x \in B(H)$. The well-known von Neumann's theorem (see [2]) says that if ω is a normal state, then there is a linear operator $\rho \in \mathfrak{S}_1,\ \rho > 0,\ \mathrm{Tr}(\rho) = 1$ such that $\omega(x) = \mathrm{Tr}(\rho x)$, $x \in B(H)$. In the present paper we are concerned mainly in the states that are not normal.

The extreme points in the set of normal states are the vector states ρ_e determined by the unit vectors $e \in H$ by the formula $\rho_e(x) = (e, xe)$, $x \in B(H)$. The set of states that are not normal was studied in [3, 4]. Nonnormal states ω can be obtained (see [3]) from a family of vector states $\{\rho_{e_n},\ n \in \mathbb{N}\}$ via integration with respect to a finitely additive measure μ on the set of natural numbers specifically, normalized by $\mu(\mathbb{N}) = 1$, that is

$$\omega(x) = \int_{\mathbb{N}} \rho_{e_n}(x)\, d\mu(n), \quad x \in B(H). \tag{1}$$

The integral in (1) is understood in the sense of Pettis. In [4], a representation of a state in the form (1) was called the expansion of that state. In [3] it was proved that any state admits an expansion as in (1), but several questions were left aside: what are systems of vector states over which a given state can be expanded? What conditions ensure that, given a system $\{e_n,\ n \in \mathbb{N}\}$, an expansion (1) is unique? What is the relationship of the properties of the state to be expanded with the geometry of the system $\{e_n\}$ and the properties of the measure μ?

In the present paper we study the impact made by the geometry of a system $\{e_n \in H,\ n \in \mathbb{N}\}$ of unit vectors on the properties of the state (1). In particular, we want to know when it will be nonnormal for sure, and when it will belong to the set $\mathrm{Extr}(\Sigma(H))$ of extreme points of the set of all states $\Sigma(H)$. Also, we study the uniqueness of the measure μ, for given ω and a system E.

Throughout, H will denote a separable Hilbert space. All measures on $\mathbb{N}$ are assumed to be nonnegative, additive, and satisfying $\mu(\mathbb{N}) = 1$ (in that sense, they may be called probability measures). The set of all such measures will be denoted by $W(\mathbb{N})$.

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§2. Ultrafilters, measures, and representation of a state as a Pettis integral

We need some information from the theory of filters. A system $\mathbb{F}$ of subsets X of $\mathbb{N}$ is called a *filter* if

1) $\mathbb{N} \in \mathbb{F}$, $\varnothing \notin \mathbb{F}$;

2) for any $F, G \in \mathbb{F}$ we have $F \cap G \in \mathbb{F}$.

A filter $\mathbb{F}$ is called an *ultrafilter* if for any subset $F \subset \mathbb{N}$ the following additional condition is fulfilled:

3) $F \in \mathbb{F}$ or $\bar{F} \in \mathbb{F}$.

An ultrafilter is said to be *principal* whenever there is $n_0 \in \mathbb{N}$ such that $F \in \mathbb{F}$ if and only if $n_0 \in F$. Each ultrafilter $\mathbb{F}$ is in one-to-one correspondence with the finitely additive two-valued measure μ on the system of all subsets X of $\mathbb{N}$ determined by the condition $\mu(X) = 1$ if $X \in \mathbb{F}$ and $\mu(X) = 0$ if $X \notin \mathbb{F}$. We say that this measure μ *generates* the ultrafilter $\mathbb{F}$.

Recall that the Stone–Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ is the set of all ultrafilters on $\mathbb{N}$. Thus, the set of all finitely additive measures on $\mathbb{N}$ may be thought of as the set of restrictions to $\mathbb{N}$ of the countably additive measures of the set of all ultrafilters [6]. In this interpretation, the two-valued measures on $\mathbb{N}$ are the restrictions of atomic measures on the set of ultrafilters. Note also that, in accordance with [7], the set of two-valued measures on $\mathbb{N}$ coincides with the set of extreme points of $W(\mathbb{N})$.

In [3] it was proved that for any state $\omega \in \Sigma(H)$ in H we can find a family $E = \{e_n,\ n \in \mathbb{N}\}$ of unit vectors and a finitely additive measure μ on $\mathbb{N}$ such that ω can be written in the form of the Pettis integral (1) with respect to μ. Note that each finitely additive measure admits a unique representation in the form $\mu = \mu^1 + \mu^2$, where μ^1 is a countably additive measure, and the measure μ^2 vanishes at any finite set $X \subset \mathbb{N}$. The measures μ^2 with this property will be called *purely finitely additive.*

A state ω is said to be purely finitely additive if $\omega(\mathbf{P}) = 0$ for every finite-dimensional orthogonal projection $\mathbf{P}$. Any $\omega \in \Sigma(H)$ can be uniquely written as

$$\omega = \lambda\omega_n + (1-\lambda)\omega_{fa}, \quad 0 \le \lambda \le 1, \tag{2}$$

where the state ω_n is normal, while ω_{fa} is purely finitely additive. If $\lambda > 0$, we shall say that ω has a normal component.

The next claim yields a partial description of the set $\mathrm{Extr}(\Sigma(H))$.

Proposition 1. *Let $\omega \in \Sigma(H)$ be an extreme point of the set of states. Then ω admits a representation in the form* (1) *with a two-valued measure μ (μ only takes values* 0 *and* 1).

Remark. Examples presented in this paper show that a state of the form (1) with a two-valued μ may fail to be an extreme point of the set of states. The question about the precise description of the set of extreme points of the entire convex set $\Sigma(H)$ remains open. Proposition 1 implies that every element $\omega \in \mathrm{Extr}(\Sigma(H))$ gives rise to an ultrafilter. Below we shall see that a representation in the form (1) is not unique; therefore, we may have several such ultrafilters.

Proof. 1. Let a state of the form (1) be an extreme point of $\Sigma(H)$. Suppose that the measure μ is not two-valued and takes positive values a_1 and $a_2 = 1 - a_1$ on the sets $\mathbb{N}_1 \subset \mathbb{N}$ and $\mathbb{N}_2 = \mathbb{N} \setminus \mathbb{N}_1$, respectively. Then $\omega = a_1\omega_1 + a_2\omega_2$, implying that $\omega_2 = \omega_1$ because ω is an extreme point. Consequently, the state ω admits a representations (1) with a measure μ concentrated on a set $\mathbb{N}_1 \subset \mathbb{N}$. Continuing by induction, we conclude that if a state ω expands in an integral of the form (1) with a measure concentrated on a set $\mathbb{N}_o \subset \mathbb{N}$, then for any set $\mathbb{N}_a \subset \mathbb{N}_o$ the state ω admits expansion (1) with a measure concentrated either on $\mathbb{N}_a \subset \mathbb{N}_o$, or or $\mathbb{N}_o \setminus \mathbb{N}_a$.

2. Let $\mathbf{G}$ be the collection of all subsets of $\mathbb{N}$ for which there is a measure supported on the subset in question and such that the state (1) can be expanded in an integral with respect to this measure. By item 1, the collection $\mathbf{G}$ contains a chain of subsets ordered by inclusion. If this chain has a minimal element that is a singleton, then we are done. If this chain has a minimal element containing at least two points, then ω is not an extreme point, which is impossible. Now, if such a chain $\mathcal{C}$ has no minimal elements, then it determines a nonprincipal ultrafilter $\mathbb{F}_{\mathcal{C}}$ of subsets of $\mathbb{N}$. Indeed, if $\mathbb{N}_o \in \mathcal{C}$ and $\mathbb{N}_a \subset \mathbb{N}_o$, then, by item 1, the state ω can be expanded in an integral either over $\mathbb{N}_a$, or over $\mathbb{N}_o \setminus \mathbb{N}_a$. Hence, for any $\mathbb{N}_a \subset \mathbb{N}$, the chain $\mathcal{C}$ contains either the set $\mathbb{N}_a$ itself, or its complement. Let $\mu_{\mathcal{C}}$ be the two-valued measure that corresponds to the ultrafilter $\mathbb{F}_{\mathcal{C}}$.

3. We show that the state ω admits expansion in an integral with respect to $\mu_{\mathcal{C}}$. In accordance with item 2, we have $\omega = \int_{\mathbb{N}_a} \rho_{e_n}\, d\mu(n)$ for any $\mathbb{N}_a \in \mathbb{F}_{\mathcal{C}}$. We prove that in fact $\omega = \int_{\mathbb{N}} \rho_{e_n}\, d\mu_{\mathcal{C}}(n)$ in the sense of Pettis, i.e., for any $A \in B(H)$ we have $\omega(A) = \int_{\mathbb{N}} \rho_{e_n}(A)\, d\mu_{\mathcal{C}}(n)$.

Let $A \in B(H)$. There is no loss of generality in assuming that $A = A^*$. Let $\{a_n\}$ be the sequence of values taken by the quadratic form of A at the vectors e_n, $n \in \mathbb{N}$. We may assume that for some segment $[-a, a]$ we have $a_n \in [-a, a]$ for all $n \in \mathbb{N}$. For each $m \in \mathbb{N}$, we split $[-a, a]$ into m segments $\{\Delta_i^m,\ i \in 1, \dots, m\}$ of equal length. Consider the corresponding partition $\tau(\mathbb{N}) = \{\mathbb{N}_1^m, \dots, \mathbb{N}_m^m\}$ of the set $\mathbb{N} = \cup_{j=1}^m \mathbb{N}_i^m$ such that $a_n \in \Delta_i^m \Leftrightarrow n \in \mathbb{N}_i^m$. Then for each $m \in \mathbb{N}$, the ultrafilter $\mathbb{F}_{\mathcal{C}}$ contains precisely one of the sets $\mathbb{N}_i^m$, $i \in 1, \dots, m$.

For some $m \in \mathbb{N}$, let $j_0 \in \{1, \dots, m\}$ be defined by the condition $\mathbb{N}_{j_0}^m \in \mathbb{F}_{\mathcal{C}}$.

By item 2, the identity $\omega = \int_{\mathbb{N}_{j_0}^m} \rho_{e_n}\, d\mu(n)$ is true for any $m \in \mathbb{N}$. Therefore, $\omega(A) = \int_{\mathbb{N}_{j_0}^m} \rho_{e_n}(A)\, d\mu_{\mathcal{C}}(n)$, and all the numbers $\rho_{e_n}(A)$ belong to $\Delta_{j_0}^m$ for $n \in \mathbb{N}_{j_0}^m$. Consequently, $\omega(A) \in \Delta_{j_0}^m$ and $\int_{\mathbb{N}} \rho_{e_n}(A)\, d\mu_{\mathcal{C}}(n) \in \Delta_{j_0}^m$ for any $m \in \mathbb{N}$. Thus,

$$\omega(A) = \int_{\mathbb{N}} \rho_{e_n}(A)\, d\mu_{\mathcal{C}}(n). \qquad \square$$

Proposition 2. *Suppose we are given a principal ultrafilter $\mathbb{F}$ and a finitely additive two-valued measure μ generating $\mathbb{F}$. Then the sequence of unit vectors $\{e_n\}$ converges with respect to the ultrafilter $\mathbb{F}$ to an element e, $\|e\| = 1$, and*

$$\omega = \int_{\mathbb{N}} \rho_{e_n}\, d\mu(n)$$

is an extreme point in the set of normal states, so that

$$\omega(x) = \rho_e(x), \quad x \in B(H).$$

Proof. Suppose that the ultrafilter $\mathbb{F}$ is determined by the condition $F \in \mathbb{F} \Leftrightarrow n_0 \in F$. Then the sequence $\{e_n\}$ converges with respect to $\mathbb{F}$ to the element e_{n_0}. This means by definition that

$$\int_{\mathbb{N}} (e_n, xe_n)\, d\mu(n) = (e_{n_0}, xe_{n_0}), \quad x \in B(H). \qquad \square$$

§3. Relationship between the geometry of a system of unit vectors and the properties of states

Consider a set $E = \{e_n,\ n \in \mathbb{N}\}$ of unit vectors in a Hilbert space H. Assume that the closed linear hull of E coincides with H. The system E is called (see [8]):

(1) a uniformly minimal system if $\inf_n \operatorname{dist}(e_n, \overline{\operatorname{lin}(e_k,\ k \neq n)}) = \delta > 0$;

(2) a Riesz basis if there exists an orthonormal basis $\{g_n\}$ in H and a bounded operator T with bounded inverse such that $e_n = Tg_n$, $n \in \mathbb{N}$;

(3) a Bari basis if it is a Riesz basis and $T = I + \Delta$ with $\Delta \in \mathfrak{S}_2$.

For our purposes, we shall use the following simple condition under which E is a Bari basis in its closed linear hull: the Gram matrix $\|(e_n, e_m)\|$ should satisfy

$$\sum_{n \neq m} |(e_n, e_m)|^2 < 1. \tag{3}$$

In [9], a system E was named overcomplete if it is complete in H and every its infinite subsystem is also complete in H. Observe that no overcomplete system can contain a subsystem $\{e_{n_k}\}$ that is a Riesz basis in its closed linear hull.

Theorem 3. *Let $\mathbb{F}$ be a nonprincipal ultrafilter, μ a finitely additive two-valued measure generating $\mathbb{F}$, and $E = \{e_n,\ n \in \mathbb{N}\}$ a system of unit vectors no subsystem of which can form a Bari basis in its closed linear hull.*

Then the state

$$\omega = \int_{\mathbb{N}} \rho_{e_n}\, d\mu(n)$$

has a normal component.

Remark. The assumptions of the theorem are fulfilled if E is a overcomplete system.

Proof. We fix $f \in H$ and consider the numerical sequence $a_n = (e_n, f)$. Being bounded, this sequence converges with respect to the ultrafilter $\mathbb{F}$. Thus, there is a vector $e \in H$ such that for any $\varepsilon > 0$ there exists a set $F_\varepsilon \in \mathbb{F}$ such that $|(e_n, f) - (e, f)| < \varepsilon$, for $n \in F_\varepsilon$. We analyze three cases separately.

1. Suppose $e = 0$. Then for any $f \in H$ and any $\varepsilon > 0$ we can find $F_\varepsilon \in \mathbb{F}$ so that $|(e_n, f)| < \varepsilon$ for all $n \in F_\varepsilon$. Consider a sequence $F_k \in \mathbb{F}$ and a subsequence $e_{n_k} \in F_k$ such that $|(e_{n_k}, e_m)| < \varepsilon^k,\ m \in F_{k+1}$. Then $|(e_{n_k}, e_{n_{k+s}})| < \varepsilon^k$, whence

$$\sum_{k \neq s} |(e_{n_k}, e_{n_s})|^2 < 1$$

for sufficiently small $\varepsilon > 0$, so that condition (3) is fulfilled. Therefore, the system (e_{n_k}) is a Bari basis in its closed linear hull, a contradiction.

2. Suppose that $\|e\| = 1$, then for any $\varepsilon > 0$ there exists a set $F_\varepsilon \in \mathbb{F}$ such that $|(e_n, e) - 1| < \varepsilon,\ n \in F_\varepsilon$, which implies that $\|e_n - e\|^2 < 2\varepsilon,\ n \in F_\varepsilon$, i.e., the e_n converge strongly to e with respect to the ultrafilter $\mathbb{F}$. It follows that

$$\begin{aligned} |(e, xe) - (e_n, xe_n)| &\le |(e - e_n, xe)| + |(e_n, x(e - e_n))| \\ &\le \|e - e_n\|\|xe\| + \|x\|\|e - e_n\| < 2\varepsilon(\|xe\| + \|x\|), \end{aligned}$$

$n \in F_\varepsilon$. By definition, this means that

$$\int_{\mathbb{N}} \rho_{e_n}\, d\mu(n) = \rho_e,$$

and this state is an extreme point in the set of normal states.

3. Suppose $0 < \|e\| < 1$. Let P_e denote the orthogonal projection to the vector e. By assumption, we have

$$\int_{\mathbb{N}} \rho_{e_n}(P_e)\, d\mu(n) = \|e\|^2 > 0,$$

so that the state in question cannot be purely finitely additive. □

Theorem 4. *Let μ be a finitely additive two-valued measure generating an ultrafilter $\mathbb{F}$, and let $E = \{e_n,\ n \in \mathbf{N}\}$ be a system of unit vectors. Suppose that*

(i) $\mathbb{F}$ *is not a principal ultrafilter*;

(ii) *there exists an element $F_0 \in \mathbb{F}$ such that the system $E_{F_0} = \{e_n,\ n \in F_0\}$ is a Riesz basis in its closed linear span.*

Then the state

$$\omega(x) = \int_{\mathbb{N}} \rho_{e_n}(x)\, d\mu(n)$$

is not normal.

Proof. Observe that the measure μ vanishes at the complement $\bar{F}$ of any set $F \in \mathbb{F}$. Hence,

$$\int_{\mathbb{N}} \rho_{e_n}(x)\, d\mu(n) = \int_{F_0} \rho_{e_n}(x)\, d\mu(n), \quad x \in B(H). \tag{4}$$

By assumption, there exists an orthonormal system $\{f_{n_k}\}$ such that

1) $\{n_k,\ k \in \mathbb{N}\} \equiv F_0 \in \mathbb{F}$;

2) on the closed linear hull H_{F_0} of the system $\{f_{n_k}\}$ we have a bounded operator $V\colon H_{F_0} \to H_{F_0}$ with bounded inverse such that

$$V f_{n_k} = e_{n_k}, \quad n_k \in N.$$

Then, by (4),

$$\omega(x) = \Omega(V^* x V),$$

where

$$\Omega(x) = \int_{F_0} \rho_{f_n}(x)\, d\mu(n).$$

By (i), the sets $F \in \mathbb{F}$ are infinite. Consequently, the state Ω is not normal, and, with it, ω is also not normal. □

§4. Expansions in integrals over systems of vector states

The vector state ρ_e given on the algebra of bounded operators $B(H)$ by the formula $\rho_e(x) = (e, xe)$, $x \in B(H)$, will be called a purely normal state.

Definition. We say that a system of purely normal states $\mathcal{S} = \{\rho_{e_k},\ k \in \mathbb{N}\}$ is pseudominimal if the identity $\int_{\mathbb{N}} \rho_{e_k}\, d\nu(k) = \int_{\mathbb{N}} \rho_{e_k}\, d\mu(k)$, where $\mu, \nu \in W(\mathbb{N})$, implies that $\mu = \nu$.

Example 1. Any orthonormal system $\{e_k\}$ determines a pseudominimal system of vector states $\{\rho_{e_k}\}$. Indeed, in the case of an orthonormal system $\{e_k\}$, the state r representable as $\int_{\mathbb{N}} \rho_{e_k}\, d\mu$ with $\mu \in W(\mathbb{N})$ uniquely determines the measure of any subset $K \subset \mathbb{N}$ by the rule $\mu(K) = r(\mathbf{P}_K)$, where $\mathbf{P}_K$ is the orthogonal projection equal to the sum of the series $\sum_{k \in K} \mathbf{P}_{e_k}$, which converges in the strong operator topology.

Example 2. Suppose that the space H is 2-dimensional, and that $\mathcal{E} = \{e_1, e_2, e_3\}$ is a system of three unit vectors in H with angles between any pair of them equal to $\frac{2\pi}{3}$. Then the system of pure states $\mathcal{S}_{\mathcal{E}}$ is pseudominimal, because otherwise one of the vector states in question is a linear combination of the two other states, which is not true. Here, the system $\mathcal{E}$ is not minimal in H.

Example 3. If a system $\{f_k\}$ is dense on the unit sphere $S_1(H)$, then the corresponding system $\{\rho_{f_k}\}$ of vector states is not pseudominimal. Indeed, in this case there exist two disjoint sequences of natural numbers $\{k'\}$, $\{f_k\}$ such that $\|f_{k'_j} - f_{k''_j}\|_H < 2^{-j}$ for all $j \in \mathbb{N}$. Then if $\nu \in W(\mathbb{N})$ is a purely finitely additive measure on $\mathbb{N}$ (not necessarily two-valued), and ν' and ν'' are the images of ν under the mappings $j \to k'_j$, $j \to k''_j$ (respectively), then $\int_{\mathbb{N}} \rho_{f_k}\, d\nu'(k) = \int_{\mathbb{N}} \rho_{f_k}\, d\nu''(k)$. Indeed, $\int_{\mathbb{N}} \rho_{f_k}\, d\nu'(k) = \int_{\mathbb{N}} \rho_{f_{k'_j}}\, d\nu(j)$ and $\int_{\mathbb{N}} \rho_{f_k}\, d\nu''(k) = \int_{\mathbb{N}} \rho_{f_{k''_j}}\, d\nu(j)$. Since ν is finitely additive, we have

$$\left| \int_{\mathbb{N}} \rho_{f_{k'_j}}\, d\nu(j) - \int_{\mathbb{N}} \rho_{f_{k''_j}}\, d\nu(j) \right| = \left| \int_{j \in \mathbb{N}\colon j > m} (\rho_{f_{k'_j}} - \rho_{f_{k''_j}})\, d\nu(j) \right|$$

for al $m \in \mathbb{N}$. Since

$$\left| \int_{j\in\mathbb{N}\,:\,j>m} (\rho_{f_{k'_j}} - \rho_{f_{k''_j}})\, d\nu(j) \right| \le \int_{k\in\mathbb{N}\,:\,j>m} |\rho_{f_{k'_j}} - \rho_{f_{k''_j}}|\, d\nu(j) \le 2^{-m}$$

for all $m \in \mathbb{N}$, the identity in question is proved. Since the measures ν' and ν'' are mutually singular, the system $\{f_k\}$ is not pseudominimal.

For any system $\mathcal{S}$ of vector states, we denote $\Sigma_{\mathcal{S}}(H)$ the set of states that can be written as $\int_{\mathbb{N}} \rho_{e_k}\, d\nu(k)$ for some $\nu \in W(\mathbb{N})$.

The definition of the pseudominimality of a system of vector states shows that an element $r \in \Sigma_{\mathcal{S}}(H)$ admits a unique expansion in the integral $\int_{\mathbb{N}} \rho_{e_k}\, d\nu(k)$, $\nu \in W(\mathbb{N})$, over the system $\mathcal{S}$ if and only if this system is pseudominimal. However, if $\mathcal{S}$ is not pseudominimal, then for any $r_1, r_2 \in \Sigma_{\mathcal{S}}(H)$ the sets of measures $W_{r_i}(\mathbb{N}) = \big\{\mu \in W(\mathbb{N}) : \int_{\mathbb{N}} \rho_{e_k}\, d\mu(k) = r_i\big\}$ are isomorphic in the sense that they can be obtained from each other by a shift in the linear space of measures. Thus, if a system $\mathcal{S}$ is not pseudominimal, then the integral expansion of any state is not unique.

Theorem 5. *Let $\mathcal{S}$ be a pseudominimal system of vector states. The state*

$$r_\mu = \int_{\mathbb{N}} \rho_{e_k}\, d\mu(k)$$

is an extreme point of the set $\Sigma_{\mathcal{S}}(H)$ if and only if the measure μ is an extreme point of the set $W(\mathbb{N})$, i.e., μ is a two-valued measure.

Proof. If μ is not an extreme point of $W(\mathbb{N})$, then $\mu = \theta\nu_1 + (1-\theta)\nu_2$, where $\theta \in (0,1)$, $\nu_{1,2} \in W(\mathbb{N})$, $\nu_1 \ne \nu_2$. Since the system $\mathcal{S}$ is pseudominimal, we have

$$\int_{\mathbb{N}} \rho_{e_k}\, d\nu_1(k) \ne \int_{\mathbb{N}} \rho_{e_k}\, d\nu_2(k),$$

so that the state r_μ is not an extreme point of $\Sigma_{\mathcal{S}}(H)$.

If μ is a two-valued measure and the state r_μ majorizes some state $r_\nu \in \Sigma_{\mathcal{S}}(H)$, i.e., there exist a number $a > 0$ such that $r_\mu - a r_\nu \ge 0$, then $\int_{\mathbb{N}} \rho_{e_k}\, d(\mu - a\nu)(k) \ge 0$. Consequently, if $\mu(K) = 0$ for some $K \subset \mathbb{N}$, then $\nu(K) = 0$, because otherwise, if $\nu(K) > 0$, then $\int_{\mathbb{N}} \rho_{e_k}\, d\nu(k) \le 0$, which is false. Thus, μ majorizes ν and, since μ is an extreme point, we have $\mu = \nu$. ☐

Theorem 6. *If a system $E = \{e_n,\ n \in \mathbb{N}\}$ of unit vectors forms a Riesz basis in H, then the system $\mathcal{S} = \{\rho_{e_k}\}$ is pseudominimal.*

Proof. Put $\omega_\mu(x) = \int_{\mathbb{N}} \rho_{e_n}(x)\, d\mu(n)$, $x \in B(H)$. Arguing as in the proof of Theorem 4, we see that there exists an orthonormal system $\{f_n\}$ and a bounded operator $V\colon H \to H$ with bounded inverse such that

$$\omega_\mu(x) = \Omega_\mu(V^* x V), \quad x \in B(H),$$

where

$$\Omega_\mu(x) = \int_{\mathbb{N}} \rho_{f_n}(x)\, d\mu(n), \quad x \in B(H).$$

Suppose that $\Omega_\mu(x) = \Omega_\nu(x)$ for all $x \in B(H)$. Then $\mu = \nu$, because the orthonormal systems are pseudominimal, which immediately implies the claim. ☐

§5. On properties of expansions

Let σ be the set of all pseudominimal systems, and let $\Sigma^1(H) = \bigcup_{\mathcal{S}\in\sigma} \Sigma_{\mathcal{S}}(H)$. Then $\rho \in \Sigma^1(H)$ if and only if there exists a pseudominimal system $\mathcal{S} = \{\rho_{e_k}\}$ and a measure $\mu \in W(\mathbb{N})$ such that $\rho = \int_{\mathbb{N}} \rho_{e_k}\, d\mu(n)$. Here, if $\rho \in \operatorname{Extr}(\Sigma^1(H))$ and $\rho \in \Sigma_{\mathcal{S}}(H)$, then $\rho \in \operatorname{Extr}(\Sigma_{\mathcal{S}}(H))$. Consequently, by Theorem 5, $\rho \in \operatorname{Extr}(\Sigma^1(H))$ if and only if in the corresponding expansion (1) we have $\mu \in \operatorname{Extr}(W(\mathbb{N}))$.

Proposition 7. *Let $\{e_n\}$ be an arbitrary system of unit vectors, and μ a nonnegative normalized finitely additive measure on the algebra of subsets $2^{\mathbb{N}}$. If the state $r = \int_{\mathbb{N}} \rho_{e_n}\, d\mu$ is purely finitely additive, then so is the measure μ. Also, if μ is countably additive, then the state r is normal.*

Proof. Since r is purely finitely additive, for any $m \in \mathbb{N}$ the state r vanishes at the orthogonal projection $\mathbf{P}_m$ onto the subspace $\overline{\operatorname{lin}(e_n,\ 1 \le n \le m)}$. Since

$$r(\mathbf{P}_m) = \int_{\mathbb{N}} (\mathbf{P}_m e_n, e_n)\, d\mu(n) \ge \mu(\{1, 2, \dots, m\})$$

and $r(\mathbf{P}_m) = 0$, we have $\mu(\{1, 2, \dots, m\}) = 0$ for all $m \in \mathbb{N}$; consequently, μ is purely finitely additive.

If μ is countably additive, then the integral $\int_{\mathbb{N}} \rho_{e_n}\, d\mu(n)$ is the sum of the series the terms of which are the normal states ρ_{e_n} with weights $\mu(n)$. Since $\sum_{n\in\mathbb{N}} \mu(n) = 1$, this series converges in the norm of the space of states to a normal state; thus, the state r is normal. □

Proposition 8. *Suppose that a system of pure states $\mathcal{S}$ is generated by a system $\{e_n\}$ of unit vectors in H such that the sequence $\{e_n\}$ converges weakly to zero. Let $\mu \in W(\mathbb{N})$ and put $r = \int_{\mathbb{N}} \rho_{e_n}\, d\mu(n)$. If μ is purely finitely additive, then so is the state r. Also, if r is normal, then μ is countably additive.*

Proof. We check that if the measure μ is purely finitely additive, then r vanishes at any finite-dimensional projection. Indeed, let $\mathbf{P}$ be a finite-dimensional orthogonal projection; we have $\lim_{n\to\infty} \|\mathbf{P}e_n\| = 0$ because the sequence $\{e_n\}$ converges to zero weakly. Therefore, for any $\epsilon > 0$ the set $\mathbb{N}_{\epsilon,\mathbf{P}} = \{n \in \mathbb{N} : (\mathbf{P}e_n, e_n) > \epsilon\}$ is finite. Consequently, $r(\mathbf{P}) \le \epsilon$ for every $\epsilon \ge 0$, so that $r(\mathbf{P}) = 0$. Since this is true for all finite-dimensional projections, we see that r is a purely finitely additive state.

If the state $r = \int_{\mathbb{N}} \rho_{e_n}\, d\mu(n)$ is normal, then the measure μ is countably additive. Suppose the contrary. Assume that μ has a nontrivial purely finitely additive component, i.e., $\mu = (1-\alpha)\mu_{ca} + \alpha\mu_p$, $\alpha \in (0, 1]$, where μ_{ca} is a countably additive measure, and μ_p is a purely finitely additive measure. By the proved above, the state $r_p = \int_{\mathbb{N}} \rho_{e_n}\, d\mu_p(n)$ is purely finitely additive. For each $m \in \mathbb{N}$, let $\mathbf{P}_m$ be the orthogonal projection onto the m-dimensional subspace $\overline{\operatorname{lin}(e_1, \dots, e_m)}$, and let $\mathbf{P}_\infty$ be the orthogonal projection onto the closure of the linear hull of the system $\{e_n,\ n \in \mathbb{N}\}$. Then, since the state r_p is purely finitely additive, we have

$$r_p(\mathbf{P}_m) = \int_{\mathbb{N}} \rho_{e_n}(\mathbf{P}_m)\, d\mu_p(n) = 0$$

for any $m \in \mathbb{N}$. We have $\mathbf{P}_m \uparrow \mathbf{P}_\infty$ and, simultaneously,

$$r(\mathbf{P}_m) = (1-\alpha)\int_{\mathbb{N}} (\mathbf{P}_m e_n, e_n)\, d\mu_{ca}(n) + \alpha \int_{\mathbb{N}} (\mathbf{P}_m e_n, e_n)\, d\mu_p(n) \le 1 - \alpha < 1 = r(\mathbf{P}_\infty)$$

for any $m \in \mathbb{N}$. This contradicts the normality of r, proving that μ is countably additive. □

Propositions 7 and 8 imply the next statement.

Corollary. *Suppose that $\mathcal{S}$ is the system of pure states generated by a system $\{e_n\}$ of unit vectors in H such that the sequence $\{e_n\}$ converges weakly to zero. Let $\mu \in W(\mathbb{N})$ and put $r = \int_{\mathbb{N}} \rho_{e_n}\, d\mu(n)$. Then the state r is normal (purely finitely additive) if and only if the measure μ is countably additive (purely finitely additive).*

Remark. If a system of pure states $\mathcal{S}$ is generated by a uniformly minimal system $\{e_n\}$ of unit vectors in H, then the pure finite additivity of a measure $\mu \in W(\mathbb{N})$ does not imply the same property of the state $r_{\mathcal{S},\mu} = \int_{\mathbb{N}} \rho_{e_n}\, d\mu(n)$. Indeed, let $\{f_n\}$ be an orthonormal basis in H, and let $e_1 = f_1$, $e_k = \frac{1}{\sqrt{2}}(f_1 + f_k)$, $k = 2, 3, \dots$. Then the system $\{e_n\}$ is complete and uniformly minimal, but for any purely finitely additive measure $\mu \in W(\mathbb{N})$ the state $r_{\mathcal{S},\mu}$ takes the value $\frac{1}{2}$ at the 1-dimensional projection $\mathbf{P}_1$, and $\lim_{m\to\infty} \rho_{\mathcal{S},\mu}(\mathbf{P}_m) = \frac{1}{2} < 1$, implying that $\rho_{\mathcal{S},\mu}$ is neither normal, nor purely finitely additive.

By Proposition 1, if $r \in \mathrm{Extr}(\Sigma_{\mathcal{S}}(H))$, then there is a two-valued measure $\mu \in W(\mathbb{N})$ such that

$$r = \int_{\mathbb{N}} \rho_{e_k}\, d\mu(k). \tag{5}$$

Proposition 9. *Suppose that, in the representation (5) for $r \in \mathrm{Extr}(\Sigma_{\mathcal{S}}(H))$, the sequence $\{e_n\}$ converges weakly to zero. Then the preimage of the unity under the action of the state r on the set of all orthogonal projections includes a decreasing linearly ordered chain of orthogonal projections. Such a chain has a minimal element if and only if the state r is normal.*

Proof. Let $\mathbb{F}$ be the ultrafilter generated by μ. For each $F \in \mathbb{F}$, we define an orthogonal projection $\mathbf{P}_F$ as the minimal projection satisfying $\|\mathbf{P}_F e_k\| = \|e_k\|$ for any $k \in F$ (i.e., $\mathbf{P}_F$ is the orthogonal projection onto the subspace $\overline{\mathrm{lin}\{e_k,\, k \in F\}}$). Since the system of projections $\mathbf{P}_F$ decreases strictly as F reduces, this system forms a chain ordered by decreasing. For any $F \in \mathbb{F}$, the projection $\mathbf{P}_F$ belongs to the preimage of the unity under the action of r on the set of all orthogonal projections.

Now we find conditions under which the chain of projections $\mathbf{P}_F$, $F \in \mathbb{F}$, has a minimal element. If r is a normal state, then the above corollary shows that the measure μ is concentrated on a single element, and the role of a minimal element of the chain of orthoprojections is played by a one-dimensional projection. Conversely, suppose that a chain of orthogonal projections in $r^{-1}(1)$ has a minimal element, which cannot be reduced by a finite-dimensional subspace; then the state r has a normal component and, hence, is normal, because it is an extreme point. □

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