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BESSEL SEQUENCES WITH FINITE UPPER DENSITY IN DE BRANGES SPACES

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Abstract. The de Branges spaces are described in which every real Bessel sequence Λ has finite upper density. The description is in terms of the spectral measure of the space in question.

§1. Introduction

Let $\mathcal{H}$ be a space of entire functions with reproducing kernel k_λ, $\lambda \in \mathbb{C}$. A sequence Λ is said to be *Bessel* if

$$\sum_{\lambda\in\Lambda}\left|\left(f, \frac{k_\lambda}{\|k_\lambda\|_{\mathcal{H}}}\right)\right|^2 = \sum_{\lambda\in\Lambda}\frac{|f(\lambda)|^2}{\|k_\lambda\|^2_{\mathcal{H}}} \le C_\Lambda \|f\|^2_{\mathcal{H}}, \quad f \in \mathcal{H}, \tag{1}$$

i.e., the system $\{k_\lambda\}_{\lambda\in\Lambda}$ of reproducing kernels is Bessel in $\mathcal{H}(E)$. Description of Bessel sequences Λ in geometric terms is a classical problem of complex analysis. For example, the Plancherel–Polya classical inequality allows one to describe all real Bessel sequences for the Paley–Wiener space $\mathcal{P}\mathrm{W}_\pi$ (this is the space of entire functions of exponential type not exceeding π that are square integrable over $\mathbb{R}$).

Theorem 1.1. *A sequence $\Lambda \subset \mathbb{R}$ is Bessel in $\mathcal{P}\mathrm{W}_\pi$ if and only if*

$$\sup_{n\in\mathbb{Z}} \#\{\Lambda \cap [n, n+1]\} < \infty. \tag{2}$$

Thus, for a sequence Λ to be Bessel in $\mathcal{P}\mathrm{W}_\pi$, the finiteness of a certain *upper density* is *necessary.* Our main goal in this paper is to describe the de Branges spaces in which an inequality like (2) is necessary for the Bessel property of a real sequence Λ.

1.1. De Branges spaces. We say that an entire function E belongs to the Hermite–Biehler class if

$$|E(z)| > |E(\bar{z})|, \quad z \in \mathbb{C}^+,$$

and E has no zeros on $\mathbb{R}$. Every such function gives rise to the *de Branges space* $\mathcal{H}(E)$ that consists of entire functions F with the property that the functions F/E and F^*/E $(= \overline{F(\bar{z})}/E(z))$ belong to the Hardy space $H^2(\mathbb{C}^+)$. The scalar product in $\mathcal{H}(E)$ is defined by the formula

$$(F, G)_{\mathcal{H}(E)} := \int_{\mathbb{R}} \frac{F(t)\overline{G(t)}}{|E(t)|^2}\, dt.$$

If $E(z) = e^{-i\pi z}$, then $\mathcal{H}(E) = \mathcal{P}\mathrm{W}_\pi$.

There are several other (equivalent) definitions of de Branges spaces (see a discussion in §2). The theory of de Branges spaces plays an important part in harmonic analysis and the spectral theory of second order differential operators. The basics of this theory

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are presented in de Branges' book [7]. Some modern results and various applications can be found in [13, 12, 14].

It is well known that the reproducing kernel of $\mathcal{H}(E)$ is given by the formula

$$k_w(z) := \frac{\overline{E(w)}E(z) - \overline{E^*(w)}E^*(z)}{2\pi i(\bar{w} - z)},$$

see [7].

We recall the notion of a *phase function*, important for the theory of de Branges spaces. Let E belong to the Hermite–Biehler class. A phase function for E is a continuous monotone increasing function φ such that $E(t)e^{i\varphi(t)} \in \mathbb{R}$, $t \in \mathbb{R}$. Clearly, φ is defined uniquely to within a constant summand of the form πl, $l \in \mathbb{Z}$.

Every de Branges space possesses an *orthogonal* basis of reproducing kernels.

Theorem 1.2 (see [7]). *Let $\alpha \in \mathbb{T}$, and let $t_{\alpha,n}$ be a (unique) solution of the equation $\varphi(t_{\alpha,n}) = \frac{1}{2}\arg\alpha + \pi n$, $n \in \mathbb{Z}$ (we only consider the values of n for which the equation is solvable). Then for all values of α except, maybe, one, the system $\{k_{t_{\alpha,n}}\}_n$ is an orthogonal basis in $\mathcal{H}(E)$.*

The existence of orthogonal bases composed of reproducing kernels is a characteristic property of the de Branges spaces. In [5] it was proved that if a Hilbert space of entire functions (with some natural properties) possesses two different orthogonal bases of reproducing kernels, then this space coincides with a certain de Branges space.

In what follows, we assume that $\alpha = 1$ is *not* an exceptional value in Theorem 1.2 (i.e., $\{k_{t_{1,n}}\}$ is an orthogonal basis).

If $\mathcal{H}(E) = \mathcal{P}\mathrm{W}_\pi$, then $\varphi(t) = \pi t$, $t_{1,n} = n$, $n \in \mathbb{Z}$. The function φ determines a metric on $\mathbb{R}$, namely, $d(x, y) = \varphi(x) - \varphi(y)$, $x \geq y$. Many properties of $\mathcal{H}(E)$ admit a nice description in terms of this metric (see [1]).

We shall compare the number of points of Λ that belong to an interval I with the number of the $t_{1,n}$ that belong to I. It could be conjectured that the finiteness of the corresponding density is equivalent to the Bessel property (see Theorem 1.1).

Conjecture. *A sequence $\Lambda \subset \mathbb{R}$ is Bessel in $\mathcal{H}(E)$ if and only if*

$$\mathrm{UD}(\Lambda) := \sup_{n\in\mathbb{Z}} \#\{\Lambda \cap [\varphi^{-1}(\pi n), \varphi^{-1}(\pi(n+1))]\} < \infty.^1 \qquad (3)$$

Unfortunately, condition (3) is neither necessary nor sufficient for the Bessel property. An example of a non-Bessel sequence satisfying (3) was constructed in [1]. The first example of a Bessel sequence with infinite upper density (with respect to $t_n = \varphi^{-1}(\pi n)$) was given in [6].

Nevertheless, for various *classes* of spaces, condition (3) is necessary. For example, if the set $|E(\bar{z})| < (1 - \varepsilon)|E(z)|$ is connected for some $\varepsilon \in (0, 1)$, then (3) is necessary and sufficient (see the paper [9] where a similar property was verified for the corresponding model spaces $K_\Theta = \mathcal{H}(E)/E$). Such functions E (in the Hermite–Biehler class) and the corresponding inner functions Θ ($= E^*/E$) are said to be *one-component.*

On the other hand, if the sequence $\{t_n\}_{n>0}$ is *lacunary*, i.e., $\inf_n t_{n+1}/t_n > 1$, then condition (3) is also necessary for the Bessel property (see [6]).

In this paper, we describe completely all de Branges spaces in which condition (3) is necessary for the Bessel property. The description will be in terms of the *spectral measure* for $\mathcal{H}(E)$.

[1] We assume implicitly that the supremum is taken over the indices n for which $\varphi^{-1}(\pi n)$ exists.

It is easily seen that for every α, $|\alpha| = 1$, the function $\operatorname{Re}\frac{\alpha E+E^*}{\alpha E-E^*}$ is positive and harmonic in $\mathbb{C}^+$. Consequently,

$$\operatorname{Re}\frac{\alpha E(z)+E^*(z)}{\alpha E(z)-E^*(z)} = p_\alpha y + \frac{y}{\pi}\int_{\mathbb{R}}\frac{d\mu_\alpha(t)}{|t-z|^2}, \quad p_\alpha \geq 0,$$

for some measure μ_α with $\int_{\mathbb{R}}\frac{d\mu_\alpha(t)}{1+t^2} < \infty$. These measures μ_α are often called the *Clark measures*[2] (see [8]). Observe that $p_\alpha = 0$ for all values of α except, maybe, one (this is the exceptional value in Theorem 1.2). The measure $\mu := \mu_1$ is called the spectral measure for $\mathcal{H}(E)$. In conclusion, we note that $\operatorname{supp}\mu = \{t_{1,n}\}$, $\mu = \sum_n \mu_n\delta_{t_{1,n}}$, $\sum_n \frac{\mu_n}{1+t_{1,n}^2} < \infty$. In fact, *an arbitrary* measure of this form is the spectral measure for some de Branges space (see the discussion in §2).

1.2. The main result. For simplicity, we assume that $\{t_n\}$ exists for every $n \in \mathbb{Z}$. We put

$$I_n = [t_n, t_{n+1}], \quad I_n^- = [t_n, (t_n+t_{n+1})/2), \quad I_n^+ = [(t_n+t_{n+1})/2, t_{n+1}),$$

$$M_n^-(x) = \mu([t_n-(x-t_n), t_n]), \quad P_n^-(x) = \int_{|y-t_n|>|x-t_n|}\frac{d\mu(y)}{|y-t_n|^2}, \quad x \in I_n^-,$$

$$M_n^+(x) = \mu([t_{n+1}, t_{n+1}+(t_{n+1}-x)]),$$

$$P_n^+(x) = \int_{|y-t_{n+1}|>|x-t_{n+1}|}\frac{d\mu(y)}{|y-t_{n+1}|^2}, \quad x \in I_n^+.$$

Clearly, $M_n^-(t_n) = \mu(\{t_n\}) = \mu_n$ and $M_n^+(t_{n+1}) = \mu(\{t_{n+1}\}) = \mu_{n+1}$. To state the main result, we need the notion of the dyadic size.

Definition 1.1. A positive sequence $\{d_n\}$ (finite or infinite) is said to have *dyadic size* m if it lies in the union of m dyadic cells of the form $[2^l, 2^{l+1})$, $l \in \mathbb{Z}$, and m is the smallest number with this property.

The dyadic size of a sequence $\{d_n\}$ will be denoted by $\mathcal{D}(\{d_n\})$ (it may be infinite). For example, $\mathcal{D}(\{n, n+1, \dots, n+k\}) = [\log_2\frac{n+k}{n}] + 1$.

We split the real axis into four sets:

$$\begin{aligned}
\mathbb{R} &= \mathcal{R}_M^- \cup \mathcal{R}_M^+ \cup \mathcal{R}_P^- \cup \mathcal{R}_P^+,\\
\mathcal{R}_M^- &= \{x : x \in I_n^-, M_n^-(x) \geq |x-t_n|^2 P_n^-(x)\},\\
\mathcal{R}_M^+ &= \{x : x \in I_n^+, M_n^+(x) \geq |x-t_{n+1}|^2 P_n^+(x)\},\\
\mathcal{R}_P^- &= \{x : x \in I_n^-, M_n^-(x) < |x-t_n|^2 P_n^-(x)\},\\
\mathcal{R}_P^+ &= \{x : x \in I_n^+, M_n^+(x) < |x-t_{n+1}|^2 P_n^+(x)\}.
\end{aligned}$$

Put

$$\mathcal{A}_n^\pm = \{M_n^\pm(x) : x \in \mathcal{R}_M^\pm, x \in I_n\}, \quad \mathcal{B}_n^\pm = \{P_n^\pm(x) : x \in \mathcal{R}_P^\pm, x \in I_n\}.$$

Observe that $\mathcal{A}_n^\pm$ and $\mathcal{B}_n^\pm$ are finite sets for every n.

Theorem 1.3. *All real Bessel sequences Λ in $\mathcal{H}(E)$ have finite upper density* $\operatorname{UD}(\Lambda)$ *if and only if*

$$\sup_n \mathcal{D}(\mathcal{A}_n^-) < \infty, \quad \sup_n \mathcal{D}(\mathcal{A}_n^+) < \infty, \quad \sup_n \mathcal{D}(\mathcal{B}_n^-) < \infty, \quad \sup_n \mathcal{D}(\mathcal{B}_n^+) < \infty, \tag{4}$$

where the sequences $\mathcal{A}_n^\pm$, $\mathcal{B}_n^\pm$ are constructed by the spectral measure μ.

If E is one-component or $\sup_n \frac{t_{n+1}}{t_n} > 1$, then the sets $\mathcal{A}_n^\pm$ are uniformly finite, $\sup_n \#\mathcal{A}_n^\pm < \infty$. Thus, (4) fulfilled automatically.

[2] Before, these measures had been introduced by de Branges [7].

Corollary. *If the spectral measure* $\mu = \sum_n \mu_n \delta_{t_n}$ *is such that*

$$C^{-1}|t_n - t_{n-1}| \le |t_{n+1} - t_n| \le C|t_n - t_{n-1}|,$$

then every Bessel sequence Λ *has finite upper density* $\mathrm{UD}(\Lambda)$.

Theorem 1.3 can be used to easily verify the existence of a Bessel sequence with infinite upper density.

Put $\mu = \sum_{k=2}^{\infty} \nu_k$, $\nu_k = \sum_{l=0}^{k} \delta_{2^k+l}$. It is readily seen that the dyadic size of the sequence $\mathcal{A}^+_{m(k)}$ (corresponding to the point 2^k) is of order of $\log k$. This counterexample is borrowed from [6].

In conclusion, we note that the description of the spectral measures μ (or functions E) such that every sequence Λ with finite $\mathrm{UD}(\Lambda)$ is Bessel presents an open problem. It can be verified that this is so if E is one-component. On the other hand, a non-one-component function with the same property can be constructed (for example, the space generated by the measure $\mu = \sum_{n\in\mathbb{N}} 2^n \delta_{2^{2^n}}$ is such, see [6, Theorem 3.1])).

Conditions (4) do not suffice. For example, if $\mu = \sum_{n>0} \delta_{2^n}$, then the sequence $\{3\cdot 2^n\}$ is not Bessel (see [6, Theorem 3.1] or [1, Example 5.2]).

The organization of the paper. In §2, we recall the construction of discrete Hilbert transforms $\mathcal{H}(T,\mu)$ and restate the problem in new terms. In §3 we prove Theorem 1.3.

Notation. We shall write $U(x) \lesssim V(x)$ (or $V(x) \gtrsim U(x)$) if there is a constant C such that $U(x) \le CV(x)$ for all x in question. If $U \lesssim V$ and $V \lesssim U$, we write $U \asymp V$. The symbol $\#Y$ denotes the cardinality of a set Y.

§2. Spaces $\mathcal{H}(T,\mu)$ of discrete Hilbert transforms

De Branges spaces admit an axiomatic description. Suppose that a nontrivial Hilbert space $\mathcal{H}$ of entire functions satisfies the following three axioms.

1. If $F \in \mathcal{H}$ and $F(w) = 0$, then the function $F\frac{z-\bar{w}}{z-w}$ belongs to $\mathcal{H}$ and has the same norm as F.
2. For every $w \in \mathbb{C}$, the linear functional $F \mapsto F(w)$ is continuous.
3. If $F \in \mathcal{H}$, then F^* belongs to $\mathcal{H}$ and has the same norm as F.

Then $\mathcal{H} = S\mathcal{H}(E)$ for some function E in the Hermite–Biehler class and some entire function S such that $S^* = S$ (see [7, Theorem 23]).

There is yet another approach to de Branges spaces, first exposed in [6]. Let $\mu = \sum_n \mu_n \delta_{t_n}$ be a discrete measure on $\mathbb{R}$ with $\int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2} < \infty$ and $|t_n| \to \infty$ as $|n| \to \infty$. Put $T = \{t_n\}$ and consider the space of discrete Hilbert transforms

$$\mathcal{H}(T,\mu) := \left\{ f \,:\, f(z) = \sum_n \frac{a_n \mu_n^{1/2}}{z - t_n}, \quad a = \{a_n\} \in \ell^2 \right\},$$

endowed with the norm $\|f\|^2_{\mathcal{H}(T,\mu)} = \sum_n |a_n|^2$. It can easily be seen that $\mathcal{H}(T,\mu)$ satisfies axioms 1–3 (if $w \in \mathbb{C} \setminus T$). Let A be an entire functions real on the real axis and having simple zeros in T. Then $A\mathcal{H}(T,\mu)$ is a space of *entire functions* satisfying axioms 1–3. Consequently, it is a Branges space.

On the other hand, the converse is also true: every de Branges space can be represented in this form (see [6]). Note that μ is the Clark measure (spectral measure) of the corresponding space $\mathcal{H}(E)$.

This approach to de Branges spaces has made it possible to better understand their structure and to solve certain open problems (see [6, 2, 3, 4]). The reproducing kernel in

$\mathcal{H}(T,\mu)$ is given by the formula

$$k_w(z) = \sum_n \frac{\mu_n}{(\bar{w}-t_n)(z-t_n)}.$$

In particular,

$$\|k_w\|^2_{\mathcal{H}(T,\mu)} = k_w(w) = \sum_n \frac{\mu_n}{|w-t_n|^2}. \tag{5}$$

Thus, it suffices to prove Theorem 1.3 in the spaces $\mathcal{H}(T,\mu)$. The Bessel inequality (1) turns into a *weighted estimate for the Hilbert transformation* (with the weight $\|k_\lambda\|^2$).

To find a condition necessary and sufficient for the boundedness of a two-weight Hilbert transform is a celebrated problem that remained open during several decades. A complete solution was obtained quite recently by M. Lacey, E. Sawyer, C. Shen, and I. Uriarte-Tuero (see [10, 11]). However, the verification of the conditions obtained in [10] is not always easy. In the present paper, we are interested in the *special weight* $\sum_\lambda \|k_\lambda\|^{-2}_{\mathcal{H}(T,\mu)}\delta_\lambda$, which is easier to work with. Our proof of Theorem 1.3 does not involve the methods of [10].

In the sequel, all calculations will be done in the space $\mathcal{H}(T,\mu)$. Therefore, we omit the indices of the norm sign, etc.

§3. Proof of Theorem 1.3

We need an estimate of $\|k_\lambda\|$ in terms of $M_n^\pm(\lambda)$ and $P_n^\pm(\lambda)$.

Lemma 3.1. *Let $\lambda \in I_n^-$, then*

$$\|k_\lambda\|^2 \asymp \max\big(M_n^-(\lambda)|\lambda-t_n|^{-2}, P_n^-(\lambda)\big).$$

If $\lambda \in I_n^+$, then

$$\|k_\lambda\|^2 \asymp \max\big(M_n^+(\lambda)|\lambda-t_{n+1}|^2, P_n^+(\lambda)\big).$$

Proof. Formula (5) implies

$$\|k_\lambda\|^2 = \sum_l \frac{\mu_l}{|\lambda-t_l|^2} \asymp \sum_{|t_n-t_l|\le\lambda-t_n} + \sum_{|t_n-t_l|>\lambda-t_n} \asymp \frac{M_n^-(\lambda)}{|\lambda-t_n|^2} + P_n^-(\lambda).$$

The second statement is proved similarly. □

3.1. Sufficiency of conditions (4)**.** We prove that there is no Bessel sequence Λ with $\mathrm{UD}(\Lambda)=\infty$. Suppose that Λ is such a sequence. Put

$$\Lambda_M^- = \Lambda\cap\mathcal{R}_M^-, \quad \Lambda_M^+ = \Lambda\cap\mathcal{R}_M^+, \quad \Lambda_P^- = \Lambda\cap\mathcal{R}_P^-, \quad \Lambda_P^+ = \Lambda\cap\mathcal{R}_P^+.$$

Then the upper density of one of the four sequences $\Lambda^\pm_{M \text{ or } P}$ is infinite. We consider the cases of $\mathrm{UD}(\Lambda_M^-)=\infty$ and of $\mathrm{UD}(\Lambda_P^-)=\infty$. The two remaining cases are treated similarly.

1. Let $\mathrm{UD}(\Lambda_M^-)=\infty$. The sequence $\mathcal{A}_n^-$ has finite dyadic size uniformly bounded in n. Therefore, there exists a set $\widetilde{\Lambda}\subset\Lambda_M^-$ with

$$\mathrm{UD}(\widetilde{\Lambda})=\infty, \quad M_n^-(\lambda_1)\asymp M_n^-(\lambda_2), \quad \lambda_1,\lambda_2\in\widetilde{\Lambda}\cap I_n^-.$$

For every n, we put $\lambda^n_{\min} = \min(\widetilde{\Lambda}\cap I_n)$, $\lambda^n_{\max} = \max(\widetilde{\Lambda}\cap I_n)$, and $m_n = \min\{s : t_n - t_s \le \lambda^n_{\min} - t_n\}$.

Consider the functions g_n,

$$g_n(z) = \sum_{m_n\le k\le n} \frac{\mu_k}{z-t_k}.$$

It is easily seen that

$$\|g_n\|^2 = \sum_{m_n \le k \le n} \mu_k = M_n^-(\lambda_{\min}).$$

On the other hand, if $\lambda \in \widetilde{\Lambda} \cap I_n^-$, then $|g_n(\lambda)| \ge \sum \frac{\mu_k}{2|\lambda - t_n|} = \frac{M_n^-(\lambda_{\min})}{|\lambda - t_n|}$. Using Lemma 3.1 and the relation $\lambda \in \Lambda_M^-$, we obtain

$$\|k_\lambda\|^2 \asymp \frac{M_n^-(\lambda)}{|\lambda - t_n|^2}.$$

Thus,

$$\sum_{\lambda \in \widetilde{\Lambda}_n} \frac{|g_n(\lambda)|^2}{\|k_\lambda\|^2} \gtrsim \#(\widetilde{\Lambda} \cap I_n^-) \cdot \frac{[M_n^-(\lambda_{\min})]^2}{M_n^-(\lambda_{\max})} \gtrsim \#(\widetilde{\Lambda} \cap I_n^-) \cdot M_n^-(\lambda_{\min}).$$

Applying the Bessel inequality to the function g_n, we arrive at a contradiction.

2. Let $\mathrm{UD}(\Lambda_P^-) = \infty$. The sequence $\mathcal{B}_n^-$ has finite dyadic diameter uniformly bounded in n. Therefore, there exists a set $\widetilde{\Lambda} \subset \Lambda_P^-$ with

$$\mathrm{UD}(\widetilde{\Lambda}) = \infty, \quad P_n^-(\lambda_1) \asymp P_n^-(\lambda_2), \quad \lambda_1, \lambda_2 \in \widetilde{\Lambda} \cap I_n.$$

For every n, we put $\lambda_{\min}^n = \min(\widetilde{\Lambda} \cap I_n)$, $\lambda_{\max}^n = \max(\widetilde{\Lambda} \cap I_n)$, and $m_n = \max\{s : t_n - t_s > \lambda_{\max} - t_n\}$. Define

$$v_n(z) = \sum_{k \notin (m_n, n]} \frac{\mu_k}{(t_n - t_k)(z - t_k)}.$$

It is easily seen that

$$\|v_n\|^2 = P_n^-(\lambda_{\max}).$$

On the other hand, if $\lambda \in \widetilde{\Lambda} \cap I_n$, then $|v_n(\lambda)| \gtrsim P_n^-(\lambda)$ and

$$\|k_\lambda\|^2 \asymp P_n^-(\lambda).$$

Thus,

$$\sum_{\lambda \in \widetilde{\Lambda}_n} \frac{|v_n(\lambda)|^2}{\|k_\lambda\|^2} \gtrsim \#(\widetilde{\Lambda} \cap I_n^-) \cdot \frac{[P_n^-(\lambda_{\max})]^2}{P_n^-(\lambda_{\min})} \gtrsim \#(\widetilde{\Lambda} \cap I_n^-) \cdot P_n^-(\lambda_{\max}).$$

Applying the Bessel inequality to the functions v_n, we arrive at a contradiction.

3.2. Necessity of conditions (4)**.** We prove that if at least one among conditions (4) fails, then there exists a Bessel sequence with infinite density.

Let $\sup_n \mathcal{D}(\mathcal{A}_n^-) = \infty$. Then there exist sequences $\Lambda_n = \{\lambda_1^{(n)}, \dots, \lambda_{k_n}^{(n)}\} \subset \mathcal{R}_M^- \cap I_n$ (maybe, Λ_n is not defined for certain indices n) with

$$k_n \to \infty, \quad M_n^-(\lambda_{l+1}^{(n)}) \ge 2M_n^-(\lambda_l^{(n)}), \quad 0 < l < k_n. \tag{6}$$

Additionally, we require that $2t_k - t_l \notin \Lambda_n$ for every k, l, n. We prove that the sequence $\cup_{n \in \mathcal{I}} \Lambda_n$ is Bessel for a sufficiently sparse set $\mathcal{I}$ of indices. For this, it suffices to show that the sequences Λ_n are *uniformly Bessel* (i.e., C in inequality (1) does not depend on n).

Indeed, suppose that the sequences $\Lambda_n \subset I_n$ are uniformly Bessel. We assume that the intervals $I_n = [t_{s_n}, t_{s_n+1}] \subset \mathbb{R}^+$ are such that $t_{s_{n+1}} > 10t_{s_n+1}$ (below, several other "sparseness" conditions will be imposed on the intervals I_n). Assume that $\widetilde{I}_n \supset 1$ and the intervals $5I_n$ are mutually disjoint.

Put $f(z) = \sum_l \frac{a_l \mu_l^{1/2}}{z - t_l}$ and write

$$f(z) = \sum_{t_l \in \widetilde{I}_n} \frac{a_l \mu_l^{1/2}}{z - t_l} + \sum_{t_l \notin \widetilde{I}_n} \frac{a_l \mu_l^{1/2}}{z - t_l} =: f_n + g_n.$$

Then

$$\sum_{\lambda \in \Lambda_n} \frac{|f(\lambda)|^2}{\|k_\lambda\|^2} \le 2 \sum_{\lambda \in \Lambda_n} \frac{|f_n(\lambda)|^2}{\|k_\lambda\|^2} + 2 \sum_{\lambda \in \Lambda_n} \frac{|g_n(\lambda)|^2}{\|k_\lambda\|^2}$$

$$\lesssim \sum_{t_l \in \widetilde{I}_n} |a_l|^2 + \Big[\sum_l |a_l|^2\Big] \sum_{\lambda \in \Lambda_n} \left(\frac{\mu(x : |x| \le \min \widetilde{I}_n)}{|\lambda|^2} + \sum_{\substack{m \,:\, t_m > \max \widetilde{I}_n \\ \text{or } t_m < -\min \widetilde{I}_n}} \frac{\mu_m}{t_m^2} \right) \|k_\lambda\|^{-2}. \tag{7}$$

We set

$$e_\lambda := \left(\frac{\mu(x : |x| \le \min \widetilde{I}_n)}{|\lambda|^2} + \sum_{\substack{m \,:\, t_m > \max \widetilde{I}_n \\ \text{or } t_m < -\min \widetilde{I}_n}} \frac{\mu_m}{t_m^2} \right) \|k_\lambda\|^{-2}.$$

It can easily be verified that we can find intervals $\widetilde{I}_n$ and a sequence $\Lambda_1 \subset \bigcup_n \Lambda_n$ such that $\mathrm{UD}(\Lambda_1) = \infty$ and $\sum_n \sum_{\lambda \in \Lambda_1 \cap I_n} e_\lambda < \infty$. Adding inequalities (7), we see that Λ_1 is a Bessel sequence.

It remains to show that the sequences Λ_n are uniformly Bessel. Let $f \in \mathcal{H}(T, \mu)$. This means that $f(z) = \sum_l \frac{a_l \mu_l^{1/2}}{z - t_l}$, $\|f\|^2 = \sum_l |a_l|^2$. We verify the Bessel inequality for f and the sequence Λ_n. First, we estimate the quantity $|f(\lambda_s)|^2 \|k_{\lambda_s}\|^{-2}$, $1 \le s \le k_n$:

$$\frac{|f(\lambda_s)|^2}{\|k_{\lambda_s}\|^2} \lesssim \left[\sum_{2t_n - \lambda_s \le t_l \le t_n} |a_l| \mu_l^{1/2} \right]^2 \frac{1}{M_n^-(\lambda_s)} + \left[\sum_{2t_n - \lambda_s \le t_l \le t_n} |a_l| \frac{\mu_l^{1/2}}{|t_n - t_l|} \right]^2 \frac{|\lambda_s - t_n|^2}{M_n^-(\lambda_s)}.$$

It remains to prove that

$$\sum_{s=1}^{k_n - 1} \frac{1}{M_n^-(\lambda_s)} \left[\sum_{2t_n - \lambda_s \le t_l \le t_n} |a_l| \mu_l^{1/2} \right]^2 \lesssim \sum_n |a_n|^2 \tag{8}$$

and

$$\sum_{s=1}^{k_n - 1} \frac{|\lambda_s - t_n|^2}{M_n^-(\lambda_s)} \left[\sum_{2t_n - \lambda_s \le t_l \le t_n} |a_l| \frac{\mu_l^{1/2}}{|t_n - t_l|} \right]^2 \lesssim \sum_n |a_n|^2. \tag{9}$$

To prove inequality (8), we use duality:

$$\sum_{s=1}^{k_n - 1} \frac{1}{M_n^-(\lambda_s)} \left[\sum_{2t_n - \lambda_s \le t_l \le t_n} |a_l| \mu_l^{1/2} \right]^2 = \sup_{\|c_s\|_2 = 1} \sum_{s=1}^{k_n - 1} \sum_{2t_n - \lambda_s \le t_l \le t_n} a_l c_s [M_n^-(\lambda_s)]^{-1/2} \mu_l^{1/2}.$$

It suffices to show that the norm of the sequence

$$\tau_l = \mu_l^{1/2} \sum_{s \,:\, |t_n - t_l| \le \lambda_s - t_n} c_s [M_n^-(\lambda_s)]^{-1/2}$$

is bounded above by a universal constant. Indeed,

$$|\tau_l|^2 \lesssim \mu_l \sum_{s \,:\, |t_n - t_l| \le \lambda_s - t_n} |c_s|^2 [M_n^-(\lambda_s)]^{-1/2} \cdot \sum_{s \,:\, |t_n - t_l| \le \lambda_s - t_n} [M_n^-(\lambda_s)]^{-1/2}.$$

We know that

$$\sum_{s\,:\,|t_n-t_l|\le\lambda_s-t_n}[M_n^-(\lambda_s)]^{-1/2}\asymp[M_n^-(\lambda_{l^*})]^{-1/2},$$

where l^* is the minimal index with $|t_n-t_l|\le\lambda_{l^*}-t_n$. Next,

$$\begin{aligned}\sum_l|\tau_l|^2&\lesssim\sum_l\sum_{s\,:\,|t_n-t_l|\le\lambda_s-t_n}|c_s|^2[M_n^-(\lambda_s)]^{-1/2}\mu_l[M_n^-(\lambda_{l^*})]^{-1/2}\\&=\sum_s|c_s|^2[M_n^-(\lambda_s)]^{-1/2}\sum_{2t_n-\lambda_s\le t_l\le t_n}\mu_l[M_n^-(\lambda_{l^*})]^{-1/2}.\end{aligned}$$

Observe that

$$[M_n^-(\lambda_s)]^{-1/2}\sum_{2t_n-\lambda_s\le t_l\le t_n}\mu_l[M_n^-(\lambda_{l^*})]^{-1/2}\le[M_n^-(\lambda_s)]^{-1/2}\int_0^{M_n^-(\lambda_s)}x^{-1/2}\,dx=2.$$

This completes the proof of (8).

We verify inequality (9). First, we show that the sequence $P_n^-(\lambda_s)$ decays as a geometric progression:

$$\begin{aligned}P_n^-(\lambda_s)&=\sum_{l\,:\,\lambda_s-t_n<|t_n-t_l|\le\lambda_{s+1}-t_n}\frac{\mu_l}{|t_n-t_l|^2}+P_n^-(\lambda_{s+1})\\&\ge\frac{M_n^-(\lambda_{s+1})-M_n^-(\lambda_s)}{|t_n-\lambda_{s+1}|^2}+P_n^-(\lambda_{s+1})\ge\frac{3}{2}P_n^-(\lambda_{s+1}).\end{aligned}$$

Observe that

$$\sum_{s=1}^{k_n-1}\frac{|\lambda_s-t_n|^2}{M_n^-(\lambda_s)}\Bigg[\sum_{2t_n-\lambda_s\le t_l\le t_n}|a_l|\frac{\mu_l^{1/2}}{|t_n-t_l|}\Bigg]^2\le\sum_{s=1}^{k_n-1}P^-(\lambda_s)\Bigg[\sum_{2t_n-\lambda_s\le t_l\le t_n}|a_l|\frac{\mu_l^{1/2}}{|t_n-t_l|}\Bigg]^2.$$

The last sum is estimated in the same way as in the proof of inequality (8).

The cases of $\sup_n\mathcal{D}(\mathcal{B}_n^\pm)=\infty$ and of $\sup_n\mathcal{D}(\mathcal{A}_n^+)=\infty$ are treated similarly. This finishes the proof of Theorem 1.3.

3.3. Concluding remarks. By the methods of this paper, a quantitative analog of Theorem 1.3 can be proved.

Theorem 3.1. *Let $\{V_n\}_{n>0}$ be a positive monotone increasing sequence, and let*

$$\mathcal{D}(\mathcal{A}_n^\pm)\le V_n,\quad \mathcal{D}(\mathcal{B}_n^\pm)\le V_n.$$

Then

$$\#(\Lambda\cap I_n)\lesssim V_n$$

for every Bessel sequence $\Lambda\subset\mathbb{R}$.

To verify this, it suffices to repeat the estimates of §4 (Subsections 1 and 2).

Let μ be an arbitrary Clark measure for $\mathcal{H}(E)$, $\mu=\sum_n\mu_n\delta_{t_n}$. Put $\widetilde{\mu}=\sum_{k\ge0}v_k\delta_{2^k}$, where

$$v_0=\mu([-1,1)),\quad v_k=\mu([2^{k-1},2^k))+\mu([-2^{|k|},2^{-|k|-1})),\quad k>0.$$

Theorem 3.2. *Let $\mathcal{H}(E)$ be an arbitrary de Branges space with spectral measure μ. A sequence $\Lambda\subset i\mathbb{R}$ is Bessel in $\mathcal{H}(E)$ if and only if it is Bessel in the de Branges space $\mathcal{H}(E_1)$ constructed by $\widetilde{\mu}$.*

This can be proved by an argument similar to that in §2 of the present paper or to that in §3 of [6].

Note that the support of $\widetilde{\mu}$ is lacunary, and all Bessel sequences in the corresponding space were described in [6].

References

[1] A. D. Baranov, *Stability of bases and frames of reproducing kernels in model subspaces*, Ann. Inst. Fourier (Grenoble) **55** (2005), no. 7, 2399–2422. MR2207388

[2] A. D. Baranov and Yu. Belov, *Systems of reproducing kernels and their biorthogonal: completeness or incompleteness?*, Int. Math. Res. Not. IMRN **2011**, no. 22, 5076–5108. MR2854723

[3] A. D. Baranov, Yu. Belov, and A. A. Borichev, *Hereditary completeness for systems of exponentials and reproducing kernels*, Adv. Math. **235** (2013), 525–554. MR3010068

[4] ———, *Spectral synthesis in de Branges spaces*, Geom. Funct. Anal. **25** (2015), no. 2, 417–452. MR3334232

[5] Yu. Belov, T. Mengestie, and K. Seip, *Unitary discrete Hilbert transforms*, J. Anal. Math. **112** (2010), 383–395. MR2763007

[6] ———, *Discrete Hilbert transforms on sparse sequences*, Proc. London Math. Soc. (3) **103** (2011), no. 1, 73–105. MR2812502

[7] L. de Branges, *Hilbert spaces of entire functions*, Prentice-Hall, Englewood Cliffs, NJ, 1968. MR0229011

[8] D. N. Clark, *One dimensional perturbations of restricted shifts*, J. Anal. Math. **25** (1972), 169–191. MR0301534

[9] W. Cohn, *Carleson measures for functions orthogonal to invariant subspaces*, Pacific J. Math. **103** (1982), no. 2, 347–364. MR705235

[10] M. Lacey, E. Sawyer, C. Shen, and I. Uriarte-Tuero, *Two-weight inequality for the Hilbert transform: A real variable characterization*. I, Duke Math. J. **163** (2014), no. 15, 2795–2820. MR3285857

[11] M. Lacey, *Two-weight inequality for the Hilbert transform: A real variable characterization*. II, Duke Math. J. **163** (2014), no. 15, 2821–2840. MR3285858

[12] N. Makarov and A. Poltoratski, *Meromorphic inner functions, Toeplitz kernels and the uncertainty principle*, Perspectives in Analysis, Math. Phys. Stud., vol. 27, Springer, Berlin, 2005, pp. 185–252. MR2215727

[13] J. Ortega-Cerdà and K. Seip, *Fourier frames*, Ann. of Math. (2) **155** (2002), no. 3, 789–806. MR1923965

[14] A. Poltoratski, *Toeplitz approach to problems of the uncertainty principle*, CBMS Reg. Conf. Ser. Math., vol. 121, Amer. Math. Soc., Providence, RI, 2015. MR3309830

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