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ON CONES TANGENT TO SCHUBERT VARIETIES OF TYPE D_n

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Abstract. It is proved that the tangent cones to Schubert subvarieties of the flag variety of a reductive group with root system of type D_n do not coincide if they correspond to different basic involutions in the Weyl group.

§1. Introduction and the main results

1.1. Let G be a complex reductive algebraic group, T a maximal torus in G, B a Borel subgroup in G containing T, and U the unipotent radical of B. Let Φ be the root system of G with respect to T, Φ^+ the set of positive roots with respect to B, Δ the set of simple roots, and W the Weyl group of Φ (see [Bo, Hu1, Hu2] for the basic facts about algebraic groups and root systems).

We denote by $\mathcal{F} = G/B$ the flag variety and by $X_w \subseteq \mathcal{F}$ the Schubert subvariety corresponding to an element w of the Weyl group W. Let $\mathcal{O} = \mathcal{O}_{p,X_w}$ denote the local ring at the point $p = eB \in X_w$, and let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$. The sequence of ideals

$$\mathcal{O} \supseteq \mathfrak{m} \supseteq \mathfrak{m}^2 \supseteq \dots$$

is a filtration on $\mathcal{O}$. We define R to be the graded algebra

$$R = \operatorname{gr} \mathcal{O} = \bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}.$$

By definition, the *tangent cone* C_w to the Schubert variety X_w at the point p is the spectrum of R: $C_w = \operatorname{Spec} R$. Obviously, C_w is a subscheme of the tangent space $T_pX_w \subseteq T_p\mathcal{F}$. A hard problem in studying the geometry of X_w is to describe C_w, see [BL, Chapter 7].

In 2011, Eliseev and Panov [EP] computed the tangent cones C_w for all $w \in W$ in the case where $G = \mathrm{SL}_n(\mathbb{C})$, $n \leq 5$. Using these computations, Panov formulated the following conjecture.

Conjecture 1.1 (A. N. Panov, 2011). *Let w_1, w_2 be involutions, i.e., $w_1^2 = w_2^2 = \mathrm{id}$. If $w_1 \neq w_2$, then $C_{w_1} \neq C_{w_2}$ as subschemes of $T_p\mathcal{F}$.*

One can easily check that it suffices to prove Conjecture 1.1 for irreducible root systems (see Remark 1.6 below). In 2013, Eliseev and the first author proved this conjecture in the types A_n, F_4, and G_2, see [EI]. In [BIS], Bochkarev and the authors proved the conjecture in the case of types B_n and C_n. In this paper, we prove that Conjecture 1.1 is true if Φ is of type D_n and w_1, w_2 are basic involutions (see Definition 2.3). Precisely, our first main result is as follows.

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Theorem 1.2. *Assume that every irreducible component of Φ is of type D_n, $n \geq 4$. Suppose that w_1, w_2 are basic involutions in the Weyl group of Φ and $w_1 \neq w_2$. Then the tangent cones C_{w_1} and C_{w_2} do not coincide as subschemes of $T_p\mathcal{F}$.*

Note that a similar question for other involutions in D_n and for the root systems E_6, E_7, E_8 remains open.

Now, let $\mathcal{A}$ be the symmetric algebra of the vector space $\mathfrak{m}/\mathfrak{m}^2$, or, equivalently, the algebra of regular functions on the tangent space T_pX_w. Since R is generated as a $\mathbb{C}$-algebra by $\mathfrak{m}/\mathfrak{m}^2$, it is a quotient ring $R = \mathcal{A}/I$. By definition, the *reduced tangent cone* C_w^{red} to X_w at the point p is the common zero locus in T_pX_w of the polynomials $f \in I \subseteq \mathcal{A}$. Clearly, if $C_{w_1}^{\text{red}} \neq C_{w_2}^{\text{red}}$, then $C_{w_1} \neq C_{w_2}$. Our second main result is as follows.

Theorem 1.3. *Assume that every irreducible component of Φ is of type D_n, $n \geq 4$. Let w_1, w_2 be basic involutions in the Weyl group of Φ, and let $w_1 \neq w_2$. Then the reduced tangent cones $C_{w_1}^{\text{red}}$ and $C_{w_2}^{\text{red}}$ do not coincide as subvarieties of $T_p\mathcal{F}$.*

In [BIS], a similar result was obtained by Bochkarev for root systems of types A_n and C_n. Our proof for D_n is based on a similar idea. Note that a similar question for other involutions in D_n and for other root systems remains open.

The paper is organized as follows. In the next subsection, we introduce the main technical tool used in the proof of Theorem 1.2. Namely, with each element $w \in W$ one can associate a polynomial d_w in the algebra of regular functions on the Lie algebra of the maximal torus T. These polynomials are called the Kostant–Kumar polynomials, see [KK1, KK2, Ku, Bi]. In [Ku], Kumar showed that if w_1 and w_2 are arbitrary elements of W and $d_{w_1} \neq d_{w_2}$, then $C_{w_1} \neq C_{w_2}$. We give three equivalent definitions of the Kostant–Kumar polynomials and formulate their properties needed for what follows.

In §2 we prove that if all irreducible components of Φ are of type D_n and w_1, w_2 are distinct basic involutions in W, then $d_{w_1} \neq d_{w_2}$, see Proposition 2.8. This implies that $C_{w_1} \neq C_{w_2}$, proving Theorem 1.2. The proofs of Conjecture 1.1 for A_n, F_4, G_2, B_n and C_n presented in [EI] and [BIS] are based on similar arguments.

§3 contains the proof of Theorem 1.3. Namely, in Subsection 3.1 we describe the relationship between the geometry of tangent cones and that of coadjoint B-orbits. Using this relationship, in Subsection 3.2 we prove the result.

Of course, Theorem 1.2 is a consequence of Theorem 1.3. Nevertheless, in §2 we give an independent proof of the former theorem, based on computation of Kostant–Kumar polynomials. The reason is that we hope to prove Theorem 1.2 for all involutions in D_n by using the same technique. On the contrary, there is no chance to prove Theorem 1.3 for nonbasic involutions in D_n with the help of arguments similar to those presented in §3, see Remark 3.3 (ii) for the details.

1.2. Let w be an element of the Weyl group W. Here we give the precise definition of the Kostant–Kumar polynomial d_w, explain how to compute it in combinatorial terms, and show that it depends only on the scheme structure of C_w.

The torus T acts on the Schubert variety X_w by left multiplications (or, equivalently, by conjugations). Since the point p is invariant under this action, on the local ring $\mathcal{O}$ we have the structure of a T-module. The action of T on $\mathcal{O}$ preserves the filtration by powers of the ideal $\mathfrak{m}$, so that we obtain the structure of a T-module on the algebra $R = \operatorname{gr}\mathcal{O}$. By [Ku, Theorem 2.2], R can be decomposed into a direct sum of its finite-dimensional weight subspaces:

$$R = \bigoplus_{\lambda \in \mathfrak{X}(T)} R_\lambda.$$

Here $\mathfrak{h}$ is the Lie algebra of the torus T, $\mathfrak{X}(T) \subseteq \mathfrak{h}^*$ is the character lattice of T, and $R_\lambda = \{f \in R \mid t.f = \lambda(t)f\}$ is the weight subspace of weight λ. Let Λ be the $\mathbb{Z}$-module consisting of all (possibly infinite) $\mathbb{Z}$-linear combinations of linearly independent elements e^λ, $\lambda \in \mathfrak{X}(T)$. The *formal character* of R is an element of Λ of the form

$$\operatorname{ch} R = \sum_{\lambda \in \mathfrak{X}(T)} m_\lambda e^\lambda,$$

where $m_\lambda = \dim R_\lambda$.

Now, pick an element $a = \sum_{\lambda \in \mathfrak{X}(T)} n_\lambda e^\lambda \in \Lambda$. Assume that there are only finitely many $\lambda \in \mathfrak{X}(T)$ such that $n_\lambda \neq 0$. Given $k \geq 0$, we define the polynomial

$$[a]_k = \sum_{\lambda \in \mathfrak{X}(T)} n_\lambda \cdot \frac{\lambda^k}{k!} \in S = \mathbb{C}[\mathfrak{h}].$$

Denote $[a] = [a]_{k_0}$, where k_0 is minimal among all nonnegative numbers k such that $[a]_k \neq 0$. For instance, if $a = 1 - e^\lambda$, then $[a]_0 = 0$ and $[a] = [a]_1 = -\lambda$ (here we denote $1 = e^0$).

Let A be the submodule of Λ consisting of all finite linear combinations. It is a commutative ring with respect to the multiplication $e^\lambda \cdot e^\mu = e^{\lambda+\mu}$. In fact, it is simply the group ring of $\mathfrak{X}(T)$. Denote the field of fractions of the ring A by $Q \subseteq \Lambda$. With each element of Q of the form $q = a/b$ with $a, b \in A$, we can associate the element

$$[q] = \frac{[a]}{[b]} \in \mathbb{C}(\mathfrak{h})$$

of the field of rational functions on $\mathfrak{h}$. Note that this element is well defined, see [Ku].

There exists an involution $q \mapsto q^*$ on Q defined by

$$e^\lambda \mapsto (e^\lambda)^* = e^{-\lambda}.$$

It turns out (see [Ku, Theorem 2.2]) that the character $\operatorname{ch} R$ belongs to Q, whence we also have $(\operatorname{ch} R)^* \in Q$. Finally, we put

$$c_w = [(\operatorname{ch} R)^*],\ d_w = (-1)^{l(w)} \cdot c_w \cdot \prod_{\alpha \in \Phi^+} \alpha.$$

Here $l(w)$ is the length of w in the Weyl group W with respect to the set of simple roots Δ. Evidently, c_w and d_w belong to $\mathbb{C}(\mathfrak{h})$; in fact, d_w is a polynomial, i.e., it belongs to the algebra $S = \mathbb{C}[\mathfrak{h}]$ of regular functions on $\mathfrak{h}$, see [KK2] and [BL, Theorem 7.2.6].

Definition 1.4. Let w be an element of the Weyl group W. The polynomial $d_w \in S$ is called the *Kostant–Kumar polynomial* associated with w.

This definition shows that c_w and d_w depend only on the canonical structure of a T-module on the algebra R of regular functions on the tangent cone C_w. Thus, to prove that the tangent cones corresponding to elements w_1, w_2 of the Weyl group are distinct, it suffices to check that $c_{w_1} \neq c_{w_2}$, or equivalently, $d_{w_1} \neq d_{w_2}$.

On the other hand, there is a purely combinatorial description of the Kostant–Kumar polynomials. To give this description, we need some more notation. Let w, v be elements of W. Fix a reduced decomposition of the element $w = s_{i_1} \dots s_{i_l}$. (Here $\alpha_1, \dots, \alpha_n \in \Delta$ are simple roots and s_i is the simple reflection corresponding to α_i.) Put

$$c_{w,v} = (-1)^{l(w)} \cdot \sum \frac{1}{s_{i_1}^{\epsilon_1} \alpha_{i_1}} \cdot \frac{1}{s_{i_1}^{\epsilon_1} s_{i_2}^{\epsilon_2} \alpha_{i_2}} \cdot \ldots \cdot \frac{1}{s_{i_1}^{\epsilon_1} \dots s_{i_l}^{\epsilon_l} \alpha_{i_l}},$$

where the sum is taken over all sequences $(\epsilon_1, \dots, \epsilon_l)$ of zeros and units such that $s_{i_1}^{\epsilon_1} \dots s_{i_l}^{\epsilon_l} = v$. Actually, the element $c_{w,v} \in \mathbb{C}(\mathfrak{h})$ depends only on w and v, but not on the choice of a reduced decomposition of w; see [Ku, §3].

Example 1.5. Let $\Phi = A_n$. Put $w = s_1s_2s_1$. To compute $c_{w,\mathrm{id}}$, we should sum over two sequences, $(0,0,0)$ and $(1,0,1)$. Hence,

$$c_{w,\mathrm{id}} = (-1)^3 \cdot \left(\frac{1}{\alpha_1\alpha_2\alpha_1} + \frac{1}{-\alpha_1(\alpha_1+\alpha_2)\alpha_1}\right) = \frac{1}{\alpha_1\alpha_2(\alpha_1+\alpha_2)}.$$

A remarkable fact is that $c_{w,\mathrm{id}} = c_w$; hence, to prove that tangent cones to Schubert varieties do not coincide as subschemes, we only need combinatorics of the Weyl group. Note also that for the classical Weyl groups, the elements $c_{w,v}$ are closely related to Schubert polynomials [Bi].

Finally, we present an original definition of the elements $c_{w,v}$, using the so-called nil-Hecke ring (see [Ku] and [BL, 7.1]). The group W acts naturally on $\mathbb{C}(\mathfrak{h})$ by automorphisms. Denote by Q_W the vector space over $\mathbb{C}(\mathfrak{h})$ with the basis $\{\delta_w,\ w \in W\}$. It is a ring with respect to the multiplication

$$f\delta_v \cdot g\delta_w = fv(g)\delta_{vw}.$$

This ring is called the *nil-Hecke ring.* For each i from 1 to n, put

$$x_i = \alpha_i^{-1}(\delta_{s_i} - \delta_{\mathrm{id}}).$$

Let $w \in W$, and let $w = s_{i_1} \dots s_{i_l}$ be a reduced decomposition of w. Then the element

$$x_w = x_{i_1} \dots x_{i_l}$$

does not depend on the choice of a reduced decomposition of w, see [KK1, Proposition 2.1].

Moreover, it turns out that $\{x_w,\ w \in W\}$ is a $\mathbb{C}(\mathfrak{h})$-basis of Q_W, see [KK1, Proposition 2.2], and

$$x_w = \sum\nolimits_{v\in W} c_{w,v}\delta_v.$$

Actually, if $w, v \in W$, then

$$(1)\quad \begin{aligned} &\text{a)}\quad x_v \cdot x_w = \begin{cases} x_{vw} & \text{if } l(vw) = l(v) + l(w), \\ 0 & \text{otherwise,} \end{cases} \\ &\text{b)}\quad c_{w,v} = -v(\alpha_i)^{-1}(c_{ws_i,v} + c_{ws_i,vs_i}) \ \text{ if } \ l(ws_i) = l(w) - 1, \\ &\text{c)}\quad c_{w,v} = \alpha_i^{-1}(s_i(c_{s_iw,s_iv}) - c_{s_iw,v}) \ \text{ if } \ l(s_iw) = l(w) - 1. \end{aligned}$$

The first property was proved in [KK1, Proposition 2.2]. The second and third properties follow immediately from the first and the definitions (see also the proof of Corollary 3.2 in [Ku]).

Remark 1.6. i) Suppose Φ is the union of subsystems Φ_1 and Φ_2 contained in mutually orthogonal subspaces. Let W_1, W_2 be the Weyl groups of Φ_1, Φ_2, respectively, so that $W = W_1 \times W_2$. Denote $\Delta_1 = \Delta \cap \Phi_1 = \{\alpha_1, \dots, \alpha_r\}$ and $\Delta_2 = \Delta \cap \Phi_2 = \{\beta_1, \dots, \beta_s\}$; then

$$\mathbb{C}[\mathfrak{h}] \cong \mathbb{C}[\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s].$$

Given $v \in W_1$, we denote by d_v^1 its Kostant–Kumar polynomial. We can view d_v^1 as an element of $\mathbb{C}(\mathfrak{h})$ depending only on $\alpha_1, \dots, \alpha_r$. We define $c_v^1 \in \mathbb{C}(\mathfrak{h})$ similarly. Given $v \in W_2$, we define $d_v^2 \in \mathbb{C}[\mathfrak{h}]$ and $c_v^2 \in \mathbb{C}(\mathfrak{h})$; they depend only on $\beta_1, \dots, \beta_s$. Suppose $w \in W$, $w_1 \in W_1$, $w_2 \in W_2$, and $w = w_1w_2$. Repeating the proof of Proposition 1.6 in [EI] word for word, we obtain

$$d_w = d_{w_1}^1 d_{w_2}^2,\ c_w = c_{w_1}^1 c_{w_2}^2.$$

Thus, to prove the claim of Theorem 1.2 it suffices to check it in the case of irreducible root systems of type D_n, because $\mathbb{C}[\mathfrak{h}]$ is a unique factorization domain.

ii) Now, suppose that $G \cong G_1 \times G_2$, where G_1, G_2 are reductive subgroups of G, $T_i = T \cap G_i$ is a maximal torus in G_i, $i = 1, 2$, and the root system of G_i with respect to T_i is isomorphic to Φ_i. Then $B_i = B \cap G_i$ is a Borel subgroup in G_i containing T_i. Denote by $\mathcal{F}_i = G_i/B_i$ the corresponding flag variety. Then $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ and $T_p\mathcal{F} = T_p\mathcal{F}_1 \times T_p\mathcal{F}_2$ as algebraic varieties. If $w \in W$ and $w = w_1w_2$, $w_i \in W_i$, $i = 1, 2$, then $C_w^{\mathrm{red}} \cong C_{w_1,G_1}^{\mathrm{red}} \times C_{w_2,G_2}^{\mathrm{red}}$ as affine varieties. Here $C_{w_i,G_i}^{\mathrm{red}}$, $i = 1, 2$, denotes the reduced tangent cone to the Schubert subvariety X_{w_i} of the flag variety $\mathcal{F}_i$. Furthermore, note that w is an involution if and only if so are w_1 and w_2. This means that it suffices to prove that Theorem 1.3 is valid for all irreducible root systems of type D_n.

§2. Nonreduced tangent cones

2.1. Throughout this section, Φ denotes an irreducible root system of type D_n, $n \geq 4$. In this subsection, we briefly recall some facts about Φ. Let $\epsilon_1, \ldots, \epsilon_n$ be the standard basis of the Euclidean space $\mathbb{R}^n$. As usual, we identify the set Φ^+ of positive roots with the following subset of $\mathbb{R}^n$:

$$D_n^+ = \{\epsilon_i - \epsilon_j,\, \epsilon_i + \epsilon_j,\, 1 \leq i < j \leq n\},$$

so W can be viewed as a subgroup of the orthogonal group $O(\mathbb{R}^n)$.

Let $S_{\pm n}$ denote the symmetric group on $2n$ letters $1, \ldots, n, -n, \ldots, -1$. The Weyl group W is isomorphic to the *even-signed hyperoctahedral group*, that is, the subgroup of $S_{\pm n}$ formed by the permutations $w \in S_{\pm n}$ such that $w(-i) = -w(i)$ for all $1 \leq i \leq n$, and $\#\{i > 0 \mid w(i) < 0\}$ is even. The isomorphism is given by

$$s_{\epsilon_i - \epsilon_j} \mapsto (i, j)(-i, -j),$$
$$s_{\epsilon_i + \epsilon_j} \mapsto (i, -j)(-i, j).$$

Here s_α is the reflection in the hyperplane orthogonal to a root α. In the sequel, we shall identify W with the even-signed hyperoctahedral group.

Remark 2.1. i) Note that every $w \in W$ is completely determined by its restriction to the subset $\{1, \ldots, n\}$. This allows us to employ the usual two-line notation: if $w(i) = w_i$ for $1 \leq i \leq n$, then we write $w = \begin{pmatrix} 1 & 2 & \ldots & n \\ w_1 & w_2 & \ldots & w_n \end{pmatrix}$. For instance, if $\Phi = D_5$, then

$$s_{\epsilon_1+\epsilon_5} s_{\epsilon_2+\epsilon_4} s_{\epsilon_2-\epsilon_4} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -5 & -2 & 3 & -4 & -1 \end{pmatrix}.$$

ii) Note also that the set of simple roots has the following form: $\Delta = \{\alpha_1, \ldots, \alpha_n\}$, where $\alpha_1 = \epsilon_1 - \epsilon_2, \ldots, \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n$, and $\alpha_n = \epsilon_{n-1} + \epsilon_n$.

We say that v is less than or equal to w with respect to the *Bruhat order*, and write $v \leq w$, if some reduced decomposition for v is a subword of some reduced decomposition for w. It is well known that this order plays the crucial role in many geometric aspects of the theory of algebraic groups. For instance, the Bruhat order encodes the incidences among Schubert varieties, i.e., X_v is contained in X_w if and only if $v \leq w$. It turns out that $c_{w,v}$ is nonzero if and only if $v \leq w$, see [Ku, Corollary 3.2]. For example, $c_w = c_{w,\mathrm{id}}$ is nonzero for *any* w, because id is the smallest element of W with respect to the Bruhat order. Note that for any v, $w \in W$ there exists $g_{w,v} \in S = \mathbb{C}[\mathfrak{h}]$ such that

$$c_{w,v} = g_{w,v} \cdot \prod_{\alpha>0,\, s_\alpha v \leq w} \alpha^{-1}, \tag{2}$$

see [Dy] and [BL, Theorem 7.1.11].

There exists a nice combinatorial description of the Bruhat order on the even-signed hyperoctahedral group. Namely, given $w \in W$, denote by X_w the $(2n \times 2n)$-matrix of the form

$$(X_w)_{i,j} = \begin{cases} 1 & \text{if } \ w(j) = i, \\ 0 & \text{otherwise.} \end{cases}$$

The rows and columns of this matrix are indexed by the numbers $1, \ldots, n, -n, \ldots, 1$. It is called a 0–1 matrix, a permutation matrix, or a rook placement for w. We define a matrix R_w by putting its (i, j)th entry to be equal to the rank of the lower left $((n-i+1) \times j)$-submatrix of X_w. In other words, $(R_w)_{i,j}$ is the number or rooks located nonstrictly to the South-West from (i, j).

Example 2.2. For $n = 4$, let $w = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -2 & 4 & 1 & -3 \end{pmatrix}$. The matrices X_w and R_w look like this (rooks are marked by $\otimes$):

$X_w =$

	1	2	3	4	−4	−3	−2	−1
1			⊗					
2								⊗
3					⊗			
4		⊗						
−4							⊗	
−3				⊗				
−2	⊗							
−1						⊗		

, $R_w =$

	1	2	3	4	−4	−3	−2	−1
1	1	2	3	4	5	6	7	8
2	1	2	2	3	4	5	6	7
3	1	2	2	3	4	5	6	6
4	1	2	2	3	3	4	5	5
−4	1	1	1	2	2	3	4	4
−3	1	1	1	2	2	3	3	3
−2	1	1	1	1	1	2	2	2
−1	0	0	0	0	0	1	1	1

.

Let $w \in W$. Given $a, b \in \{1, 2, \ldots, n\}$, we say that $[-a, a] \times [-b, b]$ is an *empty rectangle* for w if

$$\{i \in [\pm n] \mid |i| \geq b \text{ and } |w(i)| \geq a\} = \varnothing.$$

Here $[\pm n] = \{1, \ldots, n, -n, \ldots, -1\}$. For instance, in the example above, $[-4, 4] \times [-3, 3]$ and $[-4, 4] \times [-4, 4]$ are empty rectangles for w. Let X and Y be matrices with integral entries. We say that $X \leq Y$ if $X_{i,j} \leq Y_{i,j}$ for all i, j. It turns out that given $v, w \in W$, we have $v \leq w$ if and only if

(3)
i) $R_v \leq R_w$;
ii) for all $a, b \in \{1, \ldots, n\}$, if $[-a, a] \times [-b, b]$ is an empty rectangle for both v and w and $(R_v)_{-(a-1),b-1} = (R_w)_{-(a-1),b-1}$, then $(R_v)_{-(a-1),n} \equiv (R_w)_{-(a-1),n} (\mathrm{mod} 2)$.

(See, e.g., [BB, Theorem 8.2.8].)

2.2. In this subsection, we introduce some more notation and prove a technical, but crucial statement, Lemma 2.7. We introduce the maps row: $\Phi^+ \to \mathbb{Z}$ and col: $\Phi^+ \to \mathbb{Z}$ by

$$\mathrm{row}(\epsilon_i - \epsilon_j) = j, \quad \mathrm{row}(\epsilon_i + \epsilon_j) = -j,$$
$$\mathrm{col}(\epsilon_i - \epsilon_j) = \mathrm{col}(\epsilon_i + \epsilon_j) = i.$$

For any $k \in [\pm n]$, put

$$\mathcal{R}_k = \{\alpha \in \Phi^+ \mid \mathrm{row}(\alpha) = k\},$$
$$\mathcal{C}_k = \{\alpha \in \Phi^+ \mid \mathrm{col}(\alpha) = k\}.$$

The set $\mathcal{R}_k$ (respectively, $\mathcal{C}_k$) is called the kth *row* (respectively, the kth *column*) of Φ^+.

Definition 2.3. An involution $w \in W$ is said to be *basic* if

$$\{i \in \{1, \ldots, n\} \mid w(i) = -i\} = \varnothing.$$

Definition 2.4. Let $\sigma \in W$ be a basic involution. We define the *support* $\mathrm{Supp}(\sigma)$ of the involution σ by the following rule:

$$\begin{aligned} &\text{if } \ 1 \leq i < j \leq n \ \text{ and } \ \sigma(i) = j, \ \text{ then } \ \epsilon_i - \epsilon_j \in \mathrm{Supp}(\sigma), \\ &\text{if } \ 1 \leq i < j \leq n \ \text{ and } \ \sigma(i) = -j, \ \text{ then } \ \epsilon_i + \epsilon_j \in \mathrm{Supp}(\sigma). \end{aligned}$$

By definition, $\mathrm{Supp}(\sigma)$ is an orthogonal subset of Φ^+. Observe that

$$\sigma = \prod_{\beta \in \mathrm{Supp}(\sigma)} s_\beta,$$

where the product is taken in any fixed order, and that

$$|\mathrm{Supp}(\sigma) \cap \mathcal{C}_k| \leq 1, \quad |\mathrm{Supp}(\sigma) \cap \mathcal{R}_k| \leq 1$$

for any k. Note also that if w is not basic, then, in general, there are several different ways to define $\mathrm{Supp}(w)$, see Remark 3.3 (ii) below.

Example 2.5. Let $\Phi = D_6$ and $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ -6 & 2 & 5 & 4 & 3 & -1 \end{pmatrix}$. Then

$$\mathrm{Supp}(\sigma) = \{\epsilon_1 + \epsilon_6, \epsilon_3 - \epsilon_5\}.$$

Remark 2.6. i) Denote the set of involutions (respectively, of basic involutions) by $I(W)$ (respectively, by $B(W)$). By [Ig2, Proposition 2.3], if $\sigma, \tau \in I(W)$, then

$$R_\sigma \leq R_\tau \ \text{ if and only if } \ R_\sigma^* \leq R_\tau^*, \tag{4}$$

where R_w^* is the strictly lower-triangular part of R_w, i.e.,

$$(R_w^*)_{i,j} = \begin{cases} (R_w)_{i,j} & \text{if } \ i > j, \\ 0 & \text{if } \ i \leq j. \end{cases}$$

ii) By using formulas (3) or (4), it is easy to check that if $\alpha \in \mathcal{C}_1$ and $\beta \notin \mathcal{C}_1$, then $s_\alpha \nleq s_\beta$. One can also check that

$$s_{\epsilon_1 - \epsilon_2} < \cdots < s_{\epsilon_1 - \epsilon_n}, \quad s_{\epsilon_1 + \epsilon_n} < \cdots < s_{\epsilon_1 + \epsilon_2}.$$

Next, we have $s_{\epsilon_1 - \epsilon_i} < s_{\epsilon_1 + \epsilon_j}$ for all $i, j \in \{1, \ldots, n\}$ such that $i < n$ or $j < n$, but $s_{\epsilon_1 - \epsilon_n} \not< s_{\epsilon_1 + \epsilon_n}$ and $s_{\epsilon_1 + \epsilon_n} \not< s_{\epsilon_1 - \epsilon_n}$.

The following lemma plays a crucial role in the proof of Theorem 1.2 (cf. [EI, Lemmas 2.4, 2.5] and [BIS, Lemma 2.6]).

Lemma 2.7. *Let $w \in W$ be a basic involution. If $\mathrm{Supp}(w) \cap \mathcal{C}_1 = \varnothing$, then α divides d_w in the polynomial ring $\mathbb{C}[\mathfrak{h}]$ for all $\alpha \in \mathcal{C}_1$. If $\mathrm{Supp}(w) \cap \mathcal{C}_1 = \{\beta\}$, then β does not divide d_w in $\mathbb{C}[\mathfrak{h}]$.*

Proof. Let $\widetilde{W}$ denote the subgroup of W generated by $s_2, \ldots, s_n$. Suppose $\mathrm{Supp}(w) \cap \mathcal{C}_1 = \varnothing$; then $w \in \widetilde{W}$. We denote by $\widetilde{\Phi}$ the root system corresponding to $\widetilde{W}$; in fact, $\widetilde{\Phi}^+ = \Phi^+ \setminus \mathcal{C}_1$.

Let $\widetilde{d}_w \in \widetilde{S} = \mathbb{C}[\alpha_2, \ldots, \alpha_n]$ be the Kostant–Kumar polynomial of w viewed as an element of $\widetilde{W}$; we define $\widetilde{c}_w \in \mathbb{C}(\alpha_2, \ldots, \alpha_n)$ in a similar way. Since $\widetilde{W}$ is a parabolic subgroup of W, the length of w as an element of $\widetilde{W}$ is equal to the length of w as an element of W. Also, any reduced decomposition for w in $\widetilde{W}$ is a reduced decomposition for w in W. This means that $\widetilde{c}_w = c_w$, whence

$$d_w = (-1)^{l(w)} \cdot \prod_{\alpha\in\Phi^+} \alpha \cdot c_w = (-1)^{l(w)} \cdot \prod_{\alpha\in\mathcal{C}_1} \alpha \cdot \prod_{\alpha\in\widetilde{\Phi}^+} \alpha \cdot \widetilde{c}_w = \widetilde{d}_w \cdot \prod_{\alpha\in\mathcal{C}_1} \alpha.$$

In particular, α divides d_w for all $\alpha \in \mathcal{C}_1$.

Now, suppose that $\mathrm{Supp}(w) \cap \mathcal{C}_1 = \{\beta\}$. By [Hu2, Proposition 1.10], there exists a unique $v \in \widetilde{W}$ such that $w = uv$ and $l(us_i) = l(u) + 1$ for all $2 \le i \le n$ (or equivalently, $u(\alpha_i) > 0$ for all $2 \le i \le n$). Furthermore, $l(w) = l(u) + l(v)$. It is easy to check that

if $\beta = \epsilon_1 - \epsilon_j$ (i.e., $w(1) = j$), then

$$u = s_{j-1} \ldots s_2 s_1 = \begin{cases} \begin{pmatrix} 1 & 2 & 3 & \ldots & j-1 & j & j+1 & \ldots & n-1 & n \\ j & 1 & 2 & \ldots & j-2 & j-1 & j+1 & \ldots & n-1 & n \end{pmatrix} & \text{if } j < n, \\ \begin{pmatrix} 1 & 2 & 3 & \ldots & n-1 & n \\ n & 1 & 2 & \ldots & n-2 & n-1 \end{pmatrix} & \text{if } j = n; \end{cases}$$

if $\beta = \epsilon_1 + \epsilon_j$ (i.e., $w(1) = -j$), then

$$u = s_j s_{j+1} \ldots s_{n-1} s_n s_{n-2} s_{n-3} \ldots s_2 s_1 = \begin{cases} \begin{pmatrix} 1 & 2 & 3 & \ldots & j-1 & j & j+1 & \ldots & n-1 & n \\ -j & 1 & 2 & \ldots & j-2 & j-1 & j+1 & \ldots & n-1 & -n \end{pmatrix} & \text{if } j < n, \\ \begin{pmatrix} 1 & 2 & 3 & \ldots & n-1 & n \\ -n & 1 & 2 & \ldots & n-2 & -(n-1) \end{pmatrix} & \text{if } j = n. \end{cases}$$

For instance, consider the case where $\beta = \epsilon_1 + \epsilon_j$ (the case of $\beta = \epsilon_1 - \epsilon_j$ can be treated similarly). Recall that W acts on $\mathbb{C}(\mathfrak{h})$ by automorphisms. Using (1) and arguing as in the proof of [EI, Lemma 2.5], we can easily show that

$$(5) \qquad c_w = -\frac{c_{us_1,g_0} g_0(c_{v,g_0^{-1}})}{\beta} - \sum_{\substack{g\le u,\\ g^{-1}\le v,\\ g\ne g_0}} \frac{c_{us_1,g} g(c_{v,g^{-1}})}{g(\alpha_1)} = \beta^{-1} \cdot g_0(c_{v,g_0^{-1}}) \cdot \frac{K}{L} + \frac{M}{N}$$

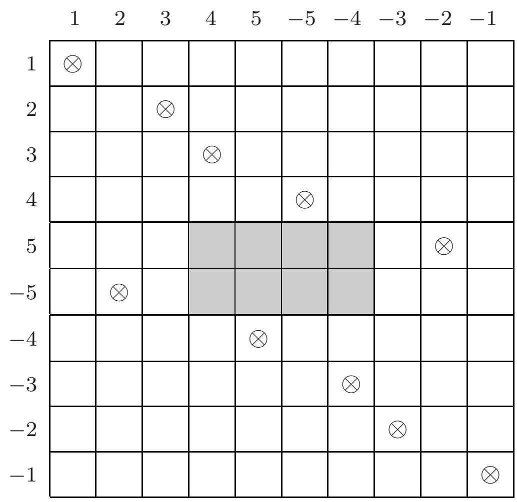
(cf. formula (7) in [EI]). Here

$$g_0 = us_1 = s_j s_{j+1} \ldots s_{n-1} s_n s_{n-2} s_{n-3} \ldots s_2 = \begin{cases} \begin{pmatrix} 1 & 2 & 3 & \ldots & j-1 & j & j+1 & \ldots & n-1 & n \\ 1 & -j & 2 & \ldots & j-2 & j-1 & j+1 & \ldots & n-1 & -n \end{pmatrix} & \text{if } j < n, \\ \begin{pmatrix} 1 & 2 & 3 & \ldots & n-1 & n \\ 1 & -n & 2 & \ldots & n-2 & -(n-1) \end{pmatrix} & \text{if } j = n, \end{cases}$$

and K, L and $M, N \in \mathbb{C}[\mathfrak{h}]$ are pairs of coprime polynomials such that β divides neither K nor N.

To prove that β does not divide d_w, it suffices to show that $c_{v,g_0^{-1}} \ne 0$, i.e., $v \ge g_0^{-1}$ (or equivalently, $v^{-1} \ge g_0$). Arguing as in the proof of [BIS, Lemma 2.6], we obtain $R_{v^{-1}} \ge R_{g_0}$. Thus, it remains to check that the second condition in the definition of the Bruhat order is satisfied. Suppose that $[-a, a] \times [-b, b]$ is an empty rectangle for both v^{-1} and g_0, and that $(R_{g_0})_{-(a-1),b-1} = (R_{v^{-1}})_{-(a-1),b-1}$. We must prove that $(R_{g_0})_{-(a-1),n} \equiv (R_{v^{-1}})_{-(a-1),n} \pmod 2$.

If $j < n$, then $g_0(n) = -n$, so that there are no empty rectangles for g_0, whence $j = n$. In this case, $a = n$ and $b \geq 3$. For example, in the figure below we draw X_{g_0} for $n = 5$, $b = 4$. The entries in the empty rectangle $[-5, 5] \times [-4, 4]$ are shadowed.



Clearly, $(R_{g_0})_{-(n-1),b-1} = 0$, we have $(R_{v^{-1}})_{-(n-1),b-1} = 0$. Also, $(R_{g_0})_{-(n-1),n} = 1$, so we must check that $(R_{v^{-1}})_{-(n-1),n}$ is odd.

By definition,

$$(R_{v^{-1}})_{-(n-1),n} = \#\{i \in \{1, \ldots, n\} \mid v^{-1}(i) \in \{-1, \ldots, -(n-1)\}\}.$$

On the other hand, $v^{-1}(2) = wu(2) = w(1) = -n$, whence

$$\#\{i \in \{1, \ldots, n\} \mid v^{-1}(i) = -n\} = \#\{n\} = 1.$$

Since the number

$$(R_{v^{-1}})_{-(n-1),n} + 1 = \#\{i \in \{1, \ldots, n\} \mid v^{-1}(i) < 0\}$$

is even (by the definition of W), we conclude that $(R_{v^{-1}})_{-(n-1),n}$ is odd, as required. □

2.3. Now we are ready to prove our first main result, Theorem 1.2. The proof immediately follows from Proposition 2.8 below (cf. [EI, Propositions 2.6, 2.7, 2.8] and [BIS, Propositions 2.7, 2.8]). Our goal is to check that if σ, τ are distinct basic involutions in W, then their Kostant–Kumar polynomials do not coincide, and, consequently, the tangent cones C_σ and C_τ do not coincide as subschemes of $T_p\mathcal{F}$. We shall proceed by induction on n (the base is trivial).

Proposition 2.8. *Let $\sigma, \tau \in W$ be distinct basic involutions. Then $d_\sigma \neq d_\tau$.*

Proof. If $\mathrm{Supp}(\sigma) \cap \mathcal{C}_1 \neq \mathrm{Supp}(\tau) \cap \mathcal{C}_1$, then we can repeat the proof of [BIS, Proposition 2.7] word for word to obtain the result. Namely, if $\mathrm{Supp}(\sigma) \cap \mathcal{C}_1 = \{\beta\}$ and $\mathrm{Supp}(\tau) \cap \mathcal{C}_1 = \varnothing$, then β does not divide d_σ by the preceding lemma. But, by formula (2), β divides d_τ, so $d_\sigma \neq d_\tau$. On the other hand, suppose that $\mathrm{Supp}(\sigma) \cap \mathcal{C}_1 = \beta$, $\mathrm{Supp}(\tau) \cap \mathcal{C}_1 = \beta'$, $\beta \not< \beta'$; then β divides d_τ (by (2)), but β does not divide d_σ (by Lemma 2.7), whence $d_\sigma \neq d_\tau$.

From now on, we may assume that $\mathrm{Supp}(\sigma) \cap \mathcal{C}_1 = \mathrm{Supp}(\tau) \cap \mathcal{C}_1$. If $\mathrm{Supp}(\sigma) \cap \mathcal{C}_1 = \mathrm{Supp}(\tau) \cap \mathcal{C}_1 = \varnothing$, then the inductive assumption completes the proof. Suppose $\mathrm{Supp}(\sigma) \cap \mathcal{C}_1 = \mathrm{Supp}(\tau) \cap \mathcal{C}_1 = \{\beta\}$. Let u be as in the proof of Lemma 2.7. There are two cases:

$$\text{i)}\ \ \beta = \epsilon_1 - \epsilon_j, \ \ \text{i.e.,}\ \ w(1) = j,$$
$$\text{ii)}\ \ \beta = \epsilon_1 + \epsilon_j, \ \ \text{i.e.,}\ \ w(1) = -j.$$

If $\beta = \epsilon_1 - \epsilon_j$, then we can repeat the proof of Case (i) of [BIS, Proposition 2.8] word for word, so we may assume that $\beta = \epsilon_1 + \epsilon_j$. Arguing as in the proof of Case (ii) of [BIS, Proposition 2.8], we obtain

$$c_{v,g_0^{-1}} = c_{v_2,\mathrm{id}} \cdot \prod_{i=3}^{n-1}(\epsilon_2 - \epsilon_i)^{-1} \cdot \prod_{i=j+1}^{n-1}(\epsilon_2 + \epsilon_i)^{-1} \cdot (\epsilon_2 + \epsilon_n)^{-2}.$$

Here $w = aw_2a^{-1}$, $a = s_2s_3 \ldots s_{n-2}s_ns_{n-1} \ldots s_{j+1}s_j$, $w_2 = u_2v_2$, $\mathrm{Supp}(w_2) \cap \mathcal{C}_1 = \{\alpha_1\}$, $u_2 = s_1$, and $v_2 \in \widetilde{W}$ is an involution.

Now, consider the involutions σ and τ. Put $\sigma = uv_\sigma$, $\tau = uv_\tau$, where u is as above. Put also $\sigma = a\sigma_2a^{-1}$, $\tau = a\tau_2a^{-1}$, $\sigma_2 = u_2v_\sigma^2$, $\tau_2 = u_2v_\tau^2$, where $u_2 = s_1$. By the inductive assumption, we have $c_{v_\sigma^2,\mathrm{id}} \neq c_{v_\tau^2,\mathrm{id}}$; hence, $c_{v_\sigma,g_0^{-1}} \neq c_{v_\tau,g_0^{-1}}$. Arguing as in the last two paragraphs of the proof of [EI, Proposition 2.8], we can conclude the proof.

Namely, formula (5) implies that if $c_\sigma = c_\tau$, then β divides $P_\sigma Q_\tau - P_\tau Q_\sigma$, where P_σ and Q_σ (respectively, P_τ and Q_τ) are coprime polynomials such that $g_0(c_{v_\sigma,g_0^{-1}}) = P_\sigma/Q_\sigma$ (respectively, $g_0(c_{v_\tau,g_0^{-1}}) = P_\tau/Q_\tau$). But these polynomials belong to the subalgebra of $\mathbb{C}[\mathfrak{h}]$ generated by $\alpha_2, \ldots, \alpha_n$, so $c_{v_\sigma,g_0^{-1}} = c_{v_\tau,g_0^{-1}}$, a contradiction. □

§3. Reduced tangent cones

3.1. In this section we prove our second main result, Theorem 1.3. Throughout the section, it is assumed that every Φ is of type D_n, $n \geq 4$. In this subsection, we briefly describe the relationship between tangent cones and coadjoint orbits of U, the unipotent radical of the Borel subgroup B.

Denote by $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{n}$ the Lie algebras of G, B, U respectively; then $T_p\mathcal{F}$ is naturally isomorphic to the quotient space $\mathfrak{g}/\mathfrak{b}$. Using the Killing form on $\mathfrak{g}$, we can identify the last space with the dual space $\mathfrak{n}^*$. The group B acts on $\mathcal{F}$ by conjugation. Since p is B-stable, B acts on the tangent space $T_p\mathcal{F} \cong \mathfrak{n}^*$. This action is said to be *coadjoint.* We denote the result of coadjoint action by $b \,.\, \lambda$, $b \in B$, $\lambda \in \mathfrak{n}^*$. In 1962, A. A. Kirillov discovered that the orbits of this action play an important role in the representation theory of B and U, see, e.g., [Ki1, Ki2]. We fix a basis $\{e_\alpha, \alpha \in \Phi^+\}$ of $\mathfrak{n}$ consisting of root vectors. Let $\{e_\alpha^*, \alpha \in \Phi^+\}$ be the dual basis of $\mathfrak{n}^*$. Let $w \in W$ be a basic involution. Put

$$f_w = \sum_{\beta \in \mathrm{Supp}(w)} e_\beta^* \in \mathfrak{n}^*.$$

Definition 3.1. We say that the B-orbit Ω_w and the U-orbit Θ_w of f_w are *associated* with the involution w.

It is easy to check that $\Theta_w \subset \Omega_w \subseteq C_w^{\mathrm{red}}$. Next, C_w^{red} is B-stable (in fact, the tangent cone to an arbitrary Schubert variety is B-stable). The orbits associated with involutions were studied by A. N. Panov [Pa] and by the second author [Ig1, Ig2, Ig3, Ig4] (see also Kostant's papers [Ko1, Ko2, Ko2] for the relationship with the center of enveloping algebra of $\mathfrak{n}$). In particular, in [Ig3, Theorem 1.2] it was shown that

$$\dim \Theta_w = l(w) - |\mathrm{Supp}(w)| \,. \tag{6}$$

We need the following corollary to this fact (cf. [Ig1, Proposition 4.1] and [Ig2, Theorem 3.1]).

Lemma 3.2. *If $w \in W$ is a basic involution, then*

$$\dim \Omega_w = l(w). \tag{7}$$

Proof. Denote $D = \mathrm{Supp}(w)$. Let $\xi\colon D \to \mathbb{C}^\times$ be a map. Denote by $\Theta_{w,\xi}$ the U-orbit of the linear form

$$f_{w,\xi} = \sum_{\beta \in D} \xi(\beta) e^*_\beta.$$

In particular, $f_w = f_{w,\xi_0}$, where $\xi_0(\beta) = 1$ for all $\beta \in D$.

Without loss of generality, we may identify G with the group $\mathrm{SO}_{2n}(\mathbb{C})$ of all invertible $(2n \times 2n)$-matrices g of determinant 1 such that $g^t J g = J$, where J is the symmetric matrix of size $(2n \times 2n)$ with 1's on the antidiagonal and 0's elsewhere. Then T (respectively, B and U) is the group of all diagonal (respectively, upper-triangular and upper-triangular with 1's on the diagonal) matrices from G. Moreover, $\mathfrak{g}$ is the algebra of $(2n \times 2n)$-matrices x of zero trace satisfying $x^t J + Jx = 0$, and $\mathfrak{h}$ (respectively, $\mathfrak{b}$ and $\mathfrak{n}$) is the algebra of all diagonal (respectively, upper-triangular and upper-triangular with 0's on the diagonal) matrices from $\mathfrak{g}$. Using the Killing form of $\mathfrak{g}$, we can identify $\mathfrak{n}^*$ with the space $\mathfrak{n}^t$ of all lower-triangular matrices from $\mathfrak{g}$ with 0's on the diagonal. Under this identification, the coadjoint action of B has a simple form:

$$b\,.\,\lambda = (b\lambda b^{-1})_{\mathrm{low}}, \quad b \in B, \quad \lambda \in \mathfrak{n}^*, \tag{8}$$

where A_{low} denotes the strictly lower-triangular part of a matrix A.

First, we claim that if $\xi_1 \neq \xi_2$, then $\Theta_{w,\xi_1} \neq \Theta_{w,\xi_2}$. Indeed, let $\widetilde{U}$ be the group of all $(2n \times 2n$-upper-triangular matrices with 1's on the diagonal. Since this group acts on the space $\widetilde{\mathfrak{n}}$ of all upper-triangular $(2n \times 2n)$-matrices with 0's on the diagonal by the adjoint action, we can consider the dual (coadjoint) action of this group on the space $\widetilde{\mathfrak{n}}^*$. Using the Killing form of $\mathfrak{gl}_{2n}(\mathbb{C})$, one can identify $\widetilde{\mathfrak{n}}^*$ with the space $\widetilde{\mathfrak{n}}^t$ of all lower-triangular $(2n \times 2n)$-matrices with 0's on the diagonal. Under this identification, the coadjoint action of $\widetilde{U}$ is given again by formula (8). Let $\widetilde{\Theta}_{w,\xi} \subset \widetilde{\mathfrak{n}}^*$ be the $\widetilde{U}$-orbit of $f_{w,\xi}$; then, clearly, $\Theta_{w,\xi} \subseteq \widetilde{\Theta}_{w,\xi}$ for any ξ. Since w is an involution in $S_{\pm n}$, from [Pa, Theorem 1.4] it follows that $\widetilde{\Theta}_{w,\xi_1} \neq \widetilde{\Theta}_{w,\xi_2}$. Thus, $\Theta_{w,\xi_1} \neq \Theta_{w,\xi_2}$, as required.

Second, we claim that $\Omega_w = \bigcup_\xi \Theta_{w,\xi}$, where the union is taken over all maps from D to $\mathbb{C}^\times$. Indeed, is is well known that the *exponential map*

$$\exp\colon \mathfrak{n} \to U, \quad x \mapsto \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

is an isomorphism of affine varieties. Given $\alpha \in \Phi^+$, $s \in \mathbb{C}^\times$, we put

$$\begin{aligned} x_\alpha(s) &= \exp s e_\alpha = 1 + s e_\alpha, \quad x_{-\alpha}(s) = x_\alpha(s)^t, \\ w_\alpha(s) &= x_\alpha(s) x_{-\alpha}(-s^{-1}) x_\alpha(s), \quad h_\alpha(s) = w_\alpha(s) w_\alpha(1)^{-1}. \end{aligned}$$

Note that $h_\alpha(s)$ belongs to T.

Let $\xi\colon D \to \mathbb{C}^\times$ a map, and $\alpha \in D$ a root. For any number $s \in \mathbb{C}^\times$, denote by $\sqrt{s}$ a complex number such that $(\sqrt{s})^2 = s$. Using direct matrix calculations, it is easy to show that

$$h_\alpha(\sqrt{s}).f_{w,\xi} = s\xi(\alpha) e_{\alpha^*} + \sum_{\beta \in D,\, \beta \neq \alpha} \xi(\beta) e^*_\beta.$$

Thus,

$$\left(\prod_{\alpha \in D} h_\alpha\big(\sqrt{\xi(\alpha)}\big) \right) .f_w = f_{w,\xi},$$

whence $\Theta_{w,\xi} \subset \Omega_w$.

On the other hand, $B = U \rtimes T$ as algebraic groups. Since T is generated by $h_\alpha(s)$, $\alpha \in \Phi^+$, $s \in \mathbb{C}^\times$, we see that if $h \in T$, then $h\,.\,f_{w,\xi} = f_{w,\xi'}$ for some map $\xi' \colon D \to \mathbb{C}^\times$. Thus, if $g \in B$ and $g = uh$, $u \in U$, $h \in T$, then $g\,.\,f_w = u\,.\,f_{w,\xi}$ for some ξ, so that $\Omega_w = \bigcup_\xi \Theta_{w,\xi}$, as required.

Third, let Z_B (respectively, Z_U and Z_T) be the stabilizer of f_w under the coadjoint action of B (respectively, of U and T). Then

$$\dim \Omega_w = \dim B - \dim Z_B,$$
$$\dim \Theta_w = \dim U - \dim Z_U.$$

If $g = uh \in Z_B$, $u \in U$, $h \in T$, then

$$g\,.\,f_w = u\,.\,(h\,.\,f_w) = u\,.\,f_{w,\xi}$$

for some ξ. If $f_w \neq f_{w,\xi}$, then $\Theta_w \neq \Theta_{w,\xi}$. Hence, $f_w = f_{w,\xi}$, implying that $h \in Z_T$ and $u \in Z_U$. It follows that the map

$$Z_U \times Z_T \to Z_B \colon (u, h) \mapsto uh$$

is an isomorphism of algebraic varieties; therefore,

$$\dim Z_B = \dim Z_U + \dim Z_T.$$

Finally, it follows that $X = \bigcup_\xi \{f_{w,\xi}\}$ is the T-orbit of f_w (the union is taken over all maps from D to $\mathbb{C}^\times$). Thus, using (6), we conclude that

$$\begin{aligned}\dim \Omega_w &= \dim B - \dim Z_B\\ &= \dim U + \dim T - \dim Z_U - \dim Z_T\\ &= \dim \Theta_w + \dim X = l(w) - |D| + |D| = l(w).\end{aligned}$$

The proof is complete. □

Remark 3.3. i) Since $\dim C_w^{\mathrm{red}} = \dim X_w = l(w)$, we see that $\overline{\Omega}_w$, the closure of Ω_w, is an irreducible component of C_w^{red} of maximal dimension. (In fact, C_w^{red} is equidimensional.)

ii) If w is not basic, then there are several different ways to define $\mathrm{Supp}(w)$. For example, if $n = 4$ and $w = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & -2 & -3 & -4 \end{pmatrix}$, then there are three subsets $D \subset \Phi^+$ such that $w = \prod_{\beta \in D} s_\beta$:

$$\{\epsilon_1 - \epsilon_2, \epsilon_1 + \epsilon_2, \epsilon_3 - \epsilon_4, \epsilon_3 + \epsilon_4\},$$
$$\{\epsilon_1 - \epsilon_3, \epsilon_1 + \epsilon_3, \epsilon_2 - \epsilon_4, \epsilon_2 + \epsilon_4\},$$
$$\{\epsilon_1 - \epsilon_4, \epsilon_1 + \epsilon_4, \epsilon_2 - \epsilon_3, \epsilon_2 + \epsilon_3\}.$$

(For some reasons, the first candidate is “the best”, see [De, S].)

So, one can define Θ_w and Ω_w by using one of the definitions of the support of w. But there is no chance that formula (7) is valid for all nonbasic involutions. Indeed, we can repeat the proof of the Lemma 3.2 word for word to obtain $\dim \Omega_w = \dim \Theta_w + |\mathrm{Supp}(w)|$. But if w is not basic, then the dimension of $\dim \Theta_w$ can be strictly less than $l(w) - |\mathrm{Supp}(w)|$, see [Ig3, Theorem 1.2]. That is the reason why we restrict our attention to the case of basic involutions.

Now, assume that G' is a reductive subgroup of G'', T' (respectively, T'') is a maximal torus of G' (respectively, of G''), $T' = T'' \cap G'$, B' (respectively, B'') is a Borel subgroup of G' (respectively, of G'') containing T' (respectively, T''), $B' = B'' \cap G'$, and Φ' (respectively, Φ'') is the root system of G' (respectively, of G'') relative to T' (respectively, to T''). We denote by W' (respectively, by W'') the Weyl group of Φ' (respectively, of Φ''). Denote by $\mathcal{F}' = G'/B'$, $\mathcal{F}'' = G''/B''$ the flag varieties. Put $p' = eB' \in \mathcal{F}'$,

$p'' = eB'' \in \mathcal{F}''$. Let U' (respectively, U'') be the unipotent radical of B' (respectively, of B''), $U' = U'' \cap B'$. Denote also by $\mathfrak{g}'$, $\mathfrak{b}'$, $\mathfrak{n}'$ the Lie algebras of G', B', U' (respectively), and define $\mathfrak{g}''$, $\mathfrak{b}''$, $\mathfrak{n}''$ in a similar way. The dual space $\mathfrak{n}'^* \cong \mathfrak{g}'/\mathfrak{b}'$ can be viewed as a subspace of $\mathfrak{n}''^* \cong \mathfrak{g}''/\mathfrak{b}''$. Hence, we can view $T_{p'}\mathcal{F}'$ as a subspace of $T_{p''}\mathcal{F}''$.

Pick involutions $w_1, w_2 \in W'$. Let C_i' be the reduced tangent cone at the point p' to the Schubert subvariety X_{w_i}' of the flag variety $\mathcal{F}'$, $i = 1, 2$. Similarly, let C_i'' be the reduced tangent cone at p'' to the Schubert subvariety X_{w_i}'' of $\mathcal{F}''$, $i = 1, 2$. Denote by l' (respectively, by l'') the length function on the Weyl group W' (respectively, on W''). Assume that $C_1' = C_2'$. Then

$$l'(w_1) = l'(w_2).$$

Since $C_i' \subseteq C_i''$, we have $B'' . C_i' \subseteq C_i''$, $i = 1, 2$. Denote by $\Omega_{w_i}' \subseteq \mathfrak{n}'^*$ the coadjoint B'-orbit associated with the involution w_i, $i = 1, 2$, and define Ω_{w_i}'' in a similar way. Formula (7) shows that

$$\begin{aligned} l''(w_i) &= \dim C_i'' \geq \dim B'' . C_i' \geq \dim B'' . \Omega_{w_i}' \\ &= \dim \Omega_{w_i}'' = l''(w_i), \end{aligned}$$

because $\Omega_{w_i}'' = B'' . \Omega_{w_i}'$. This implies $l''(w_i) = \dim C_i'' = \dim B'' . C_i'$. But $C_1' = C_2'$, so that $\dim C_1'' = \dim C_2''$. We obtain the following result:

$$\text{if } C_1' = C_2', \text{ then } l''(w_1) = l''(w_2). \tag{9}$$

3.2. In this subsection we prove Theorem 1.3: if w_1, w_2 are basic involutions in the Weyl group W of type D_n, $n \geq 4$, and $w_1 \neq w_2$, then $C_{w_1}^{\text{red}} \neq C_{w_2}^{\text{red}}$ as subvarieties of $T_p\mathcal{F}$. Let W'' be of type D_{n+2}. Let

$$D_{n+2}^+ = \{\eta_i - \eta_j,\, \eta_i + \eta_j,\ 1 \leq i < j \leq n+2\},$$

where $\{\eta_i\}_{i=1}^{n+2}$ is the standard basis of $\mathbb{R}^{n+2}$. Pick numbers k_1, k_2 such that $1 \leq k_1 < k_2 \leq n+2$. Put $P = \{k_1, k_2\}$, $Q = \{1, \ldots, n+2\} \setminus P$, and

$$\begin{aligned} \widetilde{W} &= \{w \in W'' \mid w(i) = i \text{ for all } i \in P\}, \\ \widetilde{W}_2 &= \{w \in W'' \mid w(i) = i \text{ for all } i \in Q\}, \\ W' &= \{w \in W'' \mid w(P) = P,\, w(Q) = Q\} = \widetilde{W} \times \widetilde{W}_2. \end{aligned}$$

Let Φ' (respectively, $\widetilde{\Phi}$) be the root system of W' (respectively, of $\widetilde{W}$). Clearly, Φ' (respectively, $\widetilde{\Phi}$) is of type $D_n \times A_1 \times A_1$ (respectively, of type D_n). Put $G'' = \mathrm{SO}_{2n+4}(\mathbb{C})$ and denote by G' (respectively, by $\widetilde{G}$) the subgroup of G corresponding to Φ' (respectively, to $\widetilde{\Phi}$), then $G' \cong \mathrm{SO}_n(\mathbb{C}) \times \mathrm{SO}_2(\mathbb{C})$. Put also

$$\begin{aligned} A &= \{1, \ldots, k_1 - 1\}, \\ B &= \{k_1 + 1, \ldots, k_2 - 1\}, \\ C &= \{k_2 + 1, \ldots, n+2\}. \end{aligned}$$

Now, let $\Phi = D_n$. We may assume without loss of generality that $G = \mathrm{SO}_n(\mathbb{C})$. We identify Φ with $\widetilde{\Phi}$ by the map $\epsilon_k \mapsto \eta_{k'}$, where

$$k' = \begin{cases} k & \text{if } k \leq k_1 - 1, \\ k+1 & \text{if } k_1 \leq k \leq k_2 - 2, \\ k+2 & \text{if } k_2 - 1 \leq k \leq n. \end{cases}$$

This identifies G (respectively, W) with $\widetilde{G}$ (respectively, with $\widetilde{W}$). We denote the image in $\widetilde{W}$ of an element $w \in W$ under this identification by $\widetilde{w}$. Let $w \in W$ be an involution. Arguing as in the proof of [BIS, Lemma 3.2], we obtain the following result.

Lemma 3.4. i) *If* $w' = \widetilde{w}s_{\eta_{k_1}-\eta_{k_2}}$, *then the length of* w' *in the Weyl group* W'' *equals*

$$l''(w') = 2(k_2 - k_1 - 1) + 4|\widetilde{w}(A) \cap B^-| + 4|\widetilde{w}(A) \cap A^-| + 4|\widetilde{w}(A) \cap C^{\pm}| + l(w) + 1.$$

ii) *If* $w' = \widetilde{w}s_{\eta_{k_1}+\eta_{k_2}}$, *then*

$$l''(w') = 2(k_2 - k_1 - 1) + 4|\widetilde{w}(A) \cap A^-| + 4|\widetilde{w}(A) \cap B^-| + 4|C| + l(w) + 1.$$

(By a slight abuse of notation, here we view $\widetilde{w}$ as an element of $S_{\pm(n+2)}$ and, at the same time, as an element of $\widetilde{W}$, i.e., as an element of W'' such that $\widetilde{w}(k_1) = k_1$ and $\widetilde{w}(k_2) = k_2$.)

Proof of Theorem 1.3. Assume that $C^{\mathrm{red}}_{w_1} = C^{\mathrm{red}}_{w_2}$. In particular,

$$l(w_1) = \dim C^{\mathrm{red}}_{w_1} = \dim C^{\mathrm{red}}_{w_2} = l(w_2).$$

Since $w_1 \neq w_2$, there exists $1 \le k \le n$ such that $w_1(\epsilon_i) = w_2(\epsilon_i)$ for $1 \le i \le k-1$, and $w_1(\epsilon_k) \neq w_2(\epsilon_k)$. Assume without loss of generality that $w_1(\epsilon_k) < w_2(\epsilon_k)$, i.e., $w_2(\epsilon_k) - w_1(\epsilon_k)$ is a sum of positive roots. Note that $w_1(\epsilon_k) \neq \pm\epsilon_k$, because $w_1(\epsilon_i) = w_2(\epsilon_i)$ for all i from 1 to $k-1$. Put $k_1 = k+1$, so $A = \{1, \dots, k\}$ and $\widetilde{w}_1(a) = \widetilde{w}_2(a)$ for all $a \in A \setminus \{k\}$. We consider three cases.

i) Suppose $w_1(\epsilon_k) < 0$, $w_2(\epsilon_k) > 0$. Here we put $k_2 = n+2$, so that $C = \varnothing$ and

$$(\widetilde{w}_i(A) \cap A^-) \cup (\widetilde{w}_i(A) \cap B^-) = \widetilde{w}_i(A) \cap \{-1, \dots, -(n+2)\}, \quad i = 1, 2.$$

Let $w'_i = \widetilde{w}_i s_{\eta_{k_1}-\eta_{k_2}}$, $i = 1, 2$. Since

$$\widetilde{w}_1(A) \cap \{-1, \dots, -(n+2)\} = \widetilde{w}_2(A) \cap \{-1, \dots, -(n+2)\} \cup \{k\},$$

Lemma 3.4 (i) shows that $l''(w'_1) \neq l''(w'_2)$. On the other hand, $C^{\mathrm{red}}_{w_1} = C^{\mathrm{red}}_{w_2}$ implies $C'_1 = C'_2$, which contradicts (9).

ii) Next, suppose that $w_1(\epsilon_k) = \epsilon_{m_1} > 0$, $w_2(\epsilon_k) = \epsilon_{m_2} > 0$. Observe that $m_1 > m_2 \ge k$, because $w_1(\epsilon_k) < w_2(\epsilon_k)$ and $w_1(\epsilon_i) = w_2(\epsilon_i)$ for all i from 1 to $k-1$. Here we put $k_2 = m_1 + 1$, so that $\widetilde{w}_1(k) \in C$ and $\widetilde{w}_2(k) \in B$. By Lemma 3.4 (i), we have $l''(w'_1) \neq l''(w'_2)$, where $w'_i = \widetilde{w}_i s_{\eta_{k_1}-\eta_{k_2}}$, $i = 1, 2$. But $C'_1 = C'_2$, a contradiction.

iii) Finally, suppose that $w_1(\epsilon_k) = -\epsilon_{m_1} < 0$, $w_2(\epsilon_k) = -\epsilon_{m_2} < 0$. Observe that $m_2 > m_1 > k$, because $w_1(\epsilon_k) < w_2(\epsilon_k)$ and $w_1(\epsilon_i) = w_2(\epsilon_i)$ for all i from 1 to $k-1$. Here we put $k_2 = m_2 + 1$, so that $\widetilde{w}_1(k) \in B^-$ and $\widetilde{w}_2(k) \in C^-$. By Lemma 3.4 (ii), we have $l''(w'_1) \neq l''(w'_2)$, where $w'_i = \widetilde{w}_i s_{\eta_{k_1}+\eta_{k_2}}$, $i = 1, 2$. On the other hand, $C'_1 = C'_2$, which contradicts (9). The result follows. □

Remark 3.5. Actually, for $\Phi = B_n$ one can introduce the notion of a basic involution in the same way as for D_n. It is easy to check that the preceding proposition is true for basic involutions in type B_n.

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