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EXAMPLE OF A NONRECTIFIABLE NEVANLINNA CONTOUR

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Abstract. Nevanlinna contours (and domains) were introduced by K. Yu. Fedorovskiĭ in connection with the problem of uniform approximation of continuous functions by polyanalytic polynomials; also, these contours are related to pseudocontinuation of analytic functions, to the theory of model spaces, etc. An example of a nonrectifiable Nevanlinna contour is constructed in this paper for the first time.

§1. Introduction and statement of the result

We briefly recall the definition and main properties of Nevanlinna domains (see, e.g., [1, Chapter 2, 2.3–2.4] for more details).

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk and $\mathbb{T} = \partial\mathbb{D}$ the unit circle on the complex plane $\mathbb{C}$. For every open set $E \subset \bar{\mathbb{C}}$, we denote by $H^\infty(E)$ the space of bounded analytic functions on E.

A bounded simply connected domain Ω in $\mathbb{C}$ is called a *Nevanlinna domain* (see [2, Definition 2.1]) if there exist two functions $u,\ v \in H^\infty(\Omega)$ with $v \not\equiv 0$ such that in the sense of conformal mapping we have

$$\bar{\zeta} = \frac{u(\zeta)}{v(\zeta)}$$

a.e. on Γ. This means that for almost every $\xi \in \mathbb{T}$ we have

$$\overline{\varphi(\xi)} = \frac{u(\varphi(\xi))}{v(\varphi(\xi))}, \tag{1.1}$$

for the angular boundary values, where φ is an analytic univalent function that maps $\mathbb{D}$ onto Ω.

If Ω is a Jordan domain, Γ is called a *Nevanlinna contour*.

The definition of Nevanlinna domains is consistent: clearly, it does not depend on the choice of φ; next, by the Fatou theorem, the functions $u(\varphi(\xi))$ and $v(\varphi(\xi))$ have angular boundary values a.e. on $\mathbb{T}$, and the ratio u/v is uniquely defined in Ω by the Lusin boundary uniqueness theorem.

The following criterion for being a Nevanlinna domain is important for applications.

Theorem 1 (see [2, Proposition 3.1]). *A domain Ω is a Nevanlinna domain if and only if the function φ in* (1.1) *admits analytic pseudocontinuation* (*in the sense of Nevanlinna*) *across $\mathbb{T}$, i.e., there exist two functions f_1, $f_2 \in H^\infty(\bar{\mathbb{C}} \setminus \bar{\mathbb{D}})$ such that $f_2 \not\equiv 0$ and a.e. on $\mathbb{T}$ we have $\varphi = f_1/f_2$ in the sense of angular boundary values* (*taken from inside $\mathbb{T}$ on the left and from outside on the right*).

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Let $H^p = H^p(\mathbb{D})$, $p \in (0, \infty)$, be the standard Hardy spaces (see, e.g., [3]), and let S^* be the backward shift operator on H^2, i.e.,

$$S^*\colon f \to \frac{f(z) - f(0)}{z}.$$

As was proved in [4, Theorem 2.2.1], a function $f \in H^2$ admits Nevanlinna type pseudocontinuation if and only if f is not a cyclic vector for S^* (i.e., the linear hull of the vectors $S^{*n}f$, $n \in \mathbb{Z}_+$, is not dense in H^2). This allows one to use the theory of model subspaces of H^2 (see [5, 6]) for the study of Nevanlinna domains. It should be noted that much information on the theory of model spaces is contained in the book [7].

Nevanlinna domains arise naturally in the theory of approximation by polyanalytic polynomials. The statement presented below is a partial case of Theorem 2.2 in [2]. Recall that a bounded domain Ω is called a *Carathéodory domain* if $\partial\Omega = \partial\Omega_\infty$, where Ω_∞ is the unbounded (connected) component of $\overline{\mathbb{C}} \setminus \overline{\Omega}$. *Polyanalytic polynomials of order* n (*bianalytic if* $n = 2$) are defined to be the functions of the form

$$P(z) = \sum_{m=0}^{n-1} P_m(z)\overline{z}^m,$$

where all P_m are polynomials of the complex variable z.

Theorem 2. *Let a positive integer n be fixed, and let Ω be a Carathéodory domain with boundary $\partial\Omega$. The set of polyanalytic polynomials of order n is dense in $C(\partial\Omega)$ if and only if Ω is not a Nevanlinna domain.*

Note that, unlike the case of $n = 1$ (where the Mergelyan theorem holds, see [8]), for $n \geq 2$ the connectedness of the complement to a compact set is not necessary for the uniform approximability of functions by polyanalytic polynomials.

The notions of a Nevanlinna contour and Nevanlinna domain were introduced in [9], where Theorem 2 was proved under the assumption that $\partial\Omega$ is a rectifiable contour. Despite of important advances, the problem of uniform approximation of functions by polyanalitic polynomials on plane compact sets has not been yet resolved in full generality (see [1, Subsection 2.3]).

Keeping in mind the importance of Nevanlinna domains (in approximation theory, in boundary-value problems, etc.), it is natural to ask how wide this class is, in particular, how "bad" their boundaries may be.

If Γ is an analytic contour, from Theorem 1 it easily follows (see also [10, Chapter 14]) that Γ is a Nevanlinna contour if and only if the function φ in (1.1) is rational with poles in $\overline{\mathbb{D}}$. (In particular, an ellipse different from a circle is not a Nevanlinna contour, and, by Theorem 2, every continuous function on it is approximable within any given accuracy by bianalytic polynomials).

In more complicated cases, Theorem 1 can also be applied for construction of Nevanlinna contours; in doing so, it is important to control the property for φ to be univalent in $\mathbb{D}$. The first example of a nowhere analytic Nevanlinna contour was constructed in [11]. In [5, Theorem 3], a C^1 Nevanlinna contour that is not a Lyapunov curve was constructed. In [6], a (rectifiable) Nevanlinna contour was exhibited with the property that φ' belongs to the Hardy class H^1 but does not belong to H^p with any $p > 1$.

The question about the existence of a nonrectifiable Nevanlinna contour has been known for more than 10 years (see [2, Problem 3.2], [12, Problem 2.10]); in the present paper, we answer it in the positive. The following example will be constructed.

Example 1. There exists a function

$$\varphi(z) = z + \sum_j \frac{\varepsilon_j}{z - z_j} \tag{1.2}$$

(where the sum in j is countable) such that ε_j, $z_j \in \mathbb{C}$, $\sum_j |\varepsilon_j| < \infty$, and

(1) $1 < |z_j| < 2$ for all j, the points $\{z_j\}$ accumulate only at $z = 1$ and satisfy the Blaschke condition $\sum_j (|z_j| - 1) < \infty$;

(2) φ is continuous on $\bar{\mathbb{D}}$ and analytic near every $z \in \mathbb{C}$ different from 1 and the z_j;

(3) $\operatorname{Re} \varphi'(z) > 1/2$ everywhere in $\mathbb{D}$;

(4) the image of the unit circle under the mapping φ has infinite length.

Under the conditions of Example 1, $\varphi(\mathbb{T})$ is a nonrectifiable Nevanlinna contour. Indeed, φ is univalent in $\mathbb{D}$ by (3) and, by (2), it maps $\bar{\mathbb{D}}$ onto a simply connected Jordan domain Ω bounded by the contour $\Gamma = \partial\Omega$. This contour is not rectifiable by (4). Again by (2), every open arc of Γ not containing 1 is analytic. Let $w_j = 1/z_j$, then by (1) the Blaschke condition $\sum_j (1 - |w_j|) < \infty$ is fulfilled, and, consequently, the Blaschke product

$$B(w) = \prod_j \frac{\bar{w}_j}{|w_j|} \frac{w - w_j}{\bar{w}_j w - 1}$$

is a function analytic in $\mathbb{D}$ and satisfying $|B(w)| = 1$ a.e. on $\mathbb{T}$ in the sense of angular boundary values. Thus, in the notation of Theorem 1, the function $\varphi(z)$ given by (1.2) is representable in $\bar{\mathbb{C}} \setminus \bar{\mathbb{D}}$ as the ratio of two bounded analytic functions $f_1(z) = \varphi(z) \frac{B(\frac{1}{z})}{z}$ and $f_2(z) = \frac{B(\frac{1}{z})}{z}$, i.e., φ admits a Nevanlinna-type pseudocontinuation.

Remark 1. Since φ' is a function with positive real part, it belongs to all H^p with $p < 1$ (see, e.g., [13, Chapter 2, §4.5]), but does not belong to H^1 because Γ is not rectifiable).

§2. Construction of a "Nevanlinna needle"

We begin with auxiliary considerations; they are quite elementary and can be used in future constructions of Nevanlinna domains with nonrectifiable boundaries (not necessarily of Jordan type).

Fix b, $0 < b < 1$, and $N \in \mathbb{N}$, $N \geq 2$. Consider the function

$$\Phi_{b,N}(z) = b \int_{b/N^2}^{b} \frac{dt}{t - z} = b\Big(\ln(b - z) - \ln\Big(\frac{b}{N^2} - z\Big)\Big) \tag{2.1}$$

(the integration is over the interval $[b/N^2, b]$ of the real axis, $z \notin [b/N^2, b]$, and the branch of the logarithm is chosen so that $\ln 1 = 0$). We observe the simple properties (Φ1)–(Φ3) of $\Phi_{b,N}(z)$ and of its derivative in z,

$$\Phi'_{b,N}(z) = b \int_{b/N^2}^{b} \frac{dt}{(t - z)^2} = \frac{1}{\frac{1}{N^2} - \frac{z}{b}} - \frac{1}{1 - \frac{z}{b}}. \tag{2.2}$$

(Φ1) $\Phi_{b,N}(0) = 2b \ln N$ and $|\Phi_{b,N}(z)| \leq 2b \ln N$ in the half-plane $\operatorname{Re} z \leq 0$;

(Φ2) for $|z| \geq 2b$ we have $|\Phi_{b,N}(z)| < \frac{2b^2}{|z|}$ and $|\Phi'_{b,N}(z)| < \frac{4b^2}{|z|^2}$;

(Φ3) for $\operatorname{Re} z \leq 0$ we have $\operatorname{Re} \Phi'_{b,N}(z) > -1$.

Properties (Φ1) and (Φ2) are obvious. Property (Φ3) is a consequence of the inequalities

$$\operatorname{Re}\left(\frac{1}{\frac{1}{N^2} - \frac{z}{b}}\right) > 0, \quad \left|\frac{1}{1 - \frac{z}{b}}\right| \leq 1,$$

valid in the half-plane $\operatorname{Re} z \leq 0$.

The function $\Phi_{b,N}$ is not rational. To "correct" it, we use the Simpson formula, which is well known in the theory of numerical integration. Let $[\alpha,\beta]$ be an interval on the real line, and let $g \in C^4([\alpha,\beta])$. We recall the estimate

$$\left| \int_\alpha^\beta g(x)\,dx - \frac{\beta-\alpha}{6}\Big(g(\alpha)+4g\Big(\frac{\alpha+\beta}{2}\Big)+g(\beta)\Big)\right| \le \frac{(\beta-\alpha)^5}{2880}\max_{x\in[\alpha,\beta]}|g^{(4)}(x)|. \tag{2.3}$$

Now, we split the interval $[b/N^2, b]$ into $N-1$ parts of the form $[b/k^2, b/(k-1)^2]$, where $k=2,\dots,N$, and (in accordance with the Simpson formula for the function $\Phi_{b,N}$, see (2.1)), on each part we consider the function

$$\Psi_{b,N}(z) = \frac{b}{6}\sum_{k=2}^{N}\Big(\frac{b}{(k-1)^2}-\frac{b}{k^2}\Big)\Big(\frac{1}{\frac{b}{(k-1)^2}-z}+\frac{4}{\frac{b}{2(k-1)^2}+\frac{b}{2k^2}-z}+\frac{1}{\frac{b}{k^2}-z}\Big). \tag{2.4}$$

Clearly, when applied to $\Phi'_{b,N}(z)$, the Simpson formula with the same splittings gives $\Psi'_{b,N}(z)$. It can easily be seen that relations (2.3)–(2.4) yield the following estimates (because 2880^{-1} is "sufficiently small"):

(Ψ1) For $|z| \ge 2b$, we have

$$|\Phi_{b,N}(z)-\Psi_{b,N}(z)| < \frac{b^6}{10|z|^5}, \quad |\Phi'_{b,N}(z)-\Psi'_{b,N}(z)| < \frac{b^6}{10|z|^6}.$$

(Ψ2) Despite the fact that $\Phi_{b,N}(z)$ and $\Phi'_{b,N}(z)$ are not uniformly bounded as $N\to\infty$, for $\operatorname{Re} z \le 0$ we have

$$|\Phi_{b,N}(z)-\Psi_{b,N}(z)| < \frac{b}{10}\sum_{k=1}^{\infty}\Big(\frac{b}{k^3}\Big)^5\Big(\frac{k^2}{b}\Big)^5 = \frac{b}{10}\sum_{k=1}^{\infty}\frac{1}{k^5} < b;$$

$$|\Phi'_{b,N}(z)-\Psi'_{b,N}(z)| < \frac{b}{10}\sum_{k=1}^{\infty}\Big(\frac{b}{k^3}\Big)^5\Big(\frac{k^2}{b}\Big)^6 = \frac{1}{10}\sum_{k=1}^{\infty}\frac{1}{k^3} < 1.$$

(This explains why we have used the Simpson formula: usual integral sums do not approximate $\Phi_{b,N}$ duly.)

(Ψ3) The function $\Psi_{b,N}$ has the form of a finite sum $\sum_j \frac{\varepsilon_j}{z-z_j}$, where the sum of the distances from $z=0$ to the poles z_j is at most $3b\sum_{k=1}^{\infty}1/k^2 = b\pi^2/2$, and, moreover, $z_j>0$ and $\varepsilon_j>0$ for all j, and $\sum_j \varepsilon_j < b^2$.

Now, we shift the functions $\Psi_{b,N}$ by 1 and then rotate the result about $z=0$. Specifically, let $\gamma\in[0,2\pi)$; put

$$G_{b,N,0}(z) = \Psi_{b,N}(z-1), \quad G_{b,N,\gamma}(z) = e^{i\gamma}G_{b,N,0}(z/e^{i\gamma}).$$

The properties of the functions $\Phi_{b,N}$ and $\Psi_{b,N}$ (namely, properties (Φ1)–(Φ3) and (Ψ1)–(Ψ3)) readily yield the following statement.

Lemma 1. *The functions $G_{b,N,\gamma}$ possess the following properties*:

(A1) *$G_{b,N,\gamma}$ is a finite sum of the form $\sum_j \frac{\varepsilon_j}{z-z_j}$, where the poles z_j belong to the interval $(e^{i\gamma},(1+b)e^{i\gamma}]$; moreover, $\sum_j(|z_j|-1) < 5b$ and $\sum_j|\varepsilon_j| < b^2$;*

(A2) *$G_{b,N,\gamma}(e^{i\gamma}) = e^{i\gamma}C$, where $C>0$ and $b(2\ln N-1) < C < b(2\ln N+1)$;*

(A3) *for $z\in\bar{\mathbb{D}}$ we have $|G_{b,N,\gamma}(z)| \le C < b(2\ln N+1)$, and for $|z-e^{i\gamma}|\ge 2b$ we have*

$$|G_{b,N,\gamma}(z)| < \frac{3b^2}{|z-e^{i\gamma}|}, \quad |G'_{b,N,\gamma}(z)| < \frac{5b^2}{|z-e^{i\gamma}|^2};$$

(A4) *$G_{b,N,\gamma}\bar{\mathbb{D}}$ we have $\operatorname{Re} G'_{b,N,\gamma}(z) > -2$ everywhere in $\bar{\mathbb{D}}$.*

Indeed, (A1) follows from (Ψ3); (A2) follows from (2.4), (Φ1), and (Ψ2); (A3) follows from (Φ1), (Φ2), and (Ψ1); and (A4) follows from (Φ3) and (Ψ2).

Informally, Lemma 1 means the following. For $0 < \delta < \frac{1}{2}$, the rational function $z + \delta G_{b,N,\gamma}(z)$ maps $\mathbb{D}$ conformally onto a Nevanlinna domain with analytic boundary that contains a "needle" symmetric with respect to the straight line $(0, e^{i\gamma})$; this "needle" is the "longer" the greater N is, and is the "thinner" the smaller b is; moreover, it is important that the sum of the distances of $e^{i\gamma}$ to the poles of $G_{b,N,\gamma}$ is also estimated by b from above, and the sum of the moduli of the residues is estimated by b^2.

Using Lemma 1, we construct the example in question.

§3. Construction of Example 1

For $k \in \mathbb{N}$, $k \geq 2$, put $\gamma_k = \frac{1}{2^k}$, $b_k = \frac{1}{8^k}$; in accordance with (A2) in Lemma 1, we choose N_k so that

$$\frac{1}{k} \leq \frac{G_{b_k,N_k,\gamma_k}(e^{i\gamma_k})}{e^{i\gamma_k}} < \frac{2}{k}. \tag{3.1}$$

We show that the function

$$\varphi(z) = z + \frac{1}{10}\sum_{k=2}^{\infty} G_{b_k,N_k,\gamma_k}(z) \tag{3.2}$$

satisfies the conditions of Example 1.

Indeed, φ is of the form (1.2). Condition (1) follows from the definition and property (A1) for $G_{b,N,\gamma}$, and also from the convergence of the series $\sum_{k=2}^{\infty} b_k$ and the relation $\lim_{k\to\infty} \gamma_k = 0$.

The function φ is analytic near any point except 1 and the z_j by property (1) (which has already been proved) and the relation $\sum_j |\varepsilon_j| < \infty$, which, in its turn, follows from property (A1) of $G_{b,N,\gamma}$ and the convergence of the series $\sum_{k=2}^{\infty}(b_k)^2$.

Before we prove that the function (3.2) satisfies the remaining conditions in Example 1, we observe that the disks $D_k = \{z : |z - e^{i\gamma_k}| < \gamma_k/8\}$ are mutually disjoint, $\gamma_k/8 \geq 2b_k$ for $k \geq 2$, and, by (A3), outside the disks D_k we have

$$|G_{b_k,N_k,\gamma_k}(z)| \leq 3(b_k)^2\frac{8}{\gamma_k} \leq \frac{24}{32^k}; \quad |G'_{b_k,N_k,\gamma_k}(z)| \leq 5(b_k)^2\frac{64}{(\gamma_k)^2} \leq \frac{320}{16^k}. \tag{3.3}$$

By (3.1) and (3.3), for $z \in \bar{\mathbb{D}}$ and $k_0 \in \mathbb{N}$ we have

$$\sum_{k=k_0}^{\infty} |G_{b_k,N_k,\gamma_k}(z)| < \frac{2}{k_0} + 24\sum_{k=k_0}^{\infty}\frac{1}{32^k} < \frac{3}{k_0}.$$

Thus, the series (3.2) converges absolutely and uniformly on $\bar{\mathbb{D}}$; consequently, φ is bounded and continuous on $\bar{\mathbb{D}}$. We have proved condition (2).

Condition (3) in Example 1 is a consequence of (3.3) and property (A4) of $G_{b,N,\gamma}$:

$$\operatorname{Re}\varphi'(z) > 1 - \frac{1}{10}\left(2 + 320\sum_{k=2}^{\infty}\frac{1}{16^k}\right) > \frac{1}{2}.$$

To prove (4), it suffices to show that the variation of φ on $\mathbb{T}$ is not bounded; this will follow from (3.1), (3.3), and the divergence of the series $\sum_{k=2}^{\infty} 1/k$.

Indeed, let $z_k = e^{i\gamma_k}$, $z'_k = e^{i(3/4)\gamma_k}$ (clearly, on the closed arc $[1, e^{i\gamma_2}]$ of $\mathbb{T}$, every point z'_k lies between z_{k+1} and z_k). Then $z_k \in D_k$ and $z_k \notin D_m$ for $m \neq k$; $z'_k \notin D_m$ for any $m \in \mathbb{N}$. By (3.1) and (3.3), we have

$$|G_{b_k,N_k,\gamma_k}(z_k) - G_{b_k,N_k,\gamma_k}(z'_k)| \geq \frac{1}{k} - \frac{24}{32^k}.$$

Using the estimates for derivatives in (3.3), we arrive at

$$\sum_{\{m\,:\,m\neq k\}} |G_{b_m,N_m,\gamma_m}(z_k) - G_{b_m,N_m,\gamma_m}(z'_k)| < \frac{1}{2^k}\sum_{m=2}^{\infty}\frac{320}{16^m}.$$

Thus, for the function φ in (3.2) we have

$$|\varphi(z_k) - \varphi(z'_k)| > \frac{1}{k} - \frac{4}{2^k}; \quad \sum_{k=2}^{\infty} |\varphi(z_k) - \varphi(z'_k)| = +\infty.$$

This proves (4) and completes the construction of Example 1.

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