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COINCIDENCE OF THE GYSIN HOMOMORPHISM AND THE TRANSFER

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Abstract. The coincidence of the transfer and the Gysin homomorphism is proved for homotopy invariant sheaves with transfers. For this, the category of relative motives is constructed, together with a functor from this category to the category of Voevodsky's motives.

§1. Statement of the problem

In this paper, k is a fixed field of characteristic 0. All varieties are quasiprojective (if otherwise is not stated explicitly). Often, we work in the triangulated category of Voevodsky's motives $DM(k)$ (see [SV2, V2] for the definition). For a variety Y, we denote its motive by $M(Y)$ (see [SV2, §1]). For a number l, the corresponding motivic complex is denoted by $\mathbb{Z}(l)$ (see [SV2, §2]). If X is a smooth variety, then we put $H^{i,j}(X,\mathbb{Z}) = H^i_{\mathrm{Nis}}(X,\mathbb{Z}(j))$. Next, the category Cor_k (defined in [SV2, §1]) is used systematically.

Let $f\colon X \to Y$ be a projective morphism of smooth algebraic varieties over a field k, $\operatorname{char} k = 0$, and let $l = \dim Y - \dim X$. We can construct the Gysin homomorphism $G(f)\colon M(Y) \to M(X)(l)[2l]$ for f in the category of Voevodsky's motives. Its construction is similar to that in [P2]. For this it suffices to define the first Chern class c_1 of the line bundle L (see [P1]); by definition, it coincides with the class $[L] \in \operatorname{Pic} X = H^{2,1}(X,\mathbb{Z})$. The Gysin homomorphism is defined separately for closed embeddings and projections $p\colon Y \times \mathbb{P}^n \to Y$. An arbitrary projective morphism $f\colon X \to Y$ can be factored into the composition $p \circ i$ of a closed embedding $i\colon X \hookrightarrow Y \times \mathbb{P}^n$ and the projection $p\colon Y \times \mathbb{P}^n \to Y$. The Gysin homomorphism can be defined as the composition $G(i) \circ G(p)$. It does not depend on factorization, see the proof of Theorem 2.12 in [P2].

The Gysin homomorphism for closed embeddings. Let $i\colon X \to Y$ be a regular closed embedding with normal bundle $N = N_{Y/X}$, let $\operatorname{rk} N = n$, and let $\mathrm{th}(N) \in H^{2n,n}_X(N,\mathbb{Z})$ be the Thom class of N in motivic cohomology (as defined in [P2]). By Theorem 1.5 in [SV2], we have isomorphism

$$H^{2n,n}_X(N,\mathbb{Z}) = \operatorname{Hom}_{DM(k)}(M_X(N), \mathbb{Z}(n)[2n]).$$

Thus, the Thom class $\mathrm{th}(N)$ gives rise to a unique morphism of motivic complexes $M_X(N) \to \mathbb{Z}(n)[2n]$, to be denoted also by $\mathrm{th}(N)$.

The Gysin homomorphism $G(i)\colon M(Y) \to M(X)$ is defined as the composition

$$M(Y) \to M_X(Y) \xrightarrow{M(i_0)} M_{X\times\mathbb{A}^1}(Y_t) \xrightarrow{(M(i_1))^{-1}} M_X(N) \xrightarrow{(\mathrm{th}\otimes M(p))\circ M(\Delta)} M(X)(n)[2n],$$

where Y_t is the deformation to a normal bundle, see [P1], and $\Delta\colon X \to X \times X$ is the diagonal embedding.

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The Gysin homomorphism for projections. Let $\pi\colon X\times\mathbb{P}^n\to X$ be a projection. By Theorem 4.4 in [SV2], we have canonical isomorphism

$$M(X\times\mathbb{P}^n)\xrightarrow{\sim} M(X)\oplus M(X)(1)[2]\oplus\ldots\oplus M(X)(n)[2n],$$

and the Gysin homomorphism $G(p)\colon M(X)\to M(X\times\mathbb{P}^n)(-n)[-2n]$ is simply the embedding of a direct summand.

Definition 1. Let $f\colon X\to Y$ be a finite morphism. The graph $\Gamma_f\subset X\times Y$ of f can be regarded as an element of $\operatorname{Cor}_k(X,Y)$. Then the transposed graph $(\Gamma_f)^t$ lies in $\operatorname{Cor}_k(Y,X)$, and thus, induces a morphism $M(Y)\to M(X)$ in $DM(k)$. Let $\mathcal{F}$ be a Nisnevich sheaf with transfers; the map $\operatorname{Tr}(f)\colon\mathcal{F}(X)\to\mathcal{F}(Y)$ induced by $(\Gamma_f)^t$ will be called a transfer.

The next theorem is the main result of this paper.

Theorem. *Let k be a field,* $\operatorname{char}k=0$, *let $F\subset E$ be a finite extension of fields of rational functions of algebraic varieties over k, and let $\mathcal{F}$ be an $\mathbb{A}^1$-invariant Nisnevich sheaf with transfers. Then the maps*

$$\operatorname{Tr}(f),\mathcal{F}(G(f))\colon\mathcal{F}(\operatorname{Spec}E)\to\mathcal{F}(\operatorname{Spec}F)$$

coincide.

Corollary. *Let $f\colon X\to Y$ be a finite morphism of smooth varieties over a field k,* $\operatorname{char}k=0$, *and let $\mathcal{F}$ be an $\mathbb{A}^1$-invariant Nisnevich sheaf with transfers. Then the maps*

$$\operatorname{Tr}(f),\mathcal{F}(G(f))\colon\mathcal{F}(X)\to\mathcal{F}(Y)$$

coincide.

The theorem and the corollary will be proved at the end of the paper, under the numbers 18 and 4.

In §2 we consider the category of relative motives $D_{\mathcal{M}}(S)$ for a sufficiently small variety S over k. We start with some statements similar to those in the case of motives over a field. At the end of §2 we calculate the group of morphisms between relative motives of varieties that are finite and *étale* over S. In §3 we construct Gysin homomorphisms in $D_{\mathcal{M}}(S)$ for some morphisms and prove that they coincide with transfers. Then, in §4, we construct the functor from $D_{\mathcal{M}}(S)$ to $DM(k)$ that agrres with the Gysin homomorphisms and the transfers. This allows us to prove the coincidence of the Gysin homomorphisms and the transfers in $DM(k)$.

§2. Category $D_{\mathcal{M}}(S)$ of relative motives

In this section, S is an irreducible smooth affine variety over a field k.

Definition 2. We shell denote by Cor_S the following category.

1) Its objects are varieties that are smooth over S;

2) for X, $Y\in Ob\operatorname{Cor}_S$, the morphisms $\operatorname{Cor}_S(X,Y)$ are elements of the free Abelian group generated by the irreducible subvarieties Z in $X\times_S W$ that are finite and surjective over X;

3) composition is induced by the fiber product.

Definition 3. A presheaf with S-transfers is a functor $\mathcal{F}\colon(\operatorname{Cor}_S)^{\mathrm{op}}\to Ab$. A sheaf with S-transfers is a presheaf with S-transfers whose restriction to Sm/S is a Nisnevich sheaf.

The category of Nisnevich sheafs with S-transfers is Abelian, it will be denoted by $NSwT^S$. Let $D(NShwT^S)$ be the derived category of Nisnevich sheaves with S-transfers on the category of varieties over S. Let X be an algebraic variety smooth over S. Denote $\mathbb{Z}^S_{\mathrm{tr}}[X](U)=\operatorname{Cor}_S(U,X)$.

Theorem 1. *For $X \in Sm/S$, the presheaf $\mathbb{Z}^S_{\mathrm{tr}}[X]$ is a Nisnevich sheaf on the category of varieties that are smooth over S.*

Let $A(S)$ be the full triangulated subcategory in $D(NSwT^S)$ generated by the cones of morphisms $\mathbb{Z}^S_{\mathrm{tr}}[X \times 0] \to \mathbb{Z}^S_{\mathrm{tr}}[X \times \mathbb{A}^1]$. We say that a complex A of Nisnevich sheaves with S-transfers is $\mathbb{A}^1$-local if its cohomology sheaves are strictly homotopy invariant (i.e., all their cohomology are homotopy invariant). Let $D_{\mathcal{M}}(S)$ be the full subcategory in $D(NSwT^S)$ formed by the $\mathbb{A}^1$-local objects. It is known that there exist an exact functor $L^{\mathbb{A}^1}_S\colon D(NSwT^S) \to D_{\mathcal{M}}(S)$ that satisfies $L^{\mathbb{A}^1}_S(A(S)) = 0$ and identifies the factor-category $D(NSwT^S)/A(S)$ with $D_{\mathcal{M}}(S)$. The functor $L^{\mathbb{A}^1}_S$ is the left adjoint functor to the full embedding $i\colon D_{\mathcal{M}}(S) \to D(NSwT^S)$, i.e., for $B \in D(NSwT^S)$ there exists a morphism $B \to i \circ L^{A^1}_S$ that is functorial in B and whose cone lies in $A(S)$.

Theorem 2. *Let X be a variety smooth over S, and let $Z \subset X$ be a closed subset. Then there is a distinguished triangle*

$$M^S(X - Z) \to M^S(X) \to M^S_Z(X) \to M^S(X - Z)[1]$$

in $D_{\mathcal{M}}(S)$.

Theorem 3. *Suppose that $\pi\colon X' \to X$ is an étale morphism, $Z \subset X$ is a closed subset, and $\pi\colon \pi^{-1}(Z) \to Z$ is an isomorphism. Then the map*

$$M^S(\pi)\colon M^S_Z(X') \to M^S_Z(X)$$

is an isomorphism.

Theorem 4. *The category $D(NSwT^S)$ admits a tensor product, to be denoted by $\otimes^S_{\mathrm{tr}}$, such that*
1) *the functor $A \otimes^S_{\mathrm{tr}} -$ is exact*;
2) $\mathbb{Z}^S_{\mathrm{tr}}(X) \otimes^S_{\mathrm{tr}} \mathbb{Z}^S_{\mathrm{tr}}(W) = \mathbb{Z}^S_{\mathrm{tr}}(X \times_S W)$.

Corollary 1. *The tensor product in $D(NSwT^S)$ induces a tensor product in $D_{\mathcal{M}}(S)$. The functor $L^{\mathbb{A}^1}_S$ is in agreement with the tensor structure.*

Theorem 5. *Let X be a variety smooth over S. Then we have isomorphism*

$$M^S(X)(1)[2] \simeq M^S_X(X \times \mathbb{A}^1)$$

in $D_{\mathcal{M}}(S)$.

Lemma 1. *Let $f\colon X' \to X$ be a Nisnevich covering of X. Then the sequence of Nisnevich sheaves*

$$0 \longleftarrow \mathbb{Z}^S_{\mathrm{tr}}[X] \xleftarrow{f_*} \mathbb{Z}^S_{\mathrm{tr}}[X] \xleftarrow{(p_2)_* - (p_1)_*} \mathbb{Z}^S_{\mathrm{tr}}[X' \underset{X}{\times} X'] \longleftarrow \dots$$

is exact.

Theorem 6. *Let X be a variety that is finite and étale over S. Then $M^S(X)$ is quasi-isomorphic to the Nisnevich sheaf $\mathbb{Z}^S_{\mathrm{tr}}[X]$.*

Proof. It suffices to check that the sheaf $\mathbb{Z}^S_{\mathrm{tr}}[X]$ is strictly $\mathbb{A}^1$-invariant. Let $f\colon X \to S$ be a structural morphism; then $\mathbb{Z}^S_{\mathrm{tr}}[X] = f_*\mathbb{Z}$. Observe that the sheaf $\mathbb{Z}^S_{\mathrm{tr}}[X]$ coincides with $f_*\mathbb{Z}$. Let S' be smooth over S, and let $X' = S' \times_S X$. Consider the Leray spectral sequence

$$H^p(V, R^q f_*\mathbb{Z}) \Rightarrow H^{p+q}(V, \mathbb{Z}).$$

The germ $(R^q f_*\mathbb{Z})_{s'}$ with $s' \in S'$ coincides with $H^q(X'_{s'}, \mathbb{Z})$, where $X'_{s'} = (S'_{s'})^h \times_{S'} X'$. The cohomology $H^q(X'_{s'}, \mathbb{Z})$ are zero for $i > 0$ because X' is finite over S'. Consequently, we conclude that $H^{p+q}(V, \mathbb{Z})$ coincides with $H^p(V, f_*\mathbb{Z})$. In particular, these cohomology are $\mathbb{A}^1$-invariant. ◻

Corollary 2. *Let X, Y be smooth varieties finite over S. Then*
$$\mathrm{Hom}_{D_{\mathcal{M}}(S)}(M^S(X), M^S(Y)) = \mathrm{Cor}_S(X, Y).$$

Definition 4. We define a functor $C^*\colon NSwT^S \to D(NSwT^S)$ as follows. Let $\mathcal{F}$ be a sheaf with S-transfers, then
$$(C^{-n}\mathcal{F})(U) = \mathcal{F}(\Delta^n \times U),$$
where $\Delta^n \subset \mathbb{A}^{n+1}$ is the hyperplane with the equation $\sum_{i=1}^n x_i = 1$. Let $\partial_i\colon \Delta^n \to \Delta^{n+1}$ be the embedding of the ith verge, then there is a map $\partial_i^*\colon \mathrm{Cor}_S(\Delta^{n+1} \times U, X) \to \mathrm{Cor}_S(\Delta^n \times U, X)$. The differential d in the complex $C^*\mathcal{F}$ is the sum
$$\sum_{i=0}^n (-1)^i \partial_i^*.$$

Theorem 7. 1) *If $T(X)$ is an arbitrary $\mathbb{A}^1$-local replacement* (*possibly, not functorial*), *then $T(X) \simeq L(X)$ in $D(NSwT^S)$.*

2) *If X is smooth over S, then $\mathbb{Z}_{\mathrm{tr}}^S[X] \to C^*\mathbb{Z}_{\mathrm{tr}}^S[X]$ is an isomorphism in $D_{\mathcal{M}}(S)$.*

3) *Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves in $D(NSwT^S)$. Then φ is an isomorphism in $D_{\mathcal{M}}(S)$ if and only if the map $C^*\varphi\colon C^*\mathcal{F} \to C^*\mathcal{G}$ is an isomorphism in $D(NSwT^S)$.*

Theorem 8. *Let S be an affine variety over a field k, and let X be affine, finite, and étale over S. Then the motive $M(X \times \mathbb{G}_m^{\wedge 1})[-1]$ is quasiisomorphic in $D(NSwT^S)$ to the sheaf represented in Sm/S by the variety $R_{X/S}(\mathbb{G}_{m,X})$.*

Proof. 1) First, we prove that the complex $C^*(\mathbb{Z}_{\mathrm{tr}}^S[X \times \mathbb{G}_m^{\wedge 1}])$ is quasiisomorphic to $R_{X/S}(\mathbb{G}_{m,X})$ in $D(NSwT^S)$. Let U be a smooth variety over S. Then the complex $C^*(\mathbb{Z}_{\mathrm{tr}}^S[X \times_S \mathbb{G}_m])$ evaluated on U is the following complex of Abelian groups:
$$\cdots \to \mathrm{Cor}_S(\Delta^1 \times U, X \times \mathbb{G}_m) \to \mathrm{Cor}_S(U, X \times \mathbb{G}_m) \to 0.$$
Denote $W = U \times_S X$ for brevity. For a relative curve $W \times \mathbb{G}_m \to U$, the group $C(W \times \mathbb{G}_m/U)$, in the notations of the paper [SV1], coincides with $\mathrm{Cor}_S(U, X \times \mathbb{G}_m)$. By Theorem 3.1 in [SV1], the complex $(C^*\mathbb{Z}_{\mathrm{tr}}^S[X \times \mathbb{G}_m])(U)$ is exact in all terms except zero, and $H^0((C^*\mathbb{Z}_{\mathrm{tr}}^S[X \times \mathbb{G}_m])(U))$ coincides with $\mathrm{Pic}(W \times \mathbb{P}^1, W \times \{0, \infty\})$, because the relative curve $W \times \mathbb{P}^1$ is a good compactification of $W \times \mathbb{G}_m$. The relative Picard group is included in the exact sequence
$$\begin{aligned}\Gamma(W \times \mathbb{P}^1, \mathcal{O}^*) &\to \Gamma(W \times \{0, \infty\}, \mathcal{O}^*) \\ &\to \mathrm{Pic}(W \times \mathbb{P}^1, W \times \{0, \infty\}) \to \mathrm{Pic}(W \times \mathbb{P}^1) \to \mathrm{Pic}(W \times \{0, \infty\}).\end{aligned}$$
We know four terms of this sequence, namely,
$$\mathrm{Pic}(W \times \mathbb{P}^1) \simeq \mathrm{Pic}(W) \oplus \mathbb{Z} \quad \text{and} \quad \mathrm{Pic}(W \times \{0, \infty\}) \simeq \mathrm{Pic}(W) \oplus \mathrm{Pic}(W).$$
Also, the last map in the exact sequence puts $\mathbb{Z}$ to zero, and the group $\mathrm{Pic}(W)$ is mapped diagonally. We have $\Gamma(W \times \mathbb{P}^1, \mathcal{O}^*) = k[W]^*$ and $\Gamma(W \times \{0, \infty\}, \mathcal{O}^*) = k[W]^* \oplus k[W]^*$, and the first map in the exact sequence is also diagonal. Thus, the relative Picard group is included in the short exact sequence
$$0 \to k[W]^* \to \mathrm{Pic}(W \times \mathbb{P}^1, W \times \{0, \infty\}) \to \mathbb{Z} \to 0.$$
This sequence splits because $\mathbb{Z}$ is a projective module. This splitting is functorial, so that
$$\mathrm{Pic}(U \underset{S}{\times} X \times \mathbb{P}^1, W \times \{0, \infty\}) = k[U \underset{S}{\times} X]^* \oplus \mathbb{Z}.$$
Note that divisor $U \times_S X \times 1 \subset U \times_S X \times \mathbb{G}_m$ corresponds to $1 \in \mathbb{Z} = \mathrm{Pic}(U \times_S X \times \mathbb{P}^1)$. Therefore, the complex $C^*\mathbb{Z}_{\mathrm{tr}}^S[X \times \mathbb{G}_m^{\wedge 1}]$ is quasiisomorphic to the sheaf $U \mapsto \mathcal{O}^*(U \times_S X)$, which coincides with the representable sheaf $\mathrm{Hom}_{Sm/S}(-, R_{X/S}(\mathbb{G}_{m,X}))$.

2) The sheaf $R_{X/S}(\mathbb{G}_{m,X})$ is strictly $\mathbb{A}^1$-invariant, which can be proved in the same way as in Theorem 6. Thus, $M^S(X \times \mathbb{G}_m^{\wedge 1})[-1]$ is isomorphic to $R_{X/S}(\mathbb{G}_{m,X})$ by Theorem 7. □

Corollary 3. *Let X be finite and étale over S. Then*

1) $\operatorname{Hom}_{D_{\mathcal{M}}(S)}(M^S(S)(1), M^S(X)(1)) = \operatorname{Hom}_{D_{\mathcal{M}}(S)}(M^S(S), M^S(X))$;

2) $\operatorname{Hom}_{D_{\mathcal{M}}(S)}(M^S(X)(1), M^S(X)(1)) = \operatorname{Hom}_{D_{\mathcal{M}}(S)}(M^S(X), M^S(X))$.

Proof. We prove statement 1. For simplicity, let X be irreducible. On the right-hand side the group is isomorphic $\mathbb{Z}$ by Corollary 2. By Theorem 8, the motive $M^S(X)(1)$ is quasi-isomorphic in $D(NSwT^S)$ to the sheaf representable by $R_{X/S}(\mathbb{G}_{m,X})$, and $M^S(S)(1)$ is representable by $S\times\mathbb{G}_m$ in the category of smooth varieties over S. As a result, the group of morphisms on the left-hand side is embedded in $\operatorname{Hom}_{Sm/S}(S \times \mathbb{G}_m, R_{X/S}(\mathbb{G}_{m,X}))$, which corresponds to all sheaf morphisms with neither transfer structure nor Abelian structure. By the conjugation property of the Weil restrictions functor, this set coincides with $\mathbb{Z}\oplus k[X]^*$. It is easy to understand that the group $\operatorname{Hom}_{NSwT^S}(S\times\mathbb{G}_m, R_{X/S}(\mathbb{G}_{m,X}))$ is simply $\mathbb{Z}$. Statement 2 can be proved similarly. □

Theorem 9. *In $D_{\mathcal{M}}(S)$ we have the isomorphism*

$$M^S(S \times \mathbb{P}^1) \xrightarrow{1,\tau} M^S(S) \oplus M^S(S)(1)[2],$$

where the map τ corresponds to the element $\mathcal{O}(1) \in \operatorname{Pic}\mathbb{P}^1$.

§3. Coincidence of the Gysin homomorphisms and the transfer in $D_{\mathcal{M}}(Y)$

In this section we prove the coincidence of the Gysin homomorphism and the transfer in $D_{\mathcal{M}}(Y)$ for a finite *étale* map of sufficiently small varieties in a suitable category $D_{\mathcal{M}}(Y)$. First, we fix some variety Y and construct a Gysin homomorphism in $D_{\mathcal{M}}(Y)$.

Theorem 10. *Suppose* $\operatorname{char} k = 0$ *and $f\colon X \to Y$ is a finite morphism of smooth affine varieties over k. Then there exist open subsets $U \subset X$, $V \subset Y$ such that $f|_U\colon U \to V$ and $k[U] = k[V]/F(t)$, where $F(t)$ is an irreducible monic polynomial.*

Proof. The extension $k(X)/k(Y)$ is separable and so it is simple because $\operatorname{char} k = 0$. Therefore, $k(X) = k(Y)[t]/F(t)$, where $F(t)$ is an irreducible monic polynomial. Removing from Y the zeros of the denominators of the coefficients of $F(t)$ and of the leading term, and removing their full preimages from X, we obtain the same relation for the rings of functions of the open subsets X and Y. For reducible X, we can multiply the polynomials for irreducible components. □

Theorem 11. *Let $f\colon X \to Y$ be a finite étale morphism obtained in Theorem* 10, *i.e., $k[X] = k[Y][t]/F(t)$ for an irreducible monic polynomial $F(t)$ of nth degree. Then, making Y smaller and taking $X = \operatorname{Spec} k[Y][t]/F(t)$, we may assume that there exists a finite étale morphism $g\colon Z \to Y$ such that $Z \times_Y X \xrightarrow{\sim} \bigsqcup_{i=1}^n Z$.*

Proof. Let $\alpha_1, \dots, \alpha_n$ be the roots of $F(t)$ in the algebraic closure of $k(Y)$. For any α_i we can construct Z_i, which is finite and *étale* over Y_i (Y_i is an open subset in Y) in the same way as in Theorem 10. Crossing all Y_i, we get Y. Also, we reduce the Z_i by taking the preimages of Y. We define Z as fiber product of all Z_i over Y. □

In what follows we shall assume that $f\colon X \to Y$ and Z satisfy the conditions of Theorem 11.

Remark 1. Our construction shows that $k(Y)$ and $k(X)$ can be presented as limits $k(Y) = \varinjlim_\beta k[U]$, $k(X) = \varinjlim_\beta k[U][t]/F_\beta(t)$, where $F_\beta(t)$ is an irreducible monic polynomial. Next, we may assume that the index sets in the two limits above are the same, let it be I, and that for a finite *étale* map $V = \operatorname{Spec} k[U][t]/F_\beta(t) \to U$ there exist variety Z_β such that $Z_\beta \times_U V \xrightarrow{\sim} \bigsqcup_{i=1}^n Z_\beta$.

Construction. We shall construct the Gysin homomorphism $G^Y(f)\colon M^Y(Y) \to M^Y(X)$ for a morphism $f\colon X \to Y$ in $D_{\mathcal{M}}(Y)$. By construction, f can be factored as $X \hookrightarrow Y \times \mathbb{P}^1 \to Y$, where the normal bundle for the embedding $X \hookrightarrow Y \times \mathbb{P}^1$ is trivialized. First, we define the twisted Gysin homomorphism $G^Y(f)(1)[2]$ as the composition

$$\begin{array}{ccccc}
M^Y(Y)(1)[2] & & \xrightarrow{\quad G(f)(1)[2] \quad} & & M^Y(X)(1)[2] \\
\downarrow{\scriptstyle \alpha} & & & & \\
M^Y(Y \times \mathbb{P}^1) & & & & \downarrow{\scriptstyle \varphi} \\
\downarrow & & & & \\
M^Y_X(Y \times \mathbb{P}^1) & & & & \\
\uparrow{\scriptstyle e} & & & & \\
M^Y_X(Y \times \mathbb{A}^1) & \xrightarrow{(i_1)_*} & M^Y_{X\times\mathbb{A}^1}((Y \times \mathbb{P}^1)_t) & \xleftarrow{(i_0)_*} & M^Y_{X\times 0}(X \times \mathbb{A}^1),
\end{array}$$

where α is the embedding of a direct summand as in Theorem 9, the map e is the excision isomorphism as in Theorem 3, $(Y \times \mathbb{P}^1)_t$ is deformation to a normal bundle for the embedding $X \hookrightarrow Y \times \mathbb{P}^1$ (its construction can be found in [P1]), and φ is the isomorphism occurring in Theorem 5. By part 1 of Corollary 3, we have isomorphism

$$\operatorname{Hom}_{D_{\mathcal{M}}(Y)}(M^Y(Y)(1)[2], M^Y(X)(1)[2]) = \operatorname{Hom}_{D_{\mathcal{M}}(Y)}(M^Y(Y), M^Y(X)).$$

The map $M^Y(Y) \to M^Y(X)$ that corresponds to $G^Y(f)(1)[2]$ by this isomorphism is the Gysin homomorphism $G^Y(f)$.

We shall need the Gysin homomorphism $G^Y(Z \times_Y f)\colon M^Y(Z) \to M^Y(Z \times_Y X)$ for the morphism $Z \times_Y f\colon Z \times_Y X \to Z$. Considering fiber product by Z of the factorization $X \hookrightarrow Y \times \mathbb{P}^1 \to Y$ of f in the composition of an embedding and a projection, we get a similar factorization for $Z \times_Y f$. The twisted Gysin homomorphism $G^Y(Z \times_Y f)(1)[2]$ is defined by the same construction, and an analog of the cancellation isomorphism from part 2 of Corollary 3 allows us to define $G^Y(Z \times_Y f)$, because $Z \times_Y X \xrightarrow{\sim} \bigsqcup_{i=1}^n Z$.

Theorem 12. *The Gysin homomorphism*

$$G^Y(Z \underset{Y}{\times} f)\colon M^Y(Z) \to \bigoplus_{i=1}^n M^Y(Z)$$

coincides with the diagonal map.

Proof. In the case of oriented cohomology theories [P2], a similar statement follows from the next two claims.

1) The Gysin homomorphism for a smooth reducible variety is the sum of the Gysin homomorphisms for its irreducible components,

2) The Gysin homomorphism for an isomorphism $u\colon Z \to Z$ is the pullback $(u^{-1})^*$.

Claim 1 is proved as in [P2, Subsection 2.4.9]. Claim 2 follows from the compatibility of the Gysin homomorphisms and pullbacks for the transversal Cartesian square. For our aims it suffices to prove compatibility for one Cartesian square

$$\begin{array}{ccc} Z & \xrightarrow{u} & Z \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle u^{-1}} \\ Z & \xrightarrow[\mathrm{id}]{} & Z. \end{array}$$

This can be proved as in [P2, Subsection 2.4.3]. □

We need some notation. The functor of fiber product by Z over Y induces a map $M^Y(Z) \otimes^Y_{\mathrm{tr}} -\colon D_{\mathcal{M}}(Y) \to D_{\mathcal{M}}(Y)$. For simplicity, we shall denote it by $Z \times_Y -$. Also, let $(\Gamma^Y_f)^t \in \mathrm{Cor}_Y(Y, X)$ be the transposed graph of f. It induces a map $M^Y(Y) \to M^Y(X)$.

Theorem 13. *The functor $Z \times_Y -$ induces a map*

$$\begin{array}{ccc} \mathrm{Hom}_{D_{\mathcal{M}}(Y)}(M^Y(Y), M^Y(X)), & \longrightarrow & \mathrm{Hom}_{D_{\mathcal{M}}(Y)}(M^Y(Z), M^Y(Z \underset{Y}{\times} X)). \\ \Big\| & & \Big\| \\ \mathbb{Z} & & \bigoplus_{i=1}^n \mathrm{Cor}_Y(Z, Z) \end{array}$$

and the following is true:
1) $(\Gamma^Y_f)^t = 1 \in \mathbb{Z}$;
2) $Z \times_Y (\Gamma^Y_f)^t = (\Gamma^Y_{Z \times_Y f})^t = \Gamma_{\mathrm{id}} \oplus \Gamma_{\mathrm{id}} \oplus \ldots \oplus \Gamma_{\mathrm{id}} \in \bigoplus_{i=1}^n \mathrm{Cor}_Y(Z, Z)$.

Proof. Item 1 is the statement of Corollary 2. Item 2 hold true by the choice of Z. □

Theorem 14. *The functor $Z \times_Y -$ takes the Gysin homomorphism $G^Y(f)$ to the Gysin homomorphism $G^Y(Z \times_Y f)$.*

Proof. The statement follows from our construction. The theorem is similar to [SV2, Lemma 4.9 (1)]. □

Theorem 15. *The Gysin homomorphism $G^Y(f)$ coincides with the transfer $(\Gamma^Y_f)^t$ in $D_{\mathcal{M}}(Y)$.*

Proof. We apply $Z \times_Y -$ to the two morphisms in question. By Theorem 13, on groups of morphisms the map is injective. Moreover, the Gysin homomorphism goes to the Gysin homomorphism and transfers go to transfers. After fiber product, they coincide by Theorems 12 and 14. □

§4. Coincidence of the Gysin homomorphism and transfer in $DM(k)$

Theorem 16. *There exist a functor $\Theta\colon D_{\mathcal{M}}(Y) \to DM(k)$ such that*
1) $\Theta(M^Y(X)) = M(X)$;
2) *the following diagram commutes*:

$$\begin{array}{ccc} \mathrm{Hom}_{D_{\mathcal{M}}(Y)}(M^Y(X), M^Y(Z)) & \xrightarrow{\Theta} & \mathrm{Hom}_{D_{\mathcal{M}}(Y)}(M(X), M(Z)) \\ \uparrow & & \uparrow \\ \mathrm{Cor}_Y(X, Z) & \hookrightarrow & \mathrm{Cor}(X, Z). \end{array}$$

Proof. The proof is similar to the construction of a tensor product in $DM(k)$ (it can be found in [SV2]).

1) Let $\mathcal{F}$ be a presheaf with Y-transfers. Then we have the exact sequence

$$0 \longleftarrow \mathcal{F} \longleftarrow \bigoplus_{X \in Sm/Y} \mathcal{F}(X) \otimes \mathbb{Z}^Y_{\mathrm{tr}}[X] \longleftarrow \bigoplus_{f \in \mathrm{Cor}_Y(X,Z)} \mathcal{F}(Z) \otimes \mathbb{Z}^Y_{\mathrm{tr}}[X]. \tag{1}$$

We define a functor on the representable sheaves by $\Theta(\mathbb{Z}_{\mathrm{tr}}^Y[X]) = \mathbb{Z}_{\mathrm{tr}}[X]$ and

$$\Theta\colon \operatorname{Hom}_{PreShwT^Y}(\mathbb{Z}_{\mathrm{tr}}^Y[X], \mathbb{Z}_{\mathrm{tr}}^Y[Z]) \to \operatorname{Hom}_{PreShwT}(\mathbb{Z}_{\mathrm{tr}}[X], \mathbb{Z}_{\mathrm{tr}}[Z])$$

by the embedding

$$\mathrm{Cor}_Y(X, Z) \hookrightarrow \mathrm{Cor}_k(X, Z).$$

The exact sequence (1) allows us to define the functor Θ on the category of presheaves with Y-transfers.

2) The functor $\Theta\colon PreShwT^Y \to PreShwT$ is right exact. Suppose that a sequence of presheaves $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact, i.e., the sequence of sections is exact for any $X \in Sm/Y$. We apply the functor Θ for each presheaf by using the exact sequence (1), and obtain the diagram

$$\begin{array}{ccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
\bigoplus_{f \in \mathrm{Cor}_Y(X,Z)} \mathcal{F}'(Z) \otimes \mathbb{Z}_{\mathrm{tr}}[X] & \longrightarrow & \bigoplus_{X \in Sm/Y} \mathcal{F}'(X) \otimes \mathbb{Z}_{\mathrm{tr}}[X] & \longrightarrow & \Theta(\mathcal{F}') \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{f \in \mathrm{Cor}_Y(X,Z)} \mathcal{F}(Z) \otimes \mathbb{Z}_{\mathrm{tr}}[X] & \longrightarrow & \bigoplus_{X \in Sm/Y} \mathcal{F}(X) \otimes \mathbb{Z}_{\mathrm{tr}}[X] & \longrightarrow & \Theta(\mathcal{F}) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{f \in \mathrm{Cor}_Y(X,Z)} \mathcal{F}''(Z) \otimes \mathbb{Z}_{\mathrm{tr}}[X] & \longrightarrow & \bigoplus_{X \in Sm/Y} \mathcal{F}''(X) \otimes \mathbb{Z}_{\mathrm{tr}}[X] & \longrightarrow & \Theta(\mathcal{F}''). \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}$$

Here $\Theta(\mathcal{F}')$, $\Theta(\mathcal{F})$, and $\Theta(\mathcal{F}'')$ are the corresponding cokernels. The rows are exact by definition. Note that the first two columns are exact as tensor product of exact sequence by a free Abelian group. The right exactness of Θ follows from the long exact sequence of cohomologies for the exact sequence of two-term row-complexes.

3) Let $L_i\Theta\colon PreShwT^Y \to PreShwT$ be the left derived functors of Θ. □

Claim. *Let $\mathcal{F}$ be a presheaf with Y-transfers such that $\mathcal{F}^{\sim}_{\mathrm{Nis}} = 0$. Then $L_i\Theta(\mathcal{F})^{\sim}_{\mathrm{Nis}} = 0$ for any $i \geq 0$.*

Proof of the claim. Let $f\colon X' \to X$ be a Nisnevich covering. Denote by $C_*(X'/X, Y)$ the complex of presheaves

$$0 \leftarrow \mathbb{Z}_{\mathrm{tr}}^Y[X] \xleftarrow{f_*} \mathbb{Z}_{\mathrm{tr}}^Y[X'] \xleftarrow{(p_2)_* - (p_1)_*} \mathbb{Z}_{\mathrm{tr}}^Y[X' \underset{X}{\times} X'] \leftarrow \dots,$$

and by $\mathcal{H}_i(X'/X)$ its presheaves cohomology . From Lemma 1 we know that it is exact as a complex of sheaves, i.e., $\mathcal{H}_i(X'/X)^{\sim}_{\mathrm{Nis}} = 0$ for any i. For a presheaf with Y-transfers $\mathcal{F}$ such that $\mathcal{F}^{\sim}_{\mathrm{Nis}} = 0$, there exists a surjection (in the category of presheaves) from the direct sum of presheaves of type $\mathcal{H}_i(X'/X)$ for some X', X to $\mathcal{F}$. We proceed by induction on i. For $i < 0$ the statement is obvious. Suppose the statement is true for all i with $i < j$. We prove it for j. It suffices to argue in the case where $\mathcal{F} = \mathcal{H}_0(X'/X)$ (by the existence of a surjective map from the sum of presheaves of this type).

Consider the spectral sequence

$$E^2_{pq} = L_p\Theta(\mathcal{H}_q(X'/X)) \Longrightarrow H_{p+q}(\Theta(C_*(X'/X))). \qquad (2)$$

The sequence $C_*(X'/X)$,

$$0 \leftarrow \mathbb{Z}_{\mathrm{tr}}[X] \xleftarrow{f_*} \mathbb{Z}_{\mathrm{tr}}[X'] \xleftarrow{(p_2)_*-(p_1)_*} \mathbb{Z}_{\mathrm{tr}}[X' \underset{X}{\times} X'] \leftarrow \dots,$$

is exact by Lemma 1.6 in [SV2]. Since $\Theta(C_*(X'/X,Y)) = C_*(X'/X)$, if we consider the sheafication of the spectral sequence (2), the right-hand side will become zero, i.e.,

$$E^2_{pq} = L_p\Theta(\mathcal{H}_q(X'/X))^{\sim}_{\mathrm{Nis}} \Longrightarrow 0.$$

Using the inductive assumption, we get $L_j\Theta(\mathcal{H}_0(X'/X)) = 0$.

4) We define the functor Θ on sheaves with Y-transfers as sheafication of evaluation of this functor on presheaves.

5) Let A_* be a complex bounded from above whose terms are sums of presheaves of type $\oplus_X \mathbb{Z}^Y_{\mathrm{tr}}[X]$. Suppose $H_i(A_*)^{\sim}_{\mathrm{Nis}} = 0$ for any i. Then $H_i(\Theta(A_*))^{\sim}_{\mathrm{Nis}} = 0$ for any i.

This follows from the spectral sequence (2). The statement of item 5 means that Θ induces a functor on derived categories, because every complex has canonical resolvent of presheafs of the form $\mathbb{Z}_{\mathrm{tr}}[X]$. This allows us to define the functor Θ on the category $D(NSwT^Y)$.

6) The composition $C^* \circ \Theta$ puts the subcategory $A(S)$ to 0. Thus, Θ induces a functor $D_{\mathcal{M}}(Y) \to DM(k)$ that satisfies the conditions of Theorem 16. □

Theorem 17. *The functor Θ has the following properties*:
1) $\Theta((\Gamma^Y_f)^t) = (\Gamma_f)^t$,
2) $\Theta(G^Y(f)) = G(f)$.

Proof. Statement 1 follows from the construction of Θ. The construction of the Gysin homomorphism shows that Statement 2 is a consequence of the following two claims.

a) There exist a functor $\Theta^*\colon DM(k) \to D_{\mathcal{M}}(Y)$. It is constructed much as Θ is; on the representable presheaves it is defined by the formula $\Theta^*(\mathbb{Z}_{\mathrm{tr}}[X]) = \mathbb{Z}_{\mathrm{tr}}[X \times S]$. These functors are related by the projection formula

$$\Theta\left(M^Y(W) \otimes^Y_{\mathrm{tr}} \Theta^*(M(Z))\right) = \Theta\left(M^Y(W)\right) \otimes_{\mathrm{tr}} M(Z).$$

b) The functor Θ takes the isomorphism

$$M^Y(Y \times \mathbb{P}^1) \xrightarrow{1,\tau} M^Y(Y) \oplus M^Y(Y)(1)[2]$$

in $D_{\mathcal{M}}(Y)$ described in Theorem 9, where $\tau \in \operatorname{Pic}\mathbb{P}^1$, to the isomorphism

$$M(Y \times \mathbb{P}^1) \xrightarrow{1,\tau} M(Y) \oplus M(Y)(1)[2]$$

in $DM(k)$. This can be checked straightforwardly. □

Let $f\colon X \to Y$ be a finite map and $\mathcal{F}$ an $\mathbb{A}^{-1}$-invariant sheaf with transfers. For open $U \subset Y$ and $V = f^{-1}(U)$, the morphism $f|_V\colon V \to U$ is also finite, and it induces morphisms $\mathcal{F}(G(f))$, $\operatorname{Tr}(f) = \mathcal{F}((\Gamma_f)^t)\colon \mathcal{F}(V) \to \mathcal{F}(U)$. Taking limits, we get the maps $\mathcal{F}(G(f))$ and $\operatorname{Tr}(f)$ on the fields of rational functions.

Theorem 18. *Let $\operatorname{char} k = 0$, and let $k(X)/k(Y)$ be a finite extension of fields of rational functions of varieties over k. Then the Gysin homomorphism and the transfer $\mathcal{F}(G(f)), \operatorname{Tr}(f)\colon \mathcal{F}(\operatorname{Spec} k(X)) \to \mathcal{F}(\operatorname{Spec} k(Y))$ coincide.*

Proof. As was mentioned in Remark 1, the fields $k(X)$ and $k(Y)$ are limits over the set of morphisms of open subsets $U \subset X$, $V \subset Y$ of a special type. Therefore, it suffices to prove the coincidence of $G(f)$ and $(\Gamma_f)^t$ for $f\colon X \to Y$ in the case where $k[X] = k[Y][t]/F(t)$ for some monic irreducible F and there exists Z with $Z \times X \xrightarrow{\sim} \coprod_{i=1}^n Z$. By Theorem 15, the maps $G^Y(f)$, $(\Gamma^Y_f)^t\colon M^Y(Y) \to M^Y(X)$ coincide in $D_{\mathcal{M}}(Y)$. Applying Θ, we get the coincidence of these maps in $DM(k)$. Therefore, we get a similar statement for sheaves with transfers. Taking the limit, we complete the proof. □

Corollary 4. *Let $f\colon X \to Y$ be a finite map of smooth irreducible varieties and $\mathcal{F}$ an $\mathbb{A}^1$-invariant Nisnevich sheaf with transfers. Then the maps*

$$\mathcal{F}(G(f)), \mathrm{Tr}(f)\colon \mathcal{F}(X) \to \mathcal{F}(Y)$$

coincide.

Proof. This follows from Theorem 18 and the injectivity lemma $\mathcal{F}(X) \hookrightarrow \mathcal{F}(k(X))$, $\mathcal{F}(Y) \hookrightarrow \mathcal{F}(k(Y))$ (proved in [V1]). □

References

[SV1] A. A. Suslin and V. A. Voevodsky, *Singular homology of abstract algebraic varieties*, Invent. Math. **123** (1996), no. 1, 61–94. MR1376246

[SV2] ———, *Bloch–Kato conjecture and motivic cohomology with finite coefficients*, The Arithmetics and Geometry of Algebraic Cycles, NATO Sci. Ser. C Math. Phys. Sci., vol. 548, Kluwer Acad. Publ., Dordrecht, 2000, pp. 117–189. MR1744945

[P1] I. A. Panin, *Oriented cohomology theories of algebraic varieties.* III, K-Theory **30** (2003), no. 3, 265–314. MR2064242

[P2] ———, *Oriented cohomology theories of algebraic varieties.* II, Homology, Homotopy Appl. **11** (2009), no. 1, 349–405. MR2529164

[V1] V. A. Voevodsky, *Cohomological theory of presheaves with transfers*, Cycles, Transfers and Motivic Cohomology Theories, Ann. of Math. Stud., vol. 143, Princeton Univ. Press, Princeton, NJ, 2000, pp. 138–187. MR1764200

[V2] ———, *Triangulated category of motives over a field*, Cycles, Transfers and Motivic Cohomology Theories, Ann. of Math. Stud., vol. 143, Princeton Univ. Press, Princeton, NJ, 2000, pp. 188–238. MR1764202

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