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ON m-COMMUTING MAPPINGS WITH SKEW DERIVATIONS IN PRIME RINGS

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Abstract. Let m, k be two fixed positive integers, R a prime ring with the Martindale qoutient ring Q, L a noncommutative Lie ideal of R, and δ a skew derivation of R associated with an automorphism φ, denoted by (δ, φ). If $[\delta(x), x^m]_k = 0$ for all $x \in L$, then $\mathrm{char}(R) = 2$ and $R \subseteq M_2(F)$ for some field F.

§1. Introduction

Throughout this paper, unless specifically stated, R will be an associative ring. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator. For $x, y \in R$ and each $k \geq 0$, we set $[x, y]_0 = x$, $[x, y]_1 = xy - yx$ and inductively $[x, y]_k = [[x, y]_{k-1}, y]$ for $k > 1$. Notice that an Engel condition is a polynomial $[x, y]_k = \sum_{i=0}^{k}(-1)^i\binom{k}{i}y^ixy^{k-i}$ in noncommutative indeterminates x, y. The ring R satisfies an Engel condition if there exists a positive integer k such that $[x, y]_k = 0$. Recall that a ring R is prime if for any $a, b \in R$, $aRb = \{0\}$ implies $a = 0$ or $b = 0$. An additive mapping $d\colon R \to R$ is called a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In particular, d is an inner derivation induced by an element $a \in R$ if $d(x) = [a, x]$ for all $x \in R$. An additive subgroup L of R is called a *Lie ideal* if $[u, r] \in L$ for all $u \in L$, $r \in R$. A Lie ideal L is said to be *noncommutative* if $[L, L] \neq 0$. Let L be a noncommutative Lie ideal of R. It is well known that $[R[L, L]R, L] \subseteq L$ (see the proof of [14, Lemma 1.3]). Since $[L, L] \neq 0$, we have $0 \neq [I, R] \subseteq L$, where $I = R[L, L]R$ is a nonzero ideal of R.

Given any automorphism φ of R, an additive mapping $\delta\colon R \to R$ satisfying $\delta(xy) = \delta(x)y + \varphi(x)\delta(y)$ for all $x, y \in R$ is called a *φ-derivation of R*, or a *skew derivation of R with respect to* φ, denoted by (δ, φ). It is easily seen that if $\varphi = 1_R$, the identity map of R, then a φ-derivation is merely a usual derivation. And if $\varphi \neq 1_R$, then $\varphi - 1_R$ is a skew derivation. Thus the concept of skew derivations can be regarded as a generalization of both derivations and automorphisms. For a subset S of R, a mapping $f\colon S \to R$ is said to be *commuting* (*centralizing*) on S if $[f(x), x] = 0$ (respectively, $[f(x), x] \in Z(R)$) for all $x \in S$. The study of commuting and centralizing mappings goes back to 1955 when Divinsky [12] proved that a simple Artinian ring is commutative if it has a commuting automorphism different from the identity mapping. Two years later, Posner [28] showed that a prime ring must be commutative if it possesses a nonzero centralizing derivation. In 1970, Luh [25] generalized Divinsky's result to prime rings. Later, Mayne [26] obtained an analog of Posner's result for nonidentity centralizing automorphisms. Similar results extended to the case of Lie ideals were studied by Lanski [17], Lee and Lee [21], Mayne [26], and Lee [24], and to the case of left ideals by Bell and Martindale [3], and Lanski [20]. In 1995, Deng and Bell [11] extended the notion of commuting maps to n-commuting and centralizing to n-centralizing maps. For a subset S of

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R and n a fixed positive integer, a mapping $f\colon S \to R$ is *n-centralizing* (*n-commuting*) on S if $[f(x), x^n] \in Z(R)$ (respectively, $[f(x), x^n] = 0$) for all $x \in S$. For more details on n-centralizing (n-commuting) maps we refer the reader to [10, 11, 29] and [1].

In the present paper we prove a result for a skew derivation on Lie ideals of R in a systematic way by using the theory of generalized polynomial identities with automorphism and skew derivations, as developed by Kharchenko [16], Chuang [6, 7], and recently by Chuang and Lee [8], who generalized the result of Deng and Bell [11, Theorem 2].

More precisely, we prove the following theorem.

Theorem 1.1. *Let R be a prime ring and L a noncommutative Lie ideal of R. Suppose that (δ, σ) is a nonzero skew derivation of R such that $[\delta(x), x^m]_k = 0$ for all $x \in L$, where k, m are fixed positive integers. Then* $\mathrm{char}(R) = 2$ *and $R \subseteq M_2(F)$ for some field F.*

§2. Preliminaries

Throughout this paper, R is always an associative prime ring, $Z(R)$ the center of R, Q its Martindale quotient ring, and U its Utumi quotient ring. The center of U, denoted by C, is called the *extended centroid* of R (we refer the reader to [2] for the definitions and related properties of these objects). For any two subsets A and B of R, $[A, B]$ stands for the additive subgroup of R generated by the elements $[x, y]$ for all $x \in A$ and $y \in B$.

Let φ be an automorphism of R. For $b \in R$, the map $\delta : x \in R \to \varphi(x)b - bx$ gives rise to a φ-derivation of R; then (δ, φ) is called an inner skew derivation. It is well known that any automorphism of R can be extended uniquely to an automorphism of Q. Any skew derivation (δ, φ) extends uniquely to a skew derivation of Q via extensions of each map to Q. Thus, we may assume that any skew derivation of R is the restriction of a skew derivation of Q. Recall that a skew derivation (δ, φ) of R is said to be *Q-inner* if its extension to Q is inner, that is, there exists $q \in Q$ such that $\delta(x) = \varphi(x)q - qx$ for all $x \in Q$, and otherwise it is Q-outer. Analogously, an automorphism φ of R is *inner* if when acting on Q, $\varphi(x) = gxg^{-1}$ for some invertible element $g \in Q$. When φ is not inner, then it is called an outer automorphism.

An automorphism φ of Q is *Frobenius* [7] if, in the case of $\mathrm{char}(R) = 0$, we have $\delta(\lambda) = \lambda$ for all $\lambda \in C$, and, in the case of $\mathrm{char}(R) = p \geq 2$, we have $\varphi(\lambda) = \lambda^{p^n}$ for all $\lambda \in C$, where n is a fixed integer, positive, zero, or negative.

Let V_D be a vector space over a division ring D. We denote by $\mathrm{End}(V)$ the ring of endomorphisms on V and by $\mathrm{End}(V_D)$ the ring of D-linear transformations on V_D. An additive map $T \in \mathrm{End}(V)$ is called a *semilinear transformation* if for some automorphism τ of D we have $T(v\lambda) = T(v)\tau(\lambda)$ for all $v \in V$ and $\lambda \in D$ (see [15, p. 44]).

Let $Q_{*C}C\{X\}$ be the free product of Q and the free algebra $C\{X\}$ over C on an infinite set X of indeterminates. The elements of $Q_{*C}C\{X\}$ are called generalized polynomials, and a typical element in $Q_{*C}C\{X\}$ is a finite sum of monomials of the form $\alpha a_{i_0} x_{j_1} a_{i_1} x_{j_2} \cdots x_{j_n} a_{i_n}$, where $\alpha \in C$, $a_{ik} \in Q$, and $x_{jk} \in X$. We say that R satisfies a nontrivial generalized polynomial identity (abbreviated as GPI) if there exists a nonzero polynomial $\phi(x_i) \in Q_{*C}C\{X\}$ such that $\phi(r_i) = 0$ for all $r_i \in R$. By a generalized polynomial identity with automorphisms and skew derivations, we mean an identity of R expressed as $\phi(\varphi_j(x_i), \delta_k(x_i))$, where each φ_j is an automorphism, each δ_k is a skew derivation of R, and $\phi(y_{ij}, z_{ik})$ is a generalized polynomial in distinct indeterminates y_{ij}, z_{ik}. We need some well-known facts which will be used in the sequel.

Fact 2.1 (see [16, p. 40]). Let φ be an automorphism of a prime ring R. Suppose R satisfies a nontrivial generalized polynomial identity and $\varphi(\alpha) = \alpha$ for all $\alpha \in C$. Then φ is a Q-inner automorphism.

The next fact can easily be deduced from [19] and [7].

Fact 2.2. Let L be a noncommutative Lie ideal of a prime ring R, and let $\phi(\varphi_j(x_i))$ be a generalized polynomial with automorphisms, where each φ_j is an automorphism of R. If $\phi(\varphi_j(x_i)) = 0$ for all $x_i \in L$, then $\phi(\varphi_j(x_i)) = 0$ for all $x_i \in [Q, Q]$.

Fact 2.3 (see [7, Theorem 2]). Let R be a prime ring with an automorphism φ. Suppose that φ is not a Frobenius automorphism of R. Then any generalized polynomial identity of R of the form $\phi(x_i, \varphi(x_i)) = 0$ yields the generalized polynomial identity $\phi(x_i, y_i) = 0$ of R, where the x_i, y_i are distinct indeterminates.

Fact 2.4 (see [8, Theorem 1]). Let R be a prime ring with an automorphism φ. Suppose that δ is a Q-outer φ-derivation of R. Then any generalized polynomial identity of R of the form $\phi(x_i, \delta(x_i)) = 0$ yields the generalized polynomial identity $\phi(x_i, y_i) = 0$ of R, where the x_i, y_i are distinct indeterminates.

Fact 2.5 (see [8, Theorem 1]). Let R be a prime ring with an automorphism φ. Suppose that δ is a Q-outer φ-derivation of R. Then any generalized polynomial identity of R of the form $\phi(x_i, \varphi(x_i), \delta(x_i)) = 0$ yields the generalized polynomial identity $\phi(x_i, y_i, z_i) = 0$ of R, where the x_i, y_i, z_i are distinct indeterminates.

Fact 2.6 (see [22, Proposition]). Let R be a prime algebra over an infinite field k, and let K be a field extension over k. Then R and $R \otimes_k K$ satisfy same generalized polynomial identities with coefficients in R.

The next result is a slight generalization of [18, Lemma 2], and can be obtained directly by the proof of [18, Lemma 2], and Fact 2.6. So we omit its proof here.

Fact 2.7. Let R be a noncommutative simple algebra finite-dimensional over its center $Z(R)$. Then $R \subseteq M_n(F)$ with $n > 1$ for some field F, and R and $M_n(F)$ satisfy the same generalized polynomial identities with coefficients in R.

Fact 2.8 (see [23, Lemma 1.2]). Let R be a prime ring, and let $a_i, b_i, c_j, d_j \in R$. Suppose that $\sum_{i=1}^{m} a_i z b_i + \sum_{j=1}^{n} c_j z d_j = 0$ for all $z \in R$. If $a_1, \ldots, a_m$ are C-independent, then each b_i is a C-linear combination of $d_1, \ldots, d_n$.

Fact 2.9 (see [14, Lemma 1.3, p. 4]). If L is a noncommutative Lie ideal of R, and if $I = ([L, L])$ is the ideal of R generated by $[L, L] = ([x, y] : x, y \in L)$, then $I \subseteq L + L^2$ and $[I, R] \subseteq U$.

The following lemma is crucial to our proof.

Lemma 2.1. *Let R be a dense subring of* $\mathrm{End}(V_D)$ *containing the nonzero linear transformations of finite rank, where $dimV_D \geq 2$. Let (δ, φ) be a nonzero skew derivation of R, where φ is an automorphism of R. If for every $x, y \in R$ we have $[\delta([x, y]), [x, y]^m]_k = 0$, where k is a fixed positive integer, then* $\mathrm{char}(R) = 2$ *and $R \cong M_2(D)$, the (2×2)-matrix ring over D.*

Proof. Since R is a primitive ring with nonzero socle, from Jacobson [15, p. 75] and Chuang [9, Theorem 2.8] it follows that there exists $S \in \mathrm{End}(V)$ such that $\delta(x) = \varphi(x)S - Sx$ for all $x \in R$. Again by Jacobson [15, p. 79], there exists a semilinear automorphism $T \in \mathrm{End}(V)$ such that $\varphi(x) = TxT^{-1}$ for all $x \in R$. Moreover, $T(v\lambda) = T(v)\tau(\lambda)$ for all $v \in V$ and $\lambda \in D$, where τ is an automorphism of D.

We claim that there exists $v_0 \in V$ such that v_0 and $T^{-1}S(v_0)$ are D-independent. If not, then v and $T^{-1}S(v)$ are D-dependent for all $v \in V$. As before, there exists $\alpha \in D$ such that $T^{-1}S(v)= v\alpha$ for all $v \in V$. Then

$$\begin{aligned}\delta(x)v &= (TxT^{-1}S - Sx)v = Txv\alpha - S(xv)\\ &= T((xv)\alpha) - S(xv) = T(T^{-1}S)(xv) - S(xv)\\ &= 0, \quad \text{a contradiction.}\end{aligned}$$

So we may assume that v_0 and $T^{-1}S(v_0)$ are D-independent for some $v_0 \in V$. First, assume $\dim V_D \geq 4$. Then we may choose $u_0, w_0 \in V$ such that $\{v_0, T^{-1}S(v_0), u_0, w_0\}$ are D-independent. By the density of R, there exist $x, y \in R$ such that

$$xv_0 = 0, \quad xT^{-1}S(v_0) = 0, \quad xu_0 = w_0, \quad xw_0 = T^{-1}u_0,$$
$$yv_0 = 0, \quad yT^{-1}S(v_0) = w_0, \quad yu_0 = 0, \quad yw_0 = -u_0.$$

Then $[x,y]^m v_0 = 0, [x,y]T^{-1}S(v_0) = T^{-1}u_0$ and $[x,y]^m u_0 = u_0$. Therefore,

$$0 = [\delta([x,y]), [x,y]^m]_k v_0 = (-1)^k u_0, \quad \text{a contradiction.}$$

Thus, we assume that $\dim V_D \leq 3$. Choose $u_0 \in V$ such that $\{v_0, T^{-1}S(v_0), u_0\}$ are D-independent; then $\{v_0, T^{-1}S(v_0), u_0\}$ forms a D-basis of V. If

$$T(v_0 + T^{-1}S(v_0) + u_0) \in v_0 D \quad \text{and} \quad T(T^{-1}S(v_0) + u_0) \in v_0 D,$$

then $T(v_0)$, $T(T^{-1}S(v_0) + u_0) \in v_0 D$, and then v_0, $T^{-1}S(v_0) + u_0 \in T^{-1}(v_0 D) = T^{-1}(v_0)\tau^{-1}(D) = T^{-1}(v_0)D$, contradicting the fact that v_0, $T^{-1}S(v_0) + u_0$ are D-independent. So we may pick $\alpha \in \{0, 1\}$ such that $w_0 = \alpha v_0 + T^{-1}S(v_0) + u_0$ and $T(w_0) \notin v_0 D$. Write $T(w_0) = v_0\eta + T^{-1}S(v_0)\beta + u_0\gamma$, where $\eta, \beta, \gamma \in D$ and η, β are not both zero. By the density of R, there exist $x, y \in R$ such that

$$xv_0 = 0, \quad xT^{-1}S(v_0) = u_0, \quad xu_0 = 0,$$
$$yv_0 = 0, \quad yT^{-1}S(v_0) = 0, \quad yu_0 = -w_0.$$

In particular, $xw_0 = u_0, yw_0 = w_0, xT(w_0) = u_0\beta$ and $yT(w_0) = -w_0 y$. Then

$$[x,y]^m v_0 = 0, \quad [x,y]T^{-1}S(v_0) = w_0, \quad [x,y]^m u_0 = -u_0,$$

and $\delta([x,y])v_0 = (T[x,y]T^{-1}S - S[x,y])v_0 = T(w_0)$. Also we have $[x,y]w_0 = w_0 - u_0$, $[x,y]^{2n-1}T(w_0) = w_0\beta - u_0\gamma$, and $[x,y]^{2n}T(w_0) = (w_0 - u_0)\beta + u_0\gamma$ for $n \geq 1$. Since β, γ are not both zero and w_0, u_0 are D-independent, it is easy to check that $[x,y]^n T(w_0) \neq 0$ for all $n \geq 1$. Then

$$0 = [\delta([x,y]), [x,y]^m]_k v_0 = (-1)^k [x,y]^{m+k} T(w_0), \quad \text{a contradiction.}$$

This implies that $\dim V_D = 2$. Assume $\mathrm{char}(R) \neq 2$ and $\{v_0, T^{-1}S(v_0)\}$ forms a D-basis of V_D. Using the semilinearity and invertibility of T, it is easy to show that $\{T(v_0), S(v_0)\}$ also forms a D-basis of V_D. If $T(v_0)\tau(1) + 2S(v_0) \in v_0 D$ and $T(v_0)\tau(2) + 2S(v_0) \in v_0 D$, then $T(v_0)$ and $S(v_0)$ are D-dependent, a contradiction. So we may pick $\lambda \in \{1, 2\}$ such that $T(v_0)\tau(\lambda) + 2S(v_0) = v_0\beta + T^{-1}S(v_0)\gamma$, where $\beta, \gamma \in D$ and $\gamma \neq 0$. By the density of R, there exist $x, y \in R$ such that

$$xv_0 = -v_0\lambda - T^{-1}S(v_0), \quad xT^{-1}S(v_0) = 0,$$
$$yv_0 = 0, \quad yT^{-1}S(v_0) = -v_0.$$

Then $[x,y]^m v_0 = (-1)^m v_0$, $[x,y]^{2n} T^{-1}S(v_0) = T^{-1}S(v_0)$ for all $n \geq 1$, and $\delta([x,y])v_0 = v_0\beta + T^{-1}S(v_0)\gamma$. Now we have

$$\begin{aligned} 0 = [\delta([x,y]), [x,y]^m]_k &= \sum_{i=0}^{k} (-1)^i \binom{k}{i} [x,y]^{m+i} \delta([x,y]) [x,y]^{m+k-i} v_0 \\ &= (-1)^{m+k} \sum_{i=0}^{k} \binom{k}{i} [x,y]^{m+i} \left(v_0\beta + T^{-1}S(v_0)\gamma\right) \\ &= (-1)^{m+k} \left(v_0(\lambda\mu\gamma) + 2^{m+k} T^{-1}S(v_0)\gamma\right), \end{aligned}$$

a contradiction, where $\mu = \binom{k}{1} + \binom{k}{3} + \cdots + \binom{k}{2[k+1/2]-1}$. This completes the proof. □

§3. Proof of Theorem 1.1

Lemma 3.1. *The conclusion of Theorem* 1.1 *holds true if* δ *and* φ *are both* Q*-inner.*

Proof. Since L is a noncommutative Lie ideal of R, a result of Herstein ([14], Fact 2.9) shows that there is a nonzero ideal I of R such that $[I, I] \subseteq L$. Therefore, we have $[\delta([x, y]), [x, y]^m]_k = 0$ for all $x, y \in I$. Since φ and δ are both Q-inner, there exist $q \in Q$ and $T \in Q$ such that $\delta(x) = \varphi(x)q - qx$ and $\varphi(x) = TxT^{-1}$ for all $x \in R$. If $T^{-1}q \in C$, then $\delta(x) = TxT^{-1}q - qx = T(xT^{-1}q - T^{-1}qx) = T[x, T^{-1}q] = 0$, a contradiction. Thus, $T^{-1}q \notin C$. With this, it is easily seen that

$$\begin{aligned}\phi(x, y) = [\delta([x, y]), [x, y]^m]_k &= \sum_{i=0}^{k} (-1)^i \binom{k}{i} [x, y]^{m+i}(T[x, y]T^{-1} - q[x, y])[x, y]^{m+i} \\ &= \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} [x, y]^{m+i}(T[x, y]T^{-1} - q[x, y])[x, y]^{m+i} \\ &\quad + (-1)^k [x, y]^{m+k}(-q)[x, y]^{m+1} + (-1)^k [x, y]^{m+k} T[x, y]T^{-1}q[x, y]^m\end{aligned}$$

is a nontrivial generalized polynomial identity(GPI) of R. By [5], or [2, Theorem 6.4.4], $\phi(x, y)$ is also a GPI of Q. Let $\mathcal{F}$ be the algebraic closure of C if C is infinite, and set $\mathcal{F} = C$ for C finite. By Fact 2.9, $\phi(x, y)$ is also a GPI of $Q \otimes_C \mathcal{F}$. Moreover, by [13, Theorem 3.5], $Q \otimes_C \mathcal{F}$ is a prime ring with $\mathcal{F}$ as its extended centroid. Thus, $Q \otimes_C \mathcal{F}$ is a prime ring satisfying a nontrivial GPI, and its extended centroid $\mathcal{F}$ is either an algebraically closed field or a finite field. By Martindale's theorem [2, Corollary 6.1.7], $Q \otimes_C \mathcal{F}$ is a primitive ring having nonzero socle, with the field $\mathcal{F}$ as its associated division ring. By Jacobson's theorem [15, p. 75], $Q \otimes_C \mathcal{F}$ is isomorphic to a dense subring of the ring of linear transformations on a vector space V over $\mathcal{F}$, containing nonzero linear transformations of finite rank. By assumption, R is noncommutative, so $Q \otimes_C \mathcal{F}$ is noncommutative. Hence, $\dim V_{\mathcal{F}} \geq 2$. By Lemma 2.1, $\mathrm{char}(Q \otimes_C \mathcal{F}) = 2$ and $Q \otimes_C \mathcal{F} \cong M_2(\mathcal{F})$. This implies that $\mathrm{char}(R) = 2$ and $R \subseteq Q \cong Q \otimes_C \mathcal{F} \subseteq Q \otimes_C \mathcal{F} \cong M_2(\mathcal{F})$. The proof is thereby complete. □

Lemma 3.2. *The conclusion of Theorem* 1.1 *holds true if* δ *is* Q*-outer and* φ *is* Q*-inner.*

Proof. Since L is a noncommutative Lie ideal of R, a result of Herstein ([14], Fact 2.9) shows that there is a nonzero ideal I of R such that $0 \neq [I, I] \subseteq L$. So $[\delta([x, y]), [x, y]^m]_k = 0$ for all $x, y \in I$. Thus, by Chuang [8, Theorem 2], I and Q satisfy the same differential identities, so that it is also satisfied by Q. Now for all $x, y \in Q$ we have

$$[\varphi(x)\delta(y) + \delta(x)y - \varphi(y)\delta(x) - \delta(y)x, [x, y]^m]_k = 0.$$

Since φ is Q-inner, $\varphi(x) = TxT^{-1}$ for some $T \in Q$. We have

$$[TxT^{-1}\delta(y) + \delta(x)y - TyT^{-1}\delta(x) - \delta(y)x, [x, y]^m]_k = 0.$$

By Fact 2.4, $[TxT^{-1}s + ty - TyT^{-1}t - sx, [x, y]^m]_k = 0$ for all $x, y, s, t \in Q$. Setting $t = 0$ and replacing s by Ts, we see that

$$(3.1) \qquad [T[x, s], [x, y]^m]_k = 0 \quad \text{for all} \quad x, y, s \in Q.$$

On the other hand, setting $s = 0$ and replacing t by Tt, we have

$$(3.2) \qquad [T[t, y], [x, y]^m]_k = 0 \quad \text{for all} \quad x, y, t \in Q.$$

Replacing s by $[y, t]$ in (3.1), we obtain

$$(3.3) \qquad [T[x, [y, t]], [x, y]^m]_k = 0 \quad \text{for all} \quad x, y, t \in Q.$$

Also, replacing t by $-[t,x]$ in (3.2), we get

$$[T[y,[t,x]],[x,y]^m]_k = 0 \text{ for all } x,y,t \in Q. \tag{3.4}$$

By the Jacobi identity $[x,[y,z]]+[y,[z,x]]+[z,[x,y]] = 0$ for all $x,y,z \in Q$, the sum of (3.3) and (3.4) yields

$$[T[-t,[x,y]],[x,y]^m]_k = 0 \text{ for all } x,y,t \in Q. \tag{3.5}$$

Pick $0 \neq q \in Q$, $T^{-1}q \notin C$, and let $\delta'(x) = \varphi(x)q - qx$ be a nonzero Q-inner (δ,φ) skew derivation. Then replacing t by $T^{-1}q$ in (3.5), we see that $[\delta'([x,y]),[x,y]^m]k = 0$ for all $x,y \in Q$. Now, Lemma 3.1 implies the required result. □

Lemma 3.3. *The conclusion of Theorem* 1.1 *holds true if δ and φ are both Q-outer.*

Proof. Since δ and φ are both Q-outer, the definition and Fact 2.5 show that, in any case, $[ws+ty-zt-sx,[x,y]^m]_k = 0$ for all $x,y,z,w,s,t \in Q$. Setting $t = 0$ and replacing w by x, we have

$$[[x,s],[x,y]^m]_k = 0 \text{ for all } x,y,s \in Q. \tag{3.6}$$

Similarly, setting $s = 0$ and replacing z by y, we have

$$[[t,y],[x,y]^m]_k = 0 \text{ for all } x,y,t \in Q. \tag{3.7}$$

Comparing (3.6), (3.7) with (3.1), (3.2) and continuing the same processes as before, we complete the proof. □

Now we are ready for the following proof.

Proof of Theorem 1.1. Lemma 3.1, Lemma 3.2, and Lemma 3.3 show that we only need to discuss the case where δ is Q-inner and φ is Q-outer. Since δ is Q-inner, there exists $0 \neq b \in Q$ such that $\delta(x) = \varphi(x)q - qx$ for all $x \in R$. Since $0 = [\delta(x),x^m]_k = [\varphi(x)q - qx, x^m]_k = 0$ for all $x \in L$, from Fact 2.2 it follows that

$$[\varphi(x)q - qx, x^m]_k = 0 \text{ for all } x \in [Q,Q]. \tag{3.8}$$

By Lemma 3.1, since φ is Q-outer, the formula

$$\phi(x,y) = [\varphi([x,y])q - q[x,y],[x,y]^m]_k = [(([\varphi(x),\varphi(y)])q - q[x,y],[x,y]^m]_k$$

is a nontrivial generalized polynomial identity of Q. By Chuang's [3, Theorem], Q is a primitive ring having nonzero socle, and its associated division ring D is finite-dimensional over C. Hence, Q is isomorphic to a dense subring of the ring of linear transformations on a vector space V over D, containing the nonzero linear transformations of finite rank. Then by Lemma 2.1, either $\dim V_D = 1$, that is, $Q \cong D$, or $\mathrm{char}(Q) = 2$ and $Q \cong M_2(D)$. Since D is finite-dimensional over C, Q is a finite-dimensional simple C-algebra. If C is finite, then from $\dim D_C < \infty$ it follows that D is also finite. Thus, $D \cong C$ is a field by Wedderburn's theorem [15, p. 183] on finite division rings. This implies that either $Q \cong C$, a commutative ring, or $\mathrm{char}(Q) = 2$ and $Q \cong M_2(C)$. Since R is noncommutative, we must have $\mathrm{char}(R) = 2$ and $R \subseteq Q \cong M_2(C)$. This proves our theorem. Therefore, from now on we assume that C is infinite and Q is a noncommutative simple algebra, finite-dimensional over its center C. □

Case **1.** Suppose that φ is not Frobenius. Since

$$[(([\varphi(x),\varphi(y)])q - q[x,y],[x,y]^m]_k = 0$$

for all $x,y \in Q$, by Fact 2.3 we have $[[s,t]q - q[x,y],[x,y]^m]_k = 0$ for all $x,y,s,t \in Q$. Let $s = x, t = y$, then $[\delta'([x,y]),[x,y]^m]_k = 0$ for all $x,y \in Q$, where $\delta'(x) = xq - qx$ is a Q-inner I_Q-derivation and I_Q denotes the identity map of Q. If $\delta' \neq 0$, then by Lemma 3.1 we are done. Thus, we may assume that $\delta' = 0$, implying $0 \neq q \in C$.

Then $0 = [[s,t]q - q[x,y],[x,y]^m]_k = q[[s,t]-[x,y],[x,y]^m]_k = q[[s,t],[x,y]^m]_k$ for all $x,y,s,t \in Q$. Thus, $[[s,t],[x,y]^m]_k = 0$ for all $x,y,s,t \in Q$. Pick $p \in Q$, $p \notin C$, and replace s,t by $[x,y],p$, respectively. Then $[\delta''([x,y]),[x,y]^m]_k = 0$ for all $x,y \in Q$, where $\delta''(x) = xp - px$ is a nonzero Q-inner I_Q-derivation. By Lemma 3.1, we are done.

Case **2.** Suppose that φ is Frobenius. We may assume that $\mathrm{char}(R) = p > 0$. Otherwise, if $\mathrm{char}(R) = 0$, then the Frobenius automorphism φ fixes C and hence must be Q-inner by Fact 2.1, a contradiction. So, for all $\lambda \in C$ we have $\varphi(\lambda) = \lambda^{p^n}$ for some nonzero fixed integer n. Also we may assume that $n \neq 0$ by Fact 2.1, Choose an integer ω such that $p^\omega > k$. Then, since $[[\delta(x),x^m]_k,x]_{p^\omega-k} = [\delta(x),x^m]_{p^\omega} = [\delta(x),x^{m+p^\omega}]$, (3.8) reduces to

$$[\varphi(x)q - qx, x^{m+p^\omega}] = 0 \quad \text{for all} \quad x \in [Q,Q]. \tag{3.9}$$

First, assume that $n \geq 1$. Clearly, $[Q,Q]$ is a C-space. For $\lambda \in C$ and $x \in [Q,Q]$, replacing x in (3.9) by λx, we have $0 = [\varphi(\lambda x)q - q(\lambda x),(\lambda x)^{m+p^\omega}]_k$, and we see that, in any case, $0 = \lambda^{p^n+p^\omega+m}[\varphi(x)q,x^{m+p^\omega}] - \lambda^{1+p^\omega+m}[qx,x^{m+p^\omega}]$. Since C is infinite, from a Vandermonde determinant argument it follows that

$$[\varphi(x)q, x^{m+p^\omega}] = 0 \quad \text{for all} \quad x \in [Q,Q]. \tag{3.10}$$

For $\lambda \in C$ and $x,y \in [Q,Q]$, replacing x in (3.10) by $x + \lambda y$, we have

$$0 = [\varphi(x+\lambda y)q,(x+\lambda y)^{m+p^\omega}] = \left[\varphi(x)q + \lambda^{p^n}\varphi(y)q, \sum_{i=0}^{m+p^\omega}\phi_i(x,y)\lambda^i\right]$$
$$= \sum_{i=1}^{m+p^\omega}\lambda^i[\varphi(x)q,\phi_i(x,y)] + \sum_{i=0}^{m+p^\omega-1}\lambda^{p^n+i}[\varphi(y)q,\phi_i(x,y)],$$

where $\phi_i(x,y)$ denotes the sum of all monic monomials with x-degree $m+p^\omega-1$ and y-degree i for $0 \leq i \leq m+p^\omega$. In particular,

$$\phi_1(x,y) = x^{m+p^\omega-1}y + x^{m+p^\omega-2}yx + \cdots + yx^{m+p^\omega-1} = \sum_{i=0}^{m+p^\omega-1} x^{m+p^\omega-i}yx^i.$$

Since C is infinite, a Vandermonde determinant argument shows that $[\varphi(x)q,\phi_1(x,y)] = 0$ for all $x,y \in [Q,Q]$. Note that $[x,s] \in [Q,Q]$ for $x \in [Q,Q]$ and $s \in Q$. Furthermore, $\phi_1(x,[x,s]) = x^{m+p^\omega}s - sx^{m+p^\omega}$. From $[\varphi(x)q,\phi_1(x,[x,s])] = 0$ it follows that

$$\varphi(x)qx^{m+p^\omega}s - \varphi(x)qsx^{m+p^\omega} - x^{m+p^\omega}s\varphi(x)q + sx^{m+p^\omega}\varphi(x)q = 0, \tag{3.11}$$

for all $x \in [Q,Q]$ and $s \in Q$.

Step **1.** First, we assume that $x^{m+p^\omega} \in C$ for all $x \in [Q,Q]$. Thus, for any $x,y,s \in Q$, we have $[[x,y]^{m+p^\omega},s] = 0$. By Fact 2.7, $Q \subseteq M_n(F)$, $n > 1$, for some field F, and $[[x,y]^{m+p^\omega},s] = 0$ for all $x,y,z \in M_n(F)$, where $M_n(F)$ is the set of $(n \times n)$-matrices over F. Denote e_{ij} the usual matrix unit with 1 at the (i,j)-entry and zero elsewhere. However, if $n \geq 3$, by choosing $x = e_{21}$, $y = e_{12}$, and $s = e_{31}$, we get

$$0 = \left[[x,y]^{m+p^\omega},s\right] = \left[[e_{21},e_{12}]^{m+p^\omega},e_{31}\right] = e_{31} \neq 0, \quad \text{a contradiction.}$$

Moreover, if $\mathrm{char}(R) = p \neq 2$, then for $x = e_{12}$, $y = e_{21}$, and $s = e_{21}$ we have

$$0 = \left[[x,y]^{m+p^\omega},s\right] = \left[[e_{12},e_{21}]^{m+p^\omega},e_{21}\right] = -2e_{21} \neq 0, \quad \text{again a contradiction.}$$

Thus, $\mathrm{char}(R) = 2$ and $R \subseteq M_2(F)$, as desired.

Step **2.** Second, let $x^{m+p^\omega} \notin C$ for all $x \in [Q,Q]$. It follows that 1 and x^{m+p^ω} are linearly C-independent. Applying Fact 2.8 to (3.11), we see that $\varphi(x)q$ can be expressed as a C-linear combination of 1 and x^{m+p^ω}. In particular, $[\varphi(x)q, x^m] = 0$. Recall that C is infinite. For any $y \in [Q,Q]$, there exist infinitely many $\xi \in C$ such that $(x+\xi y)^{m+p^\omega} \notin C$; otherwise it is easy to show that $x^{m+p^\omega} \in C$ by a Vandermonde determinant argument. Thus, for such $\xi \in C$, by (3.11) and Fact 2.8, we have

$$0 = [\varphi(x+\xi y)q, (x+\xi y)^m] = [\varphi(x)q + \xi^{p^n}\varphi(y)q, (x+\xi y)^m].$$

We use the binomial expansion:

$$(x+y)^m = \binom{m}{0}x^0y^m + \binom{m}{1}x^1y^{m-1} + \cdots + \binom{m}{m-1}x^{m-1}y^1 + \binom{m}{m}x^ny^0.$$

We see that

$$\begin{aligned} 0 = {} & \xi^m[\varphi(x)q, y^m] + \binom{m}{1}\xi^{m-1}[\varphi(x)q, xy^{m-1}] + \cdots + \binom{m}{m-1}\xi[\varphi(x)q, x^{m-1}y] \\ & + [\varphi(x)q, x^m] + \xi^{p^n+m}[\varphi(y)q, y^m] + \xi^{p^n}[\varphi(y)q, x^m] \\ & + \binom{m}{1}\xi^{p^n+m-1}[\varphi(y)q, xy^{m-1}] + \cdots + \binom{m}{m-1}\xi^{p^n+1}[\varphi(y)q, x^{m-1}y]. \end{aligned} \tag{3.12}$$

Applying a Vandermonde determinant argument, in any case we have $[\varphi(y)q, y^m] = 0$ for all $y \in [Q,Q]$. For any $y, z \in [Q,Q]$ and $\lambda \in C$, we have $[\varphi(y+\lambda z)q, (y+\lambda z)^m] = 0$. In view of (3.12), we obtain $[\varphi(y)q, z^m] = 0$ for all $y, z \in [Q,Q]$. Then

$$0 = [\varphi([r,s])q, [t,u]^m] = [[\varphi(r), \varphi(s)]q, [t,u]^m] \quad \text{for all} \quad r,s,t,u \in Q.$$

Thus, $[[r,s]q, [t,u]^m] = 0$ for all $r,s,t,u \in Q$ because φ is an automorphism of Q. By Fact 2.7, $Q \subseteq M_n(F)$, $n > 1$, for some field F, and $[[r,s]q, [t,u]^m] = 0$ for all $r,s,t,u \in M_n(F)$. We choose $r = e_{ii}$, $s = e_{ij}$, $t = e_{ij}$, and $u = e_{ji}$ for $i \neq j$, and let $q = \sum_{i=0}^n \lambda_i e_{ii}$. Then, by calculation, we get $0 = [[r,s]q, [t,u]^m] = -2\lambda_j e_{ii}$, which gives $\lambda_j = 0$, and hence $q = 0$, a contradiction.

Finally, assume that $n \leq -1$. In this situation, the proof is similar to the above, so we only sketch it without details. Let $n' = -n \geq 1$. Then $\varphi(\lambda^{p^{n'}}) = \lambda$ for all $\lambda \in C$. For $\lambda \in C$ and $x \in [Q,Q]$, replacing x in (3.9) by $\lambda^{p^{n'}}x$, we have $[\varphi(x)q, x^{p^\omega+m}] = 0$ for all $x \in [Q,Q]$. For $\lambda \in C$ and $x, y \in [Q,Q]$, replacing x by $x + \lambda^{p^{n'}}y$, we obtain $[\varphi(x)q, \phi_1(x,y)] = 0$ for all $x, y \in [Q,Q]$. Now proceeding with the same argument as above, we get the result.

As a natural consequence we obtain the following. Put $\delta = \varphi - I_R$, where I_R denotes the identity map of R.

Corollary 3.1. *Let R be a prime ring and L a noncommutative Lie ideal of R. If φ is a nonidentity automorphism of R and $[\varphi(x), x^m]_k = 0$ for all $x \in L$, where k, m are fixed positive integers, then* $\mathrm{char}(R) = 2$ *and $R \subseteq M_2(F)$ for some field F.*

Let R be a unital ring. For a unit $u \in R$, the map $\varphi_u\colon x \to uxu^{-1}$ determines an automorphism of R. If d is a derivation of R, then it is easy to check that the map $ud\colon x \to ud(x)$ yields a φ_u-derivation of R. So we have the following statement.

Corollary 3.2. *Let R be a prime unital ring, u a unit in R, and L a noncommutative Lie ideal of R. If d is a nonzero derivation of R such that $[\varphi_u(x), x^m]_k = 0$ for all $x \in L$, where k, m are fixed positive integers, then* $\mathrm{char}(R) = 2$ *and $R \subseteq M_2(F)$ for some field F.*

References

[1] A. Fošner and N. Rehman, *Identities with additive mappings in semiprime rings*, Bull. Korean Math. Soc. **51** (2014), no. 1, 207–211. MR3163387

[2] K. I. Beidar, W. S. Martindale III, and A. V. Mikhalev, *Rings with generalized identities*, Monogr. Textbooks Pure Appl. Math., vol. 196, Marcel Dekker, vol. 196, New York, 1996. MR1368853

[3] H. E. Bell and W. S. Martindale III, *Centralizing mappings of semiprime rings*, Canad. Math. Bull. **30** (1987), no. 1, 92–101. MR879877

[4] M. C. Chou and C. K. Liu, *An Engel condition with skew derivations*, Monatash. Math. **158** (2009), no. 3, 259–270. MR2552094

[5] C. L. Chuang, *GPIs having coefficients in Utumi quotient rings*, Proc. Amer. Math. Soc. **103** (1988), no. 3, 723–728. MR947646

[6] ———, *Differential identities with automorphisms and antiautomorphisms.* I, J. Algebra **149** (1992), no. 2, 371–404. MR1172436

[7] ———, *Differential identities with automorphisms and antiautomorphisms.* II, J. Algebra **160** (1993), no. 1, 130–171. MR1237081

[8] C. L. Chuang and T. K. Lee, *Identities with a single skew derivation*, J. Algebra **288** (2005), no. 1, 59–77. MR2138371

[9] Chuang C. L. and Liu C. K., *Extended Jacobson density theorem for rings with skew derivations*, Comm. Algebra **35** (2007), no. 4, 1391–1413. MR2313675

[10] Q. Deng, *On n-centralizing mappings of prime rings*, Proc. Roy. Irish. Acad. Sect. A **93** (1993), no. 2, 171–176. MR1368822

[11] Q. Deng and H. E. Bell, *On derivations and commutativity in semiprime rings*, Comm. Algebra **23** (1995), no. 10, 3705–3713. MR1348259

[12] N. Divinsky, *On commuting automorphisms of rings*, Trans. Roy. Soc. Canada. Sect. III **49** (1955), 19–22. MR0073573

[13] T. S. Erickson, W. S. Martindale III, and J. M. Osborn, *Prime nonassociative algebras*, Pacific. J. Math. **60** (1975), no. 1, 49–63. MR0382379

[14] I. N. Herstein, *Topics in ring theory*, Univ. Chicago Press, Chicago, 1969. MR0271135

[15] N. Jacobson N., *Structure of rings*, Amer. Math. Soc. Colloq. Pub., vol. 37, Amer. Math. Soc., Providence, RI, 1964. MR0222106

[16] V. K. Kharchenko, *Generalized identities with automorphisms*, Algebra i Logica **14** (1975), no. 2, 215–237; English transl., Algebra and Logic **14** (1975), no. 2, 132–148. MR0399153

[17] C. Lanski, *Differential identities, Lie ideals, and Posner's theorems*, Pacific J. Math. **134** (1988), no. 2, 275–297. MR961236

[18] ———, *An Engel condition with derivation*, Proc. Amer. Math. Soc. **118** (1993), no. 3, 731–734. MR1132851

[19] ———, *Derivations with nilpotent values of Lie ideals*, Proc. Amer. Math. Soc. **108** (1990), no. 1, 31–37. MR984803

[20] ———, *Left ideals and derivations in semiprime rings*, J. Algebra **277** (2004), no. 2, 658–667. MR2067624

[21] P. H. Lee and T. K. Lee, *Lie ideals of prime rings with derivations*, Bull. Inst. Math. Acad. Sinica **11** (1983), no. 1, 75–80. MR718903

[22] P. H. Lee and T. L. Wong, *Derivations cocentralizing Lie ideals*, Bull. Inst. Math. Acad. Sinica **23** (1995), no. 1, 1–5. MR1319474

[23] P. H. Lee, T. L. Wong, J. S. Lin, and R. J. Wang, *Commuting traces of multiadditive mappings*, J. Algebra **193** (1997), no. 2, 709–723. MR1458811

[24] T. K. Lee, *Posner's theorem for (σ, τ)-derivations and σ-centralizing maps*, Houston J. Math. **30** (2004), no. 2, 309–320. MR2084904

[25] J. Luh, *A note on commuting automorphisms of rings*, Amer. Math. Monthly **77** (1970), 61–62. MR0255612

[26] J. H. Mayne, *Centralizing automorphisms of prime rings*, Canad. Math. Bull. **19** (1976), no. 1, 113–115. MR0419499

[27] ———, *Centralizing automorphisms of Lie ideals in prime rings*, Canad. Math. Bull. **35** (1992), no. 4, 510–514. MR1191511

[28] E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1093–1100. MR0095863

[29] N. ur Rekhman and V. De Filippis, *On n-anticommuting mappings with generalized derivations in prime and semiprime rings*, Sibirsk. Mat. Zh. **52** (2011), no. 3, 655–664; English transl., Sib. Math. J. **52** (2011), no. 3, 516–523. MR2858650

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