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HOMOGENIZATION OF ELLIPTIC OPERATORS WITH PERIODIC COEFFICIENTS IN DEPENDENCE OF THE SPECTRAL PARAMETER

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Abstract. Differential expressions of the form $b(\mathbf{D})^* g(\mathbf{x}/\varepsilon) b(\mathbf{D})$, $\varepsilon > 0$, are considered, where a matrix-valued function $g(\mathbf{x})$ in $\mathbb{R}^d$ is assumed to be bounded, positive definite, and periodic with respect to some lattice; $b(\mathbf{D}) = \sum_{l=1}^d b_l D_l$ is a first order differential operator with constant coefficients. The symbol $b(\boldsymbol{\xi})$ is subject to some condition ensuring strong ellipticity. The operator in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ given by the expression $b(\mathbf{D})^* g(\mathbf{x}/\varepsilon) b(\mathbf{D})$ is denoted by $\mathcal{A}_\varepsilon$. Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class $C^{1,1}$. The operators $\mathcal{A}_{D,\varepsilon}$ and $\mathcal{A}_{N,\varepsilon}$ under study are generated in the space $L_2(\mathcal{O};\mathbb{C}^n)$ by the above expression with the Dirichlet or Neumann boundary conditions. Approximations in various operator norms for the resolvents $(\mathcal{A}_\varepsilon - \zeta I)^{-1}$, $(\mathcal{A}_{D,\varepsilon} - \zeta I)^{-1}$, $(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1}$ are obtained with error estimates depending on ε and ζ.

Introduction

The paper concerns homogenization theory for periodic differential operators (DO's). A broad literature is devoted to homogenization problems. First, we mention the books [BeLPa, BaPan, ZhKO].

0.1. The class of operators. We study a wide class of matrix (of size $n \times n$) strongly elliptic operators given by the differential expression $b(\mathbf{D})^* g(\mathbf{x}/\varepsilon) b(\mathbf{D})$, $\varepsilon > 0$. Here $g(\mathbf{x})$ is a Hermitian matrix-valued function on $\mathbb{R}^d$ (of size $m \times m$), which is assumed to be bounded, positive definite, and periodic with respect to a lattice Γ. The operator $b(\mathbf{D})$ is an $(m \times n)$-matrix first order DO with constant coefficients. We assume that $m \geq n$; the symbol of $b(\mathbf{D})$ is subject to a certain condition ensuring the strong ellipticity of the operator under consideration.

The selfadjoint operator in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ given by the expression $b(\mathbf{D})^* g(\mathbf{x}/\varepsilon) b(\mathbf{D})$ is denoted by $\mathcal{A}_\varepsilon$. We also study the selfadjoint operators $\mathcal{A}_{D,\varepsilon}$ and $\mathcal{A}_{N,\varepsilon}$ in $L_2(\mathcal{O};\mathbb{C}^n)$ given by the same expression with the Dirichlet or Neumann conditions on the boundary $\partial\mathcal{O}$, respectively. Here $\mathcal{O} \subset \mathbb{R}^d$ is a bounded domain of class $C^{1,1}$.

The simplest example of the operator $\mathcal{A}_\varepsilon$ is the acoustics operator $-\operatorname{div} g(\mathbf{x}/\varepsilon)\nabla$; the operator of elasticity theory can also be written in the required form. These and other examples were considered in [BSu2] in detail.

Our goal is to approximate the resolvents of the operators introduced above for small ε in various operator norms.

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0.2. A survey of results on operator error estimates. Homogenization problems for the operator $\mathcal{A}_\varepsilon$ in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ were studied in a series of papers by Birman and Suslina [BSu1, BSu3, BSu3, BSu4]. In [BSu1, BSu2] it was proved that

$$\left\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}^0 + I)^{-1}\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq C\varepsilon. \tag{0.1}$$

Here $\mathcal{A}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$ is the effective operator and g^0 is the constant positive effective matrix (g^0 is defined in §1 below.) Next, in [BSu4], the approximation of the resolvent $(\mathcal{A}_\varepsilon + I)^{-1}$ in the norm of operators acting from $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to the Sobolev space $H^1(\mathbb{R}^d;\mathbb{C}^n)$ was found:

$$\left\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}^0 + I)^{-1} - \varepsilon K(\varepsilon)\right\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leq C\varepsilon; \tag{0.2}$$

in this approximation, a corrector was taken into account. Estimates of the form (0.1), (0.2), called *operator error estimates*, are order-sharp; the constants in estimates are controlled explicitly in terms of the problem data. The method of [BSu1, BSu2, BSu3, BSu4] is based on the scaling transformation, the Floquet–Bloch theory, and the analytic perturbation theory.

A different approach to operator error estimates in homogenization problems was suggested by Zhikov. In [Zh1, Zh2, ZhPas], the acoustics operator and the operator of elasticity theory were considered. Estimates of the form (0.1), (0.2) for these operators were obtained. The method was based on the analysis of the first order approximation to the solution and on the introduction of an additional parameter. Besides problems in $\mathbb{R}^d$, in [Zh1, Zh2, ZhPas] problems in a bounded domain $\mathcal{O} \subset \mathbb{R}^d$ with the Dirichlet or Neumann boundary conditions were studied; certain analogs of estimates (0.1), (0.2) with error terms $O(\varepsilon^{1/2})$ were obtained. The error estimates deteriorate because of the boundary effects. (In the case of the Dirichlet problem for the acoustics operator, the $(L_2 \to L_2)$-estimate was improved in [ZhPas]; but still it was not order-sharp.)

Similar results for the operator $-\operatorname{div} g(\mathbf{x}/\varepsilon)\nabla$ in a bounded domain with the Dirichlet or Neumann conditions were obtained in the papers [Gr1, Gr2] by Griso via the unfolding method. In [Gr2], an analog of the sharp order estimate (0.1) for the same operator was proved for the first time.

For the matrix operators $\mathcal{A}_{D,\varepsilon}$, $\mathcal{A}_{N,\varepsilon}$ that we consider, operator error estimates were obtained in the recent papers [PSu1, PSu2, Su1, Su2, Su3]. In [PSu1, PSu2], the Dirichlet problem was studied; it was proved that

$$\left\|\mathcal{A}_{D,\varepsilon}^{-1} - (\mathcal{A}_D^0)^{-1} - \varepsilon K_D(\varepsilon)\right\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} \leq C\varepsilon^{1/2}.$$

Here $\mathcal{A}_D^0$ is the operator given by $b(\mathbf{D})^* g^0 b(\mathbf{D})$ with the Dirichlet condition and $K_D(\varepsilon)$ is the corresponding corrector. In [Su1, Su2], the following sharp order estimate was found:

$$\left\|\mathcal{A}_{D,\varepsilon}^{-1} - (\mathcal{A}_D^0)^{-1}\right\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leq C\varepsilon. \tag{0.3}$$

Similar results for the Neumann problem were obtained in [Su3]. (Note that in [Su3] the resolvent $(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1}$ at an arbitrary regular point $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$ was considered, but the optimal dependence of the constants on ζ was not searched out.) The method of [PSu1, PSu2, Su1, Su2, Su3] is based on using the results for the problem in $\mathbb{R}^d$, introduction of the boundary layer correction term, and estimates for the norms of this term in $H^1(\mathcal{O})$ and $L_2(\mathcal{O})$. Some technical tricks were borrowed from [ZhPas].

Independently and by a different method, an estimate of the form (0.3) was obtained in [KeLiS] for uniformly elliptic systems with the Dirichlet or Neumann boundary conditions under some regularity assumptions on coefficients.

0.3. Main results. In the present paper, the operators $\mathcal{A}_\varepsilon$, $\mathcal{A}_{D,\varepsilon}$, and $\mathcal{A}_{N,\varepsilon}$ are studied. *Our goal* is to find approximations for the resolvent at a regular point ζ in dependence of ε and the spectral parameter ζ. Estimates for small ε and large $|\zeta|$ are of principal interest.

Now we describe the main results. For the operator $\mathcal{A}_\varepsilon$ in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ we prove the following estimates for $\zeta = |\zeta|e^{i\varphi} \in \mathbb{C}\setminus\mathbb{R}_+$ and $\varepsilon > 0$:

$$\|(\mathcal{A}_\varepsilon - \zeta I)^{-1} - (\mathcal{A}^0 - \zeta I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le C(\varphi)|\zeta|^{-1/2}\varepsilon, \tag{0.4}$$

$$\|(\mathcal{A}_\varepsilon - \zeta I)^{-1} - (\mathcal{A}^0 - \zeta I)^{-1} - \varepsilon K(\varepsilon;\zeta)\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \le C(\varphi)(1+|\zeta|^{-1/2})\varepsilon. \tag{0.5}$$

For the operators $\mathcal{A}_{D,\varepsilon}$ and $\mathcal{A}_{N,\varepsilon}$, the following estimates are obtained for $\zeta \in \mathbb{C}\setminus\mathbb{R}_+$ and $|\zeta| \ge 1$:

$$\|(\mathcal{A}_{\dagger,\varepsilon} - \zeta I)^{-1} - (\mathcal{A}^0_\dagger - \zeta I)^{-1}\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \le C_1(\varphi)(|\zeta|^{-1/2}\varepsilon + \varepsilon^2), \tag{0.6}$$

$$\|(\mathcal{A}_{\dagger,\varepsilon} - \zeta I)^{-1} - (\mathcal{A}^0_\dagger - \zeta I)^{-1} - \varepsilon K_\dagger(\varepsilon;\zeta)\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} \le C_2(\varphi)|\zeta|^{-1/4}\varepsilon^{1/2} + C_3(\varphi)\varepsilon, \tag{0.7}$$

where $0 < \varepsilon \le \varepsilon_1$ (ε_1 is a sufficiently small number depending on the domain $\mathcal{O}$ and the lattice Γ). Here $\dagger = D, N$. The dependence of constants in estimates (0.4)–(0.7) on the angle φ is traced. Estimates (0.4)–(0.7) are two-parametric (with respect to ε and $|\zeta|$); they are uniform in any sector $\varphi \in [\varphi_0, 2\pi - \varphi_0]$ with an arbitrarily small $\varphi_0 > 0$.

In the general case, the correctors in (0.5), (0.7) involve a smoothing operator. We distinguish an additional condition under which the standard corrector can be used.

Besides the resolvent, we approximate the operator $g^\varepsilon b(\mathbf{D})(\mathcal{A}_\varepsilon - \zeta I)^{-1}$ (corresponding to the „flux") in the $(L_2 \to L_2)$-operator norm. Also, for a strictly interior subdomain $\mathcal{O}'$ of $\mathcal{O}$, we find approximation to the resolvents of $\mathcal{A}_{D,\varepsilon}$ and $\mathcal{A}_{N,\varepsilon}$ in the $(L_2(\mathcal{O}) \to H^1(\mathcal{O}'))$-norm with sharp order error estimate (with respect to ε).

For completeness, we find a different type approximation for the resolvents of $\mathcal{A}_{D,\varepsilon}$ and $\mathcal{A}_{N,\varepsilon}$, which is valid in a wider domain of the parameter ζ and may be preferable for bounded values of $|\zeta|$. Let us explain the nature of these results for the Dirichlet problem. Let $c_* > 0$ be a common lower bound of the operators $\mathcal{A}_{D,\varepsilon}$ and $\mathcal{A}^0_D$, and let $\zeta \in \mathbb{C}\setminus[c_*,\infty)$. We put $\zeta - c_* = |\zeta - c_*|e^{i\psi}$. Then for $0 < \varepsilon \le \varepsilon_1$ we have

$$\|(\mathcal{A}_{D,\varepsilon} - \zeta I)^{-1} - (\mathcal{A}^0_D - \zeta I)^{-1}\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \le \widehat{C}(\zeta)\varepsilon, \tag{0.8}$$

$$\|(\mathcal{A}_{D,\varepsilon} - \zeta I)^{-1} - (\mathcal{A}^0_D - \zeta I)^{-1} - \varepsilon K_D(\varepsilon;\zeta)\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} \le \widehat{C}(\zeta)\varepsilon^{1/2}, \tag{0.9}$$

where $\widehat{C}(\zeta) = C(\psi)|\zeta - c_*|^{-2}$ for $|\zeta - c_*| < 1$ and $\widehat{C}(\zeta) = C(\psi)$ for $|\zeta - c_*| \ge 1$. The dependence of the constants on the angle ψ is traced. Estimates (0.8), (0.9) are uniform with respect to ψ in any sector $\psi \in [\psi_0, 2\pi - \psi_0]$ with an arbitrarily small ψ_0.

The author thinks of estimates (0.6) and (0.7) as the main achievements of the paper.

0.4. The method. For the problem in $\mathbb{R}^d$, it is not difficult to deduce estimates (0.4) and (0.5) from the known estimates at the point $\zeta = -1$ (see (0.1), (0.2)) with the help of appropriate resolvent identities and a scaling transformation (see §2). However, this method is not suitable for problems in a bounded domain. In order to prove estimates (0.6) and (0.7), we are forced to repeat the entire pattern of [PSu2, Su2, Su3] anew, tracing the dependence of estimates on the spectral parameter ζ carefully. The method is based on the consideration of the associated problem in $\mathbb{R}^d$, introduction of the boundary layer correction term, and a careful analysis of this term. The Steklov smoothing operator (borrowed from [ZhPas]) and estimates in the ε-neighborhood of the boundary play an important technical role. First we prove estimate (0.7), and then we prove (0.6) by using the already proved inequality (0.7) and duality arguments.

We have achieved some technical simplifications in the approach compared with [PSu2, Su2, Su3]: the presentation is consistently given in terms of integral identities, in the Neumann problem we avoid consideration of the conormal derivative, and we introduce the correction term in a different way compared to [Su3].

It is relatively simple to deduce estimates (0.8) and (0.9) by using the already proved estimates at the point $\zeta = -1$ and appropriate resolvent identities.

0.5. Application of the results. The present investigation was stimulated by the study of homogenization for the initial boundary value problems for a parabolic equation

$$\frac{\partial \mathbf{u}_\varepsilon(\mathbf{x},t)}{\partial t} = -b(\mathbf{D})^* g(\mathbf{x}/\varepsilon) b(\mathbf{D}) \mathbf{u}_\varepsilon(\mathbf{x},t), \quad \mathbf{x} \in \mathcal{O}, \ t \geq 0,$$

with the initial condition $\mathbf{u}_\varepsilon(\mathbf{x},0) = \boldsymbol{\phi}(\mathbf{x})$, $\boldsymbol{\phi} \in L_2(\mathcal{O};\mathbb{C}^n)$, and the Dirichlet or Neumann condition on $\partial\mathcal{O}$. The solution can be written in terms of the operator exponential, $\mathbf{u}_\varepsilon = e^{-\mathcal{A}_{\dagger,\varepsilon} t}\boldsymbol{\phi}$, and therefore the problem reduces to approximation of the operator exponential $e^{-\mathcal{A}_{\dagger,\varepsilon} t}$ in various operator norms. To study this problem, one may use the representation of the operator exponential as the integral of the resolvent over an appropriate contour γ in the complex plane:

$$e^{-\mathcal{A}_{\dagger,\varepsilon} t} = -\frac{1}{2\pi i} \int_\gamma e^{-\zeta t} (\mathcal{A}_{\dagger,\varepsilon} - \zeta I)^{-1} \, d\zeta.$$

It turns out that, in order to obtain two-parametric approximations of the exponential $e^{-\mathcal{A}_{\dagger,\varepsilon} t}$ of right order with respect to ε and t, approximations of the form (0.6)–(0.9) (where the dependence on both parameters ε and ζ is traced) are required. The separate paper [MSu2] was devoted to the parabolic problems (see also the brief communication [MSu1]).

0.6. Organization of the paper. The paper consists of three chapters. Chapter 1 (§§1 and 2) is devoted to the problem in $\mathbb{R}^d$. In §1, we introduce the class of operators to be studied, describe the effective operator, and define the Steklov smoothing operator S_ε. In §2, we deduce estimates (0.4) and (0.5) from the known estimates for $\zeta = -1$ with the help of the resolvent identities and the scaling transformation. In the general case, the corrector in (0.5) involves the operator S_ε. It is shown that, under an additional condition (the solution $\Lambda(\mathbf{x})$ of the auxiliary problem (1.7) should be bounded), the operator S_ε can be removed and the standard corrector can be used.

Chapter 2 (§§3–8) is devoted to the Dirichlet problem. In §3, the statement of the problem is given, the effective operator is described, and some auxiliary statements are given. §4 contains the formulations of the main results for the Dirichlet problem (Theorems 4.1 and 4.2) and the first two steps of the proof: the associated problem in $\mathbb{R}^d$ is considered, the boundary layer correction term $\mathbf{w}_\varepsilon$ is introduced, and the problem is reduced to estimates of $\mathbf{w}_\varepsilon$ in $H^1(\mathcal{O})$ and $L_2(\mathcal{O})$. In §5, the required estimates of the correction term are obtained and the proof of Theorems 4.1 and 4.2 is completed. In §6, the case where $\Lambda \in L_\infty$ and some special cases are treated. §7 is devoted to approximation of the solutions of the Dirichlet problem in a strictly interior subdomain of the domain $\mathcal{O}$. In §8, a different type approximation (estimates (0.8), (0.9)) for the resolvent of $\mathcal{A}_{D,\varepsilon}$ is obtained.

Chapter 3 (§§9–14) is devoted to the Neumann problem. In §9, the statement of the problem is given and the effective operator is described. In §10, the main results for the Neumann problem (Theorems 10.1 and 10.2) are formulated and the first two steps of the proof are done. In §11, the required estimates of the correction term are obtained and the proof of Theorems 10.1 and 10.2 is completed. In §12, the case where $\Lambda \in L_\infty$ and the special cases are considered; also, estimates in a strictly interior subdomain of $\mathcal{O}$

are obtained. In §13, we find a different type approximation for the resolvent of $\mathcal{A}_{N,\varepsilon}$ for $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$. Finally, in §14 we consider the operator $\mathcal{B}_{N,\varepsilon}$ that is the part of $\mathcal{A}_{N,\varepsilon}$ in the invariant subspace orthogonal to the kernel of $b(\mathbf{D})$. For the resolvent of $\mathcal{B}_{N,\varepsilon}$ we find approximations (similar to (0.8), (0.9)) for $\zeta \in \mathbb{C} \setminus [c_\flat, \infty)$, where $c_\flat > 0$ is a common lower bound of the operators $\mathcal{B}_{N,\varepsilon}$ and $\mathcal{B}_N^0$. Next, with the help of these results we find approximations of the resolvent $(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1}$ at a regular point $\zeta \in \mathbb{C} \setminus [c_\flat, \infty)$, $\zeta \neq 0$.

0.7. Notation. Let $\mathfrak{H}$ and $\mathfrak{H}_*$ be complex separable Hilbert spaces. The symbols $(\cdot, \cdot)_{\mathfrak{H}}$ and $\|\cdot\|_{\mathfrak{H}}$ stand for the inner product and the norm in $\mathfrak{H}$; the symbol $\|\cdot\|_{\mathfrak{H}\to\mathfrak{H}_*}$ denotes the norm of a linear continuous operator acting from $\mathfrak{H}$ to $\mathfrak{H}_*$.

The symbols $\langle \cdot, \cdot \rangle$ and $|\cdot|$ stand for the inner product and the norm in $\mathbb{C}^n$; $\mathbf{1} = \mathbf{1}_n$ is the identity $(n\times n)$-matrix. If a is an $(m\times n)$-matrix, the symbol $|a|$ denotes the norm of a as an operator from $\mathbb{C}^n$ to $\mathbb{C}^m$. We denote $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$, $iD_j = \partial_j = \partial/\partial x_j$, $j = 1, \dots, d$, and $\mathbf{D} = -i\nabla = (D_1, \dots, D_d)$. The L_p-classes of $\mathbb{C}^n$-valued functions in a domain $\mathcal{O} \subset \mathbb{R}^d$ are denoted by $L_p(\mathcal{O}; \mathbb{C}^n)$, $1 \le p \le \infty$. The Sobolev classes of $\mathbb{C}^n$-valued functions in a domain $\mathcal{O} \subset \mathbb{R}^d$ are denoted by $H^s(\mathcal{O}; \mathbb{C}^n)$. Next, $H^1_0(\mathcal{O}; \mathbb{C}^n)$ is the closure of $C_0^\infty(\mathcal{O}; \mathbb{C}^n)$ in $H^1(\mathcal{O}; \mathbb{C}^n)$. If $n = 1$, we write simply $L_p(\mathcal{O})$, $H^s(\mathcal{O})$, etc., but sometimes we use this abbreviated notation also for spaces of vector-valued or matrix-valued functions.

We denote $\mathbb{R}_+ = [0, \infty)$. Various constants in estimates are denoted by the symbols c, C, $\mathcal{C}$, $\mathfrak{C}$ (possibly, with indices and marks).

A brief communication on the results of the present paper was published in [Su4].

Chapter 1. The problem in $\mathbb{R}^d$

§1. Periodic elliptic operators in $L_2(\mathbb{R}^d; \mathbb{C}^n)$

1.1. Lattices in $\mathbb{R}^d$. Let $\Gamma \subset \mathbb{R}^d$ be the lattice generated by a basis $\mathbf{a}_1, \dots, \mathbf{a}_d \in \mathbb{R}^d$:

$$\Gamma = \Big\{ \mathbf{a} \in \mathbb{R}^d \,:\, \mathbf{a} = \sum_{j=1}^d \nu_j \mathbf{a}_j,\ \ \nu_j \in \mathbb{Z} \Big\},$$

and let Ω be the (elementary) cell of the lattice Γ:

$$\Omega := \Big\{ \mathbf{x} \in \mathbb{R}^d \,:\, \mathbf{x} = \sum_{j=1}^d \tau_j \mathbf{a}_j,\ \ -\frac{1}{2} < \tau_j < \frac{1}{2} \Big\}.$$

We use the notation $|\Omega| = \operatorname{meas}\Omega$.

The basis $\mathbf{b}_1, \dots, \mathbf{b}_d$ in $\mathbb{R}^d$ dual to $\mathbf{a}_1, \dots, \mathbf{a}_d$ is defined by the relations $\langle \mathbf{b}_i, \mathbf{a}_j \rangle = 2\pi\delta_{ij}$. This basis generates a lattice $\widetilde{\Gamma}$ dual to the lattice Γ. Below we use the notation

$$r_0 = \frac{1}{2} \min_{0 \neq \mathbf{b} \in \widetilde{\Gamma}} |\mathbf{b}|, \quad r_1 = \frac{1}{2} \operatorname{diam}\Omega.$$

By $\widetilde{H}^1(\Omega)$ we denote the subspace of all functions in $H^1(\Omega)$ whose Γ-periodic extension to $\mathbb{R}^d$ belongs to $H^1_{\mathrm{loc}}(\mathbb{R}^d)$. If $\varphi(\mathbf{x})$ is a Γ-periodic function on $\mathbb{R}^d$, we put

$$\varphi^\varepsilon(\mathbf{x}) := \varphi(\varepsilon^{-1}\mathbf{x}), \quad \varepsilon > 0.$$

1.2. The class of operators. In $L_2(\mathbb{R}^d; \mathbb{C}^n)$, consider the second order DO $\mathcal{A}_\varepsilon$ formally given by the differential expression

$$\mathcal{A}_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}), \quad \varepsilon > 0. \tag{1.1}$$

Here $g(\mathbf{x})$ is a measurable Hermitian $(m \times m)$-matrix-valued function (in general, with complex entries). It is assumed that $g(\mathbf{x})$ is periodic with respect to the lattice Γ, uniformly positive definite, and bounded. Next, $b(\mathbf{D})$ is an $(m \times n)$-matrix DO of the form

$$b(\mathbf{D}) = \sum_{l=1}^{d} b_l D_l, \tag{1.2}$$

where the b_l are constant matrices (in general, with complex entries). The symbol $b(\boldsymbol{\xi}) = \sum_{l=1}^{d} b_l \xi_l$, $\boldsymbol{\xi} \in \mathbb{R}^d$, corresponds to the operator $b(\mathbf{D})$. It is assumed that $m \geq n$ and

$$\operatorname{rank} b(\boldsymbol{\xi}) = n, \quad 0 \neq \boldsymbol{\xi} \in \mathbb{R}^d. \tag{1.3}$$

This condition is equivalent to the inequalities

$$\alpha_0 \mathbf{1}_n \leq b(\boldsymbol{\theta})^* b(\boldsymbol{\theta}) \leq \alpha_1 \mathbf{1}_n, \quad \boldsymbol{\theta} \in \mathbb{S}^{d-1}, \quad 0 < \alpha_0 \leq \alpha_1 < \infty, \tag{1.4}$$

with some positive constants α_0 and α_1. Note that (1.4) implies

$$|b_l| \leq \alpha_1^{1/2}, \quad l = 1, \dots, d. \tag{1.5}$$

The precise definition of the operator $\mathcal{A}_\varepsilon$ is given in terms of the quadratic form

$$a_\varepsilon[\mathbf{u}, \mathbf{u}] = \int_{\mathbb{R}^d} \langle g^\varepsilon(\mathbf{x}) b(\mathbf{D})\mathbf{u}, b(\mathbf{D})\mathbf{u} \rangle \, d\mathbf{x}, \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n).$$

Under the above assumptions, this form is closed in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ and nonnegative. Using the Fourier transformation and condition (1.4), it is easy to check that

$$\begin{gathered} c_0 \int_{\mathbb{R}^d} |\mathbf{D}\mathbf{u}|^2 \, d\mathbf{x} \leq a_\varepsilon[\mathbf{u}, \mathbf{u}] \leq c_1 \int_{\mathbb{R}^d} |\mathbf{D}\mathbf{u}|^2 \, d\mathbf{x}, \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n), \\ c_0 = \alpha_0 \|g^{-1}\|_{L_\infty}^{-1}, \quad c_1 = \alpha_1 \|g\|_{L_\infty}. \end{gathered} \tag{1.6}$$

1.3. The effective operator. In order to formulate the results, we need to introduce the effective operator $\mathcal{A}^0$. Let an $(n \times m)$-matrix-valued function $\Lambda \in \widetilde{H}^1(\Omega)$ be the (weak) Γ-periodic solution of the problem

$$b(\mathbf{D})^* g(\mathbf{x}) \left(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m \right) = 0, \quad \int_\Omega \Lambda(\mathbf{x}) \, d\mathbf{x} = 0. \tag{1.7}$$

The so-called *effective matrix* g^0 of size $m \times m$ is defined as follows:

$$g^0 = |\Omega|^{-1} \int_\Omega \widetilde{g}(\mathbf{x}) \, d\mathbf{x}, \tag{1.8}$$

where

$$\widetilde{g}(\mathbf{x}) := g(\mathbf{x}) \left(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m \right). \tag{1.9}$$

It turns out that the matrix g^0 is positive. The *effective operator* $\mathcal{A}^0$ for the operator (1.1) is given by the differential expression $\mathcal{A}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$ on the domain $H^2(\mathbb{R}^d; \mathbb{C}^n)$.

Below we need the following estimates for $\Lambda(\mathbf{x})$ proved in [BSu3], see formula (6.28) and Subsection 7.3 therein:

$$\|\mathbf{D}\Lambda\|_{L_2(\Omega)} \leq |\Omega|^{1/2} m^{1/2} \alpha_0^{-1/2} \|g\|_{L_\infty}^{1/2} \|g^{-1}\|_{L_\infty}^{1/2}, \tag{1.10}$$

$$\|\Lambda\|_{L_2(\Omega)} \leq |\Omega|^{1/2} m^{1/2} (2r_0)^{-1} \alpha_0^{-1/2} \|g\|_{L_\infty}^{1/2} \|g^{-1}\|_{L_\infty}^{1/2}. \tag{1.11}$$

1.4. Properties of the effective matrix. The following properties of the effective matrix were checked in [BSu2, Chapter 3, Theorem 1.5].

Proposition 1.1. *The effective matrix satisfies the estimates*

$$\underline{g} \le g^0 \le \overline{g}. \tag{1.12}$$

Here

$$\overline{g} = |\Omega|^{-1} \int_\Omega g(\mathbf{x})\, d\mathbf{x}, \quad \underline{g} = \left(|\Omega|^{-1} \int_\Omega g(\mathbf{x})^{-1}\, d\mathbf{x} \right)^{-1}.$$

If $m = n$, the effective matrix g^0 coincides with $\underline{g}$.

In homogenization theory for specific DO's, estimates (1.12) are known as the Voight–Reuss bracketing. We distinguish the cases where one of the inequalities in (1.12) becomes an identity. The following statements were obtained in [BSu2, Chapter 3, Propositions 1.6 and 1.7].

Proposition 1.2. *The identity $g^0 = \overline{g}$ is equivalent to the relations*

$$b(\mathbf{D})^* \mathbf{g}_k(\mathbf{x}) = 0, \quad k = 1, \dots, m, \tag{1.13}$$

where the $\mathbf{g}_k(\mathbf{x})$, $k = 1, \dots, m$, are the columns of the matrix $g(\mathbf{x})$.

Proposition 1.3. *The identity $g^0 = \underline{g}$ is equivalent to the relations*

$$\mathbf{l}_k(\mathbf{x}) = \mathbf{l}_k^0 + b(\mathbf{D})\mathbf{w}_k, \ \ \mathbf{l}_k^0 \in \mathbb{C}^m, \quad \mathbf{w}_k \in \widetilde{H}^1(\Omega; \mathbb{C}^n), \ \ k = 1, \dots, m, \tag{1.14}$$

where $\mathbf{l}_k(\mathbf{x})$, $k = 1, \dots, m$, are the columns of the matrix $g(\mathbf{x})^{-1}$.

Obviously, (1.12) implies the following estimates for the norms of the matrices g^0 and $(g^0)^{-1}$:

$$|g^0| \le \|g\|_{L_\infty}, \quad |(g^0)^{-1}| \le \|g^{-1}\|_{L_\infty}. \tag{1.15}$$

By (1.4) and (1.15), the symbol of the effective operator $\mathcal{A}^0$ satisfies the inequalities

$$c_0 |\boldsymbol{\xi}|^2 \mathbf{1}_n \le b(\boldsymbol{\xi})^* g^0 b(\boldsymbol{\xi}) \le c_1 |\boldsymbol{\xi}|^2 \mathbf{1}_n, \quad \boldsymbol{\xi} \in \mathbb{R}^d, \tag{1.16}$$

where c_0 and c_1 are the same as in (1.6).

1.5. The Steklov smoothing. We introduce an auxiliary smoothing operator S_ε that acts in $L_2(\mathbb{R}^d; \mathbb{C}^m)$ and is defined by

$$(S_\varepsilon \mathbf{u})(\mathbf{x}) = |\Omega|^{-1} \int_\Omega \mathbf{u}(\mathbf{x} - \varepsilon \mathbf{z})\, d\mathbf{z}. \tag{1.17}$$

It is called the *Steklov smoothing operator*. Note that

$$\|S_\varepsilon\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \le 1. \tag{1.18}$$

Obviously, $D^\alpha S_\varepsilon \mathbf{u} = S_\varepsilon D^\alpha \mathbf{u}$ for $\mathbf{u} \in H^s(\mathbb{R}^d; \mathbb{C}^m)$ and for any multiindex α such that $|\alpha| \le s$. We need some properties of the operator (1.17) (see [ZhPas, Lemmas 1.1 and 1.2] or [PSu2, Propositions 3.1 and 3.2]).

Proposition 1.4. *For any $\mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^m)$ we have*

$$\|S_\varepsilon \mathbf{u} - \mathbf{u}\|_{L_2(\mathbb{R}^d)} \le \varepsilon r_1 \|\mathbf{D}\mathbf{u}\|_{L_2(\mathbb{R}^d)}, \quad \varepsilon > 0.$$

Proposition 1.5. *Let $f(\mathbf{x})$ be a Γ-periodic function in $\mathbb{R}^d$ such that $f \in L_2(\Omega)$. Let $[f^\varepsilon]$ be the operator of multiplication by the function $f^\varepsilon(\mathbf{x})$. Then the operator $[f^\varepsilon]S_\varepsilon$ is continuous in $L_2(\mathbb{R}^d; \mathbb{C}^m)$, and*

$$\|[f^\varepsilon] S_\varepsilon\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \le |\Omega|^{-1/2} \|f\|_{L_2(\Omega)}, \quad \varepsilon > 0.$$

From Proposition 1.5 and estimates (1.10), (1.11) it follows that

$$\|[\Lambda^\varepsilon]S_\varepsilon\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le m^{1/2}(2r_0)^{-1}\alpha_0^{-1/2}\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty}^{1/2} =: M_1, \ \ \varepsilon > 0, \tag{1.19}$$

$$\|[(\mathbf{D}\Lambda)^\varepsilon]S_\varepsilon\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le m^{1/2}\alpha_0^{-1/2}\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty}^{1/2} =: M_2, \ \ \varepsilon > 0. \tag{1.20}$$

§2. Results for the homogenization problem in $\mathbb{R}^d$

In this section, we study homogenization for the operator $\mathcal{A}_\varepsilon$ in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. Precisely, we deduce theorems about approximation of the resolvent $(\mathcal{A}_\varepsilon - \zeta I)^{-1}$ for any $\zeta \in \mathbb{C}\setminus\mathbb{R}_+$ from the known results about approximation of the resolvent $(\mathcal{A}_\varepsilon + I)^{-1}$, obtained in [BSu2, BSu4], and [PSu2].

2.1. Approximation of the resolvent in the operator norm in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. A point $\zeta \in \mathbb{C}\setminus\mathbb{R}_+$ is regular for both $\mathcal{A}_\varepsilon$ and $\mathcal{A}^0$. We put $\zeta = |\zeta|e^{i\varphi}$, $\varphi \in (0,2\pi)$, and denote

$$c(\varphi) = \begin{cases} |\sin\varphi|^{-1} & \text{if } \varphi \in (0,\pi/2)\cup(3\pi/2,2\pi), \\ 1 & \text{if } \varphi \in [\pi/2,3\pi/2]. \end{cases} \tag{2.1}$$

Let $\mathbf{F} \in L_2(\mathbb{R}^d;\mathbb{C}^n)$. Consider the following elliptic equation in $\mathbb{R}^d$:

$$\mathcal{A}_\varepsilon \mathbf{u}_\varepsilon - \zeta\mathbf{u}_\varepsilon = \mathbf{F}. \tag{2.2}$$

Lemma 2.1. *The solution $\mathbf{u}_\varepsilon$ of problem* (2.2) *satisfies the following estimates for $\varepsilon > 0$:*

$$\|\mathbf{u}_\varepsilon\|_{L_2(\mathbb{R}^d)} \le c(\varphi)|\zeta|^{-1}\|\mathbf{F}\|_{L_2(\mathbb{R}^d)}, \tag{2.3}$$

$$\|\mathbf{D}\mathbf{u}_\varepsilon\|_{L_2(\mathbb{R}^d)} \le \mathcal{C}_0 c(\varphi)|\zeta|^{-1/2}\|\mathbf{F}\|_{L_2(\mathbb{R}^d)}, \tag{2.4}$$

where $\mathcal{C}_0 = \sqrt{2}c_0^{-1/2}$. In operator terms,

$$\begin{aligned} &\left\|(\mathcal{A}_\varepsilon - \zeta I)^{-1}\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le c(\varphi)|\zeta|^{-1}, \\ &\left\|\mathbf{D}(\mathcal{A}_\varepsilon - \zeta I)^{-1}\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \mathcal{C}_0 c(\varphi)|\zeta|^{-1/2}. \end{aligned} \tag{2.5}$$

Proof. The spectrum of $\mathcal{A}_\varepsilon$ is contained in $\mathbb{R}_+$. The norm of the resolvent $(\mathcal{A}_\varepsilon - \zeta I)^{-1}$ does not exceed the inverse distance from the point ζ to $\mathbb{R}_+$. This implies (2.5).

In order to check (2.4), we write the following integral identity for $\mathbf{u}_\varepsilon$:

$$(g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon, b(\mathbf{D})\boldsymbol{\eta})_{L_2(\mathbb{R}^d)} - \zeta(\mathbf{u}_\varepsilon,\boldsymbol{\eta})_{L_2(\mathbb{R}^d)} = (\mathbf{F},\boldsymbol{\eta})_{L_2(\mathbb{R}^d)}, \quad \boldsymbol{\eta} \in H^1(\mathbb{R}^d;\mathbb{C}^n). \tag{2.6}$$

Putting $\boldsymbol{\eta} = \mathbf{u}_\varepsilon$ in (2.6) and using (2.3), we obtain

$$a_\varepsilon[\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon] \le 2c(\varphi)^2|\zeta|^{-1}\|\mathbf{F}\|_{L_2(\mathbb{R}^d)}^2.$$

By (1.6), this yields (2.4). □

It is known that, as $\varepsilon \to 0$, the solution $\mathbf{u}_\varepsilon$ converges in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to the solution of the "homogenized" equation

$$\mathcal{A}^0\mathbf{u}_0 - \zeta\mathbf{u}_0 = \mathbf{F}. \tag{2.7}$$

Theorem 2.2. *Suppose that $\zeta = |\zeta|e^{i\varphi} \in \mathbb{C}\setminus\mathbb{R}_+$. Let $c(\varphi)$ be defined by* (2.1), *let $\mathbf{u}_\varepsilon$ be the solution of equation* (2.2) *and $\mathbf{u}_0$ the solution of* (2.7). *Then*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathbb{R}^d)} \le C_1 c(\varphi)^2|\zeta|^{-1/2}\varepsilon\|\mathbf{F}\|_{L_2(\mathbb{R}^d)}, \quad \varepsilon > 0,$$

or, in operator terms,

$$\left\|(\mathcal{A}_\varepsilon - \zeta I)^{-1} - (\mathcal{A}^0 - \zeta I)^{-1}\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le C_1 c(\varphi)^2|\zeta|^{-1/2}\varepsilon, \quad \varepsilon > 0. \tag{2.8}$$

The constant C_1 depends only on the norms $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the constants α_0, α_1 from (1.4), *and the parameters of the lattice Γ.*

Proof. In [BSu2, Chapter 4, Theorem 2.1], estimate (2.8) was proved in the case where $\zeta = -1$ and $0 < \varepsilon \le 1$. Obviously, for $\zeta = -1$ and $\varepsilon > 1$ the left-hand side of (2.8) does not exceed 2, and then also does not exceed 2ε. So, we start with the estimate

$$\big\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}^0 + I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \check{C}_1\varepsilon, \quad \varepsilon > 0, \tag{2.9}$$

where $\check{C}_1$ depends only on $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, α_0, α_1, and the parameters of the lattice Γ.

Now, we carry this estimate over to the case where $\zeta = \widehat{\zeta} = e^{i\varphi}$ by using the identity

$$\begin{aligned}(\mathcal{A}_\varepsilon - \zeta I)^{-1} &- (\mathcal{A}^0 - \zeta I)^{-1} \\ &= (\mathcal{A}_\varepsilon + I)(\mathcal{A}_\varepsilon - \zeta I)^{-1}\big((\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}^0 + I)^{-1}\big)(\mathcal{A}^0 + I)(\mathcal{A}^0 - \zeta I)^{-1}.\end{aligned} \tag{2.10}$$

We have

$$\big\|(\mathcal{A}_\varepsilon + I)(\mathcal{A}_\varepsilon - \widehat{\zeta} I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \sup_{x\ge 0}(x+1)|x - \widehat{\zeta}|^{-1} \le 2c(\varphi). \tag{2.11}$$

The norm of the operator $(\mathcal{A}^0 + I)(\mathcal{A}^0 - \widehat{\zeta} I)^{-1}$ satisfies a similar estimate:

$$\big\|(\mathcal{A}^0 + I)(\mathcal{A}^0 - \widehat{\zeta} I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le 2c(\varphi). \tag{2.12}$$

From (2.9)–(2.12) it follows that

$$\big\|(\mathcal{A}_\varepsilon - \widehat{\zeta} I)^{-1} - (\mathcal{A}^0 - \widehat{\zeta} I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le 4\check{C}_1 c(\varphi)^2\varepsilon, \quad \varepsilon > 0. \tag{2.13}$$

By the scaling transformation, (2.13) is equivalent to

$$\big\|(\mathcal{A} - \varepsilon^2\widehat{\zeta} I)^{-1} - (\mathcal{A}^0 - \varepsilon^2\widehat{\zeta} I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le 4\check{C}_1 c(\varphi)^2\varepsilon^{-1}, \quad \varepsilon > 0. \tag{2.14}$$

Here $\mathcal{A} = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D})$. Replacing ε by $\varepsilon|\zeta|^{1/2}$ in (2.14) and applying the inverse transformation, we obtain (2.8) with $C_1 = 4\check{C}_1$. ☐

2.2. Approximation of the resolvent in the $(L_2 \to H^1)$-norm. In order to approximate the solution $\mathbf{u}_\varepsilon$ in $H^1(\mathbb{R}^d;\mathbb{C}^n)$, we need to take the first order corrector into account. We put

$$K(\varepsilon;\zeta) = [\Lambda^\varepsilon] S_\varepsilon b(\mathbf{D})(\mathcal{A}^0 - \zeta I)^{-1}. \tag{2.15}$$

Here $[\Lambda^\varepsilon]$ denotes the operator of multiplication by the matrix-valued function $\Lambda(\varepsilon^{-1}\mathbf{x})$, and S_ε is the smoothing operator (1.17). The operator (2.15) is a continuous mapping of $L_2(\mathbb{R}^d;\mathbb{C}^n)$ into $H^1(\mathbb{R}^d;\mathbb{C}^n)$; this is a consequence of the identity

$$K(\varepsilon;\zeta) = K(\varepsilon;-1)(\mathcal{A}^0 + I)(\mathcal{A}^0 - \zeta I)^{-1} \tag{2.16}$$

and the following lemma.

Lemma 2.3. *For $\varepsilon > 0$ and $\zeta = -1$, the operator* (2.15) *is continuous from $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to $H^1(\mathbb{R}^d;\mathbb{C}^n)$, and we have*

$$\|K(\varepsilon;-1)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le C_K^{(1)}, \tag{2.17}$$

$$\|\varepsilon\mathbf{D}K(\varepsilon;-1)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le C_K^{(2)}\varepsilon + C_K^{(3)}. \tag{2.18}$$

The constants $C_K^{(j)}$, $j = 1,2,3$, depend only on m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.

Proof. First, we estimate the $(L_2 \to L_2)$-norm of $K(\varepsilon;-1)$:

$$\|K(\varepsilon;-1)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \|[\Lambda^\varepsilon]S_\varepsilon\|_{L_2\to L_2}\|b(\mathbf{D})(\mathcal{A}^0 + I)^{-1}\|_{L_2\to L_2}. \tag{2.19}$$

Using the Fourier transformation and (1.4), (1.16), we obtain

$$\begin{aligned}\|b(\mathbf{D})(\mathcal{A}^0+I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} &\le \sup_{\boldsymbol{\xi}\in\mathbb{R}^d} |b(\boldsymbol{\xi})\left(b(\boldsymbol{\xi})^*g^0b(\boldsymbol{\xi})+\mathbf{1}\right)^{-1}| \\ &\le \alpha_1^{1/2}\sup_{\boldsymbol{\xi}\in\mathbb{R}^d}|\boldsymbol{\xi}|(c_0|\boldsymbol{\xi}|^2+1)^{-1} \le \frac{1}{2}\alpha_1^{1/2}c_0^{-1/2}.\end{aligned} \tag{2.20}$$

Relations (1.19), (2.19), and (2.20) imply (2.17) with the constant $C_K^{(1)}=\frac{1}{2}\alpha_1^{1/2}c_0^{-1/2}M_1$.

Now, consider the operators

$$\varepsilon D_jK(\varepsilon;-1)=[(D_j\Lambda)^\varepsilon]S_\varepsilon b(\mathbf{D})(\mathcal{A}^0+I)^{-1}+\varepsilon[\Lambda^\varepsilon]S_\varepsilon D_jb(\mathbf{D})(\mathcal{A}^0+I)^{-1}. \tag{2.21}$$

Like in (2.20), we have

$$\|\mathbf{D}b(\mathbf{D})(\mathcal{A}^0+I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \alpha_1^{1/2}\sup_{\boldsymbol{\xi}\in\mathbb{R}^d}|\boldsymbol{\xi}|^2(c_0|\boldsymbol{\xi}|^2+1)^{-1}\le \alpha_1^{1/2}c_0^{-1}. \tag{2.22}$$

Relations (1.19), (1.20), and (2.20)–(2.22) imply inequality (2.18) with the constants $C_K^{(2)}=(2\alpha_1)^{1/2}c_0^{-1}M_1$ and $C_K^{(3)}=\alpha_1^{1/2}(2c_0)^{-1/2}M_2$. □

The "first order approximation" to the solution $\mathbf{u}_\varepsilon$ is given by

$$\mathbf{v}_\varepsilon=\mathbf{u}_0+\varepsilon\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\mathbf{u}_0=(\mathcal{A}^0-\zeta I)^{-1}\mathbf{F}+\varepsilon K(\varepsilon;\zeta)\mathbf{F}. \tag{2.23}$$

Theorem 2.4. *In the assumptions of Theorem* 2.2, *suppose that* $\mathbf{v}_\varepsilon$ *is defined by* (2.23). *Then for* $\varepsilon>0$ *we have*

$$\|\mathbf{D}(\mathbf{u}_\varepsilon-\mathbf{v}_\varepsilon)\|_{L_2(\mathbb{R}^d)}\le C_2c(\varphi)^2\varepsilon\|\mathbf{F}\|_{L_2(\mathbb{R}^d)}, \tag{2.24}$$

$$\|\mathbf{u}_\varepsilon-\mathbf{v}_\varepsilon\|_{L_2(\mathbb{R}^d)}\le C_3c(\varphi)^2|\zeta|^{-1/2}\varepsilon\|\mathbf{F}\|_{L_2(\mathbb{R}^d)}, \tag{2.25}$$

or, in operator terms,

$$\|\mathbf{D}\left((\mathcal{A}_\varepsilon-\zeta I)^{-1}-(\mathcal{A}^0-\zeta I)^{-1}-\varepsilon K(\varepsilon;\zeta)\right)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\le C_2c(\varphi)^2\varepsilon, \tag{2.26}$$

$$\|(\mathcal{A}_\varepsilon-\zeta I)^{-1}-(\mathcal{A}^0-\zeta I)^{-1}-\varepsilon K(\varepsilon;\zeta)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\le C_3c(\varphi)^2|\zeta|^{-1/2}\varepsilon. \tag{2.27}$$

Here $c(\varphi)$ *is defined by* (2.1), *and the constants* C_2, C_3 *depend only on* m, d, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, α_0, α_1, *and the parameters of the lattice* Γ.

Theorem 2.4 directly implies the following corollary.

Corollary 2.5. *Under the assumptions of Theorem* 2.4 *we have*

$$\|\mathbf{u}_\varepsilon-\mathbf{v}_\varepsilon\|_{H^1(\mathbb{R}^d)}\le c(\varphi)^2(C_2+C_3|\zeta|^{-1/2})\varepsilon\|\mathbf{F}\|_{L_2(\mathbb{R}^d)},\quad \varepsilon>0.$$

Proof of Theorem 2.4. In [BSu4, Theorem 10.6], similar estimates were proved for $\zeta=-1$, but with a different smoothing operator in place of S_ε. In [PSu2, Theorem 3.3], it was shown that the passage to the smoothing operator S_ε is justified, and inequalities (2.24), (2.25) were proved for $\zeta=-1$ and $0<\varepsilon\le 1$. Thus, we start with the estimates

$$\|\mathbf{D}\left((\mathcal{A}_\varepsilon+I)^{-1}-(\mathcal{A}^0+I)^{-1}-\varepsilon K(\varepsilon;-1)\right)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\le \check{C}_2\varepsilon,\ \ 0<\varepsilon\le 1, \tag{2.28}$$

$$\|(\mathcal{A}_\varepsilon+I)^{-1}-(\mathcal{A}^0+I)^{-1}-\varepsilon K(\varepsilon;-1)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\le \check{C}_3\varepsilon,\ \ 0<\varepsilon\le 1. \tag{2.29}$$

The constants $\check{C}_2$, $\check{C}_3$ depend only on m, d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.

For $\varepsilon>1$ estimates are trivial: it suffices to estimate each term under the norm sign in (2.28), (2.29) separately. By (2.17), the left-hand side of (2.29) does not exceed $2+C_K^{(1)}\varepsilon$, and so does not exceed $(2+C_K^{(1)})\varepsilon$ if $\varepsilon>1$. Combining this with (2.29), we obtain

$$\|(\mathcal{A}_\varepsilon+I)^{-1}-(\mathcal{A}^0+I)^{-1}-\varepsilon K(\varepsilon;-1)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\le \widehat{C}_3\varepsilon,\ \ \varepsilon>0, \tag{2.30}$$

where $\widehat{C}_3 = \max\{\check{C}_3, 2 + C_K^{(1)}\}$.

Next, by (1.6) and (1.15),

$$\|\mathbf{D}(\mathcal{A}_\varepsilon + I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le c_0^{-1/2}, \tag{2.31}$$

$$\|\mathbf{D}(\mathcal{A}^0 + I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le c_0^{-1/2}. \tag{2.32}$$

From (2.18), (2.31), and (2.32) it is seen that the left-hand side of (2.28) does not exceed $2c_0^{-1/2} + C_K^{(2)}\varepsilon + C_K^{(3)}$, and so does not exceed $(2c_0^{-1/2} + C_K^{(2)} + C_K^{(3)})\varepsilon$ if $\varepsilon > 1$. Together with (2.28), this yields

$$\big\|\mathbf{D}\big((\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}^0 + I)^{-1} - \varepsilon K(\varepsilon; -1)\big)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \widehat{C}_2\varepsilon, \ \ \varepsilon > 0, \tag{2.33}$$

where $\widehat{C}_2 = \max\{\check{C}_2, 2c_0^{-1/2} + C_K^{(2)} + C_K^{(3)}\}$.

Now, we carry estimates (2.30), (2.33) over to the case where $\zeta = \widehat{\zeta} = e^{i\varphi}$ by using the identity

$$\begin{aligned} &(\mathcal{A}_\varepsilon - \zeta I)^{-1} - (\mathcal{A}^0 - \zeta I)^{-1} - \varepsilon K(\varepsilon;\zeta) \\ &\quad = (\mathcal{A}_\varepsilon + I)(\mathcal{A}_\varepsilon - \zeta I)^{-1}\big((\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}^0 + I)^{-1} - \varepsilon K(\varepsilon; -1)\big) \\ &\qquad \times (\mathcal{A}^0 + I)(\mathcal{A}^0 - \zeta I)^{-1} + \varepsilon(\zeta + 1)(\mathcal{A}_\varepsilon - \zeta I)^{-1} K(\varepsilon;\zeta). \end{aligned} \tag{2.34}$$

By (2.5),

$$\big\|(\mathcal{A}_\varepsilon - \widehat{\zeta} I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le c(\varphi). \tag{2.35}$$

From (2.12), (2.16), and (2.17) it follows that

$$\|K(\varepsilon;\widehat{\zeta})\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le 2C_K^{(1)} c(\varphi). \tag{2.36}$$

Combining (2.30) and (2.34)–(2.36) and taking (2.11), (2.12) into account, we obtain

$$\big\|(\mathcal{A}_\varepsilon - \widehat{\zeta} I)^{-1} - (\mathcal{A}^0 - \widehat{\zeta} I)^{-1} - \varepsilon K(\varepsilon;\widehat{\zeta})\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le C_3 c(\varphi)^2\varepsilon, \ \ \varepsilon > 0, \tag{2.37}$$

where $C_3 = 4\widehat{C}_3 + 4C_K^{(1)}$.

Next, relations (2.11), (2.12), (2.34), and (2.36) imply that

$$\begin{aligned} &\big\|\mathcal{A}_\varepsilon^{1/2}\big((\mathcal{A}_\varepsilon - \widehat{\zeta} I)^{-1} - (\mathcal{A}^0 - \widehat{\zeta} I)^{-1} - \varepsilon K(\varepsilon;\widehat{\zeta})\big)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\ &\quad \le 4c(\varphi)^2\big\|\mathcal{A}_\varepsilon^{1/2}\big((\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}^0 + I)^{-1} - \varepsilon K(\varepsilon; -1)\big)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\ &\qquad + 2\varepsilon|\widehat{\zeta} + 1| C_K^{(1)} c(\varphi)\big\|\mathcal{A}_\varepsilon^{1/2}(\mathcal{A}_\varepsilon - \widehat{\zeta} I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}. \end{aligned} \tag{2.38}$$

We have

$$\big\|\mathcal{A}_\varepsilon^{1/2}(\mathcal{A}_\varepsilon - \widehat{\zeta} I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \sup_{x\ge 0} x^{1/2}|x - \widehat{\zeta}|^{-1} \le c(\varphi). \tag{2.39}$$

From (1.6), (2.33), (2.38), and (2.39) it follows that

$$\big\|\mathcal{A}_\varepsilon^{1/2}\big((\mathcal{A}_\varepsilon - \widehat{\zeta} I)^{-1} - (\mathcal{A}^0 - \widehat{\zeta} I)^{-1} - \varepsilon K(\varepsilon;\widehat{\zeta})\big)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \widetilde{C}_2 c(\varphi)^2\varepsilon \tag{2.40}$$

for $\varepsilon > 0$, where $\widetilde{C}_2 = 4(c_1^{1/2}\widehat{C}_2 + C_K^{(1)})$. Combining this with (1.6), we obtain

$$\big\|\mathbf{D}\big((\mathcal{A}_\varepsilon - \widehat{\zeta} I)^{-1} - (\mathcal{A}^0 - \widehat{\zeta} I)^{-1} - \varepsilon K(\varepsilon;\widehat{\zeta})\big)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le C_2 c(\varphi)^2\varepsilon \tag{2.41}$$

for $\varepsilon > 0$, where $C_2 = c_0^{-1/2}\widetilde{C}_2$.

By the scaling transformation, (2.37) is equivalent to the inequality

$$\big\|(\mathcal{A} - \widehat{\zeta}\varepsilon^2 I)^{-1} - (\mathcal{A}^0 - \widehat{\zeta}\varepsilon^2 I)^{-1} - \Lambda S_1 b(\mathbf{D})(\mathcal{A}^0 - \widehat{\zeta}\varepsilon^2 I)^{-1}\big\|_{L_2\to L_2} \le C_3 c(\varphi)^2\varepsilon^{-1} \tag{2.42}$$

for $\varepsilon > 0$, and (2.41) is equivalent to

$$\big\|\mathbf{D}\big((\mathcal{A}-\widehat{\zeta}\varepsilon^2 I)^{-1}-\mathcal{A}^0-\widehat{\zeta}\varepsilon^2 I)^{-1}-\Lambda S_1 b(\mathbf{D})(\mathcal{A}^0-\widehat{\zeta}\varepsilon^2 I)^{-1}\big)\big\|_{L_2\to L_2}\leq C_2 c(\varphi)^2, \quad \varepsilon>0. \tag{2.43}$$

Replacing ε by $\varepsilon|\zeta|^{1/2}$ in (2.42) and (2.43) and applying the inverse transformation, we arrive at (2.26), (2.27). □

Now we approximate the flux $\mathbf{p}_\varepsilon := g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$ in $L_2(\mathbb{R}^d;\mathbb{C}^m)$.

Theorem 2.6. *Under the assumptions of Theorem 2.2, let $\mathbf{p}_\varepsilon := g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$. Then*

$$\|\mathbf{p}_\varepsilon-\widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathbb{R}^d)}\leq C_4 c(\varphi)^2\varepsilon\|\mathbf{F}\|_{L_2(\mathbb{R}^d)},\quad \varepsilon>0,$$

or, in operator terms,

$$\big\|g^\varepsilon b(\mathbf{D})(\mathcal{A}_\varepsilon-\zeta I)^{-1}-\widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})(\mathcal{A}^0-\zeta I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\leq C_4 c(\varphi)^2\varepsilon. \tag{2.44}$$

Here $\widetilde{g}(\mathbf{x})$ is the matrix (1.9). *The constant C_4 depends only on d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.*

Proof. We start with the case where $\zeta=\widehat{\zeta}=e^{i\varphi}$. From (2.40) it follows that

$$\big\|g^\varepsilon b(\mathbf{D})\big((\mathcal{A}_\varepsilon-\widehat{\zeta}I)^{-1}-(\mathcal{A}^0-\widehat{\zeta}I)^{-1}-\varepsilon K(\varepsilon;\widehat{\zeta})\big)\big\|_{L_2\to L_2}\leq \|g\|^{1/2}_{L_\infty}\widetilde{C}_2 c(\varphi)^2\varepsilon \tag{2.45}$$

for $\varepsilon>0$. Taking (1.2) into account, we have

$$\begin{aligned}&\varepsilon g^\varepsilon b(\mathbf{D})K(\varepsilon;\widehat{\zeta})\\ &\quad= g^\varepsilon(b(\mathbf{D})\Lambda)^\varepsilon S_\varepsilon b(\mathbf{D})(\mathcal{A}^0-\widehat{\zeta}I)^{-1}+\varepsilon\sum_{l=1}^d g^\varepsilon b_l\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})D_l(\mathcal{A}^0-\widehat{\zeta}I)^{-1}.\end{aligned} \tag{2.46}$$

From (2.12) and (2.22) it follows that

$$\big\|\mathbf{D}b(\mathbf{D})(\mathcal{A}^0-\widehat{\zeta}I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\leq 2\alpha_1^{1/2}c_0^{-1}c(\varphi). \tag{2.47}$$

The second term on the right-hand side of (2.46) is estimated with the help of (1.5), (1.19), and (2.47):

$$\varepsilon\left\|\sum_{l=1}^d g^\varepsilon b_l\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})D_l(\mathcal{A}^0-\widehat{\zeta}I)^{-1}\right\|_{L_2\to L_2}\leq 2\varepsilon\|g\|_{L_\infty}\alpha_1 c_0^{-1}M_1 d^{1/2}c(\varphi). \tag{2.48}$$

Next, by Proposition 1.4 and (2.47) we have

$$\big\|g^\varepsilon(I-S_\varepsilon)b(\mathbf{D})(\mathcal{A}^0-\widehat{\zeta}I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\leq 2\varepsilon\|g\|_{L_\infty}r_1\alpha_1^{1/2}c_0^{-1}c(\varphi). \tag{2.49}$$

As a result, relations (2.45), (2.46), (2.48), and (2.49) together with (1.9) imply that

$$\big\|g^\varepsilon b(\mathbf{D})(\mathcal{A}_\varepsilon-\widehat{\zeta}I)^{-1}-\widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})(\mathcal{A}^0-\widehat{\zeta}I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\leq C_4 c(\varphi)^2\varepsilon \tag{2.50}$$

for $\varepsilon>0$, where $C_4=\|g\|^{1/2}_{L_\infty}\widetilde{C}_2+2\|g\|_{L_\infty}c_0^{-1}\alpha_1^{1/2}((d\alpha_1)^{1/2}M_1+r_1)$.

By the scaling transformation, (2.50) is equivalent to the inequality

$$\big\|gb(\mathbf{D})(\mathcal{A}-\widehat{\zeta}\varepsilon^2 I)^{-1}-\widetilde{g}S_1 b(\mathbf{D})(\mathcal{A}^0-\widehat{\zeta}\varepsilon^2 I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\leq C_4 c(\varphi)^2 \tag{2.51}$$

for $\varepsilon>0$. Replacing ε by $\varepsilon|\zeta|^{1/2}$ in (2.51) and applying the inverse transformation, we arrive at (2.44). □

Now we distinguish the case where the corrector is equal to zero. The next statement follows from Theorem 2.4, Proposition 1.2, and relations (1.7).

Proposition 2.7. *Let $\mathbf{u}_\varepsilon$ be the solution of equation* (2.2), *and let $\mathbf{u}_0$ be the solution of* (2.7). *If $g^0 = \underline{g}$, i.e., relations* (1.13) *are satisfied, then $\Lambda = 0$, $K(\varepsilon;\zeta) = 0$, and we have*

$$\|\mathbf{D}(\mathbf{u}_\varepsilon - \mathbf{u}_0)\|_{L_2(\mathbb{R}^d)} \leq C_2 c(\varphi)^2 \varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^d)}, \quad \varepsilon > 0.$$

2.3. The results for the homogenization problem in $\mathbb{R}^d$ in the case where $\Lambda \in L_\infty$. It turns out that, under some additional assumptions on the solution of problem (1.7), the smoothing operator S_ε in the corrector (2.15) can be removed (replaced by the identity operator). We impose the following condition.

Condition 2.8. *Suppose that the Γ-periodic solution $\Lambda(\mathbf{x})$ of problem* (1.7) *is bounded: $\Lambda \in L_\infty$.*

We need the following multiplicative property of Λ (see [PSu2, Corollary 2.4]).

Proposition 2.9. *Under Condition* 2.8, *for any $u \in H^1(\mathbb{R}^d)$ and any $\varepsilon > 0$ we have*

$$\int_{\mathbb{R}^d} |(\mathbf{D}\Lambda)^\varepsilon(\mathbf{x})|^2 |u|^2 \, d\mathbf{x} \leq \beta_1 \|u\|^2_{L_2(\mathbb{R}^d)} + \beta_2 \|\Lambda\|^2_{L_\infty} \varepsilon^2 \int_{\mathbb{R}^d} |\mathbf{D}u|^2 d\mathbf{x}.$$

The constants β_1 and β_2 are given by

$$\beta_1 = 16 m \alpha_0^{-1} \|g\|_{L_\infty} \|g^{-1}\|_{L_\infty},$$
$$\beta_2 = 2 + 4d\alpha_0^{-1}\alpha_1 + 40 d \alpha_0^{-1} \alpha_1 \|g\|_{L_\infty} \|g^{-1}\|_{L_\infty}.$$

We put

$$K^0(\varepsilon;\zeta) = [\Lambda^\varepsilon] b(\mathbf{D})(\mathcal{A}^0 - \zeta I)^{-1}. \tag{2.52}$$

Under Condition 2.8 the operator (2.52) is a continuous mapping of $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to $H^1(\mathbb{R}^d;\mathbb{C}^n)$; this is seen from the identity

$$K^0(\varepsilon;\zeta) = K^0(\varepsilon;-1)(\mathcal{A}^0 + I)(\mathcal{A}^0 - \zeta I)^{-1} \tag{2.53}$$

and the following lemma.

Lemma 2.10. *Suppose that Condition* 2.8 *is satisfied. For $\varepsilon > 0$ and $\zeta = -1$ the operator* (2.52) *is continuous from $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to $H^1(\mathbb{R}^d;\mathbb{C}^n)$, and we have*

$$\|K^0(\varepsilon;-1)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \check{C}_K^{(1)}, \tag{2.54}$$

$$\|\varepsilon \mathbf{D} K^0(\varepsilon;-1)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \check{C}_K^{(2)} \varepsilon + \check{C}_K^{(3)}. \tag{2.55}$$

The constants $\check{C}_K^{(j)}$, $j = 1,2,3$, depend only on d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, and $\|\Lambda\|_{L_\infty}$.

Proof. By (2.20),

$$\|K^0(\varepsilon;-1)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \frac{1}{2} \|\Lambda\|_{L_\infty} \alpha_1^{1/2} c_0^{-1/2} =: \check{C}_K^{(1)},$$

which implies (2.54). Now consider the operators

$$\varepsilon D_j K^0(\varepsilon;-1) = [(D_j\Lambda)^\varepsilon] b(\mathbf{D})(\mathcal{A}^0 + I)^{-1} + \varepsilon [\Lambda^\varepsilon] D_j b(\mathbf{D})(\mathcal{A}^0 + I)^{-1}. \tag{2.56}$$

The second term in (2.56) is estimated with the help of (2.22):

$$\sum_{j=1}^d \left\| \varepsilon \Lambda^\varepsilon b(\mathbf{D}) D_j (\mathcal{A}^0 + I)^{-1} \right\|^2_{L_2\to L_2} \leq \varepsilon^2 \|\Lambda\|^2_{L_\infty} \alpha_1 c_0^{-2}. \tag{2.57}$$

To estimate the first term, we apply Proposition 2.9 and (2.20), (2.22):

$$\sum_{j=1}^d \left\| (D_j\Lambda)^\varepsilon b(\mathbf{D})(\mathcal{A}^0 + I)^{-1} \right\|^2_{L_2\to L_2} \leq \beta_1 \alpha_1 (4c_0)^{-1} + \beta_2 \varepsilon^2 \|\Lambda\|^2_{L_\infty} \alpha_1 c_0^{-2}. \tag{2.58}$$

Relations (2.56)–(2.58) imply estimate (2.55) with $\check{C}_K^{(2)} = (2\alpha_1)^{1/2} c_0^{-1} (\beta_2+1)^{1/2} \|\Lambda\|_{L_\infty}$ and $\check{C}_K^{(3)} = (\beta_1\alpha_1)^{1/2}(2c_0)^{-1/2}$. $\square$

Instead of (2.23), consider another approximation of the solution $\mathbf{u}_\varepsilon$:

$$\check{\mathbf{v}}_\varepsilon = \mathbf{u}_0 + \varepsilon\Lambda^\varepsilon b(\mathbf{D})\mathbf{u}_0 = (\mathcal{A}^0 - \zeta I)^{-1}\mathbf{F} + \varepsilon K^0(\varepsilon;\zeta)\mathbf{F}. \tag{2.59}$$

Theorem 2.11. *Suppose that the assumptions of Theorem* 2.2 *and Condition* 2.8 *are satisfied. Let* $\check{\mathbf{v}}_\varepsilon$ *be defined by* (2.59). *Let* $\mathbf{p}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$. *Then for* $\varepsilon > 0$ *we have*

$$\|\mathbf{D}(\mathbf{u}_\varepsilon - \check{\mathbf{v}}_\varepsilon)\|_{L_2(\mathbb{R}^d)} \le C_5 c(\varphi)^2 \varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^d)},$$
$$\|\mathbf{u}_\varepsilon - \check{\mathbf{v}}_\varepsilon\|_{L_2(\mathbb{R}^d)} \le C_6 c(\varphi)^2 |\zeta|^{-1/2} \varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^d)},$$
$$\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathbb{R}^d)} \le C_7 c(\varphi)^2 \varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^d)},$$

or, in operator terms,

$$\big\|\mathbf{D}\big((\mathcal{A}_\varepsilon - \zeta I)^{-1} - (\mathcal{A}^0 - \zeta I)^{-1} - \varepsilon K^0(\varepsilon;\zeta)\big)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le C_5 c(\varphi)^2 \varepsilon, \tag{2.60}$$

$$\big\|(\mathcal{A}_\varepsilon - \zeta I)^{-1} - (\mathcal{A}^0 - \zeta I)^{-1} - \varepsilon K^0(\varepsilon;\zeta)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le C_6 c(\varphi)^2 |\zeta|^{-1/2} \varepsilon, \tag{2.61}$$

$$\big\|g^\varepsilon b(\mathbf{D})(\mathcal{A}_\varepsilon - \zeta I)^{-1} - \widetilde{g}^\varepsilon b(\mathbf{D})(\mathcal{A}^0 - \zeta I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le C_7 c(\varphi)^2 \varepsilon. \tag{2.62}$$

The constants C_5, C_6, *and* C_7 *depend on* d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the parameters of the lattice* Γ, *and the norm* $\|\Lambda\|_{L_\infty}$.

Proof. The proof is similar to that of Theorem 2.4. In [BSu4] it was shown that under Condition 2.8 we have

$$\big\|\mathbf{D}\big((\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}^0 + I)^{-1} - \varepsilon K^0(\varepsilon;-1)\big)\big\|_{L_2\to L_2} \le \check{C}_5 \varepsilon, \ \ 0 < \varepsilon \le 1, \tag{2.63}$$

$$\big\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}^0 + I)^{-1} - \varepsilon K^0(\varepsilon;-1)\big\|_{L_2\to L_2} \le \check{C}_6 \varepsilon, \quad 0 < \varepsilon \le 1. \tag{2.64}$$

The constants $\check{C}_5$, $\check{C}_6$ depend only on m, d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, and the norm $\|\Lambda\|_{L_\infty}$.

For $\varepsilon > 1$, the estimates are trivial. As in the proof of (2.30) and (2.33), using (2.63), (2.64), and Lemma 2.10, we obtain

$$\big\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}^0 + I)^{-1} - \varepsilon K^0(\varepsilon;-1)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \widehat{C}_6 \varepsilon, \quad \varepsilon > 0, \tag{2.65}$$

$$\big\|\mathbf{D}\big((\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}^0 + I)^{-1} - \varepsilon K^0(\varepsilon;-1)\big)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \widehat{C}_5 \varepsilon, \quad \varepsilon > 0, \tag{2.66}$$

where $\widehat{C}_6 = \max\{\check{C}_6, 2 + \check{C}_K^{(1)}\}$ and $\widehat{C}_5 = \max\{\check{C}_5, 2c_0^{-1/2} + \check{C}_K^{(2)} + \check{C}_K^{(3)}\}$.

Using the analog of identity (2.34) with $K(\varepsilon;\zeta)$ replaced by $K^0(\varepsilon;\zeta)$, we carry estimates (2.65) and (2.66) over to the point $\widehat{\zeta} = e^{i\varphi}$. Like in the proof of (2.37) and (2.40), we deduce

$$\big\|(\mathcal{A}_\varepsilon - \widehat{\zeta} I)^{-1} - (\mathcal{A}^0 - \widehat{\zeta} I)^{-1} - \varepsilon K^0(\varepsilon;\widehat{\zeta})\big\|_{L_2\to L_2} \le C_6 c(\varphi)^2 \varepsilon, \ \ \varepsilon > 0, \tag{2.67}$$

$$\big\|\mathcal{A}_\varepsilon^{1/2}\big((\mathcal{A}_\varepsilon - \widehat{\zeta} I)^{-1} - (\mathcal{A}^0 - \widehat{\zeta} I)^{-1} - \varepsilon K^0(\varepsilon;\widehat{\zeta})\big)\big\|_{L_2\to L_2} \le \widetilde{C}_5 c(\varphi)^2 \varepsilon, \ \ \varepsilon > 0, \tag{2.68}$$

where $C_6 = 4\widehat{C}_6 + 4\check{C}_K^{(1)}$ and $\widetilde{C}_5 = 4(c_1^{1/2}\widehat{C}_5 + \check{C}_K^{(1)})$. Combining this with (1.6), we see that

$$\big\|\mathbf{D}\big((\mathcal{A}_\varepsilon - \widehat{\zeta} I)^{-1} - (\mathcal{A}^0 - \widehat{\zeta} I)^{-1} - \varepsilon K^0(\varepsilon;\widehat{\zeta})\big)\big\|_{L_2\to L_2} \le C_5 c(\varphi)^2 \varepsilon, \ \ \varepsilon > 0, \tag{2.69}$$

where $C_5 = \widetilde{C}_5 c_0^{-1/2}$.

Now, by the scaling transformation, estimate (2.61) is deduced from (2.67), and (2.60) follows from (2.69). (Cf. the proof of Theorem 2.4.)

It remains to check (2.62). By (2.68),

$$\big\| g^\varepsilon b(\mathbf{D})\big((\mathcal{A}_\varepsilon - \widehat{\zeta} I)^{-1} - (\mathcal{A}^0 - \widehat{\zeta} I)^{-1} - \varepsilon K^0(\varepsilon; \widehat{\zeta})\big) \big\|_{L_2 \to L_2} \le \|g\|^{1/2}_{L_\infty} \widetilde{C}_5 c(\varphi)^2 \varepsilon \tag{2.70}$$

for $\varepsilon > 0$. We have

$$\varepsilon g^\varepsilon b(\mathbf{D}) K^0(\varepsilon; \widehat{\zeta}) = g^\varepsilon (b(\mathbf{D})\Lambda)^\varepsilon b(\mathbf{D})(\mathcal{A}^0 - \widehat{\zeta} I)^{-1} + \varepsilon \sum_{l=1}^d g^\varepsilon b_l \Lambda^\varepsilon b(\mathbf{D}) D_l (\mathcal{A}^0 - \widehat{\zeta} I)^{-1}. \tag{2.71}$$

The second term in (2.71) is estimated with the help of (1.5) and (2.47):

$$\varepsilon \left\| \sum_{l=1}^d g^\varepsilon b_l \Lambda^\varepsilon b(\mathbf{D}) D_l (\mathcal{A}^0 - \widehat{\zeta} I)^{-1} \right\|_{L_2 \to L_2} \le 2\varepsilon \|g\|_{L_\infty} \|\Lambda\|_{L_\infty} \alpha_1 c_0^{-1} d^{1/2} c(\varphi). \tag{2.72}$$

Relations (2.70)–(2.72) together with (1.9) imply that

$$\big\| g^\varepsilon b(\mathbf{D}) (\mathcal{A}_\varepsilon - \widehat{\zeta} I)^{-1} - \widetilde{g}^\varepsilon b(\mathbf{D}) (\mathcal{A}^0 - \widehat{\zeta} I)^{-1} \big\|_{L_2 \to L_2} \le C_7 c(\varphi)^2 \varepsilon, \ \ \varepsilon > 0, \tag{2.73}$$

where $C_7 = \|g\|^{1/2}_{L_\infty} \widetilde{C}_5 + 2\|g\|_{L_\infty} \|\Lambda\|_{L_\infty} \alpha_1 c_0^{-1} d^{1/2}$.

Inequality (2.62) is deduced from (2.73) by the scaling transformation. (Cf. the proof of Theorem 2.6.) □

In some cases Condition 2.8 is valid automatically. The next statement was proved in [BSu4, Lemma 8.7].

Proposition 2.12. *Condition* 2.8 *is valid a fortiori if at least one of the following assumptions is fulfilled*:

1° *the dimension does not exceed* 2, *i.e.*, $d \le 2$;

2° *the operator* $\mathcal{A}_\varepsilon$ *acts in* $L_2(\mathbb{R}^d)$, $d \ge 1$, *and is given by* $\mathbf{D}^* g^\varepsilon(\mathbf{x}) \mathbf{D}$, *where* $g(\mathbf{x})$ *has real entries*;

3° *the dimension is arbitrary and* $g^0 = \underline{g}$, *i.e., relations* (1.14) *are satisfied.*

Note that Condition 2.8 is also ensured if $g(\mathbf{x})$ is sufficiently smooth.

We distinguish the special case where $g^0 = \underline{g}$. In this case the matrix (1.9) is constant: $\widetilde{g}(\mathbf{x}) = g^0 = \underline{g}$; moreover, Condition 2.8 is satisfied. Applying the statement of Theorem 2.11 concerning the fluxes, we arrive at the following proposition.

Proposition 2.13. *Under the assumptions of Theorem* 2.2, *let* $\mathbf{p}_\varepsilon = g^\varepsilon b(\mathbf{D}) \mathbf{u}_\varepsilon$. *Suppose* $g^0 = \underline{g}$, *i.e., relations* (1.14) *are satisfied. Then*

$$\|\mathbf{p}_\varepsilon - g^0 b(\mathbf{D}) \mathbf{u}_0\|_{L_2(\mathbb{R}^d)} \le C_7 c(\varphi)^2 \varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^d)}, \ \ \varepsilon > 0.$$

Chapter 2. The Dirichlet problem

§3. The Dirichlet problem in a bounded domain: preliminaries

3.1. The operator $\mathcal{A}_{D,\varepsilon}$. Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class $C^{1,1}$. In $L_2(\mathcal{O}; \mathbb{C}^n)$, consider the operator $\mathcal{A}_{D,\varepsilon}$ formally given by the differential expression $b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})$ with the Dirichlet condition on $\partial\mathcal{O}$. More precisely, $\mathcal{A}_{D,\varepsilon}$ is the selfadjoint operator in $L_2(\mathcal{O}; \mathbb{C}^n)$ generated by the quadratic form

$$a_{D,\varepsilon}[\mathbf{u}, \mathbf{u}] = \int_{\mathcal{O}} \langle g^\varepsilon(\mathbf{x}) b(\mathbf{D}) \mathbf{u}, b(\mathbf{D}) \mathbf{u} \rangle \, d\mathbf{x}, \quad \mathbf{u} \in H^1_0(\mathcal{O}; \mathbb{C}^n).$$

This form is closed and positive definite. Indeed, extending $\mathbf{u}$ by zero to $\mathbb{R}^d \setminus \mathcal{O}$ and applying (1.6), we obtain

$$c_0 \|\mathbf{D}\mathbf{u}\|^2_{L_2(\mathcal{O})} \le a_{D,\varepsilon}[\mathbf{u}, \mathbf{u}] \le c_1 \|\mathbf{D}\mathbf{u}\|^2_{L_2(\mathcal{O})}, \quad \mathbf{u} \in H^1_0(\mathcal{O}; \mathbb{C}^n). \tag{3.1}$$

It remains to take into account that $\|\mathbf{Du}\|_{L_2(\mathcal{O})}$ determines a norm in $H_0^1(\mathcal{O};\mathbb{C}^n)$ equivalent to the standard one. By the Friedrichs inequality, (3.1) implies that

$$a_{D,\varepsilon}[\mathbf{u},\mathbf{u}] \geq c_2\|\mathbf{u}\|^2_{L_2(\mathcal{O})}, \quad \mathbf{u}\in H_0^1(\mathcal{O};\mathbb{C}^n),\ \ c_2 = c_0(\operatorname{diam}\mathcal{O})^{-2}. \tag{3.2}$$

Our goal in Chapter 2 is to approximate (for small ε) the generalized solution $\mathbf{u}_\varepsilon \in H_0^1(\mathcal{O};\mathbb{C}^n)$ of the Dirichlet problem

$$b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})\mathbf{u}_\varepsilon(\mathbf{x}) - \zeta \mathbf{u}_\varepsilon(\mathbf{x}) = \mathbf{F}(\mathbf{x}), \quad \mathbf{x}\in\mathcal{O};\ \ \mathbf{u}_\varepsilon|_{\partial\mathcal{O}} = 0, \tag{3.3}$$

where $\mathbf{F}\in L_2(\mathcal{O};\mathbb{C}^n)$. As in §2, we assume that $\zeta\in\mathbb{C}\backslash\mathbb{R}_+$. (The case of other admissible values of ζ is studied in §8.) We have $\mathbf{u}_\varepsilon = (\mathcal{A}_{D,\varepsilon} - \zeta I)^{-1}\mathbf{F}$. In operator terms, we study the behavior of the resolvent $(\mathcal{A}_{D,\varepsilon} - \zeta I)^{-1}$.

Lemma 3.1. *Let $\zeta\in\mathbb{C}\setminus\mathbb{R}_+$. Let $\mathbf{u}_\varepsilon$ be the generalized solution of problem* (3.3). *Then for $\varepsilon>0$ we have*

$$\|\mathbf{u}_\varepsilon\|_{L_2(\mathcal{O})} \leq c(\varphi)|\zeta|^{-1}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{3.4}$$

$$\|\mathbf{D}\mathbf{u}_\varepsilon\|_{L_2(\mathcal{O})} \leq \mathcal{C}_0 c(\varphi)|\zeta|^{-1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{3.5}$$

Here $\mathcal{C}_0 = \sqrt{2}\alpha_0^{-1/2}\|g^{-1}\|^{1/2}_{L_\infty}$. *In operator terms,*

$$\begin{aligned} &\|(\mathcal{A}_{D,\varepsilon} - \zeta I)^{-1}\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leq c(\varphi)|\zeta|^{-1}, \\ &\|\mathbf{D}(\mathcal{A}_{D,\varepsilon} - \zeta I)^{-1}\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leq \mathcal{C}_0 c(\varphi)|\zeta|^{-1/2}. \end{aligned} \tag{3.6}$$

Proof. By (3.2), the spectrum of $\mathcal{A}_{D,\varepsilon}$ is contained in $[c_2,\infty)\subset\mathbb{R}_+$. The norm of the resolvent $(\mathcal{A}_{D,\varepsilon} - \zeta I)^{-1}$ does not exceed the inverse distance from ζ to $\mathbb{R}_+$. This yields (3.6).

To check (3.5), we write the integral identity for the solution $\mathbf{u}_\varepsilon\in H_0^1(\mathcal{O};\mathbb{C}^n)$ of problem (3.3):

$$(g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon, b(\mathbf{D})\boldsymbol{\eta})_{L_2(\mathcal{O})} - \zeta(\mathbf{u}_\varepsilon,\boldsymbol{\eta})_{L_2(\mathcal{O})} = (\mathbf{F},\boldsymbol{\eta})_{L_2(\mathcal{O})}, \quad \boldsymbol{\eta}\in H_0^1(\mathcal{O};\mathbb{C}^n). \tag{3.7}$$

Substituting $\boldsymbol{\eta} = \mathbf{u}_\varepsilon$ in (3.7) and using (3.4), we see that

$$(g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon, b(\mathbf{D})\mathbf{u}_\varepsilon)_{L_2(\mathcal{O})} \leq 2c(\varphi)^2|\zeta|^{-1}\|\mathbf{F}\|^2_{L_2(\mathcal{O})}.$$

Taking (3.1) into account, we obtain (3.5) with the constant $\mathcal{C}_0 = \sqrt{2}c_0^{-1/2}$. □

3.2. The effective operator $\mathcal{A}_D^0$. In $L_2(\mathcal{O};\mathbb{C}^n)$, consider the selfadjoint operator $\mathcal{A}_D^0$ generated by the quadratic form

$$a_D^0[\mathbf{u},\mathbf{u}] = \int_{\mathcal{O}} \langle g^0 b(\mathbf{D})\mathbf{u}, b(\mathbf{D})\mathbf{u}\rangle\, d\mathbf{x}, \quad \mathbf{u}\in H_0^1(\mathcal{O};\mathbb{C}^n). \tag{3.8}$$

Here g^0 is the effective matrix defined by (1.8). Taking (1.15) into account, we conclude that the form (3.8) satisfies estimates like in (3.1) and (3.2) with the same constants.

Since $\partial\mathcal{O}\in C^{1,1}$, the operator $\mathcal{A}_D^0$ is given by $b(\mathbf{D})^* g^0 b(\mathbf{D})$ on the domain $H^2(\mathcal{O};\mathbb{C}^n)\cap H_0^1(\mathcal{O};\mathbb{C}^n)$. We have

$$\|(\mathcal{A}_D^0)^{-1}\|_{L_2(\mathcal{O})\to H^2(\mathcal{O})} \leq \widehat{c}. \tag{3.9}$$

The constant $\widehat{c}$ depends only on α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the domain $\mathcal{O}$. To justify this, we note that the operator $b(\mathbf{D})^* g^0 b(\mathbf{D})$ is a *strongly elliptic* matrix operator and refer to the theorems about regularity of solutions for strongly elliptic equations (see, e.g., [McL, Chapter 4]).

Remark 3.2. Instead of condition $\partial\mathcal{O} \in C^{1,1}$, the following implicit condition could be imposed: a bounded domain $\mathcal{O}$ with Lipschitz boundary is such that estimate (3.9) holds true. For such domain the results of Chapter 2 remain valid. In the case of scalar elliptic operators, broad sufficient conditions on $\partial\mathcal{O}$ ensuring estimate (3.9) can be found in [KoE] and [MaSh, Chapter 7] (in particular, it suffices to assume that $\partial\mathcal{O} \in C^{\alpha}$, $\alpha > 3/2$).

Let $\mathbf{u}_0 = (\mathcal{A}_D^0 - \zeta I)^{-1}\mathbf{F}$, i.e., $\mathbf{u}_0 \in H_0^1(\mathcal{O};\mathbb{C}^n)$ is the generalized solution of the problem

$$b(\mathbf{D})^* g^0 b(\mathbf{D})\mathbf{u}_0(\mathbf{x}) - \zeta\mathbf{u}_0(\mathbf{x}) = \mathbf{F}(\mathbf{x}),\ \ \mathbf{x} \in \mathcal{O}; \quad \mathbf{u}_0|_{\partial\mathcal{O}} = 0. \tag{3.10}$$

Lemma 3.3. *Let $\zeta \in \mathbb{C}\setminus\mathbb{R}_+$. Let $\mathbf{u}_0$ be the generalized solution of problem* (3.10). *Then*

$$\begin{aligned} \|\mathbf{u}_0\|_{L_2(\mathcal{O})} &\le c(\varphi)|\zeta|^{-1}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \\ \|\mathbf{D}\mathbf{u}_0\|_{L_2(\mathcal{O})} &\le \mathcal{C}_0 c(\varphi)|\zeta|^{-1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \\ \|\mathbf{u}_0\|_{H^2(\mathcal{O})} &\le \widehat{c}c(\varphi)\|\mathbf{F}\|_{L_2(\mathcal{O})}, \end{aligned} \tag{3.11}$$

or, in operator terms,

$$\|(\mathcal{A}_D^0 - \zeta I)^{-1}\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \le c(\varphi)|\zeta|^{-1}, \tag{3.12}$$

$$\|\mathbf{D}(\mathcal{A}_D^0 - \zeta I)^{-1}\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \le \mathcal{C}_0 c(\varphi)|\zeta|^{-1/2}, \tag{3.13}$$

$$\|(\mathcal{A}_D^0 - \zeta I)^{-1}\|_{L_2(\mathcal{O})\to H^2(\mathcal{O})} \le \widehat{c}c(\varphi). \tag{3.14}$$

Proof. Estimates (3.12), (3.13) are checked as in the proof of Lemma 3.1. Estimate (3.14) follows from (3.9) and the inequality

$$\|\mathcal{A}_D^0(\mathcal{A}_D^0 - \zeta I)^{-1}\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \le \sup_{x\ge 0} x|x-\zeta|^{-1} \le c(\varphi). \qquad \square$$

3.3. Estimates in a neighborhood of the boundary. This subsection contains two simple auxiliary statements, which are valid for bounded domains $\mathcal{O}$ with Lipschitz boundary. Precisely, we impose the following condition.

Condition 3.4. *Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain. We put $(\partial\mathcal{O})_\varepsilon = \{\mathbf{x} \in \mathbb{R}^d : \operatorname{dist}\{\mathbf{x};\partial\mathcal{O}\} < \varepsilon\}$. Suppose that there exists a number $\varepsilon_0 \in (0,1]$ such that the strip $(\partial\mathcal{O})_{\varepsilon_0}$ can be covered by a finite number of open sets admitting diffeomorphisms of class $C^{0,1}$ that rectify the boundary $\partial\mathcal{O}$. Denote $\varepsilon_1 = \varepsilon_0(1+r_1)^{-1}$, where $2r_1 = \operatorname{diam}\Omega$.*

Obviously, Condition 3.4 is less restrictive than the assumption $\partial\mathcal{O} \in C^{1,1}$ made above.

Lemma 3.5. *Suppose that Condition* 3.4 *is satisfied. Denote $B_\varepsilon = (\partial\mathcal{O})_\varepsilon \cap \mathcal{O}$. Then the following statements are true.*

1°. *For any function $u \in H^1(\mathcal{O})$ we have*

$$\int_{B_\varepsilon} |u|^2\,d\mathbf{x} \le \beta\varepsilon\|u\|_{H^1(\mathcal{O})}\|u\|_{L_2(\mathcal{O})}, \quad 0 < \varepsilon \le \varepsilon_0.$$

2°. *For any function $u \in H^1(\mathbb{R}^d)$ we have*

$$\int_{(\partial\mathcal{O})_\varepsilon} |u|^2\,d\mathbf{x} \le \beta\varepsilon\|u\|_{H^1(\mathbb{R}^d)}\|u\|_{L_2(\mathbb{R}^d)}, \quad 0 < \varepsilon \le \varepsilon_0.$$

The constant β depends only on the domain $\mathcal{O}$.

Lemma 3.6. *Suppose that Condition* 3.4 *is satisfied. Let $f(\mathbf{x})$ be a Γ-periodic function on $\mathbb{R}^d$ such that $f \in L_2(\Omega)$. Let S_ε be the operator* (1.17). *Denote $\beta_* = \beta(1+r_1)$, where $2r_1 = \operatorname{diam}\Omega$. Then for any $0 < \varepsilon \le \varepsilon_1$ and any $\mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^m)$ we have*

$$\int_{(\partial\mathcal{O})_\varepsilon} |f^\varepsilon(\mathbf{x})|^2|(S_\varepsilon\mathbf{u})(\mathbf{x})|^2\,d\mathbf{x} \le \beta_*\varepsilon|\Omega|^{-1}\|f\|^2_{L_2(\Omega)}\|\mathbf{u}\|_{H^1(\mathbb{R}^d)}\|\mathbf{u}\|_{L_2(\mathbb{R}^d)}.$$

Lemmas 3.5 and 3.6 were proved in [PSu2, §5] under the condition $\partial\mathcal{O} \in C^1$, but the proofs can be automatically carried over to the case of Condition 3.4.

§4. The results for the Dirichlet problem

Assume that the numbers $\varepsilon_0, \varepsilon_1 \in (0,1]$ are chosen in accordance with Condition 3.4. Clearly, ε_1 depends only on the domain $\mathcal{O}$ and the lattice Γ.

4.1. Approximation of the resolvent $(\mathcal{A}_{D,\varepsilon} - \zeta I)^{-1}$ for $|\zeta| \geq 1$. Now we formulate our main results for the operator $\mathcal{A}_{D,\varepsilon}$.

Theorem 4.1. *Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class $C^{1,1}$. Let $\zeta = |\zeta|e^{i\varphi} \in \mathbb{C} \setminus \mathbb{R}_+$. Suppose that $|\zeta| \geq 1$, $\mathbf{u}_\varepsilon$ is the solution of problem* (3.3), *and $\mathbf{u}_0$ is the solution of problem* (3.10) *with $\mathbf{F} \in L_2(\mathcal{O};\mathbb{C}^n)$. Suppose that ε_1 is subject to Condition* 3.4. *Then for $0 < \varepsilon \leq \varepsilon_1$ we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O})} \leq \mathcal{C}_1 c(\varphi)^5 (|\zeta|^{-1/2}\varepsilon + \varepsilon^2)\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{4.1}$$

where $c(\varphi)$ is defined by (2.1). *In operator terms,*

$$\left\|(\mathcal{A}_{D,\varepsilon} - \zeta I)^{-1} - (\mathcal{A}_D^0 - \zeta I)^{-1}\right\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \leq \mathcal{C}_1 c(\varphi)^5 (|\zeta|^{-1/2}\varepsilon + \varepsilon^2). \tag{4.2}$$

The constant $\mathcal{C}_1$ depends on d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, and the domain $\mathcal{O}$.

In order to approximate the solution in $H^1(\mathcal{O};\mathbb{C}^n)$, we need to introduce a corrector. We fix a continuous linear extension operator $P_\mathcal{O} : H^2(\mathcal{O};\mathbb{C}^n) \to H^2(\mathbb{R}^d;\mathbb{C}^n)$, requiring that $P_\mathcal{O}$ be simultaneously continuous from $L_2(\mathcal{O};\mathbb{C}^n)$ to $L_2(\mathbb{R}^d;\mathbb{C}^n)$ and from $H^1(\mathcal{O};\mathbb{C}^n)$ to $H^1(\mathbb{R}^d;\mathbb{C}^n)$. Such an operator exists for any bounded domain $\mathcal{O}$ with Lipschitz boundary (see, e.g., [St]). Denote

$$\|P_\mathcal{O}\|_{H^s(\mathcal{O}) \to H^s(\mathbb{R}^d)} =: C_\mathcal{O}^{(s)}, \quad s = 0,1,2. \tag{4.3}$$

The constants $C_\mathcal{O}^{(s)}$ depend only on the domain $\mathcal{O}$. Next, let $R_\mathcal{O}$ denote the operator of restriction of functions in $\mathbb{R}^d$ to $\mathcal{O}$. We introduce the corrector

$$K_D(\varepsilon;\zeta) = R_\mathcal{O}[\Lambda^\varepsilon]S_\varepsilon b(\mathbf{D})P_\mathcal{O}(\mathcal{A}_D^0 - \zeta I)^{-1}. \tag{4.4}$$

The operator $K_D(\varepsilon;\zeta)$ is a continuous mapping of $L_2(\mathcal{O};\mathbb{C}^n)$ to $H^1(\mathcal{O};\mathbb{C}^n)$. Indeed, the operator $b(\mathbf{D})P_\mathcal{O}(\mathcal{A}_D^0 - \zeta I)^{-1}$ is continuous from $L_2(\mathcal{O};\mathbb{C}^n)$ to $H^1(\mathbb{R}^d;\mathbb{C}^m)$, and the operator $[\Lambda^\varepsilon]S_\varepsilon$ is a continuous mapping of $H^1(\mathbb{R}^d;\mathbb{C}^m)$ to $H^1(\mathbb{R}^d;\mathbb{C}^n)$ (this follows from Proposition 1.5 and the fact that $\Lambda \in \widetilde{H}^1(\Omega)$).

Let $\mathbf{u}_0$ be the solution of problem (3.10). Denote $\widetilde{\mathbf{u}}_0 := P_\mathcal{O}\mathbf{u}_0$. We put

$$\widetilde{\mathbf{v}}_\varepsilon(\mathbf{x}) = \widetilde{\mathbf{u}}_0(\mathbf{x}) + \varepsilon\Lambda^\varepsilon(\mathbf{x})(S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0)(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d, \tag{4.5}$$

$$\mathbf{v}_\varepsilon := \widetilde{\mathbf{v}}_\varepsilon|_\mathcal{O}. \tag{4.6}$$

Then

$$\mathbf{v}_\varepsilon = \mathbf{u}_0 + \varepsilon K_D(\varepsilon;\zeta)\mathbf{F} = (\mathcal{A}_D^0 - \zeta I)^{-1}\mathbf{F} + \varepsilon K_D(\varepsilon;\zeta)\mathbf{F}. \tag{4.7}$$

Theorem 4.2. *Under the assumptions of Theorem* 4.1, *let $\mathbf{v}_\varepsilon$ be defined by* (4.4), (4.7). *Then for $0 < \varepsilon \leq \varepsilon_1$ we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon\|_{H^1(\mathcal{O})} \leq \big(\mathcal{C}_2 c(\varphi)^2 |\zeta|^{-1/4}\varepsilon^{1/2} + \mathcal{C}_3 c(\varphi)^4 \varepsilon\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{4.8}$$

or, in operator terms,

$$\begin{aligned} &\|(\mathcal{A}_{D,\varepsilon} - \zeta I)^{-1} - (\mathcal{A}_D^0 - \zeta I)^{-1} - \varepsilon K_D(\varepsilon;\zeta)\|_{L_2(\mathcal{O}) \to H^1(\mathcal{O})} \\ &\qquad\qquad \leq \mathcal{C}_2 c(\varphi)^2 |\zeta|^{-1/4}\varepsilon^{1/2} + \mathcal{C}_3 c(\varphi)^4 \varepsilon. \end{aligned} \tag{4.9}$$

For the flux $\mathbf{p}_\varepsilon := g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$ *we have*

$$\big\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\big\|_{L_2(\mathcal{O})} \le \big(\widetilde{\mathcal{C}}_2 c(\varphi)^2|\zeta|^{-1/4}\varepsilon^{1/2} + \widetilde{\mathcal{C}}_3 c(\varphi)^4\varepsilon\big)\|\mathbf{F}\|_{L_2(\mathcal{O})} \tag{4.10}$$

for $0<\varepsilon\le\varepsilon_1$. *Here* $\widetilde{g}(\mathbf{x})$ *is the matrix* (1.9). *The constants* $\mathcal{C}_2$, $\mathcal{C}_3$, $\widetilde{\mathcal{C}}_2$, *and* $\widetilde{\mathcal{C}}_3$ *depend on* d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the parameters of the lattice* Γ, *and the domain* $\mathcal{O}$.

4.2. The first step of the proof. Associated problem in $\mathbb{R}^d$. The proof of Theorems 4.1 and 4.2 relies on application of the results for the problem in $\mathbb{R}^d$ and introduction of the boundary layer correction term.

By Lemma 3.3 and (4.3), we have

$$\|\widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} \le C_{\mathcal{O}}^{(0)} c(\varphi)|\zeta|^{-1}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{4.11}$$

$$\|\widetilde{\mathbf{u}}_0\|_{H^1(\mathbb{R}^d)} \le C_{\mathcal{O}}^{(1)}(\mathcal{C}_0+1) c(\varphi)|\zeta|^{-1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{4.12}$$

$$\|\widetilde{\mathbf{u}}_0\|_{H^2(\mathbb{R}^d)} \le C_{\mathcal{O}}^{(2)}\widehat{c}\, c(\varphi)\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{4.13}$$

We have taken into account that $|\zeta|\ge 1$. We put

$$\widetilde{\mathbf{F}} := \mathcal{A}^0\widetilde{\mathbf{u}}_0 - \zeta\widetilde{\mathbf{u}}_0. \tag{4.14}$$

Then $\widetilde{\mathbf{F}}\in L_2(\mathbb{R}^d;\mathbb{C}^n)$ and $\widetilde{\mathbf{F}}|_{\mathcal{O}} = \mathbf{F}$. From (1.16), (4.11), and (4.13) it follows that

$$\|\widetilde{\mathbf{F}}\|_{L_2(\mathbb{R}^d)} \le c_1\|\widetilde{\mathbf{u}}_0\|_{H^2(\mathbb{R}^d)} + |\zeta|\|\widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} \le \mathcal{C}_4 c(\varphi)\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{4.15}$$

where $\mathcal{C}_4 = c_1 C_{\mathcal{O}}^{(2)}\widehat{c} + C_{\mathcal{O}}^{(0)}$. Let $\widetilde{\mathbf{u}}_\varepsilon\in H^1(\mathbb{R}^d;\mathbb{C}^n)$ be the solution of the following equation in $\mathbb{R}^d$:

$$\mathcal{A}_\varepsilon\widetilde{\mathbf{u}}_\varepsilon - \zeta\widetilde{\mathbf{u}}_\varepsilon = \widetilde{\mathbf{F}}, \tag{4.16}$$

i.e., $\widetilde{\mathbf{u}}_\varepsilon = (\mathcal{A}_\varepsilon - \zeta I)^{-1}\widetilde{\mathbf{F}}$.

We can apply theorems of §2. Combining Theorems 2.2, 2.4 and (4.5), (4.14)–(4.16), we see that, for $\varepsilon>0$,

$$\|\widetilde{\mathbf{u}}_\varepsilon - \widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} \le C_1 c(\varphi)^2|\zeta|^{-1/2}\varepsilon\|\widetilde{\mathbf{F}}\|_{L_2(\mathbb{R}^d)} \le C_1\mathcal{C}_4 c(\varphi)^3|\zeta|^{-1/2}\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{4.17}$$

$$\|\mathbf{D}(\widetilde{\mathbf{u}}_\varepsilon - \widetilde{\mathbf{v}}_\varepsilon)\|_{L_2(\mathbb{R}^d)} \le C_2 c(\varphi)^2\varepsilon\|\widetilde{\mathbf{F}}\|_{L_2(\mathbb{R}^d)} \le C_2\mathcal{C}_4 c(\varphi)^3\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{4.18}$$

$$\|\widetilde{\mathbf{u}}_\varepsilon - \widetilde{\mathbf{v}}_\varepsilon\|_{L_2(\mathbb{R}^d)} \le C_3 c(\varphi)^2|\zeta|^{-1/2}\varepsilon\|\widetilde{\mathbf{F}}\|_{L_2(\mathbb{R}^d)} \le C_3\mathcal{C}_4 c(\varphi)^3|\zeta|^{-1/2}\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{4.19}$$

4.3. The second step of the proof. Introduction of the correction term $\mathbf{w}_\varepsilon$. The first order approximation $\mathbf{v}_\varepsilon$ of the solution $\mathbf{u}_\varepsilon$ does not satisfy the Dirichlet condition. We have $\mathbf{v}_\varepsilon|_{\partial\mathcal{O}} = \varepsilon\Lambda^\varepsilon(S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0)|_{\partial\mathcal{O}}$. Consider the "correction term" $\mathbf{w}_\varepsilon$ that is the generalized solution of the problem

$$b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})\mathbf{w}_\varepsilon - \zeta\mathbf{w}_\varepsilon = 0 \ \text{ in } \ \mathcal{O}; \quad \mathbf{w}_\varepsilon|_{\partial\mathcal{O}} = \varepsilon\Lambda^\varepsilon(S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0)|_{\partial\mathcal{O}}. \tag{4.20}$$

Here the equation is understood in the weak sense: $\mathbf{w}_\varepsilon\in H^1(\mathcal{O};\mathbb{C}^n)$ satisfies the identity

$$(g^\varepsilon b(\mathbf{D})\mathbf{w}_\varepsilon, b(\mathbf{D})\boldsymbol{\eta})_{L_2(\mathcal{O})} - \zeta(\mathbf{w}_\varepsilon,\boldsymbol{\eta})_{L_2(\mathcal{O})} = 0, \quad \boldsymbol{\eta}\in H_0^1(\mathcal{O};\mathbb{C}^n). \tag{4.21}$$

Lemma 4.3. *Let* $\mathbf{u}_\varepsilon$ *be the solution of problem* (3.3), *and let* $\mathbf{v}_\varepsilon$ *be given by* (4.7). *Let* $\mathbf{w}_\varepsilon$ *be the solution of problem* (4.20). *Then for* $\varepsilon>0$ *we have*

$$\|\mathbf{D}(\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon + \mathbf{w}_\varepsilon)\|_{L_2(\mathcal{O})} \le \mathcal{C}_5 c(\varphi)^4\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{4.22}$$

$$\|\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon + \mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})} \le \mathcal{C}_6 c(\varphi)^4|\zeta|^{-1/2}\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{4.23}$$

The constants $\mathcal{C}_5$, $\mathcal{C}_6$ *depend on* d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the parameters of the lattice* Γ, *and the domain* $\mathcal{O}$.

Proof. Denote $\mathbf{V}_\varepsilon := \mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon + \mathbf{w}_\varepsilon$. By (3.7), (4.20), and (4.21), the function $\mathbf{V}_\varepsilon$ belongs to $H^1_0(\mathcal{O};\mathbb{C}^n)$ and satisfies the identity

$$(g^\varepsilon b(\mathbf{D})\mathbf{V}_\varepsilon, b(\mathbf{D})\boldsymbol{\eta})_{L_2(\mathcal{O})} - \zeta(\mathbf{V}_\varepsilon,\boldsymbol{\eta})_{L_2(\mathcal{O})} = J_\varepsilon[\boldsymbol{\eta}], \quad \boldsymbol{\eta}\in H^1_0(\mathcal{O};\mathbb{C}^n), \tag{4.24}$$

where $J_\varepsilon[\boldsymbol{\eta}] := (\mathbf{F},\boldsymbol{\eta})_{L_2(\mathcal{O})} - (g^\varepsilon b(\mathbf{D})\mathbf{v}_\varepsilon, b(\mathbf{D})\boldsymbol{\eta})_{L_2(\mathcal{O})} + \zeta(\mathbf{v}_\varepsilon,\boldsymbol{\eta})_{L_2(\mathcal{O})}$. We extend $\boldsymbol{\eta}$ by zero to $\mathbb{R}^d\setminus\mathcal{O}$ keeping the same notation; then $\boldsymbol{\eta}\in H^1(\mathbb{R}^d;\mathbb{C}^n)$. Recall that $\widetilde{\mathbf{F}}$ is an extension of $\mathbf{F}$, and $\widetilde{\mathbf{v}}_\varepsilon$ is an extension of $\mathbf{v}_\varepsilon$. Hence,

$$J_\varepsilon[\boldsymbol{\eta}] = (\widetilde{\mathbf{F}},\boldsymbol{\eta})_{L_2(\mathbb{R}^d)} - (g^\varepsilon b(\mathbf{D})\widetilde{\mathbf{v}}_\varepsilon, b(\mathbf{D})\boldsymbol{\eta})_{L_2(\mathbb{R}^d)} + \zeta(\widetilde{\mathbf{v}}_\varepsilon,\boldsymbol{\eta})_{L_2(\mathbb{R}^d)}.$$

Taking (4.16) into account, we obtain

$$J_\varepsilon[\boldsymbol{\eta}] = (g^\varepsilon b(\mathbf{D})(\widetilde{\mathbf{u}}_\varepsilon - \widetilde{\mathbf{v}}_\varepsilon), b(\mathbf{D})\boldsymbol{\eta})_{L_2(\mathbb{R}^d)} - \zeta(\widetilde{\mathbf{u}}_\varepsilon - \widetilde{\mathbf{v}}_\varepsilon,\boldsymbol{\eta})_{L_2(\mathbb{R}^d)}.$$

From (1.4), (4.18), and (4.19) it follows that

$$|J_\varepsilon[\boldsymbol{\eta}]| \le \varepsilon c(\varphi)^3\|\mathbf{F}\|_{L_2(\mathcal{O})}\big(\mathcal{C}_7\|(g^\varepsilon)^{1/2}b(\mathbf{D})\boldsymbol{\eta}\|_{L_2(\mathcal{O})} + \mathcal{C}_8|\zeta|^{1/2}\|\boldsymbol{\eta}\|_{L_2(\mathcal{O})}\big),\ \ \varepsilon>0, \tag{4.25}$$

where $\mathcal{C}_7 = \|g\|^{1/2}_{L_\infty}\alpha_1^{1/2}C_2\mathcal{C}_4$ and $\mathcal{C}_8 = C_3\mathcal{C}_4$.

We substitute $\boldsymbol{\eta} = \mathbf{V}_\varepsilon$ in (4.24), take the imaginary part of the corresponding relation, and apply (4.25):

$$|\operatorname{Im}\zeta|\|\mathbf{V}_\varepsilon\|^2_{L_2(\mathcal{O})} \le \varepsilon c(\varphi)^3\|\mathbf{F}\|_{L_2(\mathcal{O})}\big(\mathcal{C}_7\|(g^\varepsilon)^{1/2}b(\mathbf{D})\mathbf{V}_\varepsilon\|_{L_2(\mathcal{O})} + \mathcal{C}_8|\zeta|^{1/2}\|\mathbf{V}_\varepsilon\|_{L_2(\mathcal{O})}\big). \tag{4.26}$$

If $\operatorname{Re}\zeta\ge 0$ (and then $\operatorname{Im}\zeta\ne 0$), this implies the inequality

$$\begin{aligned}\|\mathbf{V}_\varepsilon\|^2_{L_2(\mathcal{O})} \le\ & 2\varepsilon|\zeta|^{-1}c(\varphi)^4\mathcal{C}_7\|\mathbf{F}\|_{L_2(\mathcal{O})}\|(g^\varepsilon)^{1/2}b(\mathbf{D})\mathbf{V}_\varepsilon\|_{L_2(\mathcal{O})}\\ &+ \varepsilon^2|\zeta|^{-1}c(\varphi)^8\mathcal{C}_8^2\|\mathbf{F}\|^2_{L_2(\mathcal{O})}.\end{aligned} \tag{4.27}$$

If $\operatorname{Re}\zeta<0$, we take the real part of the corresponding relation and apply (4.25). Then

$$|\operatorname{Re}\zeta|\|\mathbf{V}_\varepsilon\|^2_{L_2(\mathcal{O})} \le \varepsilon c(\varphi)^3\|\mathbf{F}\|_{L_2(\mathcal{O})}\big(\mathcal{C}_7\|(g^\varepsilon)^{1/2}b(\mathbf{D})\mathbf{V}_\varepsilon\|_{L_2(\mathcal{O})} + \mathcal{C}_8|\zeta|^{1/2}\|\mathbf{V}_\varepsilon\|_{L_2(\mathcal{O})}\big). \tag{4.28}$$

Adding (4.26) and (4.28), we deduce an inequality similar to (4.27). As a result, for all values of ζ under consideration we obtain

$$\begin{aligned}\|\mathbf{V}_\varepsilon\|^2_{L_2(\mathcal{O})} \le\ & 4\varepsilon|\zeta|^{-1}c(\varphi)^4\mathcal{C}_7\|\mathbf{F}\|_{L_2(\mathcal{O})}\|(g^\varepsilon)^{1/2}b(\mathbf{D})\mathbf{V}_\varepsilon\|_{L_2(\mathcal{O})}\\ &+ 4\varepsilon^2|\zeta|^{-1}c(\varphi)^8\mathcal{C}_8^2\|\mathbf{F}\|^2_{L_2(\mathcal{O})}.\end{aligned} \tag{4.29}$$

Now, (4.24) with $\boldsymbol{\eta} = \mathbf{V}_\varepsilon$, (4.25), and (4.29) yield

$$a_{D,\varepsilon}[\mathbf{V}_\varepsilon,\mathbf{V}_\varepsilon] \le 9\mathcal{C}_7 c(\varphi)^4\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}\|(g^\varepsilon)^{1/2}b(\mathbf{D})\mathbf{V}_\varepsilon\|_{L_2(\mathcal{O})} + 9\mathcal{C}_8^2 c(\varphi)^8\varepsilon^2\|\mathbf{F}\|^2_{L_2(\mathcal{O})}.$$

From this we deduce the inequality

$$a_{D,\varepsilon}[\mathbf{V}_\varepsilon,\mathbf{V}_\varepsilon] \le \check{\mathcal{C}}_5^2 c(\varphi)^8\varepsilon^2\|\mathbf{F}\|^2_{L_2(\mathcal{O})}, \tag{4.30}$$

where $\check{\mathcal{C}}_5^2 = 18\mathcal{C}_8^2 + 81\mathcal{C}_7^2$. By (4.30) and (3.1), we obtain (4.22) with the constant $\mathcal{C}_5 = \check{\mathcal{C}}_5 c_0^{-1/2}$. Finally, (4.29) and (4.30) imply (4.23) with $\mathcal{C}_6 = 2\left(\mathcal{C}_7\check{\mathcal{C}}_5 + \mathcal{C}_8^2\right)^{1/2}$. $\square$

Conclusions. 1) From (4.22) and (4.23) it follows that

$$\|\mathbf{D}(\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon)\|_{L_2(\mathcal{O})} \le \mathcal{C}_5 c(\varphi)^4\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})} + \|\mathbf{D}\mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})},\ \ \varepsilon>0, \tag{4.31}$$

$$\|\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon\|_{L_2(\mathcal{O})} \le \mathcal{C}_6 c(\varphi)^4|\zeta|^{-1/2}\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})} + \|\mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})},\ \ \varepsilon>0. \tag{4.32}$$

Thus, for the proof of Theorem 4.2 it suffices to obtain an appropriate estimate for the norm $\|\mathbf{w}_\varepsilon\|_{H^1(\mathcal{O})}$.

2) Note that the difference $\mathbf{v}_\varepsilon - \mathbf{u}_0 = \varepsilon(\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0)|_{\mathcal{O}}$ can be estimated by using (1.4), (1.19), and (4.12):

$$\|\mathbf{v}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O})} \le \|\widetilde{\mathbf{v}}_\varepsilon - \widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} \le \mathcal{C}_9 c(\varphi)|\zeta|^{-1/2}\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{4.33}$$

where $\mathcal{C}_9 = M_1\alpha_1^{1/2}C_{\mathcal{O}}^{(1)}(\mathcal{C}_0+1)$. From (4.32) and (4.33) it follows that

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O})} \le (\mathcal{C}_6+\mathcal{C}_9)c(\varphi)^4|\zeta|^{-1/2}\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})} + \|\mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})}, \ \ \varepsilon>0. \tag{4.34}$$

Therefore, for the proof of Theorem 4.1 we must obtain an appropriate estimate for the norm $\|\mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})}$.

§5. Estimates for the correction term. Proof of Theorems 4.1 and 4.2

First we estimate the H^1-norm of $\mathbf{w}_\varepsilon$ and complete the proof of Theorem 4.2. Next, using the already proved Theorem 4.2 and duality arguments, we estimate the L_2-norm of $\mathbf{w}_\varepsilon$ and prove Theorem 4.1.

5.1. Localization near the boundary. Suppose that the number $\varepsilon_0 \in (0,1]$ is chosen in accordance with Condition 3.4. Let $0<\varepsilon\le\varepsilon_0$. We fix a smooth cut-off function $\theta_\varepsilon(\mathbf{x})$ on $\mathbb{R}^d$ such that

$$\begin{aligned} &\theta_\varepsilon \in C_0^\infty(\mathbb{R}^d), \quad \operatorname{supp}\theta_\varepsilon \subset (\partial\mathcal{O})_\varepsilon, \quad 0\le\theta_\varepsilon(\mathbf{x})\le 1,\\ &\theta_\varepsilon(\mathbf{x}) = 1 \ \text{ for } \ \mathbf{x}\in\partial\mathcal{O}; \quad \varepsilon|\nabla\theta_\varepsilon(\mathbf{x})|\le\kappa=\text{const}. \end{aligned} \tag{5.1}$$

The constant κ depends only on d and the domain $\mathcal{O}$. Consider the following function in $\mathbb{R}^d$:

$$\boldsymbol{\phi}_\varepsilon(\mathbf{x}) = \varepsilon\theta_\varepsilon(\mathbf{x})\Lambda^\varepsilon(\mathbf{x})(S_\varepsilon b(\mathbf{D})\tilde{\mathbf{u}}_0)(\mathbf{x}). \tag{5.2}$$

Lemma 5.1. *Let $\mathbf{w}_\varepsilon$ be the solution of problem* (4.20), *and let $\boldsymbol{\phi}_\varepsilon$ be defined by* (5.2). *Then for $0<\varepsilon\le\varepsilon_0$ and $\zeta\in\mathbb{C}\setminus\mathbb{R}_+$, $|\zeta|\ge 1$, we have*

$$\|\mathbf{w}_\varepsilon\|_{H^1(\mathcal{O})} \le c(\varphi)\big(\mathcal{C}_{10}|\zeta|^{1/2}\|\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})} + \mathcal{C}_{11}\|\mathbf{D}\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})}\big). \tag{5.3}$$

The constants $\mathcal{C}_{10}$ and $\mathcal{C}_{11}$ depend on d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, and the domain $\mathcal{O}$.

Proof. By (4.20), (5.1), and (5.2), we have $\mathbf{w}_\varepsilon|_{\partial\mathcal{O}} = \boldsymbol{\phi}_\varepsilon|_{\partial\mathcal{O}}$. By (4.21), the function $\boldsymbol{\rho}_\varepsilon := \mathbf{w}_\varepsilon - \boldsymbol{\phi}_\varepsilon \in H_0^1(\mathcal{O};\mathbb{C}^n)$ satisfies the identity

$$\begin{aligned} &(g^\varepsilon b(\mathbf{D})\boldsymbol{\rho}_\varepsilon, b(\mathbf{D})\boldsymbol{\eta})_{L_2(\mathcal{O})} - \zeta(\boldsymbol{\rho}_\varepsilon,\boldsymbol{\eta})_{L_2(\mathcal{O})}\\ &\qquad = -(g^\varepsilon b(\mathbf{D})\boldsymbol{\phi}_\varepsilon, b(\mathbf{D})\boldsymbol{\eta})_{L_2(\mathcal{O})} + \zeta(\boldsymbol{\phi}_\varepsilon,\boldsymbol{\eta})_{L_2(\mathcal{O})}, \quad \boldsymbol{\eta}\in H_0^1(\mathcal{O};\mathbb{C}^n). \end{aligned} \tag{5.4}$$

We substitute $\boldsymbol{\eta} = \boldsymbol{\rho}_\varepsilon$ in (5.4) and take the imaginary part of the corresponding relation. Using (1.2) and (1.5), we arrive at

$$|\operatorname{Im}\zeta|\|\boldsymbol{\rho}_\varepsilon\|^2_{L_2(\mathcal{O})} \le \mathcal{C}_{12}\|\mathbf{D}\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})}\|(g^\varepsilon)^{1/2}b(\mathbf{D})\boldsymbol{\rho}_\varepsilon\|_{L_2(\mathcal{O})} + |\zeta|\|\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})}\|\boldsymbol{\rho}_\varepsilon\|_{L_2(\mathcal{O})}, \tag{5.5}$$

where $\mathcal{C}_{12} = \|g\|^{1/2}_{L_\infty}(d\alpha_1)^{1/2}$. If $\operatorname{Re}\zeta\ge 0$ (and then $\operatorname{Im}\zeta\ne 0$), this yields

$$\|\boldsymbol{\rho}_\varepsilon\|^2_{L_2(\mathcal{O})} \le 2\mathcal{C}_{12}c(\varphi)|\zeta|^{-1}\|\mathbf{D}\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})}\|(g^\varepsilon)^{1/2}b(\mathbf{D})\boldsymbol{\rho}_\varepsilon\|_{L_2(\mathcal{O})} + c(\varphi)^2\|\boldsymbol{\phi}_\varepsilon\|^2_{L_2(\mathcal{O})}. \tag{5.6}$$

If $\operatorname{Re}\zeta<0$, we take the real part of the corresponding relation, obtaining

$$|\operatorname{Re}\zeta|\|\boldsymbol{\rho}_\varepsilon\|^2_{L_2(\mathcal{O})} \le \mathcal{C}_{12}\|\mathbf{D}\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})}\|(g^\varepsilon)^{1/2}b(\mathbf{D})\boldsymbol{\rho}_\varepsilon\|_{L_2(\mathcal{O})} + |\zeta|\|\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})}\|\boldsymbol{\rho}_\varepsilon\|_{L_2(\mathcal{O})}. \tag{5.7}$$

Adding (5.5) and (5.7), we deduce an inequality similar to (5.6). As a result, for all ζ under consideration we obtain

$$\|\boldsymbol{\rho}_\varepsilon\|^2_{L_2(\mathcal{O})} \le 4\mathcal{C}_{12}c(\varphi)|\zeta|^{-1}\|\mathbf{D}\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})}\|(g^\varepsilon)^{1/2}b(\mathbf{D})\boldsymbol{\rho}_\varepsilon\|_{L_2(\mathcal{O})} + 4c(\varphi)^2\|\boldsymbol{\phi}_\varepsilon\|^2_{L_2(\mathcal{O})}. \tag{5.8}$$

Now, from (5.4) with $\boldsymbol{\eta} = \boldsymbol{\rho}_\varepsilon$ and (5.8) it follows that

$$a_{D,\varepsilon}[\boldsymbol{\rho}_\varepsilon,\boldsymbol{\rho}_\varepsilon] \le 9c(\varphi)^2|\zeta|\|\boldsymbol{\phi}_\varepsilon\|^2_{L_2(\mathcal{O})} + 9\mathcal{C}_{12}c(\varphi)\|\mathbf{D}\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})}\|(g^\varepsilon)^{1/2}b(\mathbf{D})\boldsymbol{\rho}_\varepsilon\|_{L_2(\mathcal{O})}.$$

This implies the inequality

$$a_{D,\varepsilon}[\boldsymbol{\rho}_\varepsilon,\boldsymbol{\rho}_\varepsilon] \le 18c(\varphi)^2|\zeta|\|\boldsymbol{\phi}_\varepsilon\|^2_{L_2(\mathcal{O})} + 81\mathcal{C}_{12}^2 c(\varphi)^2\|\mathbf{D}\boldsymbol{\phi}_\varepsilon\|^2_{L_2(\mathcal{O})}. \tag{5.9}$$

By (5.9) and (3.1),

$$\|\mathbf{D}\boldsymbol{\rho}_\varepsilon\|_{L_2(\mathcal{O})} \le c(\varphi)\big(\check{\mathcal{C}}_{10}|\zeta|^{1/2}\|\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})} + \check{\mathcal{C}}_{11}\|\mathbf{D}\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})}\big), \tag{5.10}$$

where $\check{\mathcal{C}}_{10} = \sqrt{18}c_0^{-1/2}$ and $\check{\mathcal{C}}_{11} = 9c_0^{-1/2}\mathcal{C}_{12}$. Next, by (5.8) and (5.9), we have

$$\|\boldsymbol{\rho}_\varepsilon\|_{L_2(\mathcal{O})} \le c(\varphi)\big(\sqrt{22}\|\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})} + \sqrt{85}\,\mathcal{C}_{12}|\zeta|^{-1/2}\|\mathbf{D}\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})}\big). \tag{5.11}$$

Relations (5.10), (5.11) together with the condition $|\zeta| \ge 1$ imply the required estimate (5.3) with $\mathcal{C}_{10} = \check{\mathcal{C}}_{10} + \sqrt{22} + 1$ and $\mathcal{C}_{11} = \check{\mathcal{C}}_{11} + \sqrt{85}\mathcal{C}_{12} + 1$. □

5.2. Estimation of the function $\boldsymbol{\phi}_\varepsilon$. Now, we estimate the function (5.2).

Lemma 5.2. *Suppose that ε_1 satisfies Condition* 3.4. *Let $\boldsymbol{\phi}_\varepsilon$ be given by* (5.2). *For $0 < \varepsilon \le \varepsilon_1$ and $\zeta \in \mathbb{C}\setminus\mathbb{R}_+$, $|\zeta| \ge 1$, we have*

$$\|\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})} \le \mathcal{C}_{13}c(\varphi)|\zeta|^{-1/2}\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{5.12}$$

$$\|\mathbf{D}\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})} \le c(\varphi)\big(\mathcal{C}_{14}|\zeta|^{-1/4}\varepsilon^{1/2} + \mathcal{C}_{15}\varepsilon\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{5.13}$$

The constants $\mathcal{C}_{13}$, $\mathcal{C}_{14}$, and $\mathcal{C}_{15}$ depend on d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, and the domain $\mathcal{O}$.

Proof. Assume that $0 < \varepsilon \le \varepsilon_1$. We start with an estimate for the L_2-norm of the function (5.2). Relations (1.4), (1.19), (4.12), and (5.1) imply (5.12) with the constant $\mathcal{C}_{13} = M_1\alpha_1^{1/2}C_{\mathcal{O}}^{(1)}(\mathcal{C}_0+1)$.

Now we consider the derivatives

$$\partial_j\boldsymbol{\phi}_\varepsilon = \varepsilon(\partial_j\theta_\varepsilon)\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0 + \theta_\varepsilon(\partial_j\Lambda)^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0 + \varepsilon\theta_\varepsilon\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\partial_j\widetilde{\mathbf{u}}_0,$$

$j = 1,\dots,d$. Hence,

$$\begin{aligned}\|\mathbf{D}\boldsymbol{\phi}_\varepsilon\|^2_{L_2(\mathcal{O})} \le{}& 3\varepsilon^2\|(\nabla\theta_\varepsilon)\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|^2_{L_2(\mathbb{R}^d)} + 3\|\theta_\varepsilon(\mathbf{D}\Lambda)^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|^2_{L_2(\mathbb{R}^d)}\\ &+ 3\varepsilon^2\sum_{j=1}^d \|\theta_\varepsilon\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\partial_j\widetilde{\mathbf{u}}_0\|^2_{L_2(\mathbb{R}^d)}.\end{aligned} \tag{5.14}$$

We denote the consecutive terms on the right-hand side of (5.14) by $J_1(\varepsilon)$, $J_2(\varepsilon)$, and $J_3(\varepsilon)$, respectively. By (5.1) and Lemma 3.6,

$$\begin{aligned}J_1(\varepsilon) &\le 3\kappa^2\int_{(\partial\mathcal{O})_\varepsilon}|\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0|^2\,d\mathbf{x}\\ &\le 3\kappa^2\beta_*\varepsilon|\Omega|^{-1}\|\Lambda\|^2_{L_2(\Omega)}\|b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{H^1(\mathbb{R}^d)}\|b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)}.\end{aligned}$$

Combining this with (1.4), (1.11), (4.12), and (4.13), we arrive at

$$J_1(\varepsilon) \le \gamma_1 c(\varphi)^2|\zeta|^{-1/2}\varepsilon\|\mathbf{F}\|^2_{L_2(\mathcal{O})}, \tag{5.15}$$

where $\gamma_1 = 3\kappa^2\beta_* M_1^2\alpha_1 C_{\mathcal{O}}^{(2)}\widehat{c}C_{\mathcal{O}}^{(1)}(\mathcal{C}_0+1)$. Similarly, for the second term we have

$$J_2(\varepsilon) \le 3\beta_*\varepsilon|\Omega|^{-1}\|\mathbf{D}\Lambda\|^2_{L_2(\Omega)}\|b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{H^1(\mathbb{R}^d)}\|b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)}.$$

Using (1.4), (1.10), (4.12), and (4.13), we deduce that

$$J_2(\varepsilon) \le \gamma_2 c(\varphi)^2|\zeta|^{-1/2}\varepsilon\|\mathbf{F}\|^2_{L_2(\mathcal{O})}, \tag{5.16}$$

where $\gamma_2 = 3\beta_* M_2^2\alpha_1 C_{\mathcal{O}}^{(2)}\widehat{c}C_{\mathcal{O}}^{(1)}(\mathcal{C}_0+1)$. The third term in (5.14) is estimated with the help of (1.4), (1.19), (4.13), and (5.1):

$$J_3(\varepsilon) \le \gamma_3 c(\varphi)^2\varepsilon^2\|\mathbf{F}\|^2_{L_2(\mathcal{O})}, \tag{5.17}$$

where $\gamma_3 = 3M_1^2\alpha_1(C_{\mathcal{O}}^{(2)}\widehat{c})^2$. Now, relations (5.14)–(5.17) imply (5.13) with the constants $\mathcal{C}_{14} = (\gamma_1+\gamma_2)^{1/2}$ and $\mathcal{C}_{15} = \gamma_3^{1/2}$. □

5.3. Completion of the proof of Theorem 4.2. By Lemmas 5.1 and 5.2, we have

$$\|\mathbf{w}_\varepsilon\|_{H^1(\mathcal{O})} \le c(\varphi)^2\big(\mathcal{C}_{16}|\zeta|^{-1/4}\varepsilon^{1/2} + \mathcal{C}_{17}\varepsilon\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}, \quad 0<\varepsilon\le\varepsilon_1,$$

where $\mathcal{C}_{16} = \mathcal{C}_{11}\mathcal{C}_{14}$ and $\mathcal{C}_{17} = \mathcal{C}_{10}\mathcal{C}_{13} + \mathcal{C}_{11}\mathcal{C}_{15}$. Together with (4.31) and (4.32), this implies the required estimate (4.8) with the constants $\mathcal{C}_2 = \sqrt{2}\mathcal{C}_{16}$ and $\mathcal{C}_3 = \mathcal{C}_5 + \mathcal{C}_6 + \sqrt{2}\mathcal{C}_{17}$.

It remains to check (4.10). From (4.8) and (1.2), (1.5) it follows that

$$\|\mathbf{p}_\varepsilon - g^\varepsilon b(\mathbf{D})\mathbf{v}_\varepsilon\|_{L_2(\mathcal{O})} \le \|g\|_{L_\infty}(d\alpha_1)^{1/2}\big(\mathcal{C}_2 c(\varphi)^2|\zeta|^{-1/4}\varepsilon^{1/2} + \mathcal{C}_3 c(\varphi)^4\varepsilon\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{5.18}$$

We have

$$g^\varepsilon b(\mathbf{D})\mathbf{v}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{u}_0 + g^\varepsilon(b(\mathbf{D})\Lambda)^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0 + \varepsilon\sum_{l=1}^d g^\varepsilon b_l \Lambda^\varepsilon S_\varepsilon b(\mathbf{D})D_l\widetilde{\mathbf{u}}_0. \tag{5.19}$$

The third term in (5.19) is estimated with the help of (1.4), (1.5), and (1.19):

$$\Big\|\varepsilon\sum_{l=1}^d g^\varepsilon b_l\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})D_l\widetilde{\mathbf{u}}_0\Big\|_{L_2(\mathcal{O})} \le \mathcal{C}'\varepsilon\|\widetilde{\mathbf{u}}_0\|_{H^2(\mathbb{R}^d)}, \tag{5.20}$$

where $\mathcal{C}' = \|g\|_{L_\infty}\alpha_1 d^{1/2}M_1$. Note that, by Proposition 1.4 and (1.4), we have

$$\|g^\varepsilon(I-S_\varepsilon)b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathcal{O})} \le \mathcal{C}''\varepsilon\|\widetilde{\mathbf{u}}_0\|_{H^2(\mathbb{R}^d)}, \tag{5.21}$$

where $\mathcal{C}'' = \|g\|_{L_\infty} r_1\alpha_1^{1/2}$. From (5.19)–(5.21) and (1.9) it follows that

$$\|g^\varepsilon b(\mathbf{D})\mathbf{v}_\varepsilon - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathcal{O})} \le (\mathcal{C}'+\mathcal{C}'')\varepsilon\|\widetilde{\mathbf{u}}_0\|_{H^2(\mathbb{R}^d)}. \tag{5.22}$$

Now, inequalities (4.13), (5.18), and (5.22) imply (4.10) with the following constants: $\widetilde{\mathcal{C}}_2 = \|g\|_{L_\infty}(d\alpha_1)^{1/2}\mathcal{C}_2$ and $\widetilde{\mathcal{C}}_3 = \|g\|_{L_\infty}(d\alpha_1)^{1/2}\mathcal{C}_3 + \mathcal{C}_{18}$, where $\mathcal{C}_{18} = (\mathcal{C}'+\mathcal{C}'')C_{\mathcal{O}}^{(2)}\widehat{c}$. □

5.4. Proof of Theorem 4.1. Let us estimate the L_2-norm of $\mathbf{w}_\varepsilon$.

Lemma 5.3. *Let* $\mathbf{w}_\varepsilon$ *be the solution of problem* (4.20). *Suppose that the number* ε_1 *satisfies Condition* 3.4. *Then for* $0<\varepsilon\le\varepsilon_1$ *and* $\zeta\in\mathbb{C}\setminus\mathbb{R}_+$, $|\zeta|\ge 1$, *we have*

$$\|\mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})} \le c(\varphi)^5\big(\mathcal{C}_{19}|\zeta|^{-1/2}\varepsilon + \mathcal{C}_{20}\varepsilon^2\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{5.23}$$

The constants $\mathcal{C}_{19}$ *and* $\mathcal{C}_{20}$ *depend on* d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the parameters of the lattice* Γ, *and the domain* $\mathcal{O}$.

Proof. Assume that $0<\varepsilon\le\varepsilon_1$. Consider identity (5.4) and take the test function $\boldsymbol{\eta}$ equal to $\boldsymbol{\eta}_\varepsilon = (\mathcal{A}_{D,\varepsilon} - \overline{\zeta}I)^{-1}\boldsymbol{\Phi}$, where $\boldsymbol{\Phi}\in L_2(\mathcal{O};\mathbb{C}^n)$. Then the left-hand side of (5.4) takes the form $(\mathbf{w}_\varepsilon - \boldsymbol{\phi}_\varepsilon, \boldsymbol{\Phi})_{L_2(\mathcal{O})}$. Hence,

$$(\mathbf{w}_\varepsilon - \boldsymbol{\phi}_\varepsilon, \boldsymbol{\Phi})_{L_2(\mathcal{O})} = -(g^\varepsilon b(\mathbf{D})\boldsymbol{\phi}_\varepsilon, b(\mathbf{D})\boldsymbol{\eta}_\varepsilon)_{L_2(\mathcal{O})} + \zeta(\boldsymbol{\phi}_\varepsilon, \boldsymbol{\eta}_\varepsilon)_{L_2(\mathcal{O})}. \tag{5.24}$$

To approximate $\boldsymbol{\eta}_\varepsilon$ in $H^1(\mathcal{O};\mathbb{C}^n)$, we apply the already proved Theorem 4.2. We put $\boldsymbol{\eta}_0 = (\mathcal{A}_D^0 - \overline{\zeta}I)^{-1}\boldsymbol{\Phi}$ and $\widetilde{\boldsymbol{\eta}}_0 = P_{\mathcal{O}}\boldsymbol{\eta}_0$. Then an approximation of $\boldsymbol{\eta}_\varepsilon$ is given by $\boldsymbol{\eta}_0 + \varepsilon\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\boldsymbol{\eta}}_0$. Rewrite (5.24) as

$$\begin{aligned}(\mathbf{w}_\varepsilon - \boldsymbol{\phi}_\varepsilon, \boldsymbol{\Phi})_{L_2(\mathcal{O})} =& -(g^\varepsilon b(\mathbf{D})\boldsymbol{\phi}_\varepsilon, b(\mathbf{D})(\boldsymbol{\eta}_\varepsilon - \boldsymbol{\eta}_0 - \varepsilon\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\boldsymbol{\eta}}_0))_{L_2(\mathcal{O})}\\ &- (g^\varepsilon b(\mathbf{D})\boldsymbol{\phi}_\varepsilon, b(\mathbf{D})\boldsymbol{\eta}_0)_{L_2(\mathcal{O})}\\ &- (g^\varepsilon b(\mathbf{D})\boldsymbol{\phi}_\varepsilon, b(\mathbf{D})(\varepsilon\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\boldsymbol{\eta}}_0))_{L_2(\mathcal{O})} + \zeta(\boldsymbol{\phi}_\varepsilon, \boldsymbol{\eta}_\varepsilon)_{L_2(\mathcal{O})}.\end{aligned} \tag{5.25}$$

We denote the consecutive terms on the right-hand side of (5.25) by $\mathcal{I}_j(\varepsilon)$, $j=1,2,3,4$.

It is easy to estimate $\mathcal{I}_4(\varepsilon)$, using Lemma 3.1 and (5.12):

$$|\mathcal{I}_4(\varepsilon)| \le |\zeta| \|\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})} \|\boldsymbol{\eta}_\varepsilon\|_{L_2(\mathcal{O})} \le \mathcal{C}_{13} c(\varphi)^2 |\zeta|^{-1/2} \varepsilon \|\mathbf{F}\|_{L_2(\mathcal{O})} \|\boldsymbol{\Phi}\|_{L_2(\mathcal{O})}. \tag{5.26}$$

Now, we estimate the first term in (5.25). By (1.2) and (1.5), we have

$$|\mathcal{I}_1(\varepsilon)| \le \|g\|_{L_\infty} d\alpha_1 \|\mathbf{D}\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})} \|\mathbf{D}(\boldsymbol{\eta}_\varepsilon - \boldsymbol{\eta}_0 - \varepsilon \Lambda^\varepsilon S_\varepsilon b(\mathbf{D}) \widetilde{\boldsymbol{\eta}}_0)\|_{L_2(\mathcal{O})}.$$

Applying Theorem 4.2 and (5.13), we arrive at

$$\begin{aligned} |\mathcal{I}_1(\varepsilon)| \le \|g\|_{L_\infty} d\alpha_1 c(\varphi) \big(\mathcal{C}_{14} |\zeta|^{-1/4} \varepsilon^{1/2} + \mathcal{C}_{15} \varepsilon\big) \\ \times \big(\mathcal{C}_2 c(\varphi)^2 |\zeta|^{-1/4} \varepsilon^{1/2} + \mathcal{C}_3 c(\varphi)^4 \varepsilon\big) \|\mathbf{F}\|_{L_2(\mathcal{O})} \|\boldsymbol{\Phi}\|_{L_2(\mathcal{O})}. \end{aligned}$$

Hence,

$$|\mathcal{I}_1(\varepsilon)| \le c(\varphi)^5 \big(\check{\gamma}_1 |\zeta|^{-1/2} \varepsilon + \check{\gamma}_2 \varepsilon^2\big) \|\mathbf{F}\|_{L_2(\mathcal{O})} \|\boldsymbol{\Phi}\|_{L_2(\mathcal{O})}, \tag{5.27}$$

where

$$\begin{aligned} \check{\gamma}_1 &= \|g\|_{L_\infty} d\alpha_1 \big(\mathcal{C}_2 \mathcal{C}_{14} + \mathcal{C}_2 \mathcal{C}_{15} + \mathcal{C}_3 \mathcal{C}_{14}\big), \\ \check{\gamma}_2 &= \|g\|_{L_\infty} d\alpha_1 \left(\mathcal{C}_3 \mathcal{C}_{14} + \mathcal{C}_3 \mathcal{C}_{15} + \mathcal{C}_2 \mathcal{C}_{15}\right). \end{aligned}$$

In order to estimate the second term in (5.25), recall that $\boldsymbol{\phi}_\varepsilon$ is supported on the ε-neighborhood of the boundary and apply Lemma 3.5(1°). Taking (1.2) and (1.5) into account, we have

$$\begin{aligned} |\mathcal{I}_2(\varepsilon)| &\le \|g\|_{L_\infty} (d\alpha_1)^{1/2} \|\mathbf{D}\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})} \left(\int_{B_\varepsilon} |b(\mathbf{D})\boldsymbol{\eta}_0|^2 \, d\mathbf{x} \right)^{1/2} \\ &\le \|g\|_{L_\infty} (d\alpha_1)^{1/2} \|\mathbf{D}\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})} \beta^{1/2} \varepsilon^{1/2} \|b(\mathbf{D})\boldsymbol{\eta}_0\|_{H^1(\mathcal{O})}^{1/2} \|b(\mathbf{D})\boldsymbol{\eta}_0\|_{L_2(\mathcal{O})}^{1/2}. \end{aligned}$$

Together with Lemma 3.3 and (5.13), this yields

$$\begin{aligned} |\mathcal{I}_2(\varepsilon)| \le \|g\|_{L_\infty} d\alpha_1 c(\varphi) \big(\mathcal{C}_{14} |\zeta|^{-1/4} \varepsilon^{1/2} + \mathcal{C}_{15} \varepsilon\big) \\ \times \beta^{1/2} \varepsilon^{1/2} (\widehat{c} c(\varphi))^{1/2} (\mathcal{C}_0 c(\varphi) |\zeta|^{-1/2})^{1/2} \|\mathbf{F}\|_{L_2(\mathcal{O})} \|\boldsymbol{\Phi}\|_{L_2(\mathcal{O})}. \end{aligned}$$

Hence,

$$|\mathcal{I}_2(\varepsilon)| \le c(\varphi)^2 \big(\check{\gamma}_3 |\zeta|^{-1/2} \varepsilon + \check{\gamma}_4 \varepsilon^2\big) \|\mathbf{F}\|_{L_2(\mathcal{O})} \|\boldsymbol{\Phi}\|_{L_2(\mathcal{O})}, \tag{5.28}$$

where $\check{\gamma}_3 = \|g\|_{L_\infty} d\alpha_1 (\beta \widehat{c} \mathcal{C}_0)^{1/2} (\mathcal{C}_{14} + \mathcal{C}_{15})$ and $\check{\gamma}_4 = \|g\|_{L_\infty} d\alpha_1 (\beta \widehat{c} \mathcal{C}_0)^{1/2} \mathcal{C}_{15}$.

It remains to estimate the third term in (5.25). By (1.2), we have

$$\mathcal{I}_3(\varepsilon) = \mathcal{I}_3^{(1)}(\varepsilon) + \mathcal{I}_3^{(2)}(\varepsilon), \tag{5.29}$$

$$\mathcal{I}_3^{(1)}(\varepsilon) = -\left(g^\varepsilon b(\mathbf{D})\boldsymbol{\phi}_\varepsilon, (b(\mathbf{D})\Lambda)^\varepsilon S_\varepsilon b(\mathbf{D}) \widetilde{\boldsymbol{\eta}}_0\right)_{L_2(\mathcal{O})}, \tag{5.30}$$

$$\mathcal{I}_3^{(2)}(\varepsilon) = -\Big(g^\varepsilon b(\mathbf{D})\boldsymbol{\phi}_\varepsilon, \varepsilon \sum_{l=1}^d b_l \Lambda^\varepsilon S_\varepsilon b(\mathbf{D}) D_l \widetilde{\boldsymbol{\eta}}_0\Big)_{L_2(\mathcal{O})}. \tag{5.31}$$

The term (5.30) is estimated with the help of (1.2), (1.5), and Lemma 3.6:

$$\begin{aligned} |\mathcal{I}_3^{(1)}(\varepsilon)| &\le \|g\|_{L_\infty} (d\alpha_1)^{1/2} \|\mathbf{D}\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})} \left(\int_{(\partial\mathcal{O})_\varepsilon} |(b(\mathbf{D})\Lambda)^\varepsilon S_\varepsilon b(\mathbf{D}) \widetilde{\boldsymbol{\eta}}_0|^2 \, d\mathbf{x} \right)^{1/2} \\ &\le \|g\|_{L_\infty} (d\alpha_1)^{1/2} \|\mathbf{D}\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})} \\ &\qquad \times \big(\beta_* \varepsilon |\Omega|^{-1} \|b(\mathbf{D})\Lambda\|_{L_2(\Omega)}^2 \|b(\mathbf{D})\widetilde{\boldsymbol{\eta}}_0\|_{H^1(\mathbb{R}^d)} \|b(\mathbf{D})\widetilde{\boldsymbol{\eta}}_0\|_{L_2(\mathbb{R}^d)}\big)^{1/2}. \end{aligned} \tag{5.32}$$

As in (4.12) and (4.13), we have

$$\|\widetilde{\boldsymbol{\eta}}_0\|_{H^1(\mathbb{R}^d)} \le C_{\mathcal{O}}^{(1)}(\mathcal{C}_0+1)c(\varphi)|\zeta|^{-1/2}\|\mathbf{\Phi}\|_{L_2(\mathcal{O})}, \tag{5.33}$$

$$\|\widetilde{\boldsymbol{\eta}}_0\|_{H^2(\mathbb{R}^d)} \le C_{\mathcal{O}}^{(2)}\widehat{c}c(\varphi)\|\mathbf{\Phi}\|_{L_2(\mathcal{O})}. \tag{5.34}$$

From (5.32)–(5.34) and (1.10), (5.13) it follows that

$$|\mathcal{I}_3^{(1)}(\varepsilon)| \le \|g\|_{L_\infty} d\alpha_1^{3/2} M_2 c(\varphi)\big(\mathcal{C}_{14}|\zeta|^{-1/4}\varepsilon^{1/2}+\mathcal{C}_{15}\varepsilon\big)\beta_*^{1/2}\varepsilon^{1/2} \times (C_{\mathcal{O}}^{(2)}\widehat{c}c(\varphi))^{1/2}(C_{\mathcal{O}}^{(1)}(\mathcal{C}_0+1)c(\varphi)|\zeta|^{-1/2})^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\mathbf{\Phi}\|_{L_2(\mathcal{O})}.$$

Hence,

$$|\mathcal{I}_3^{(1)}(\varepsilon)| \le c(\varphi)^2\big(\breve{\gamma}_5|\zeta|^{-1/2}\varepsilon+\breve{\gamma}_6\varepsilon^2\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\mathbf{\Phi}\|_{L_2(\mathcal{O})}, \tag{5.35}$$

where

$$\breve{\gamma}_5 = \|g\|_{L_\infty} d\alpha_1^{3/2} M_2(\beta_* C_{\mathcal{O}}^{(2)}\widehat{c}C_{\mathcal{O}}^{(1)}(\mathcal{C}_0+1))^{1/2}(\mathcal{C}_{14}+\mathcal{C}_{15}),$$
$$\breve{\gamma}_6 = \|g\|_{L_\infty} d\alpha_1^{3/2} M_2(\beta_* C_{\mathcal{O}}^{(2)}\widehat{c}C_{\mathcal{O}}^{(1)}(\mathcal{C}_0+1))^{1/2}\mathcal{C}_{15}.$$

Finally, the term (5.31) is estimated by using (1.2), (1.4), (1.5), and (1.19):

$$|\mathcal{I}_3^{(2)}(\varepsilon)| \le \varepsilon\|g\|_{L_\infty} d\alpha_1^{3/2} M_1\|\mathbf{D}\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})}\|\widetilde{\boldsymbol{\eta}}_0\|_{H^2(\mathbb{R}^d)}.$$

Combining this with (5.13) and (5.34), we obtain

$$|\mathcal{I}_3^{(2)}(\varepsilon)| \le c(\varphi)^2\big(\breve{\gamma}_7|\zeta|^{-1/2}\varepsilon+\breve{\gamma}_8\varepsilon^2\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\mathbf{\Phi}\|_{L_2(\mathcal{O})}, \tag{5.36}$$

where $\breve{\gamma}_7 = \|g\|_{L_\infty} d\alpha_1^{3/2} M_1 C_{\mathcal{O}}^{(2)}\widehat{c}\mathcal{C}_{14}$ and $\breve{\gamma}_8 = \|g\|_{L_\infty} d\alpha_1^{3/2} M_1 C_{\mathcal{O}}^{(2)}\widehat{c}(\mathcal{C}_{14}+\mathcal{C}_{15})$.

As a result, relations (5.25)–(5.29), (5.35), and (5.36) imply that

$$|(\mathbf{w}_\varepsilon-\boldsymbol{\phi}_\varepsilon,\mathbf{\Phi})_{L_2(\mathcal{O})}| \le c(\varphi)^5\big(\widetilde{\gamma}|\zeta|^{-1/2}\varepsilon+\widehat{\gamma}\varepsilon^2\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\mathbf{\Phi}\|_{L_2(\mathcal{O})}$$

for any $\mathbf{\Phi}\in L_2(\mathcal{O};\mathbb{C}^n)$, where $\widetilde{\gamma} = \mathcal{C}_{13}+\breve{\gamma}_1+\breve{\gamma}_3+\breve{\gamma}_5+\breve{\gamma}_7$ and $\widehat{\gamma} = \breve{\gamma}_2+\breve{\gamma}_4+\breve{\gamma}_6+\breve{\gamma}_8$. Consequently,

$$\|\mathbf{w}_\varepsilon-\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})} \le c(\varphi)^5\big(\widetilde{\gamma}|\zeta|^{-1/2}\varepsilon+\widehat{\gamma}\varepsilon^2\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{5.37}$$

Now, the required estimate (5.23) follows directly from (5.37) and (5.12), and we have $\mathcal{C}_{19} = \widetilde{\gamma}+\mathcal{C}_{13}$ and $\mathcal{C}_{20} = \widehat{\gamma}$. □

Completion of the proof of Theorem 4.1. By (4.34) and (5.23),

$$\|\mathbf{u}_\varepsilon-\mathbf{u}_0\|_{L_2(\mathcal{O})} \le (\mathcal{C}_6+\mathcal{C}_9)c(\varphi)^4|\zeta|^{-1/2}\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})} + c(\varphi)^5\big(\mathcal{C}_{19}|\zeta|^{-1/2}\varepsilon+\mathcal{C}_{20}\varepsilon^2\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}, \quad 0<\varepsilon\le\varepsilon_1.$$

This implies estimate (4.1) with the constant $\mathcal{C}_1 = \max\{\mathcal{C}_6+\mathcal{C}_9+\mathcal{C}_{19};\mathcal{C}_{20}\}$. □

§6. Results about the Dirichlet problem: the case where $\Lambda\in L_\infty$ and special cases

6.1. The case where $\Lambda\in L_\infty$. As for the problem in $\mathbb{R}^d$ (see Subsection 2.3), under Condition 2.8 it is possible to remove the smoothing operator from the corrector, i.e., instead of the corrector (4.4) we can use the following simpler operator:

$$K_D^0(\varepsilon;\zeta) = [\Lambda^\varepsilon]b(\mathbf{D})(\mathcal{A}_D^0-\zeta I)^{-1}. \tag{6.1}$$

Using Proposition 2.9, it is easy to check that under Condition 2.8 the operator (6.1) is a continuous mapping of $L_2(\mathcal{O};\mathbb{C}^n)$ to $H^1(\mathcal{O};\mathbb{C}^n)$. Instead of (4.7), now we use another approximation of the solution $\mathbf{u}_\varepsilon$ of problem (3.3):

$$\check{\mathbf{v}}_\varepsilon := (\mathcal{A}_D^0-\zeta I)^{-1}\mathbf{F}+\varepsilon K_D^0(\varepsilon;\zeta)\mathbf{F} = \mathbf{u}_0+\varepsilon\Lambda^\varepsilon b(\mathbf{D})\mathbf{u}_0. \tag{6.2}$$

Theorem 6.1. *Suppose that the assumptions of Theorem* 4.1 *and Condition* 2.8 *are satisfied. Let* $\check{\mathbf{v}}_\varepsilon$ *be defined by* (6.2). *Then for* $0 < \varepsilon \le \varepsilon_1$ *we have*

$$\|\mathbf{u}_\varepsilon - \check{\mathbf{v}}_\varepsilon\|_{H^1(\mathcal{O})} \le \big(\mathcal{C}_2 c(\varphi)^2|\zeta|^{-1/4}\varepsilon^{1/2} + \mathcal{C}_3^\circ c(\varphi)^4\varepsilon\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{6.3}$$

or, in operator terms,

$$\big\|(\mathcal{A}_{D,\varepsilon}-\zeta I)^{-1}-(\mathcal{A}_D^0-\zeta I)^{-1}-\varepsilon K_D^0(\varepsilon;\zeta)\big\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} \le \mathcal{C}_2 c(\varphi)^2|\zeta|^{-1/4}\varepsilon^{1/2}+\mathcal{C}_3^\circ c(\varphi)^4\varepsilon.$$

For the flux $\mathbf{p}_\varepsilon := g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$ *we have*

$$\big\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0\big\|_{L_2(\mathcal{O})} \le \big(\widetilde{\mathcal{C}}_2 c(\varphi)^2|\zeta|^{-1/4}\varepsilon^{1/2} + \widetilde{\mathcal{C}}_3^\circ c(\varphi)^4\varepsilon\big)\|\mathbf{F}\|_{L_2(\mathcal{O})} \tag{6.4}$$

for $0 < \varepsilon \le \varepsilon_1$. *The constants* $\mathcal{C}_2$ *and* $\widetilde{\mathcal{C}}_2$ *are the same as in Theorem* 4.2. *The constants* $\mathcal{C}_3^\circ$ *and* $\widetilde{\mathcal{C}}_3^\circ$ *depend on* d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the parameters of the lattice* Γ, *the domain* $\mathcal{O}$, *and the norm* $\|\Lambda\|_{L_\infty}$.

Proof. In order to deduce (6.3) from (4.8), it suffices to estimate the $H^1(\mathcal{O};\mathbb{C}^n)$-norm of the function $\mathbf{v}_\varepsilon - \check{\mathbf{v}}_\varepsilon = \varepsilon\,(\Lambda^\varepsilon(S_\varepsilon - I)b(\mathbf{D})\widetilde{\mathbf{u}}_0)\,|_{\mathcal{O}}$. We start with estimation of the L_2-norm. By Condition 2.8 and estimates (1.4), (1.18), we have

$$\|\mathbf{v}_\varepsilon - \check{\mathbf{v}}_\varepsilon\|_{L_2(\mathcal{O})} \le \varepsilon\|\Lambda\|_{L_\infty}\|(S_\varepsilon - I)b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} \le 2\|\Lambda\|_{L_\infty}\alpha_1^{1/2}\varepsilon\|\widetilde{\mathbf{u}}_0\|_{H^1(\mathbb{R}^d)}. \tag{6.5}$$

Now, consider the derivatives

$$\partial_j(\mathbf{v}_\varepsilon - \check{\mathbf{v}}_\varepsilon) = ((\partial_j\Lambda)^\varepsilon(S_\varepsilon - I)b(\mathbf{D})\widetilde{\mathbf{u}}_0)\,|_{\mathcal{O}} + \varepsilon\,(\Lambda^\varepsilon(S_\varepsilon - I)b(\mathbf{D})\partial_j\widetilde{\mathbf{u}}_0)\,|_{\mathcal{O}},$$

$j = 1,\dots,d$. Hence,

$$\begin{aligned}&\|\mathbf{D}(\mathbf{v}_\varepsilon - \check{\mathbf{v}}_\varepsilon)\|^2_{L_2(\mathcal{O})}\\ &\quad\le 2\|(\mathbf{D}\Lambda)^\varepsilon(S_\varepsilon - I)b(\mathbf{D})\widetilde{\mathbf{u}}_0\|^2_{L_2(\mathbb{R}^d)} + 2\varepsilon^2\sum_{j=1}^d\|\Lambda^\varepsilon(S_\varepsilon - I)b(\mathbf{D})\partial_j\widetilde{\mathbf{u}}_0\|^2_{L_2(\mathbb{R}^d)}.\end{aligned} \tag{6.6}$$

By Proposition 2.9, this yields

$$\begin{aligned}\|\mathbf{D}(\mathbf{v}_\varepsilon - \check{\mathbf{v}}_\varepsilon)\|^2_{L_2(\mathcal{O})} &\le 2\beta_1\|(S_\varepsilon - I)b(\mathbf{D})\widetilde{\mathbf{u}}_0\|^2_{L_2(\mathbb{R}^d)}\\ &\quad + 2\varepsilon^2\|\Lambda\|^2_{L_\infty}(\beta_2+1)\sum_{j=1}^d\|(S_\varepsilon - I)b(\mathbf{D})\partial_j\widetilde{\mathbf{u}}_0\|^2_{L_2(\mathbb{R}^d)}.\end{aligned} \tag{6.7}$$

To estimate the first term on the right, we apply Proposition 1.4. The second term is estimated with the help of (1.18). Taking (1.4) into account, we arrive at the inequality

$$\|\mathbf{D}(\mathbf{v}_\varepsilon - \check{\mathbf{v}}_\varepsilon)\|_{L_2(\mathcal{O})} \le \mathcal{C}_{21}\varepsilon\|\widetilde{\mathbf{u}}_0\|_{H^2(\mathbb{R}^d)}, \tag{6.8}$$

where $\mathcal{C}_{21} = \alpha_1^{1/2}(2\beta_1 r_1^2 + 8(\beta_2+1)\|\Lambda\|^2_{L_\infty})^{1/2}$. From (6.5) and (6.8) it follows that

$$\|\mathbf{v}_\varepsilon - \check{\mathbf{v}}_\varepsilon\|_{H^1(\mathcal{O})} \le \mathcal{C}'''\varepsilon\|\widetilde{\mathbf{u}}_0\|_{H^2(\mathbb{R}^d)}, \tag{6.9}$$

where $\mathcal{C}''' = \alpha_1^{1/2}(2\beta_1 r_1^2 + (8\beta_2+12)\|\Lambda\|^2_{L_\infty})^{1/2}$. Now (4.8), (4.13), and (6.9) imply the required estimate (6.3) with the constant $\mathcal{C}_3^\circ = \mathcal{C}_3 + \mathcal{C}''' C_{\mathcal{O}}^{(2)}\widehat{c}$.

It remains to check (6.4). From (6.3), (1.2), and (1.5) it follows that

$$\|\mathbf{p}_\varepsilon - g^\varepsilon b(\mathbf{D})\check{\mathbf{v}}_\varepsilon\|_{L_2(\mathcal{O})} \le \|g\|_{L_\infty}(d\alpha_1)^{1/2}\big(\mathcal{C}_2 c(\varphi)^2|\zeta|^{-1/4}\varepsilon^{1/2}+\mathcal{C}_3^\circ c(\varphi)^4\varepsilon\big)\|\mathbf{F}\|_{L_2(\mathcal{O})} \tag{6.10}$$

for $0 < \varepsilon \le \varepsilon_1$. We have

$$g^\varepsilon b(\mathbf{D})\check{\mathbf{v}}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{u}_0 + g^\varepsilon(b(\mathbf{D})\Lambda)^\varepsilon b(\mathbf{D})\mathbf{u}_0 + \varepsilon\sum_{l=1}^d g^\varepsilon b_l\Lambda^\varepsilon b(\mathbf{D})D_l\mathbf{u}_0. \tag{6.11}$$

The third term in (6.11) can be estimated by using (1.2) and (1.5):

$$\left\| \varepsilon \sum_{l=1}^{d} g^\varepsilon b_l \Lambda^\varepsilon b(\mathbf{D}) D_l \mathbf{u}_0 \right\|_{L_2(\mathcal{O})} \leq \widetilde{\mathcal{C}}' \varepsilon \|\mathbf{u}_0\|_{H^2(\mathcal{O})}, \tag{6.12}$$

where $\widetilde{\mathcal{C}}' = \|g\|_{L_\infty} \alpha_1 d \|\Lambda\|_{L_\infty}$. From (6.11), (6.12), and (1.9) it follows that

$$\|g^\varepsilon b(\mathbf{D}) \check{\mathbf{v}}_\varepsilon - \widetilde{g}^\varepsilon b(\mathbf{D}) \mathbf{u}_0\|_{L_2(\mathcal{O})} \leq \widetilde{\mathcal{C}}' \varepsilon \|\mathbf{u}_0\|_{H^2(\mathcal{O})}. \tag{6.13}$$

Combining relations (6.10), (6.13), and (3.11), we obtain (6.4) with the constant $\widetilde{\mathcal{C}}_3^\circ = \|g\|_{L_\infty} (d\alpha_1)^{1/2} \mathcal{C}_3^\circ + \widetilde{\mathcal{C}}' \widehat{c}$. □

6.2. Special cases. The next statement follows from Theorem 4.2, Proposition 1.2, and equation (1.7). (Cf. Proposition 2.7.)

Proposition 6.2. *Under the assumptions of Theorem* 4.2, *if* $g^0 = \overline{g}$, *i.e.*, (1.13) *is true, then* $\Lambda = 0$, $\mathbf{v}_\varepsilon = \mathbf{u}_0$, *and for* $0 < \varepsilon \leq \varepsilon_1$ *we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{H^1(\mathcal{O})} \leq \big(\mathcal{C}_2 c(\varphi)^2 |\zeta|^{-1/4} \varepsilon^{1/2} + \mathcal{C}_3 c(\varphi)^4 \varepsilon \big) \|\mathbf{F}\|_{L_2(\mathcal{O})}.$$

As has already been mentioned (see Subsection 2.3), under the condition $g^0 = \underline{g}$ the matrix (1.9) is constant: $\widetilde{g}(\mathbf{x}) = g^0 = \underline{g}$. Moreover, in this case Condition 2.8 is satisfied (cf. Proposition 2.12). Applying the statement of Theorem 6.1 concerning the fluxes, we arrive at the following result.

Proposition 6.3. *Under the assumptions of Theorem* 4.2, *if* $g^0 = \underline{g}$, *i.e.*, (1.14) *is true, then for* $0 < \varepsilon \leq \varepsilon_1$ *we have*

$$\|\mathbf{p}_\varepsilon - g^0 b(\mathbf{D}) \mathbf{u}_0\|_{L_2(\mathcal{O})} \leq \big(\widetilde{\mathcal{C}}_2 c(\varphi)^2 |\zeta|^{-1/4} \varepsilon^{1/2} + \widetilde{\mathcal{C}}_3^\circ c(\varphi)^4 \varepsilon \big) \|\mathbf{F}\|_{L_2(\mathcal{O})}.$$

§7. Approximation of solutions of the Dirichlet problem in a strictly interior subdomain

7.1. The general case. Using Theorem 4.1 and the results for the homogenization problem in $\mathbb{R}^d$, it is not difficult to improve the error estimates in $H^1(\mathcal{O}')$ for a strictly interior subdomain $\mathcal{O}'$ of $\mathcal{O}$.

Theorem 7.1. *Under the assumptions of Theorem* 4.2, *let* $\mathcal{O}'$ *be a strictly interior subdomain of the domain* $\mathcal{O}$. *Denote* $\delta := \operatorname{dist}\{\mathcal{O}'; \partial\mathcal{O}\}$. *Then for* $0 < \varepsilon \leq \varepsilon_1$ *we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon\|_{H^1(\mathcal{O}')} \leq (\mathcal{C}_{22}' \delta^{-1} + \mathcal{C}_{22}'') c(\varphi)^6 \varepsilon \|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{7.1}$$

$$\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D}) \widetilde{\mathbf{u}}_0\|_{L_2(\mathcal{O}')} \leq (\widetilde{\mathcal{C}}_{22}' \delta^{-1} + \widetilde{\mathcal{C}}_{22}'') c(\varphi)^6 \varepsilon \|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{7.2}$$

The constants $\mathcal{C}_{22}'$, $\mathcal{C}_{22}''$, $\widetilde{\mathcal{C}}_{22}'$, *and* $\widetilde{\mathcal{C}}_{22}''$ *depend on* d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the parameters of the lattice* Γ, *and the domain* $\mathcal{O}$.

Proof. We fix a smooth cut-off function $\chi(\mathbf{x})$ such that

$$\chi \in C_0^\infty(\mathcal{O}); \ 0 \leq \chi(\mathbf{x}) \leq 1; \ \chi(\mathbf{x}) = 1 \ \text{ for } \ \mathbf{x} \in \mathcal{O}'; \ |\nabla \chi(\mathbf{x})| \leq \kappa' \delta^{-1}. \tag{7.3}$$

The constant κ' depends only on d and the domain $\mathcal{O}$. Let $\mathbf{u}_\varepsilon$ be the solution of problem (3.3), and let $\widetilde{\mathbf{u}}_\varepsilon$ be the solution of equation (4.16). Then $(\mathcal{A}_\varepsilon - \zeta)(\mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon) = 0$ in $\mathcal{O}$. Hence, we have

$$(g^\varepsilon b(\mathbf{D})(\mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon), b(\mathbf{D})\boldsymbol{\eta})_{L_2(\mathcal{O})} - \zeta(\mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon, \boldsymbol{\eta})_{L_2(\mathcal{O})} = 0 \quad \text{for all} \quad \boldsymbol{\eta} \in H_0^1(\mathcal{O}; \mathbb{C}^n). \tag{7.4}$$

We substitute $\boldsymbol{\eta} = \chi^2(\mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon)$ in (7.4) and denote

$$\mathfrak{A}(\varepsilon) := (g^\varepsilon b(\mathbf{D})(\chi(\mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon)), b(\mathbf{D})(\chi(\mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon)))_{L_2(\mathcal{O})}. \tag{7.5}$$

The corresponding identity can easily be transformed to

$$\mathfrak{A}(\varepsilon) - \zeta\|\chi(\mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon)\|^2_{L_2(\mathcal{O})} = -\,(g^\varepsilon b(\mathbf{D})(\chi(\mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon)), \mathbf{z}_\varepsilon)_{L_2(\mathcal{O})} + (g^\varepsilon \mathbf{z}_\varepsilon, b(\mathbf{D})(\chi(\mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon)))_{L_2(\mathcal{O})} + (g^\varepsilon \mathbf{z}_\varepsilon, \mathbf{z}_\varepsilon)_{L_2(\mathcal{O})}, \tag{7.6}$$

where

$$\mathbf{z}_\varepsilon := \sum_{l=1}^d b_l (D_l \chi)(\mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon). \tag{7.7}$$

The right-hand side of (7.6) is dominated by $2\mathfrak{A}(\varepsilon)^{1/2}\|g\|^{1/2}_{L_\infty}\|\mathbf{z}_\varepsilon\|_{L_2(\mathcal{O})} + \|g\|_{L_\infty}\|\mathbf{z}_\varepsilon\|^2_{L_2(\mathcal{O})}$. Taking the imaginary part in (7.6), we obtain

$$|\operatorname{Im}\zeta|\|\chi(\mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon)\|^2_{L_2(\mathcal{O})} \le 2\mathfrak{A}(\varepsilon)^{1/2}\|g\|^{1/2}_{L_\infty}\|\mathbf{z}_\varepsilon\|_{L_2(\mathcal{O})} + \|g\|_{L_\infty}\|\mathbf{z}_\varepsilon\|^2_{L_2(\mathcal{O})}. \tag{7.8}$$

In the case where $\operatorname{Re}\zeta \ge 0$ (and then $\operatorname{Im}\zeta \ne 0$), it follows that

$$\|\chi(\mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon)\|^2_{L_2(\mathcal{O})} \le c(\varphi)|\zeta|^{-1}\big(2\mathfrak{A}(\varepsilon)^{1/2}\|g\|^{1/2}_{L_\infty}\|\mathbf{z}_\varepsilon\|_{L_2(\mathcal{O})} + \|g\|_{L_\infty}\|\mathbf{z}_\varepsilon\|^2_{L_2(\mathcal{O})}\big). \tag{7.9}$$

If $\operatorname{Re}\zeta < 0$, we take the real part in (7.6), obtaining

$$|\operatorname{Re}\zeta|\|\chi(\mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon)\|^2_{L_2(\mathcal{O})} \le 2\mathfrak{A}(\varepsilon)^{1/2}\|g\|^{1/2}_{L_\infty}\|\mathbf{z}_\varepsilon\|_{L_2(\mathcal{O})} + \|g\|_{L_\infty}\|\mathbf{z}_\varepsilon\|^2_{L_2(\mathcal{O})}. \tag{7.10}$$

Adding (7.8) and (7.10), we deduce an inequality similar to (7.9). As a result, for all values of ζ under consideration we have

$$\|\chi(\mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon)\|^2_{L_2(\mathcal{O})} \le 2c(\varphi)|\zeta|^{-1}\big(2\mathfrak{A}(\varepsilon)^{1/2}\|g\|^{1/2}_{L_\infty}\|\mathbf{z}_\varepsilon\|_{L_2(\mathcal{O})} + \|g\|_{L_\infty}\|\mathbf{z}_\varepsilon\|^2_{L_2(\mathcal{O})}\big). \tag{7.11}$$

Now, (7.6) and (7.11) imply that

$$\mathfrak{A}(\varepsilon) \le 42c(\varphi)^2\|g\|_{L_\infty}\|\mathbf{z}_\varepsilon\|^2_{L_2(\mathcal{O})}. \tag{7.12}$$

By (1.5) and (7.3), the function (7.7) satisfies

$$\|\mathbf{z}_\varepsilon\|_{L_2(\mathcal{O})} \le (d\alpha_1)^{1/2}\kappa'\delta^{-1}\|\mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon\|_{L_2(\mathcal{O})}. \tag{7.13}$$

From (7.5), (7.12), (7.13), and (3.1) it follows that

$$\|\mathbf{D}(\chi(\mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon))\|_{L_2(\mathcal{O})} \le \mathcal{C}_{23}c(\varphi)\delta^{-1}\|\mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon\|_{L_2(\mathcal{O})}, \tag{7.14}$$

where $\mathcal{C}_{23} = c_0^{-1/2}\|g\|^{1/2}_{L_\infty}(42d\alpha_1)^{1/2}\kappa'$.

Relations (4.1) and (4.17) imply

$$\|\mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon\|_{L_2(\mathcal{O})} \le \mathcal{C}_{24}c(\varphi)^5(|\zeta|^{-1/2}\varepsilon + \varepsilon^2)\|\mathbf{F}\|_{L_2(\mathcal{O})}, \quad 0 < \varepsilon \le \varepsilon_1, \tag{7.15}$$

where $\mathcal{C}_{24} = C_1\mathcal{C}_4 + \mathcal{C}_1$. By (7.14) and (7.15),

$$\|\mathbf{D}(\chi(\mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon))\|_{L_2(\mathcal{O})} \le \mathcal{C}_{23}\mathcal{C}_{24}\delta^{-1}c(\varphi)^6(|\zeta|^{-1/2}\varepsilon + \varepsilon^2)\|\mathbf{F}\|_{L_2(\mathcal{O})}, \quad 0 < \varepsilon \le \varepsilon_1.$$

Hence,

$$\|\mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon\|_{H^1(\mathcal{O}')} \le \mathcal{C}_{24}(\mathcal{C}_{23}\delta^{-1} + 1)c(\varphi)^6(|\zeta|^{-1/2}\varepsilon + \varepsilon^2)\|\mathbf{F}\|_{L_2(\mathcal{O})}, \quad 0 < \varepsilon \le \varepsilon_1. \tag{7.16}$$

From (4.6), (4.18), and (4.19) it follows that

$$\|\widetilde{\mathbf{u}}_\varepsilon - \mathbf{v}_\varepsilon\|_{H^1(\mathcal{O})} \le (C_2 + C_3)\mathcal{C}_4 c(\varphi)^3\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{7.17}$$

Relations (7.16) and (7.17) imply the required estimate (7.1) with the constants $\mathcal{C}'_{22} = 2\mathcal{C}_{23}\mathcal{C}_{24}$ and $\mathcal{C}''_{22} = 2\mathcal{C}_{24} + (C_2 + C_3)\mathcal{C}_4$.

Now we check (7.2). By (7.1) and (1.2), (1.5), we have

$$\|\mathbf{p}_\varepsilon - g^\varepsilon b(\mathbf{D})\mathbf{v}_\varepsilon\|_{L_2(\mathcal{O}')} \le \|g\|_{L_\infty}(d\alpha_1)^{1/2}(\mathcal{C}'_{22}\delta^{-1} + \mathcal{C}''_{22})c(\varphi)^6\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}.$$

Combining this with (5.22) and (4.13), we obtain (7.2) with $\widetilde{\mathcal{C}}'_{22} = \|g\|_{L_\infty}(d\alpha_1)^{1/2}\mathcal{C}'_{22}$ and $\widetilde{\mathcal{C}}''_{22} = \|g\|_{L_\infty}(d\alpha_1)^{1/2}\mathcal{C}''_{22} + \mathcal{C}_{18}$. $\square$

7.2. The case of $\Lambda \in L_\infty$. Similarly, in the case where Condition 2.8 is satisfied, we obtain the following result.

Theorem 7.2. *Under the assumptions of Theorem* 6.1, *let $\mathcal{O}'$ be a strictly interior subdomain of the domain $\mathcal{O}$, and let $\delta := \operatorname{dist}\{\mathcal{O}'; \partial\mathcal{O}\}$. Then for $0 < \varepsilon \le \varepsilon_1$ we have*

$$\|\mathbf{u}_\varepsilon - \check{\mathbf{v}}_\varepsilon\|_{H^1(\mathcal{O}')} \le (\mathcal{C}'_{22}\delta^{-1} + \check{\mathcal{C}}''_{22}) c(\varphi)^6 \varepsilon \|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{7.18}$$

$$\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathcal{O}')} \le (\widetilde{\mathcal{C}}'_{22}\delta^{-1} + \widehat{\mathcal{C}}''_{22}) c(\varphi)^6 \varepsilon \|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{7.19}$$

The constants $\mathcal{C}'_{22}$ and $\widetilde{\mathcal{C}}'_{22}$ are the same as in Theorem 7.1. *The constants $\check{\mathcal{C}}''_{22}$ and $\widehat{\mathcal{C}}''_{22}$ depend on d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, the domain $\mathcal{O}$, and the norm $\|\Lambda\|_{L_\infty}$.*

Proof. Relations (7.1), (6.9), and (4.13) imply (7.18) with the constant $\check{\mathcal{C}}''_{22} = \mathcal{C}''_{22} + \mathcal{C}''' C^{(2)}_{\mathcal{O}} \widehat{c}$.

Next, using (7.18) and (1.2), (1.5), we get

$$\|\mathbf{p}_\varepsilon - g^\varepsilon b(\mathbf{D})\check{\mathbf{v}}_\varepsilon\|_{L_2(\mathcal{O}')} \le \|g\|_{L_\infty} (d\alpha_1)^{1/2} (\mathcal{C}'_{22}\delta^{-1} + \check{\mathcal{C}}''_{22}) c(\varphi)^6 \varepsilon \|\mathbf{F}\|_{L_2(\mathcal{O})}$$

for $0 < \varepsilon \le \varepsilon_1$. Combining this with (6.13) and (3.11), we obtain (7.19) with the constant $\widehat{\mathcal{C}}''_{22} = \|g\|_{L_\infty}(d\alpha_1)^{1/2}\check{\mathcal{C}}''_{22} + \widetilde{\mathcal{C}}'\widehat{c}$. $\square$

§8. Another approximation of the resolvent $(\mathcal{A}_{D,\varepsilon} - \zeta I)^{-1}$

8.1. Approximation of the resolvent $(\mathcal{A}_{D,\varepsilon} - \zeta I)^{-1}$ for $\zeta \in \mathbb{C} \setminus [c_*, \infty)$. In the theorems of §4, 6, 7 it was assumed that $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$ and $|\zeta| \ge 1$. For completeness, we obtain yet another result about approximation of the resolvent $(\mathcal{A}_{D,\varepsilon} - \zeta I)^{-1}$, which is valid for a wider domain of the parameter ζ. This result may be preferable for bounded $|\zeta|$, and for points ζ with small φ or $2\pi - \varphi$.

Theorem 8.1. *Suppose that $\mathcal{O} \subset \mathbb{R}^d$ is a bounded domain of class $C^{1,1}$. Let $\zeta \in \mathbb{C} \setminus [c_*, \infty)$, where $c_* > 0$ is a common lower bound of the operators $\mathcal{A}_{D,\varepsilon}$ and $\mathcal{A}^0_D$. We put $\zeta - c_* = |\zeta - c_*| e^{i\psi}$ and denote*

$$\rho_*(\zeta) = \begin{cases} c(\psi)^2 |\zeta - c_*|^{-2} & \text{if} \quad |\zeta - c_*| < 1, \\ c(\psi)^2 & \text{if} \quad |\zeta - c_*| \ge 1. \end{cases} \tag{8.1}$$

Let $\mathbf{u}_\varepsilon$ be the solution of problem (3.3), *and let $\mathbf{u}_0$ be the solution of problem* (3.10). *Let $\mathbf{v}_\varepsilon$ be defined by* (4.7). *Suppose that the number ε_1 satisfies Condition* 3.4. *Then for $0 < \varepsilon \le \varepsilon_1$ we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O})} \le \mathcal{C}_{25} \rho_*(\zeta) \varepsilon \|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{8.2}$$

$$\|\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon\|_{H^1(\mathcal{O})} \le \mathcal{C}_{26} \rho_*(\zeta) \varepsilon^{1/2} \|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{8.3}$$

or, in operator terms,

$$\big\|(\mathcal{A}_{D,\varepsilon} - \zeta I)^{-1} - (\mathcal{A}^0_D - \zeta I)^{-1}\big\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \le \mathcal{C}_{25} \rho_*(\zeta) \varepsilon, \tag{8.4}$$

$$\big\|(\mathcal{A}_{D,\varepsilon} - \zeta I)^{-1} - (\mathcal{A}^0_D - \zeta I)^{-1} - \varepsilon K_D(\varepsilon; \zeta)\big\|_{L_2(\mathcal{O}) \to H^1(\mathcal{O})} \le \mathcal{C}_{26} \rho_*(\zeta) \varepsilon^{1/2}. \tag{8.5}$$

For the flux $\mathbf{p}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$, we have

$$\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathcal{O})} \le \mathcal{C}_{27} \rho_*(\zeta) \varepsilon^{1/2} \|\mathbf{F}\|_{L_2(\mathcal{O})}, \quad 0 < \varepsilon \le \varepsilon_1. \tag{8.6}$$

The constants $\mathcal{C}_{25}$, $\mathcal{C}_{26}$, and $\mathcal{C}_{27}$ depend on d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, and the domain $\mathcal{O}$.

Remark 8.2. 1) The expression $c(\psi)^2|\zeta - c_*|^{-2}$ in (8.1) is the inverse of the square of the distance from ζ to $[c_*, \infty)$.

2) One can take $c_* = c_2$, where c_2 is defined by (3.2).

3) Let $\nu > 0$ be an arbitrarily small number. If ε is sufficiently small, one can take $c_* = \lambda_1^0(D) - \nu$, where $\lambda_1^0(D)$ is the first eigenvalue of the operator $\mathcal{A}_D^0$.

4) It is easy to find an upper bound for c_*: from (3.1) it is seen that $c_* \le c_1 \mu_1^0(D)$, where $\mu_1^0(D)$ is the first eigenvalue of the operator $-\Delta$ with the Dirichlet condition. Therefore, c_* is bounded by a number depending only on α_1, $\|g\|_{L_\infty}$, and the domain $\mathcal{O}$.

Proof. We apply Theorem 4.1 with $\zeta = -1$. By (4.2),

$$\big\|(\mathcal{A}_{D,\varepsilon} + I)^{-1} - (\mathcal{A}_D^0 + I)^{-1}\big\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \le \mathcal{C}_1(\varepsilon + \varepsilon^2) \le 2\mathcal{C}_1 \varepsilon, \quad 0 < \varepsilon \le \varepsilon_1.$$

Using an analog of identity (2.10) (with $\mathcal{A}_\varepsilon$ replaced by $\mathcal{A}_{D,\varepsilon}$ and $\mathcal{A}^0$ replaced by $\mathcal{A}_D^0$), we see that

$$\big\|(\mathcal{A}_{D,\varepsilon} - \zeta I)^{-1} - (\mathcal{A}_D^0 - \zeta I)^{-1}\big\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \le 2\mathcal{C}_1 \varepsilon \sup_{x \ge c_*} (x+1)^2 |x - \zeta|^{-2} \tag{8.7}$$

for $0 < \varepsilon \le \varepsilon_1$. A calculation shows that

$$\sup_{x \ge c_*} (x+1)^2 |x - \zeta|^{-2} \le \check{c} \rho_*(\zeta), \quad \zeta \in \mathbb{C} \setminus [c_*, \infty), \tag{8.8}$$

where $\check{c} = (c_* + 2)^2$. By Remark 8.2(4), $\check{c}$ is bounded by a number depending only on α_1, $\|g\|_{L_\infty}$, and the domain $\mathcal{O}$. Now (8.7) and (8.8) imply (8.4) with the constant $\mathcal{C}_{25} = 2\mathcal{C}_1 \check{c}$.

We apply Theorem 4.2 with $\zeta = -1$. By (4.9),

$$\big\|(\mathcal{A}_{D,\varepsilon} + I)^{-1} - (\mathcal{A}_D^0 + I)^{-1} - \varepsilon K_D(\varepsilon; -1)\big\|_{L_2(\mathcal{O}) \to H^1(\mathcal{O})} \le (\mathcal{C}_2 + \mathcal{C}_3)\varepsilon^{1/2} \tag{8.9}$$

for $0 < \varepsilon \le \varepsilon_1$. From Lemma 5.2 with $\zeta = -1$ it follows that

$$\|\varepsilon \theta_\varepsilon K_D(\varepsilon; -1)\|_{L_2(\mathcal{O}) \to H^1(\mathcal{O})} \le (\mathcal{C}_{13} + \mathcal{C}_{14} + \mathcal{C}_{15})\varepsilon^{1/2}, \quad 0 < \varepsilon \le \varepsilon_1. \tag{8.10}$$

By (8.9) and (8.10), we have

$$\big\|(\mathcal{A}_{D,\varepsilon} + I)^{-1} - (\mathcal{A}_D^0 + I)^{-1} - \varepsilon(1 - \theta_\varepsilon) K_D(\varepsilon; -1)\big\|_{L_2(\mathcal{O}) \to H^1(\mathcal{O})} \le \mathcal{C}_{28} \varepsilon^{1/2} \tag{8.11}$$

for $0 < \varepsilon \le \varepsilon_1$, where $\mathcal{C}_{28} = \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_{13} + \mathcal{C}_{14} + \mathcal{C}_{15}$. We use the following analog of identity (2.34):

$$\begin{aligned} &(\mathcal{A}_{D,\varepsilon} - \zeta I)^{-1} - (\mathcal{A}_D^0 - \zeta I)^{-1} - \varepsilon(1 - \theta_\varepsilon) K_D(\varepsilon; \zeta) \\ &\quad = (\mathcal{A}_{D,\varepsilon} + I)(\mathcal{A}_{D,\varepsilon} - \zeta I)^{-1} \\ &\qquad \times \big((\mathcal{A}_{D,\varepsilon} + I)^{-1} - (\mathcal{A}_D^0 + I)^{-1} - \varepsilon(1 - \theta_\varepsilon) K_D(\varepsilon; -1)\big) \\ &\qquad \times (\mathcal{A}_D^0 + I)(\mathcal{A}_D^0 - \zeta I)^{-1} + \varepsilon(\zeta + 1)(\mathcal{A}_{D,\varepsilon} - \zeta I)^{-1}(1 - \theta_\varepsilon) K_D(\varepsilon; \zeta). \end{aligned} \tag{8.12}$$

Since the range of the operators in (8.12) is contained in $H_0^1(\mathcal{O}; \mathbb{C}^n)$, we can multiply (8.12) by $\mathcal{A}_{D,\varepsilon}^{1/2}$ from the left. Taking (8.8) into account, we obtain

$$\begin{aligned} &\big\|\mathcal{A}_{D,\varepsilon}^{1/2}\big((\mathcal{A}_{D,\varepsilon} - \zeta I)^{-1} - (\mathcal{A}_D^0 - \zeta I)^{-1} - \varepsilon(1 - \theta_\varepsilon) K_D(\varepsilon; \zeta)\big)\big\|_{L_2 \to L_2} \\ &\quad \le \check{c} \rho_*(\zeta) \big\|\mathcal{A}_{D,\varepsilon}^{1/2}\big((\mathcal{A}_{D,\varepsilon} + I)^{-1} - (\mathcal{A}_D^0 + I)^{-1} - \varepsilon(1 - \theta_\varepsilon) K_D(\varepsilon; -1)\big)\big\|_{L_2 \to L_2} \\ &\qquad + \varepsilon|\zeta + 1| \sup_{x \ge c_*} x^{1/2} |x - \zeta|^{-1} \|(1 - \theta_\varepsilon) K_D(\varepsilon; \zeta)\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})}. \end{aligned} \tag{8.13}$$

Denote the summands on the right-hand side of (8.13) by $\mathcal{L}_1(\varepsilon)$ and $\mathcal{L}_2(\varepsilon)$. The first term is estimated with the help of (8.11) and (3.1):

$$\mathcal{L}_1(\varepsilon) \le c_1^{1/2} \check{c} \mathcal{C}_{28} \rho_*(\zeta) \varepsilon^{1/2}, \quad 0 < \varepsilon \le \varepsilon_1. \tag{8.14}$$

Since $K_D(\varepsilon;\zeta) = R_{\mathcal{O}}[\Lambda^\varepsilon]S_\varepsilon b(\mathbf{D})P_{\mathcal{O}}(\mathcal{A}_D^0)^{-1/2}(\mathcal{A}_D^0)^{1/2}(\mathcal{A}_D^0-\zeta I)^{-1}$, relations (1.4), (1.19), (4.3), and (5.1) imply that

$$\mathcal{L}_2(\varepsilon) \le \varepsilon|\zeta+1|\big(\sup_{x\ge c_*} x|x-\zeta|^{-2}\big)M_1\alpha_1^{1/2}C_{\mathcal{O}}^{(1)}\|(\mathcal{A}_D^0)^{-1/2}\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})}. \tag{8.15}$$

Using the analogs of (3.1) and (3.2) for $\mathcal{A}_D^0$, we have

$$\|(\mathcal{A}_D^0)^{-1/2}\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} \le (c_0^{-1}+c_2^{-1})^{1/2}. \tag{8.16}$$

A direct calculation shows that

$$\sup_{x\ge c_*} x|x-\zeta|^{-2} \le \begin{cases}(c_*+1)c(\psi)^2|\zeta-c_*|^{-2} & \text{if} \quad |\zeta-c_*|<1,\\ (c_*+1)c(\psi)^2|\zeta-c_*|^{-1} & \text{if} \quad |\zeta-c_*|\ge 1.\end{cases}$$

Note that $|\zeta+1|\le 2+c_*$ for $|\zeta-c_*|<1$, and $|\zeta+1||\zeta-c_*|^{-1}\le 2+c_*$ for $|\zeta-c_*|\ge 1$, whence

$$|\zeta+1|\sup_{x\ge c_*} x|x-\zeta|^{-2} \le (c_*+2)(c_*+1)\rho_*(\zeta). \tag{8.17}$$

From (8.15)–(8.17) it follows that

$$\mathcal{L}_2(\varepsilon) \le \mathcal{C}_{29}\rho_*(\zeta)\varepsilon, \tag{8.18}$$

where $\mathcal{C}_{29} = (c_*+2)(c_*+1)M_1\alpha_1^{1/2}C_{\mathcal{O}}^{(1)}(c_0^{-1}+c_2^{-1})^{1/2}$.

As a result, inequalities (8.13), (8.14), and (8.18) yield

$$\begin{aligned}\big\|\mathcal{A}_{D,\varepsilon}^{1/2}\big((\mathcal{A}_{D,\varepsilon}-\zeta I)^{-1}-(\mathcal{A}_D^0-\zeta I)^{-1}-\varepsilon(1-\theta_\varepsilon)K_D(\varepsilon;\zeta)\big)\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})}&\\ \le (c_1^{1/2}\check{c}\,\mathcal{C}_{28}+\mathcal{C}_{29})\rho_*(\zeta)\varepsilon^{1/2}, \quad 0<\varepsilon\le\varepsilon_1.&\end{aligned}$$

Together with (3.1) and (3.2), this implies

$$\begin{aligned}\big\|(\mathcal{A}_{D,\varepsilon}-\zeta I)^{-1}-(\mathcal{A}_D^0-\zeta I)^{-1}-\varepsilon(1-\theta_\varepsilon)K_D(\varepsilon;\zeta)\big\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})}&\\ \le (c_0^{-1}+c_2^{-1})^{1/2}(c_1^{1/2}\check{c}\,\mathcal{C}_{28}+\mathcal{C}_{29})\rho_*(\zeta)\varepsilon^{1/2}, \quad 0<\varepsilon\le\varepsilon_1.&\end{aligned} \tag{8.19}$$

Finally, by (8.10) and (8.8), we have

$$\begin{aligned}&\|\varepsilon\theta_\varepsilon K_D(\varepsilon;\zeta)\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} \le \|\varepsilon\theta_\varepsilon K_D(\varepsilon;-1)\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})}\\ &\quad\times\big\|(\mathcal{A}_D^0+I)(\mathcal{A}_D^0-\zeta I)^{-1}\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \le (\mathcal{C}_{13}+\mathcal{C}_{14}+\mathcal{C}_{15})\check{c}^{1/2}\rho_*(\zeta)^{1/2}\varepsilon^{1/2}\end{aligned} \tag{8.20}$$

for $0<\varepsilon\le\varepsilon_1$. As a result, relations (8.19) and (8.20) imply (8.5) with the constant $\mathcal{C}_{26} = (c_0^{-1}+c_2^{-1})^{1/2}(c_1^{1/2}\check{c}\,\mathcal{C}_{28}+\mathcal{C}_{29})+(\mathcal{C}_{13}+\mathcal{C}_{14}+\mathcal{C}_{15})\check{c}^{1/2}$.

It remains to check (8.6). From (8.3) and (1.2), (1.5) it is seen that

$$\|\mathbf{p}_\varepsilon - g^\varepsilon b(\mathbf{D})\mathbf{v}_\varepsilon\|_{L_2(\mathcal{O})} \le \|g\|_{L_\infty}(d\alpha_1)^{1/2}\mathcal{C}_{26}\rho_*(\zeta)\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})} \tag{8.21}$$

for $0<\varepsilon\le\varepsilon_1$. Next, as in (5.19)–(5.22) we obtain

$$\|g^\varepsilon b(\mathbf{D})\mathbf{v}_\varepsilon - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathcal{O})} \le (\mathcal{C}'+\mathcal{C}'')\varepsilon\|\widetilde{\mathbf{u}}_0\|_{H^2(\mathbb{R}^d)}. \tag{8.22}$$

From (3.9) and (8.8) it follows that

$$\|(\mathcal{A}_D^0-\zeta I)^{-1}\|_{L_2(\mathcal{O})\to H^2(\mathcal{O})} \le \widehat{c}\sup_{x\ge c_*} x|x-\zeta|^{-1} \le \widehat{c}\,\check{c}^{1/2}\rho_*(\zeta)^{1/2}.$$

Hence,

$$\|\mathbf{u}_0\|_{H^2(\mathcal{O})} \le \widehat{c}\,\check{c}^{1/2}\rho_*(\zeta)^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{8.23}$$

Together with (4.3) and (8.22), this yields

$$\|g^\varepsilon b(\mathbf{D})\mathbf{v}_\varepsilon - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathcal{O})} \le \widetilde{\mathcal{C}}_{27}\rho_*(\zeta)^{1/2}\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{8.24}$$

where $\widetilde{\mathcal{C}}_{27} = (\mathcal{C}' + \mathcal{C}'')C_{\mathcal{O}}^{(2)}\widehat{c}\,\check{c}^{1/2}$. Combining this with (8.21), we obtain (8.6) with $\mathcal{C}_{27} = \|g\|_{L_\infty}(d\alpha_1)^{1/2}\mathcal{C}_{26} + \widetilde{\mathcal{C}}_{27}$. $\square$

8.2. The case where $\Lambda \in L_\infty$. Under Condition 2.8 we obtain the following result.

Theorem 8.3. *Suppose that the assumptions of Theorem 8.1 and Condition 2.8 are satisfied. Let $\check{\mathbf{v}}_\varepsilon$ be defined by (6.2). Then for $0 < \varepsilon \le \varepsilon_1$ we have*

$$\|\mathbf{u}_\varepsilon - \check{\mathbf{v}}_\varepsilon\|_{H^1(\mathcal{O})} \le \mathcal{C}_{26}^\circ \rho_*(\zeta)\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{8.25}$$

or, in operator terms,

$$\big\|(\mathcal{A}_{D,\varepsilon} - \zeta I)^{-1} - (\mathcal{A}_D^0 - \zeta I)^{-1} - \varepsilon K_D^0(\varepsilon;\zeta)\big\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} \le \mathcal{C}_{26}^\circ \rho_*(\zeta)\varepsilon^{1/2}.$$

For the flux $\mathbf{p}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$ we have

$$\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathcal{O})} \le \mathcal{C}_{27}^\circ \rho_*(\zeta)\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \quad 0 < \varepsilon \le \varepsilon_1. \tag{8.26}$$

The constants $\mathcal{C}_{26}^\circ$ and $\mathcal{C}_{27}^\circ$ depend on d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, the domain $\mathcal{O}$, and the norm $\|\Lambda\|_{L_\infty}$.

Proof. Arguing as in the proof of Theorem 6.1 (see (6.5)–(6.9)), it is easy to obtain an analog of estimate (6.9) with the same constant. Together with (8.23) and (4.3), this yields

$$\|\mathbf{v}_\varepsilon - \check{\mathbf{v}}_\varepsilon\|_{H^1(\mathcal{O})} \le \mathcal{C}''' C_{\mathcal{O}}^{(2)}\widehat{c}\,\check{c}^{1/2}\rho_*(\zeta)^{1/2}\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{8.27}$$

Now relations (8.3) and (8.27) imply (8.25) with the constant

$$\mathcal{C}_{26}^\circ = \mathcal{C}_{26} + \mathcal{C}''' C_{\mathcal{O}}^{(2)}\widehat{c}\,\check{c}^{1/2}.$$

Like in (6.11)–(6.13) we check an analog of estimate (6.13) with the same constant. Together with (8.23), (8.25), and (1.2), (1.5), this implies (8.26) with the constant $\mathcal{C}_{27}^\circ = \|g\|_{L_\infty}(d\alpha_1)^{1/2}\mathcal{C}_{26}^\circ + \widetilde{\mathcal{C}}'\widehat{c}\check{c}^{1/2}$. $\square$

8.3. Special cases. The proof of the following statements is similar to that of Propositions 6.2 and 6.3.

Proposition 8.4. *Under the assumptions of Theorem 8.1, if $g^0 = \overline{g}$, i.e., (1.13) is true, then $\Lambda = 0$, $\mathbf{v}_\varepsilon = \mathbf{u}_0$, and for $0 < \varepsilon \le \varepsilon_1$ we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{H^1(\mathcal{O})} \le \mathcal{C}_{26}\rho_*(\zeta)\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}.$$

Proposition 8.5. *Under the assumptions of Theorem 8.1, if $g^0 = \underline{g}$, i.e., (1.14) is true, then for $0 < \varepsilon \le \varepsilon_1$ we have*

$$\|\mathbf{p}_\varepsilon - g^0 b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathcal{O})} \le \mathcal{C}_{27}^\circ \rho_*(\zeta)\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}.$$

8.4. Estimates in a strictly interior subdomain. As in Theorem 7.1, we can obtain an error estimate of order ε in $H^1(\mathcal{O}')$ in a strictly interior subdomain $\mathcal{O}'$ of the domain $\mathcal{O}$, using Theorem 8.1 and the results for the problem in $\mathbb{R}^d$.

Theorem 8.6. *Under the assumptions of Theorem 8.1, let $\mathcal{O}'$ be a strictly interior subdomain of the domain $\mathcal{O}$, and let $\delta := \operatorname{dist}\{\mathcal{O}';\partial\mathcal{O}\}$. Denote $\widehat{\rho}(\zeta) := c(\psi)\rho_*(\zeta) + c(\psi)^{5/2}\rho_*(\zeta)^{3/4}$. Then for $0 < \varepsilon \le \varepsilon_1$ we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon\|_{H^1(\mathcal{O}')} \le \big(\mathcal{C}_{30}'\delta^{-1}\widehat{\rho}(\zeta) + \mathcal{C}_{30}'' c(\psi)^{1/2}\rho_*(\zeta)^{5/4}\big)\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{8.28}$$

$$\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathcal{O}')} \le \big(\widetilde{\mathcal{C}}_{30}'\delta^{-1}\widehat{\rho}(\zeta) + \widetilde{\mathcal{C}}_{30}'' c(\psi)^{1/2}\rho_*(\zeta)^{5/4}\big)\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{8.29}$$

The constants $\mathcal{C}_{30}'$, $\mathcal{C}_{30}''$, $\widetilde{\mathcal{C}}_{30}'$, and $\widetilde{\mathcal{C}}_{30}''$ depend on d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, and the domain $\mathcal{O}$.

Proof. Largely, the proof is similar to that of Theorem 7.1. However, the associated problem in $\mathbb{R}^d$ is chosen in a different way. Since

$$\|(\mathcal{A}_D^0 - \zeta I)^{-1}\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leq c(\psi)|\zeta - c_*|^{-1},$$

we can use (4.3) to obtain

$$\|\widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} \leq C_{\mathcal{O}}^{(0)}\|\mathbf{u}_0\|_{L_2(\mathcal{O})} \leq C_{\mathcal{O}}^{(0)} c(\psi)|\zeta - c_*|^{-1}\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{8.30}$$

By (4.3) and (8.23), we have

$$\|\widetilde{\mathbf{u}}_0\|_{H^2(\mathbb{R}^d)} \leq C_{\mathcal{O}}^{(2)}\widehat{c}\,\check{c}^{1/2}\rho_*(\zeta)^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{8.31}$$

We put

$$\widehat{\mathbf{F}} := \mathcal{A}^0\widetilde{\mathbf{u}}_0 - (\zeta - c_*)\widetilde{\mathbf{u}}_0. \tag{8.32}$$

As in (4.15), from (8.30) and (8.31) we deduce that

$$\|\widehat{\mathbf{F}}\|_{L_2(\mathbb{R}^d)} \leq \mathcal{C}_{31}\rho_*(\zeta)^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{8.33}$$

where $\mathcal{C}_{31} = c_1 C_{\mathcal{O}}^{(2)}\widehat{c}\,\check{c}^{1/2} + C_{\mathcal{O}}^{(0)}$. Note that $(\widehat{\mathbf{F}} - \mathbf{F})|_{\mathcal{O}} = c_*\mathbf{u}_0$.

Under our assumptions we have $\zeta \in \mathbb{C} \setminus [c_*, \infty)$. Then the point $\zeta - c_* \in \mathbb{C} \setminus [0, \infty)$ is regular for the operator $\mathcal{A}_\varepsilon$. Let $\widehat{\mathbf{u}}_\varepsilon$ be the solution of the equation

$$\mathcal{A}_\varepsilon\widehat{\mathbf{u}}_\varepsilon - (\zeta - c_*)\widehat{\mathbf{u}}_\varepsilon = \widehat{\mathbf{F}} \tag{8.34}$$

in $\mathbb{R}^d$. Then the function $\mathbf{u}_\varepsilon - \widehat{\mathbf{u}}_\varepsilon$ satisfies the identity

$$\begin{aligned}(g^\varepsilon b(\mathbf{D})(\mathbf{u}_\varepsilon - \widehat{\mathbf{u}}_\varepsilon), b(\mathbf{D})\boldsymbol{\eta})_{L_2(\mathcal{O})} - (\zeta - c_*)(\mathbf{u}_\varepsilon - \widehat{\mathbf{u}}_\varepsilon, \boldsymbol{\eta})_{L_2(\mathcal{O})} &= c_*(\mathbf{u}_\varepsilon - \mathbf{u}_0, \boldsymbol{\eta})_{L_2(\mathcal{O})}, \\ \text{for all} \quad \boldsymbol{\eta} &\in H_0^1(\mathcal{O}; \mathbb{C}^n).\end{aligned} \tag{8.35}$$

Let $\chi(\mathbf{x})$ be a cut-off function satisfying (7.3). We substitute $\boldsymbol{\eta} = \chi^2(\mathbf{u}_\varepsilon - \widehat{\mathbf{u}}_\varepsilon)$ in (8.35) and denote

$$\mathfrak{B}(\varepsilon) := (g^\varepsilon b(\mathbf{D})(\chi(\mathbf{u}_\varepsilon - \widehat{\mathbf{u}}_\varepsilon)), b(\mathbf{D})(\chi(\mathbf{u}_\varepsilon - \widehat{\mathbf{u}}_\varepsilon)))_{L_2(\mathcal{O})}.$$

As in (7.6), the corresponding relation can easily be rewritten as

$$\begin{aligned}&\mathfrak{B}(\varepsilon) - (\zeta - c_*)\|\chi(\mathbf{u}_\varepsilon - \widehat{\mathbf{u}}_\varepsilon)\|^2_{L_2(\mathcal{O})} \\ &\quad = c_*(\mathbf{u}_\varepsilon - \mathbf{u}_0, \chi^2(\mathbf{u}_\varepsilon - \widehat{\mathbf{u}}_\varepsilon))_{L_2(\mathcal{O})} - (g^\varepsilon b(\mathbf{D})(\chi(\mathbf{u}_\varepsilon - \widehat{\mathbf{u}}_\varepsilon)), \mathbf{r}_\varepsilon)_{L_2(\mathcal{O})} \\ &\qquad + (g^\varepsilon\mathbf{r}_\varepsilon, b(\mathbf{D})(\chi(\mathbf{u}_\varepsilon - \widehat{\mathbf{u}}_\varepsilon)))_{L_2(\mathcal{O})} + (g^\varepsilon\mathbf{r}_\varepsilon, \mathbf{r}_\varepsilon)_{L_2(\mathcal{O})},\end{aligned} \tag{8.36}$$

where $\mathbf{r}_\varepsilon := \sum_{l=1}^d b_l(D_l\chi)(\mathbf{u}_\varepsilon - \widehat{\mathbf{u}}_\varepsilon)$.

Take the imaginary part in (8.36). Then

$$\begin{aligned}|\operatorname{Im}\zeta|\|\chi(\mathbf{u}_\varepsilon - \widehat{\mathbf{u}}_\varepsilon)\|^2_{L_2(\mathcal{O})} &\leq c_*\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O})}\|\chi(\mathbf{u}_\varepsilon - \widehat{\mathbf{u}}_\varepsilon)\|_{L_2(\mathcal{O})} \\ &\quad + 2\mathfrak{B}(\varepsilon)^{1/2}\|g\|^{1/2}_{L_\infty}\|\mathbf{r}_\varepsilon\|_{L_2(\mathcal{O})} + \|g\|_{L_\infty}\|\mathbf{r}_\varepsilon\|^2_{L_2(\mathcal{O})}.\end{aligned} \tag{8.37}$$

If $\operatorname{Re}\zeta \geq c_*$ (and then $\operatorname{Im}\zeta \neq 0$), we deduce that

$$\begin{aligned}\|\chi(\mathbf{u}_\varepsilon - \widehat{\mathbf{u}}_\varepsilon)\|^2_{L_2(\mathcal{O})} &\leq c(\psi)^2|\zeta - c_*|^{-2}c_*^2\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|^2_{L_2(\mathcal{O})} \\ &\quad + c(\psi)|\zeta - c_*|^{-1}\big(4\mathfrak{B}(\varepsilon)^{1/2}\|g\|^{1/2}_{L_\infty}\|\mathbf{r}_\varepsilon\|_{L_2(\mathcal{O})} + 2\|g\|_{L_\infty}\|\mathbf{r}_\varepsilon\|^2_{L_2(\mathcal{O})}\big).\end{aligned} \tag{8.38}$$

If $\operatorname{Re}\zeta < c_*$, we take the real part in (8.36), obtaining

$$\begin{aligned}|\operatorname{Re}\zeta - c_*|\|\chi(\mathbf{u}_\varepsilon - \widehat{\mathbf{u}}_\varepsilon)\|^2_{L_2(\mathcal{O})} &\leq c_*\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O})}\|\chi(\mathbf{u}_\varepsilon - \widehat{\mathbf{u}}_\varepsilon)\|_{L_2(\mathcal{O})} \\ &\quad + 2\mathfrak{B}(\varepsilon)^{1/2}\|g\|^{1/2}_{L_\infty}\|\mathbf{r}_\varepsilon\|_{L_2(\mathcal{O})} + \|g\|_{L_\infty}\|\mathbf{r}_\varepsilon\|^2_{L_2(\mathcal{O})}.\end{aligned} \tag{8.39}$$

Adding (8.37) and (8.39), we deduce an inequality similar to (8.38). As a result, for all values of ζ under consideration we obtain

$$\|\chi(\mathbf{u}_\varepsilon - \widehat{\mathbf{u}}_\varepsilon)\|^2_{L_2(\mathcal{O})} \le 4c(\psi)^2|\zeta - c_*|^{-2}c_*^2\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|^2_{L_2(\mathcal{O})} + c(\psi)|\zeta - c_*|^{-1}\big(8\mathfrak{B}(\varepsilon)^{1/2}\|g\|^{1/2}_{L_\infty}\|\mathbf{r}_\varepsilon\|_{L_2(\mathcal{O})} + 4\|g\|_{L_\infty}\|\mathbf{r}_\varepsilon\|^2_{L_2(\mathcal{O})}\big). \tag{8.40}$$

Now, relations (8.36) and (8.40) imply that

$$\mathfrak{B}(\varepsilon) \le 342c(\psi)^2\|g\|_{L_\infty}\|\mathbf{r}_\varepsilon\|^2_{L_2(\mathcal{O})} + 18c_*^2c(\psi)^2|\zeta - c_*|^{-1}\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|^2_{L_2(\mathcal{O})}. \tag{8.41}$$

By (1.5) and (7.3),

$$\|\mathbf{r}_\varepsilon\|_{L_2(\mathcal{O})} \le (d\alpha_1)^{1/2}\kappa'\delta^{-1}\|\mathbf{u}_\varepsilon - \widehat{\mathbf{u}}_\varepsilon\|_{L_2(\mathcal{O})}. \tag{8.42}$$

From (8.41), (8.42), and (3.1) it follows that

$$\|\mathbf{D}(\chi(\mathbf{u}_\varepsilon - \widehat{\mathbf{u}}_\varepsilon))\|_{L_2(\mathcal{O})} \le \mathcal{C}_{32}c(\psi)\delta^{-1}\|\mathbf{u}_\varepsilon - \widehat{\mathbf{u}}_\varepsilon\|_{L_2(\mathcal{O})} + \sqrt{18}c_0^{-1/2}c_*c(\psi)|\zeta - c_*|^{-1/2}\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O})}, \tag{8.43}$$

where $\mathcal{C}_{32}^2 = 342c_0^{-1}\|g\|_{L_\infty}d\alpha_1(\kappa')^2$.

The norm $\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O})}$ satisfies (8.2). Using Theorem 2.2 and relations (8.32)–(8.34), we obtain

$$\|\widehat{\mathbf{u}}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O})} \le \|\widehat{\mathbf{u}}_\varepsilon - \widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} \le C_1c(\psi)^2|\zeta - c_*|^{-1/2}\varepsilon\|\widehat{\mathbf{F}}\|_{L_2(\mathbb{R}^d)} \le C_1\mathcal{C}_{31}c(\psi)^2\rho_*(\zeta)^{1/2}|\zeta - c_*|^{-1/2}\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{8.44}$$

By (8.1), (8.2), and (8.44),

$$\|\mathbf{u}_\varepsilon - \widehat{\mathbf{u}}_\varepsilon\|_{L_2(\mathcal{O})} \le \big(\mathcal{C}_{25}\rho_*(\zeta) + C_1\mathcal{C}_{31}c(\psi)^{3/2}\rho_*(\zeta)^{3/4}\big)\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{8.45}$$

Now relations (8.2), (8.43), and (8.45) yield

$$\|\mathbf{D}(\chi(\mathbf{u}_\varepsilon - \widehat{\mathbf{u}}_\varepsilon))\|_{L_2(\mathcal{O})} \le \big(\mathcal{C}'_{30}\delta^{-1}\widehat{\rho}(\zeta) + \mathcal{C}_{33}c(\psi)^{1/2}\rho_*(\zeta)^{5/4}\big)\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})},$$

where $\mathcal{C}'_{30} = \mathcal{C}_{32}\max\{\mathcal{C}_{25}, C_1\mathcal{C}_{31}\}$ and $\mathcal{C}_{33} = \sqrt{18}c_0^{-1/2}c_*\mathcal{C}_{25}$. Together with (8.45), this implies that

$$\|\mathbf{u}_\varepsilon - \widehat{\mathbf{u}}_\varepsilon\|_{H^1(\mathcal{O}')} \le \big(\mathcal{C}'_{30}\delta^{-1}\widehat{\rho}(\zeta) + \mathcal{C}_{34}c(\psi)^{1/2}\rho_*(\zeta)^{5/4}\big)\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{8.46}$$

where $\mathcal{C}_{34} = \mathcal{C}_{33} + \mathcal{C}_{25} + C_1\mathcal{C}_{31}$.

By Corollary 2.5 and (8.32)–(8.34), we have

$$\|\widehat{\mathbf{u}}_\varepsilon - \widetilde{\mathbf{u}}_0 - \varepsilon\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{H^1(\mathbb{R}^d)} \le c(\psi)^2(C_2 + C_3|\zeta - c_*|^{-1/2})\varepsilon\|\widehat{\mathbf{F}}\|_{L_2(\mathbb{R}^d)} \le \mathcal{C}_{31}(C_2 + C_3)c(\psi)^{3/2}\rho_*(\zeta)^{3/4}\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{8.47}$$

As a result, relations (8.46) and (8.47) imply the required estimate (8.28) with the constant $\mathcal{C}''_{30} = \mathcal{C}_{34} + \mathcal{C}_{31}(C_2 + C_3)$.

Estimate (8.29) can easily be deduced from (1.2), (1.5), (8.28), and (8.24); we have $\widetilde{\mathcal{C}}'_{30} = \|g\|_{L_\infty}(d\alpha_1)^{1/2}\mathcal{C}'_{30}$ and $\widetilde{\mathcal{C}}''_{30} = \|g\|_{L_\infty}(d\alpha_1)^{1/2}\mathcal{C}''_{30} + \widetilde{\mathcal{C}}_{27}$. □

Similarly, in the case where Condition 2.8 is satisfied we obtain the following result.

Theorem 8.7. *Under the assumptions of Theorem* 8.3, *let $\mathcal{O}'$ be a strictly interior subdomain of the domain $\mathcal{O}$, and let $\delta := \operatorname{dist}\{\mathcal{O}'; \partial\mathcal{O}\}$. Suppose that $\widehat{\rho}(\zeta)$ is defined as in Theorem* 8.6. *Then for $0 < \varepsilon \le \varepsilon_1$ we have*

$$\|\mathbf{u}_\varepsilon - \check{\mathbf{v}}_\varepsilon\|_{H^1(\mathcal{O}')} \le \big(\mathcal{C}'_{30}\delta^{-1}\widehat{\rho}(\zeta) + \check{\mathcal{C}}''_{30}c(\psi)^{1/2}\rho_*(\zeta)^{5/4}\big)\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{8.48}$$

$$\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathcal{O}')} \le \big(\widetilde{\mathcal{C}}'_{30}\delta^{-1}\widehat{\rho}(\zeta) + \widehat{\mathcal{C}}''_{30}c(\psi)^{1/2}\rho_*(\zeta)^{5/4}\big)\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{8.49}$$

The constants $\mathcal{C}'_{30}$ *and* $\widetilde{\mathcal{C}}'_{30}$ *are the same as in Theorem* 8.6. *The constants* $\check{\mathcal{C}}''_{30}$ *and* $\widehat{\mathcal{C}}''_{30}$ *depend on* d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the parameters of the lattice* Γ, *the domain* $\mathcal{O}$, *and* $\|\Lambda\|_{L_\infty}$.

Proof. Relations (8.27) and (8.28) imply (8.48) with the constant $\check{\mathcal{C}}''_{30} = \mathcal{C}''_{30} + \mathcal{C}''' C^{(2)}_{\mathcal{O}} \widetilde{\widehat{c}}^{1/2}$.

Combining (8.48), (8.23), and an analog of (6.13), we obtain (8.49) with the constant $\widehat{\mathcal{C}}''_{30} = \|g\|_{L_\infty}(d\alpha_1)^{1/2}\check{\mathcal{C}}''_{30} + \widetilde{\mathcal{C}}'\widetilde{\widehat{c}}^{1/2}$. □

Chapter 3. The Neumann problem

§9. The Neumann problem in a bounded domain: Preliminaries

9.1. Coercivity. As in Chapter 2, we assume that $\mathcal{O} \subset \mathbb{R}^d$ is a bounded domain of class $C^{1,1}$. We impose an additional condition on the symbol $b(\boldsymbol{\xi})$ of the operator (1.2).

Condition 9.1. *The matrix-valued function* $b(\boldsymbol{\xi}) = \sum_{l=1}^d b_l \xi_l$ *satisfies*

$$\operatorname{rank} b(\boldsymbol{\xi}) = n, \quad 0 \neq \boldsymbol{\xi} \in \mathbb{C}^d. \tag{9.1}$$

Note that condition (9.1) is more restrictive than (1.3). The following statement was proved in the book [Ne] (see Theorem 7.8 in §3.7 therein; note that this statement remains valid even in the case where the boundary $\partial\mathcal{O}$ is Lipschitz).

Proposition 9.2 (see [Ne]). *Condition* 9.1 *is necessary and sufficient for the existence of constants* $k_1 > 0$ *and* $k_2 \geq 0$ *such that the Gårding type inequality*

$$\|b(\mathbf{D})\mathbf{u}\|^2_{L_2(\mathcal{O})} + k_2\|\mathbf{u}\|^2_{L_2(\mathcal{O})} \geq k_1\|\mathbf{D}\mathbf{u}\|^2_{L_2(\mathcal{O})}, \quad \mathbf{u} \in H^1(\mathcal{O};\mathbb{C}^n), \tag{9.2}$$

holds true.

Remark 9.3. The constants k_1 and k_2 depend on the matrix $b(\boldsymbol{\xi})$ and the domain $\mathcal{O}$, but in the general case it is difficult to control these constants explicitly. However, for particular operators they are often known. Therefore, in what follows we indicate the dependence of other constants on k_1 and k_2.

9.2. The operator $\mathcal{A}_{N,\varepsilon}$. In $L_2(\mathcal{O};\mathbb{C}^n)$, consider the operator $\mathcal{A}_{N,\varepsilon}$ formally given by the differential expression $b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})$ with the Neumann condition on $\partial\mathcal{O}$. More precisely, $\mathcal{A}_{N,\varepsilon}$ is the selfadjoint operator in $L_2(\mathcal{O};\mathbb{C}^n)$ generated by the quadratic form

$$a_{N,\varepsilon}[\mathbf{u},\mathbf{u}] := \int_{\mathcal{O}} \langle g^\varepsilon(\mathbf{x}) b(\mathbf{D})\mathbf{u}, b(\mathbf{D})\mathbf{u}\rangle \, d\mathbf{x}, \quad \mathbf{u} \in H^1(\mathcal{O};\mathbb{C}^n). \tag{9.3}$$

Relations (1.2) and (1.5) imply the upper estimate

$$a_{N,\varepsilon}[\mathbf{u},\mathbf{u}] \leq d\alpha_1 \|g\|_{L_\infty} \|\mathbf{D}\mathbf{u}\|^2_{L_2(\mathcal{O})}, \quad \mathbf{u} \in H^1(\mathcal{O};\mathbb{C}^n). \tag{9.4}$$

The lower estimate follows from (9.2):

$$a_{N,\varepsilon}[\mathbf{u},\mathbf{u}] \geq \|g^{-1}\|^{-1}_{L_\infty}\left(k_1\|\mathbf{D}\mathbf{u}\|^2_{L_2(\mathcal{O})} - k_2\|\mathbf{u}\|^2_{L_2(\mathcal{O})}\right), \quad \mathbf{u} \in H^1(\mathcal{O};\mathbb{C}^n). \tag{9.5}$$

By (9.3)–(9.5), the form (9.3) is closed and nonnegative.

The spectrum of the operator $\mathcal{A}_{N,\varepsilon}$ is contained in $\mathbb{R}_+$. The point $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$ is regular for this operator. *Our goal* in Chapter 3 is to approximate the generalized solution $\mathbf{u}_\varepsilon \in H^1(\mathcal{O};\mathbb{C}^n)$ of the Neumann problem

$$b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})\mathbf{u}_\varepsilon(\mathbf{x}) - \zeta\mathbf{u}_\varepsilon(\mathbf{x}) = \mathbf{F}(\mathbf{x}), \ \ \mathbf{x} \in \mathcal{O}; \quad \partial^\varepsilon_{\boldsymbol{\nu}}\mathbf{u}_\varepsilon|_{\partial\mathcal{O}} = 0, \tag{9.6}$$

for small ε. Here $\mathbf{F} \in L_2(\mathcal{O};\mathbb{C}^n)$. Then $\mathbf{u}_\varepsilon = (\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1}\mathbf{F}$. We have used the notation $\partial^\varepsilon_{\boldsymbol{\nu}}$ for the "conormal derivative". Let $\boldsymbol{\nu}(\mathbf{x})$ be the outer normal vector of unit length to $\partial\mathcal{O}$ at the point $\mathbf{x} \in \partial\mathcal{O}$. Then, formally, the conormal derivative is given by $\partial^\varepsilon_{\boldsymbol{\nu}}\mathbf{u}(\mathbf{x}) := b(\boldsymbol{\nu}(\mathbf{x}))^* g^\varepsilon(\mathbf{x}) b(\nabla)\mathbf{u}(\mathbf{x})$.

The following lemma is an analog of Lemma 3.1.

Lemma 9.4. *Let $\zeta \in \mathbb{C}\setminus\mathbb{R}_+$, $|\zeta| \geq 1$. Let $\mathbf{u}_\varepsilon$ be the generalized solution of problem* (9.6). *Then for $\varepsilon > 0$ we have*

$$\|\mathbf{u}_\varepsilon\|_{L_2(\mathcal{O})} \leq c(\varphi)|\zeta|^{-1}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{9.7}$$

$$\|\mathbf{D}\mathbf{u}_\varepsilon\|_{L_2(\mathcal{O})} \leq \mathfrak{C}_0 c(\varphi)|\zeta|^{-1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{9.8}$$

or, in operator terms,

$$\begin{aligned} &\|(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1}\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leq c(\varphi)|\zeta|^{-1}, \\ &\|\mathbf{D}(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1}\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leq \mathfrak{C}_0 c(\varphi)|\zeta|^{-1/2}. \end{aligned} \tag{9.9}$$

The constant $\mathfrak{C}_0$ depends only on $\|g^{-1}\|_{L_\infty}$ and the constants k_1 and k_2 from (9.2).

Proof. Since the norm of the resolvent $(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1}$ does not exceed the inverse distance from the point ζ to $\mathbb{R}_+$, we obtain (9.9). In order to check (9.8), we write the integral identity for the solution $\mathbf{u}_\varepsilon \in H^1(\mathcal{O};\mathbb{C}^n)$ of problem (9.6):

$$(g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon, b(\mathbf{D})\boldsymbol{\eta})_{L_2(\mathcal{O})} - \zeta(\mathbf{u}_\varepsilon, \boldsymbol{\eta})_{L_2(\mathcal{O})} = (\mathbf{F}, \boldsymbol{\eta})_{L_2(\mathcal{O})}, \quad \boldsymbol{\eta} \in H^1(\mathcal{O};\mathbb{C}^n). \tag{9.10}$$

Substituting $\boldsymbol{\eta} = \mathbf{u}_\varepsilon$ in (9.10) and taking (9.7) into account, we obtain

$$a_{N,\varepsilon}[\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon] \leq 2c(\varphi)^2|\zeta|^{-1}\|\mathbf{F}\|^2_{L_2(\mathcal{O})}.$$

Recalling (9.5) and (9.7), we arrive at (9.8) with $\mathfrak{C}_0 = (2k_1^{-1}\|g^{-1}\|_{L_\infty} + k_2 k_1^{-1})^{1/2}$. $\square$

9.3. The effective operator $\mathcal{A}_N^0$. In $L_2(\mathcal{O};\mathbb{C}^n)$, consider the selfadjoint operator $\mathcal{A}_N^0$ generated by the quadratic form

$$a_N^0[\mathbf{u}, \mathbf{u}] = \int_{\mathcal{O}} \langle g^0 b(\mathbf{D})\mathbf{u}, b(\mathbf{D})\mathbf{u}\rangle \, d\mathbf{x}, \quad \mathbf{u} \in H^1(\mathcal{O};\mathbb{C}^n). \tag{9.11}$$

Here g^0 is the effective matrix defined by (1.8). Taking (1.15) and (9.2) into account, we see that the form (9.11) satisfies estimates like (9.4) and (9.5) with the same constants. Since $\partial\mathcal{O} \in C^{1,1}$, the operator $\mathcal{A}_N^0$ is given by the differential expression $b(\mathbf{D})^* g^0 b(\mathbf{D})$ on the domain $\{\mathbf{u} \in H^2(\mathcal{O};\mathbb{C}^n) : \partial^0_{\boldsymbol{\nu}}\mathbf{u}|_{\partial\mathcal{O}} = 0\}$, where $\partial^0_{\boldsymbol{\nu}}$ is the conormal derivative corresponding to the operator $b(\mathbf{D})^* g^0 b(\mathbf{D})$, i.e., $\partial^0_{\boldsymbol{\nu}}\mathbf{u}(\mathbf{x}) = b(\boldsymbol{\nu}(\mathbf{x}))^* g^0 b(\nabla)\mathbf{u}(\mathbf{x})$. We have

$$\|(\mathcal{A}_N^0 + I)^{-1}\|_{L_2(\mathcal{O})\to H^2(\mathcal{O})} \leq c^\circ. \tag{9.12}$$

Here the constant c° depends only on the constants k_1, k_2 occurring in (9.2) and on α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the domain $\mathcal{O}$. To justify this, we refer the reader to theorems about regularity of solutions for strongly elliptic systems (see, e.g., [McL, Chapter 4]).

The following remark is similar to Remark 3.2.

Remark 9.5. Instead of condition $\partial\mathcal{O} \in C^{1,1}$, the following implicit condition could be imposed: a bounded domain $\mathcal{O}$ with Lipschitz boundary is such that estimate (9.12) holds true. For such a domain, the results of Chapter 3 remain valid. In the case of scalar elliptic operators, wide conditions on $\partial\mathcal{O}$ that ensure estimate (9.12) can be found in [KoE] and [MaSh, Chapter 7] (in particular, it suffices to assume that $\partial\mathcal{O} \in C^\alpha$, $\alpha > 3/2$).

Let $\mathbf{u}_0$ be the generalized solution of the problem

$$b(\mathbf{D})^* g^0 b(\mathbf{D})\mathbf{u}_0(\mathbf{x}) - \zeta\mathbf{u}_0(\mathbf{x}) = \mathbf{F}(\mathbf{x}), \ \mathbf{x} \in \mathcal{O}; \quad \partial^0_{\boldsymbol{\nu}}\mathbf{u}_0|_{\partial\mathcal{O}} = 0, \tag{9.13}$$

where $\mathbf{F} \in L_2(\mathcal{O};\mathbb{C}^n)$. Then $\mathbf{u}_0 = (\mathcal{A}_N^0 - \zeta I)^{-1}\mathbf{F}$.

Lemma 9.6. *Let $\zeta \in \mathbb{C}\backslash\mathbb{R}_+$, $|\zeta| \geq 1$. Let $\mathbf{u}_0$ be the generalized solution of problem* (9.13). *Then*

$$\|\mathbf{u}_0\|_{L_2(\mathcal{O})} \leq c(\varphi)|\zeta|^{-1}\|\mathbf{F}\|_{L_2(\mathcal{O})},$$

$$\|\mathbf{D}\mathbf{u}_0\|_{L_2(\mathcal{O})} \leq \mathfrak{C}_0 c(\varphi)|\zeta|^{-1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})},$$

$$\|\mathbf{u}_0\|_{H^2(\mathcal{O})} \leq 2c^\circ c(\varphi)\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{9.14}$$

or, in operator terms,

$$\|(\mathcal{A}_N^0 - \zeta I)^{-1}\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leq c(\varphi)|\zeta|^{-1}, \tag{9.15}$$

$$\|\mathbf{D}(\mathcal{A}_N^0 - \zeta I)^{-1}\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leq \mathfrak{C}_0 c(\varphi)|\zeta|^{-1/2}, \tag{9.16}$$

$$\|(\mathcal{A}_N^0 - \zeta I)^{-1}\|_{L_2(\mathcal{O})\to H^2(\mathcal{O})} \leq 2c^\circ c(\varphi). \tag{9.17}$$

Proof. Estimates (9.15) and (9.16) can be checked as in the proof of Lemma 9.4. Estimate (9.17) is a consequence of (9.12) and the inequality

$$\|(\mathcal{A}_N^0 + I)(\mathcal{A}_N^0 - \zeta I)^{-1}\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leq \sup_{x\geq 0}(x+1)|x-\zeta|^{-1} \leq 2c(\varphi),$$

which is valid for $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, $|\zeta| \geq 1$. □

§10. Results for the Neumann problem

10.1. Approximation of the resolvent $(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1}$ for $|\zeta| \geq 1$. Now we formulate our main results for the operator $\mathcal{A}_{N,\varepsilon}$.

Theorem 10.1. *Let $\mathcal{O}$ be a bounded domain of class $C^{1,1}$, and let $\zeta \in \mathbb{C}\backslash\mathbb{R}_+$ and $|\zeta| \geq 1$. Let $\mathbf{u}_\varepsilon$ be the solution of problem* (9.6), *and let $\mathbf{u}_0$ be the solution of problem* (9.13) *with $\mathbf{F} \in L_2(\mathcal{O};\mathbb{C}^n)$. Suppose that the number ε_1 satisfies Condition* 3.4. *Then for $0 < \varepsilon \leq \varepsilon_1$ we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O})} \leq \mathfrak{C}_1 c(\varphi)^5(|\zeta|^{-1/2}\varepsilon + \varepsilon^2)\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{10.1}$$

or, in operator terms,

$$\|(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1} - (\mathcal{A}_N^0 - \zeta I)^{-1}\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leq \mathfrak{C}_1 c(\varphi)^5(|\zeta|^{-1/2}\varepsilon + \varepsilon^2).$$

The constant $\mathfrak{C}_1$ depends on d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the constants k_1, k_2 from (9.2), *the parameters of the lattice Γ, and the domain $\mathcal{O}$.*

To approximate the solution in $H^1(\mathcal{O};\mathbb{C}^n)$, we introduce a corrector similar to (4.4):

$$K_N(\varepsilon;\zeta) = R_{\mathcal{O}}[\Lambda^\varepsilon]S_\varepsilon b(\mathbf{D})P_{\mathcal{O}}(\mathcal{A}_N^0 - \zeta I)^{-1}. \tag{10.2}$$

The operator $K_N(\varepsilon;\zeta)$ is a continuous mapping of $L_2(\mathcal{O};\mathbb{C}^n)$ to $H^1(\mathcal{O};\mathbb{C}^n)$.

Let $\mathbf{u}_0$ be the solution of problem (9.13). Denote $\widetilde{\mathbf{u}}_0 := P_{\mathcal{O}}\mathbf{u}_0$. As in (4.5) and (4.6), we put

$$\widetilde{\mathbf{v}}_\varepsilon(\mathbf{x}) = \widetilde{\mathbf{u}}_0(\mathbf{x}) + \varepsilon\Lambda^\varepsilon(\mathbf{x})(S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0)(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d,$$

$$\mathbf{v}_\varepsilon := \widetilde{\mathbf{v}}_\varepsilon|_{\mathcal{O}}. \tag{10.3}$$

Then

$$\mathbf{v}_\varepsilon = (\mathcal{A}_N^0 - \zeta I)^{-1}\mathbf{F} + \varepsilon K_N(\varepsilon;\zeta)\mathbf{F}. \tag{10.4}$$

Theorem 10.2. *Under the assumptions of Theorem* 10.1, *let $\mathbf{v}_\varepsilon$ be defined by* (10.2), (10.4). *Then for $0 < \varepsilon \leq \varepsilon_1$ we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon\|_{H^1(\mathcal{O})} \leq \big(\mathfrak{C}_2 c(\varphi)^2|\zeta|^{-1/4}\varepsilon^{1/2} + \mathfrak{C}_3 c(\varphi)^4\varepsilon\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{10.5}$$

or, in operator terms,

$$\big\|(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1} - (\mathcal{A}_N^0 - \zeta I)^{-1} - \varepsilon K_N(\varepsilon;\zeta)\big\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} \leq \mathfrak{C}_2 c(\varphi)^2|\zeta|^{-1/4}\varepsilon^{1/2} + \mathfrak{C}_3 c(\varphi)^4\varepsilon.$$

For the flux $\mathbf{p}_\varepsilon := g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$ *we have*

$$\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathcal{O})} \le \big(\widetilde{\mathfrak{C}}_2 c(\varphi)^2|\zeta|^{-1/4}\varepsilon^{1/2} + \widetilde{\mathfrak{C}}_3 c(\varphi)^4\varepsilon\big)\|\mathbf{F}\|_{L_2(\mathcal{O})} \tag{10.6}$$

for $0 < \varepsilon \le \varepsilon_1$. *The constants* $\mathfrak{C}_2$, $\mathfrak{C}_3$, $\widetilde{\mathfrak{C}}_2$, *and* $\widetilde{\mathfrak{C}}_3$ *depend on* d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the constants* k_1, k_2 *in* (9.2), *the parameters of the lattice* Γ, *and the domain* $\mathcal{O}$.

10.2. The first step of the proof. Associated problem in $\mathbb{R}^d$. As in Subsection 4.2, we start with the associated problem in $\mathbb{R}^d$. By Lemma 9.6 and (4.3),

$$\|\widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} \le C_{\mathcal{O}}^{(0)} c(\varphi)|\zeta|^{-1}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{10.7}$$

$$\|\widetilde{\mathbf{u}}_0\|_{H^1(\mathbb{R}^d)} \le C_{\mathcal{O}}^{(1)} (\mathfrak{C}_0 + 1)c(\varphi)|\zeta|^{-1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{10.8}$$

$$\|\widetilde{\mathbf{u}}_0\|_{H^2(\mathbb{R}^d)} \le 2C_{\mathcal{O}}^{(2)} c^\circ c(\varphi)\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{10.9}$$

We put

$$\widetilde{\mathbf{F}} := \mathcal{A}^0\widetilde{\mathbf{u}}_0 - \zeta\widetilde{\mathbf{u}}_0. \tag{10.10}$$

Then $\widetilde{\mathbf{F}} \in L_2(\mathbb{R}^d;\mathbb{C}^n)$ and $\widetilde{\mathbf{F}}|_{\mathcal{O}} = \mathbf{F}$. Like in (4.15), from (1.16), (10.7), and (10.9) it follows that

$$\|\widetilde{\mathbf{F}}\|_{L_2(\mathbb{R}^d)} \le \mathfrak{C}_4 c(\varphi)\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{10.11}$$

where $\mathfrak{C}_4 = 2c_1 C_{\mathcal{O}}^{(2)} c^\circ + C_{\mathcal{O}}^{(0)}$.

Let $\widetilde{\mathbf{u}}_\varepsilon \in H^1(\mathbb{R}^d;\mathbb{C}^n)$ be the solution of the following equation in $\mathbb{R}^d$:

$$\mathcal{A}_\varepsilon\widetilde{\mathbf{u}}_\varepsilon - \zeta\widetilde{\mathbf{u}}_\varepsilon = \widetilde{\mathbf{F}}, \tag{10.12}$$

i.e., $\widetilde{\mathbf{u}}_\varepsilon = (\mathcal{A}_\varepsilon - \zeta I)^{-1}\widetilde{\mathbf{F}}$. We can apply theorems of §2. Using Theorems 2.2, 2.4, 2.6 and relations (10.10)–(10.12), we see that, for $\varepsilon > 0$, we have

$$\|\widetilde{\mathbf{u}}_\varepsilon - \widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} \le \mathfrak{C}_4 C_1 c(\varphi)^3|\zeta|^{-1/2}\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{10.13}$$

$$\|\mathbf{D}(\widetilde{\mathbf{u}}_\varepsilon - \widetilde{\mathbf{v}}_\varepsilon)\|_{L_2(\mathbb{R}^d)} \le \mathfrak{C}_4 C_2 c(\varphi)^3\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{10.14}$$

$$\|\widetilde{\mathbf{u}}_\varepsilon - \widetilde{\mathbf{v}}_\varepsilon\|_{L_2(\mathbb{R}^d)} \le \mathfrak{C}_4 C_3 c(\varphi)^3|\zeta|^{-1/2}\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{10.15}$$

$$\|g^\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_\varepsilon - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} \le \mathfrak{C}_4 C_4 c(\varphi)^3\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{10.16}$$

10.3. The second step of the proof. Introduction of the correction term $\mathbf{w}_\varepsilon$. Now we introduce the "correction term" $\mathbf{w}_\varepsilon \in H^1(\mathcal{O};\mathbb{C}^n)$ as a function satisfying the integral identity

$$\begin{aligned}&(g^\varepsilon b(\mathbf{D})\mathbf{w}_\varepsilon, b(\mathbf{D})\boldsymbol{\eta})_{L_2(\mathcal{O})} - \zeta(\mathbf{w}_\varepsilon,\boldsymbol{\eta})_{L_2(\mathcal{O})}\\ &\quad = (\widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0, b(\mathbf{D})\boldsymbol{\eta})_{L_2(\mathcal{O})} - (\zeta\mathbf{u}_0 + \mathbf{F},\boldsymbol{\eta})_{L_2(\mathcal{O})}, \ \ \boldsymbol{\eta} \in H^1(\mathcal{O};\mathbb{C}^n).\end{aligned} \tag{10.17}$$

The right-hand side of (10.17) is an antilinear continuous functional of $\boldsymbol{\eta} \in H^1(\mathcal{O};\mathbb{C}^n)$; therefore, we can use a standard argument to show that the solution $\mathbf{w}_\varepsilon$ exists and is unique.

Lemma 10.3. *Let* $\zeta \in \mathbb{C}\setminus\mathbb{R}_+$ *and* $|\zeta| \ge 1$. *Let* $\mathbf{u}_\varepsilon$ *be the solution of problem* (9.6), *and let* $\widetilde{\mathbf{u}}_\varepsilon$ *be the solution of equation* (10.12). *Suppose that* $\mathbf{w}_\varepsilon$ *satisfies* (10.17). *Then for* $\varepsilon > 0$ *we have*

$$\|\mathbf{D}(\mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon + \mathbf{w}_\varepsilon)\|_{L_2(\mathcal{O})} \le \mathfrak{C}_5 c(\varphi)^4\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{10.18}$$

$$\|\mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon + \mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})} \le \mathfrak{C}_6 c(\varphi)^4|\zeta|^{-1/2}\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{10.19}$$

The constants $\mathfrak{C}_5$ *and* $\mathfrak{C}_6$ *depend on* d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the constants* k_1, k_2 *in* (9.2), *the parameters of the lattice* Γ, *and the domain* $\mathcal{O}$.

Proof. Denote $\mathbf{U}_\varepsilon := \mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon + \mathbf{w}_\varepsilon$. By (9.10) and (10.17), the function $\mathbf{U}_\varepsilon \in H^1(\mathcal{O};\mathbb{C}^n)$ satisfies the identity

$$\begin{aligned}(g^\varepsilon b(\mathbf{D})\mathbf{U}_\varepsilon, b(\mathbf{D})\boldsymbol{\eta})_{L_2(\mathcal{O})} - \zeta(\mathbf{U}_\varepsilon,\boldsymbol{\eta})_{L_2(\mathcal{O})} \\ = -(g^\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_\varepsilon - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0, b(\mathbf{D})\boldsymbol{\eta})_{L_2(\mathcal{O})} + \zeta(\widetilde{\mathbf{u}}_\varepsilon - \mathbf{u}_0,\boldsymbol{\eta})_{L_2(\mathcal{O})}, \\ \boldsymbol{\eta}\in H^1(\mathcal{O};\mathbb{C}^n).\end{aligned} \tag{10.20}$$

From (10.13) and (10.16) it follows that the right-hand side of (10.20) does not exceed $\mathfrak{C}_4 c(\varphi)^3\varepsilon\left(C_4\|b(\mathbf{D})\boldsymbol{\eta}\|_{L_2(\mathcal{O})} + C_1|\zeta|^{1/2}\|\boldsymbol{\eta}\|_{L_2(\mathcal{O})}\right)\|\mathbf{F}\|_{L_2(\mathcal{O})}$.

Like in (4.26)–(4.29), we can substitute $\boldsymbol{\eta} = \mathbf{U}_\varepsilon$ in (10.20) to deduce the inequality

$$\begin{aligned}\|\mathbf{U}_\varepsilon\|^2_{L_2(\mathcal{O})} \le 4\mathfrak{C}_4 C_4 c(\varphi)^4|\zeta|^{-1}\varepsilon\|b(\mathbf{D})\mathbf{U}_\varepsilon\|_{L_2(\mathcal{O})}\|\mathbf{F}\|_{L_2(\mathcal{O})} \\ + 4\mathfrak{C}_4^2 C_1^2 c(\varphi)^8|\zeta|^{-1}\varepsilon^2\|\mathbf{F}\|^2_{L_2(\mathcal{O})}.\end{aligned} \tag{10.21}$$

Now, combining (10.20) with $\boldsymbol{\eta} = \mathbf{U}_\varepsilon$ and (10.21), we obtain

$$\|b(\mathbf{D})\mathbf{U}_\varepsilon\|_{L_2(\mathcal{O})} \le \widetilde{\mathfrak{C}}_5 c(\varphi)^4\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{10.22}$$

where $\widetilde{\mathfrak{C}}_5^2 = 9\mathfrak{C}_4^2(2C_1^2\|g^{-1}\|_{L_\infty} + 9C_4^2\|g^{-1}\|^2_{L_\infty})$. Relations (10.21) and (10.22) imply (10.19) with the constant $\mathfrak{C}_6 = 2(\mathfrak{C}_4\widetilde{\mathfrak{C}}_5 C_4 + \mathfrak{C}_4^2 C_1^2)^{1/2}$. Finally, combining (9.2), (10.19), and (10.22), we arrive at (10.18) with $\mathfrak{C}_5^2 = k_1^{-1}(\widetilde{\mathfrak{C}}_5^2 + k_2\mathfrak{C}_6^2)$. □

Conclusions. 1) From (10.3), (10.14), (10.15), (10.18), and (10.19) it follows that

$$\|\mathbf{D}(\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon)\|_{L_2(\mathcal{O})} \le \mathfrak{C}_7 c(\varphi)^4\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})} + \|\mathbf{D}\mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})}, \quad \varepsilon > 0, \tag{10.23}$$

$$\|\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon\|_{L_2(\mathcal{O})} \le \mathfrak{C}_8 c(\varphi)^4|\zeta|^{-1/2}\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})} + \|\mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})}, \quad \varepsilon > 0, \tag{10.24}$$

where $\mathfrak{C}_7 = \mathfrak{C}_4 C_2 + \mathfrak{C}_5$ and $\mathfrak{C}_8 = \mathfrak{C}_4 C_3 + \mathfrak{C}_6$. Thus, in order to prove Theorem 10.2, we need to obtain an appropriate estimate for $\|\mathbf{w}_\varepsilon\|_{H^1(\mathcal{O})}$.

2) From (10.13) and (10.19) it follows that

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O})} \le \mathfrak{C}_9 c(\varphi)^4|\zeta|^{-1/2}\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})} + \|\mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})}, \quad \varepsilon > 0, \tag{10.25}$$

where $\mathfrak{C}_9 = \mathfrak{C}_6 + \mathfrak{C}_4 C_1$. Hence, for the proof of Theorem 10.1 we need to find an appropriate estimate for $\|\mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})}$.

§11. Estimates of the correction term. Proof of Theorems 10.1 and 10.2

As in §5, first we estimate the H^1-norm of $\mathbf{w}_\varepsilon$ and prove Theorem 10.2. Next, using the already proved Theorem 10.2 and duality arguments, we estimate the L_2-norm of $\mathbf{w}_\varepsilon$ and prove Theorem 10.1.

11.1. Estimation of the norm of $\mathbf{w}_\varepsilon$ in $H^1(\mathcal{O};\mathbb{C}^n)$. Denote

$$\mathcal{I}_\varepsilon[\boldsymbol{\eta}] := (\widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0 - g^0 b(\mathbf{D})\mathbf{u}_0, b(\mathbf{D})\boldsymbol{\eta})_{L_2(\mathcal{O})}, \quad \boldsymbol{\eta}\in H^1(\mathcal{O};\mathbb{C}^n). \tag{11.1}$$

Note that the solution $\mathbf{u}_0$ of problem (9.13) satisfies the identity

$$(g^0 b(\mathbf{D})\mathbf{u}_0, b(\mathbf{D})\boldsymbol{\eta})_{L_2(\mathcal{O})} - (\zeta\mathbf{u}_0 + \mathbf{F},\boldsymbol{\eta})_{L_2(\mathcal{O})} = 0, \quad \boldsymbol{\eta}\in H^1(\mathcal{O};\mathbb{C}^n). \tag{11.2}$$

By (10.17) and (11.2), $\mathbf{w}_\varepsilon$ satisfies the identity

$$(g^\varepsilon b(\mathbf{D})\mathbf{w}_\varepsilon, b(\mathbf{D})\boldsymbol{\eta})_{L_2(\mathcal{O})} - \zeta(\mathbf{w}_\varepsilon,\boldsymbol{\eta})_{L_2(\mathcal{O})} = \mathcal{I}_\varepsilon[\boldsymbol{\eta}], \quad \boldsymbol{\eta}\in H^1(\mathcal{O};\mathbb{C}^n). \tag{11.3}$$

Lemma 11.1. *Let $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$ and $|\zeta| \geq 1$, and let ε_1 be the number defined in Condition 3.4. Then for $0 < \varepsilon \leq \varepsilon_1$ the functional (11.1) satisfies the following estimate:*

$$|\mathcal{I}_\varepsilon[\boldsymbol{\eta}]| \leq c(\varphi)\big(\mathfrak{C}_{10}|\zeta|^{-1/4}\varepsilon^{1/2} + \mathfrak{C}_{11}\varepsilon\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\mathbf{D}\boldsymbol{\eta}\|_{L_2(\mathcal{O})}, \quad \boldsymbol{\eta} \in H^1(\mathcal{O};\mathbb{C}^n). \tag{11.4}$$

The constants $\mathfrak{C}_{10}$ and $\mathfrak{C}_{11}$ depend on d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the constants k_1, k_2 in (9.2), the parameters of the lattice Γ, and the domain $\mathcal{O}$.

Proof. The functional (11.1) can be represented as

$$\mathcal{I}_\varepsilon[\boldsymbol{\eta}] = \mathcal{I}_\varepsilon^{(1)}[\boldsymbol{\eta}] + \mathcal{I}_\varepsilon^{(2)}[\boldsymbol{\eta}], \tag{11.5}$$

$$\mathcal{I}_\varepsilon^{(1)}[\boldsymbol{\eta}] := (g^0 S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0 - g^0 b(\mathbf{D})\mathbf{u}_0, b(\mathbf{D})\boldsymbol{\eta})_{L_2(\mathcal{O})}, \tag{11.6}$$

$$\mathcal{I}_\varepsilon^{(2)}[\boldsymbol{\eta}] := ((\widetilde{g}^\varepsilon - g^0)S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0, b(\mathbf{D})\boldsymbol{\eta})_{L_2(\mathcal{O})}. \tag{11.7}$$

The term (11.6) is estimated with the help of Proposition 1.4 and relations (1.2), (1.4), (1.5), (1.15), and (10.9):

$$\begin{aligned} |\mathcal{I}_\varepsilon^{(1)}[\boldsymbol{\eta}]| &\leq |g^0|\|(S_\varepsilon - I)b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)}\|b(\mathbf{D})\boldsymbol{\eta}\|_{L_2(\mathcal{O})} \\ &\leq \mathfrak{C}^{(1)}c(\varphi)\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\mathbf{D}\boldsymbol{\eta}\|_{L_2(\mathcal{O})}, \quad \varepsilon > 0, \end{aligned} \tag{11.8}$$

where $\mathfrak{C}^{(1)} = 2\|g\|_{L_\infty} r_1 \alpha_1 d^{1/2} C_{\mathcal{O}}^{(2)} c^\circ$.

Using (1.2), we transform the term (11.7):

$$\mathcal{I}_\varepsilon^{(2)}[\boldsymbol{\eta}] = \sum_{l=1}^d (f_l^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0, D_l\boldsymbol{\eta})_{L_2(\mathcal{O})}, \tag{11.9}$$

where $f_l(\mathbf{x}) := b_l^*(\widetilde{g}(\mathbf{x}) - g^0)$, $l = 1, \dots, d$. In accordance with [Su3, (5.13)], we have

$$\begin{aligned} &\|f_l\|_{L_2(\Omega)} \leq |\Omega|^{1/2}\check{\mathfrak{C}}, \quad l = 1, \dots, d, \\ &\check{\mathfrak{C}} = \alpha_1^{1/2}\|g\|_{L_\infty}(1 + (dm)^{1/2}\alpha_1^{1/2}\alpha_0^{-1/2}\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty}^{1/2}). \end{aligned} \tag{11.10}$$

As was checked in [Su3, Subsection 5.2], there exist Γ-periodic $(n \times m)$-matrix-valued functions $M_{lj}(\mathbf{x})$ in $\mathbb{R}^d$, $l, j = 1, \dots, d$, such that

$$\begin{aligned} &M_{lj} \in \widetilde{H}^1(\Omega), \quad \int_\Omega M_{lj}(\mathbf{x})\,d\mathbf{x} = 0, \quad M_{lj}(\mathbf{x}) = -M_{jl}(\mathbf{x}), \quad l, j = 1, \dots, d, \\ &f_l(\mathbf{x}) = \sum_{j=1}^d \partial_j M_{lj}(\mathbf{x}), \quad l = 1, \dots, d, \end{aligned} \tag{11.11}$$

and

$$\|M_{lj}\|_{L_2(\Omega)} \leq r_0^{-1}|\Omega|^{1/2}\check{\mathfrak{C}}, \quad l, j = 1, \dots, d. \tag{11.12}$$

By (11.11), we have $f_l^\varepsilon(\mathbf{x}) = \varepsilon\sum_{j=1}^d \partial_j M_{lj}^\varepsilon(\mathbf{x})$, $l = 1, \dots, d$, whence the term (11.9) can be written as

$$\mathcal{I}_\varepsilon^{(2)}[\boldsymbol{\eta}] = \widetilde{\mathcal{I}}_\varepsilon^{(2)}[\boldsymbol{\eta}] + \widehat{\mathcal{I}}_\varepsilon^{(2)}[\boldsymbol{\eta}], \tag{11.13}$$

$$\widetilde{\mathcal{I}}_\varepsilon^{(2)}[\boldsymbol{\eta}] := \varepsilon \sum_{l,j=1}^d \big(\partial_j(M_{lj}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0), D_l\boldsymbol{\eta}\big)_{L_2(\mathcal{O})}, \tag{11.14}$$

$$\widehat{\mathcal{I}}_\varepsilon^{(2)}[\boldsymbol{\eta}] := -\varepsilon \sum_{l,j=1}^d \big(M_{lj}^\varepsilon S_\varepsilon b(\mathbf{D})\partial_j\widetilde{\mathbf{u}}_0, D_l\boldsymbol{\eta}\big)_{L_2(\mathcal{O})}. \tag{11.15}$$

The term (11.15) is estimated by using Proposition 1.5 and (1.4), (10.9), (11.12):

$$\begin{aligned}(11.16)\qquad |\widehat{\mathcal{I}}_\varepsilon^{(2)}[\boldsymbol{\eta}]| &\leq \varepsilon \sum_{l,j=1}^{d} |\Omega|^{-1/2} \|M_{lj}\|_{L_2(\Omega)} \|b(\mathbf{D})\partial_j \widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} \|D_l \boldsymbol{\eta}\|_{L_2(\mathcal{O})} \\ &\leq \mathfrak{C}^{(2)} c(\varphi)\varepsilon \|\mathbf{F}\|_{L_2(\mathcal{O})} \|\mathbf{D}\boldsymbol{\eta}\|_{L_2(\mathcal{O})}, \quad \varepsilon > 0,\end{aligned}$$

where $\mathfrak{C}^{(2)} = 2dr_0^{-1}\check{\mathfrak{C}}\alpha_1^{1/2} C_{\mathcal{O}}^{(2)} c^\circ$.

Now, we address the term (11.14). Suppose that $0 < \varepsilon \leq \varepsilon_1$. Let θ_ε be a cut-off function satisfying (5.1). We have

$$\sum_{l,j=1}^{d} \big(\partial_j((1-\theta_\varepsilon)M_{lj}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0), D_l \boldsymbol{\eta}\big)_{L_2(\mathcal{O})} = 0, \quad \boldsymbol{\eta} \in H^1(\mathcal{O};\mathbb{C}^n),$$

which can be checked by integration by parts and using the fact that $M_{lj} = -M_{jl}$. Consequently, the term (11.14) can be written as

$$(11.17)\qquad \widetilde{\mathcal{I}}_\varepsilon^{(2)}[\boldsymbol{\eta}] = \sum_{l=1}^{d} \left(\boldsymbol{\psi}_l(\varepsilon), D_l \boldsymbol{\eta}\right)_{L_2(\mathcal{O})},$$

where $\boldsymbol{\psi}_l(\varepsilon) := \varepsilon \sum_{j=1}^{d} \partial_j(\theta_\varepsilon M_{lj}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0)$. We represent $\boldsymbol{\psi}_l(\varepsilon)$ as

$$\boldsymbol{\psi}_l(\varepsilon) = \varepsilon\theta_\varepsilon \sum_{j=1}^{d} M_{lj}^\varepsilon S_\varepsilon (b(\mathbf{D})\partial_j \widetilde{\mathbf{u}}_0) + \varepsilon \sum_{j=1}^{d} (\partial_j \theta_\varepsilon) M_{lj}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0 + \theta_\varepsilon f_l^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0.$$

Then

$$(11.18)\qquad \|\boldsymbol{\psi}_l(\varepsilon)\|_{L_2(\mathcal{O})} \leq J_l^{(1)}(\varepsilon) + J_l^{(2)}(\varepsilon) + J_l^{(3)}(\varepsilon),$$

$$(11.19)\qquad J_l^{(1)}(\varepsilon) := \varepsilon \sum_{j=1}^{d} \|\theta_\varepsilon M_{lj}^\varepsilon S_\varepsilon (b(\mathbf{D})\partial_j \widetilde{\mathbf{u}}_0)\|_{L_2(\mathbb{R}^d)},$$

$$(11.20)\qquad J_l^{(2)}(\varepsilon) := \varepsilon \sum_{j=1}^{d} \|(\partial_j \theta_\varepsilon) M_{lj}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)},$$

$$(11.21)\qquad J_l^{(3)}(\varepsilon) := \|\theta_\varepsilon f_l^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)}.$$

To estimate the term (11.19), we apply (5.1), Proposition 1.5, and (1.4), (10.9), (11.12):

$$(11.22)\qquad J_l^{(1)}(\varepsilon) \leq \varepsilon \sum_{j=1}^{d} |\Omega|^{-1/2} \|M_{lj}\|_{L_2(\Omega)} \|b(\mathbf{D})\partial_j \widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} \leq \nu_1 c(\varphi)\varepsilon \|\mathbf{F}\|_{L_2(\mathcal{O})},$$

where $\nu_1 = 2r_0^{-1}\check{\mathfrak{C}}(d\alpha_1)^{1/2} C_{\mathcal{O}}^{(2)} c^\circ$. The term (11.20) is estimated with the help of (5.1) and Lemma 3.6:

$$\begin{aligned}(J_l^{(2)}(\varepsilon))^2 &\leq d\kappa^2 \sum_{j=1}^{d} \int_{(\partial\mathcal{O})_\varepsilon} |M_{lj}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0|^2 \, d\mathbf{x} \\ &\leq \varepsilon d\kappa^2 \beta_* |\Omega|^{-1} \sum_{j=1}^{d} \|M_{lj}\|_{L_2(\Omega)}^2 \|b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{H^1(\mathbb{R}^d)} \|b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)}, \\ &\qquad 0 < \varepsilon \leq \varepsilon_1.\end{aligned}$$

Combining this with relations (1.4), (10.8), (10.9), and (11.12), we obtain

$$(11.23)\qquad J_l^{(2)}(\varepsilon) \leq \nu_2 c(\varphi)|\zeta|^{-1/4}\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \quad 0 < \varepsilon \leq \varepsilon_1,$$

where $\nu_2 = d\kappa r_0^{-1}\check{\mathfrak{C}}\big(2\beta_*\alpha_1 C_{\mathcal{O}}^{(2)} C_{\mathcal{O}}^{(1)} c^\circ(\mathfrak{C}_0+1)\big)^{1/2}$. Similarly, using (5.1) and Lemma 3.6, we see that the term (11.21) satisfies

$$(J_l^{(3)}(\varepsilon))^2 \le \int_{(\partial\mathcal{O})_\varepsilon} |f_l^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0|^2\, d\mathbf{x} \le \varepsilon\beta_*|\Omega|^{-1}\|f_l\|^2_{L_2(\Omega)}\|b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{H^1(\mathbb{R}^d)}\|b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)}$$

for $0 < \varepsilon \le \varepsilon_1$. Taking (1.4), (10.8), (10.9), and (11.10) into account, we deduce the inequality

$$J_l^{(3)}(\varepsilon) \le \nu_3 c(\varphi)|\zeta|^{-1/4}\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \quad 0<\varepsilon\le\varepsilon_1, \tag{11.24}$$

where $\nu_3 = \check{\mathfrak{C}}\big(2\beta_*\alpha_1 C_{\mathcal{O}}^{(2)} C_{\mathcal{O}}^{(1)} c^\circ(\mathfrak{C}_0+1)\big)^{1/2}$.

Now, relations (11.18), (11.22)–(11.24) lead to the estimate

$$\|\boldsymbol{\psi}_l(\varepsilon)\|_{L_2(\mathcal{O})} \le c(\varphi)\big(\nu_1\varepsilon + (\nu_2+\nu_3)|\zeta|^{-1/4}\varepsilon^{1/2}\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}, \quad 0<\varepsilon\le\varepsilon_1. \tag{11.25}$$

Combining (11.17) and (11.25), we see that

$$|\widetilde{\mathcal{I}}_\varepsilon^{(2)}[\boldsymbol{\eta}]| \le d^{1/2} c(\varphi)\big(\nu_1\varepsilon + (\nu_2+\nu_3)|\zeta|^{-1/4}\varepsilon^{1/2}\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\mathbf{D}\boldsymbol{\eta}\|_{L_2(\mathcal{O})} \tag{11.26}$$

for $0 < \varepsilon \le \varepsilon_1$. As a result, relations (11.5), (11.8), (11.13), (11.16), and (11.26) imply the required estimate (11.4) with the constants $\mathfrak{C}_{10} = d^{1/2}(\nu_2+\nu_3)$ and $\mathfrak{C}_{11} = \mathfrak{C}^{(1)} + \mathfrak{C}^{(2)} + d^{1/2}\nu_1$. □

Lemma 11.2. *Let $\zeta \in \mathbb{C}\setminus\mathbb{R}_+$ and $|\zeta|\ge 1$. Suppose that $\mathbf{w}_\varepsilon$ satisfies* (10.17). *If ε_1 is the number occurring in Condition* 3.4, *then for $0<\varepsilon\le\varepsilon_1$ we have*

$$\|\mathbf{w}_\varepsilon\|_{H^1(\mathcal{O})} \le c(\varphi)^2\big(\mathfrak{C}_{12}|\zeta|^{-1/4}\varepsilon^{1/2} + \mathfrak{C}_{13}\varepsilon\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{11.27}$$

The constants $\mathfrak{C}_{12}$ and $\mathfrak{C}_{13}$ depend on d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the constants k_1, k_2 in (9.2), *the parameters of the lattice Γ, and the domain $\mathcal{O}$.*

Proof. We substitute $\boldsymbol{\eta} = \mathbf{w}_\varepsilon$ in (11.3) and calculate the imaginary part of the corresponding relation. Taking (11.4) into account, for $0<\varepsilon\le\varepsilon_1$ we have

$$|\operatorname{Im}\zeta|\|\mathbf{w}_\varepsilon\|^2_{L_2(\mathcal{O})} \le c(\varphi)\big(\mathfrak{C}_{10}|\zeta|^{-1/4}\varepsilon^{1/2} + \mathfrak{C}_{11}\varepsilon\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\mathbf{D}\mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})}. \tag{11.28}$$

If $\operatorname{Re}\zeta\ge 0$ (and then $\operatorname{Im}\zeta\ne 0$), we get

$$\|\mathbf{w}_\varepsilon\|^2_{L_2(\mathcal{O})} \le c(\varphi)^2|\zeta|^{-1}\big(\mathfrak{C}_{10}|\zeta|^{-1/4}\varepsilon^{1/2} + \mathfrak{C}_{11}\varepsilon\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\mathbf{D}\mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})}. \tag{11.29}$$

If $\operatorname{Re}\zeta<0$, we take the real part of the corresponding relation, obtaining

$$|\operatorname{Re}\zeta|\|\mathbf{w}_\varepsilon\|^2_{L_2(\mathcal{O})} \le c(\varphi)\big(\mathfrak{C}_{10}|\zeta|^{-1/4}\varepsilon^{1/2} + \mathfrak{C}_{11}\varepsilon\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\mathbf{D}\mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})} \tag{11.30}$$

for $0<\varepsilon\le\varepsilon_1$. Adding (11.28) and (11.30), we deduce an inequality similar to (11.29). As a result, for all values of ζ under consideration we have

$$\|\mathbf{w}_\varepsilon\|^2_{L_2(\mathcal{O})} \le 2c(\varphi)^2|\zeta|^{-1}\big(\mathfrak{C}_{10}|\zeta|^{-1/4}\varepsilon^{1/2} + \mathfrak{C}_{11}\varepsilon\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\mathbf{D}\mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})} \tag{11.31}$$

for $0<\varepsilon\le\varepsilon_1$. Now from (11.3) with $\boldsymbol{\eta}=\mathbf{w}_\varepsilon$, (11.4), and (11.31) it follows that

$$a_{N,\varepsilon}[\mathbf{w}_\varepsilon,\mathbf{w}_\varepsilon] \le 3c(\varphi)^2\big(\mathfrak{C}_{10}|\zeta|^{-1/4}\varepsilon^{1/2} + \mathfrak{C}_{11}\varepsilon\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\mathbf{D}\mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})}$$

for $0<\varepsilon\le\varepsilon_1$. Together with (9.5) and (11.31) this implies

$$\|\mathbf{D}\mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})} \le c(\varphi)^2\mathfrak{C}_{14}\big(\mathfrak{C}_{10}|\zeta|^{-1/4}\varepsilon^{1/2} + \mathfrak{C}_{11}\varepsilon\big)\|\mathbf{F}\|_{L_2(\mathcal{O})} \tag{11.32}$$

for $0<\varepsilon\le\varepsilon_1$, where $\mathfrak{C}_{14} = k_1^{-1}(3\|g^{-1}\|_{L_\infty} + 2k_2)$. Comparing (11.31) and (11.32), we arrive at (11.27) with $\mathfrak{C}_{12} = (2\mathfrak{C}_{14}+\mathfrak{C}_{14}^2)^{1/2}\mathfrak{C}_{10}$ and $\mathfrak{C}_{13} = (2\mathfrak{C}_{14}+\mathfrak{C}_{14}^2)^{1/2}\mathfrak{C}_{11}$. □

11.2. Completion of the proof of Theorem 10.2. Relations (10.23), (10.24), and (11.27) imply the required estimate (10.5) with the constants $\mathfrak{C}_2 = \sqrt{2}\mathfrak{C}_{12}$ and $\mathfrak{C}_3 = \mathfrak{C}_7 + \mathfrak{C}_8 + \sqrt{2}\mathfrak{C}_{13}$.

Arguing as in (5.19)–(5.22), one can check an analog of estimate (5.22) with the same constant. Together with (10.9), (10.5), (1.2), and (1.5), this implies (10.6) with the constants $\widetilde{\mathfrak{C}}_2 = \|g\|_{L_\infty}(d\alpha_1)^{1/2}\mathfrak{C}_2$ and $\widetilde{\mathfrak{C}}_3 = \|g\|_{L_\infty}(d\alpha_1)^{1/2}\mathfrak{C}_3 + \mathfrak{C}_{15}$, where $\mathfrak{C}_{15} = 2(\mathcal{C}' + \mathcal{C}'')C_{\mathcal{O}}^{(2)}c^\circ$. $\square$

11.3. Proof of Theorem 10.1. Now, we estimate the L_2-norm of the correction term $\mathbf{w}_\varepsilon$.

Lemma 11.3. *Under the assumptions of Lemma* 11.2, *for* $0 < \varepsilon \le \varepsilon_1$ *we have*

$$\|\mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})} \le c(\varphi)^5\big(\mathfrak{C}_{16}|\zeta|^{-1/2}\varepsilon + \mathfrak{C}_{17}\varepsilon^2\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{11.33}$$

The constants $\mathfrak{C}_{16}$ *and* $\mathfrak{C}_{17}$ *depend on* d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the constants* k_1, k_2 *in* (9.2), *the parameters of the lattice* Γ, *and the domain* $\mathcal{O}$.

Proof. Consider identity (11.3) with the test function $\boldsymbol{\eta}_\varepsilon = (\mathcal{A}_{N,\varepsilon} - \bar{\zeta}I)^{-1}\mathbf{\Phi}$, where $\mathbf{\Phi} \in L_2(\mathcal{O};\mathbb{C}^n)$. Then the left-hand side of (11.3) coincides with $(\mathbf{w}_\varepsilon, \mathbf{\Phi})_{L_2(\mathcal{O})}$, and the identity can be written as

$$(\mathbf{w}_\varepsilon, \mathbf{\Phi})_{L_2(\mathcal{O})} = \mathcal{I}_\varepsilon[\boldsymbol{\eta}_\varepsilon]. \tag{11.34}$$

Assume that $0 < \varepsilon \le \varepsilon_1$. To estimate $\mathcal{I}_\varepsilon[\boldsymbol{\eta}_\varepsilon]$, we use relations (11.5), (11.8), (11.13), and (11.16). We have

$$|\mathcal{I}_\varepsilon[\boldsymbol{\eta}_\varepsilon]| \le (\mathfrak{C}^{(1)} + \mathfrak{C}^{(2)})c(\varphi)\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\mathbf{D}\boldsymbol{\eta}_\varepsilon\|_{L_2(\mathcal{O})} + |\widetilde{\mathcal{I}}_\varepsilon^{(2)}[\boldsymbol{\eta}_\varepsilon]|.$$

Applying Lemma 9.4 to estimate $\|\mathbf{D}\boldsymbol{\eta}_\varepsilon\|_{L_2(\mathcal{O})}$, we obtain

$$|\mathcal{I}_\varepsilon[\boldsymbol{\eta}_\varepsilon]| \le \mathfrak{C}_0(\mathfrak{C}^{(1)} + \mathfrak{C}^{(2)})c(\varphi)^2|\zeta|^{-1/2}\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\mathbf{\Phi}\|_{L_2(\mathcal{O})} + |\widetilde{\mathcal{I}}_\varepsilon^{(2)}[\boldsymbol{\eta}_\varepsilon]|. \tag{11.35}$$

Consider the term $\widetilde{\mathcal{I}}_\varepsilon^{(2)}[\boldsymbol{\eta}_\varepsilon]$. By (11.17), we have

$$\widetilde{\mathcal{I}}_\varepsilon^{(2)}[\boldsymbol{\eta}_\varepsilon] = \sum_{l=1}^d \left(\boldsymbol{\psi}_l(\varepsilon), D_l\boldsymbol{\eta}_\varepsilon\right)_{L_2(\mathcal{O})}. \tag{11.36}$$

To approximate the function $\boldsymbol{\eta}_\varepsilon$, we apply the already proved Theorem 10.2. We put $\boldsymbol{\eta}_0 = (\mathcal{A}_N^0 - \bar{\zeta}I)^{-1}\mathbf{\Phi}$ and $\widetilde{\boldsymbol{\eta}}_0 = P_{\mathcal{O}}\boldsymbol{\eta}_0$. The approximation of $\boldsymbol{\eta}_\varepsilon$ is given by $\boldsymbol{\eta}_0 + \varepsilon\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\boldsymbol{\eta}}_0$. We rewrite the term (11.36) as

$$\widetilde{\mathcal{I}}_\varepsilon^{(2)}[\boldsymbol{\eta}_\varepsilon] = \mathcal{J}_1(\varepsilon) + \mathcal{J}_2(\varepsilon) + \mathcal{J}_3(\varepsilon), \tag{11.37}$$

$$\mathcal{J}_1(\varepsilon) := \sum_{l=1}^d \left(\boldsymbol{\psi}_l(\varepsilon), D_l(\boldsymbol{\eta}_\varepsilon - \boldsymbol{\eta}_0 - \varepsilon\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\boldsymbol{\eta}}_0)\right)_{L_2(\mathcal{O})},$$

$$\mathcal{J}_2(\varepsilon) := \sum_{l=1}^d \left(\boldsymbol{\psi}_l(\varepsilon), D_l\boldsymbol{\eta}_0\right)_{L_2(\mathcal{O})}, \tag{11.38}$$

$$\mathcal{J}_3(\varepsilon) := \sum_{l=1}^d \left(\boldsymbol{\psi}_l(\varepsilon), D_l(\varepsilon\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\boldsymbol{\eta}}_0)\right)_{L_2(\mathcal{O})}. \tag{11.39}$$

Applying Theorem 10.2 and (11.25), we see that

$$|\mathcal{J}_1(\varepsilon)| \le c(\varphi)\big(\nu_1\varepsilon + (\nu_2 + \nu_3)|\zeta|^{-1/4}\varepsilon^{1/2}\big) \times d^{1/2}\big(\mathfrak{C}_2 c(\varphi)^2|\zeta|^{-1/4}\varepsilon^{1/2} + \mathfrak{C}_3 c(\varphi)^4\varepsilon\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\mathbf{\Phi}\|_{L_2(\mathcal{O})}.$$

Hence,

$$|\mathcal{J}_1(\varepsilon)| \le c(\varphi)^5\big(\breve{\nu}_1|\zeta|^{-1/2}\varepsilon + \breve{\nu}_2\varepsilon^2\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\mathbf{\Phi}\|_{L_2(\mathcal{O})}, \tag{11.40}$$

where

$$\breve{\nu}_1 = d^{1/2}\big((\nu_1+\nu_2+\nu_3)\mathfrak{C}_2 + (\nu_2+\nu_3)\mathfrak{C}_3\big),$$
$$\breve{\nu}_2 = d^{1/2}\big(\nu_1(\mathfrak{C}_2+\mathfrak{C}_3) + (\nu_2+\nu_3)\mathfrak{C}_3\big).$$

By (5.1) and (11.25), the term (11.38) satisfies the estimate

$$|\mathcal{J}_2(\varepsilon)| \le c(\varphi)\big(\nu_1\varepsilon + (\nu_2+\nu_3)|\zeta|^{-1/4}\varepsilon^{1/2}\big)\|\mathbf{F}\|_{L_2(\mathcal{O})} d^{1/2}\left(\int_{B_\varepsilon}|\mathbf{D}\boldsymbol{\eta}_0|^2\,d\mathbf{x}\right)^{1/2}.$$

Applying Lemma 3.5(1°) and Lemma 9.6, we obtain

$$|\mathcal{J}_2(\varepsilon)| \le c(\varphi)^2\big(\breve{\nu}_3|\zeta|^{-1/2}\varepsilon + \breve{\nu}_4\varepsilon^2\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\mathbf{\Phi}\|_{L_2(\mathcal{O})}, \tag{11.41}$$

where $\breve{\nu}_3 = (2d\beta c^\circ\mathfrak{C}_0)^{1/2}(\nu_1+\nu_2+\nu_3)$ and $\breve{\nu}_4 = (2d\beta c^\circ\mathfrak{C}_0)^{1/2}\nu_1$.

It remains to estimate the term (11.39), which can be represented as

$$\mathcal{J}_3(\varepsilon) = \mathcal{J}_3^{(1)}(\varepsilon) + \mathcal{J}_3^{(2)}(\varepsilon), \tag{11.42}$$

$$\mathcal{J}_3^{(1)}(\varepsilon) := \sum_{l=1}^{d}\big(\boldsymbol{\psi}_l(\varepsilon), (D_l\Lambda)^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\boldsymbol{\eta}}_0\big)_{L_2(\mathcal{O})}, \tag{11.43}$$

$$\mathcal{J}_3^{(2)}(\varepsilon) := \sum_{l=1}^{d}\big(\boldsymbol{\psi}_l(\varepsilon), \varepsilon\Lambda^\varepsilon S_\varepsilon b(\mathbf{D}) D_l\widetilde{\boldsymbol{\eta}}_0\big)_{L_2(\mathcal{O})}. \tag{11.44}$$

By (5.1) and (11.25), the term (11.43) satisfies the estimate

$$|\mathcal{J}_3^{(1)}(\varepsilon)| \le c(\varphi)\big(\nu_1\varepsilon+(\nu_2+\nu_3)|\zeta|^{-1/4}\varepsilon^{1/2}\big)\|\mathbf{F}\|_{L_2(\mathcal{O})} d^{1/2}\left(\int_{(\partial\mathcal{O})_\varepsilon}|(\mathbf{D}\Lambda)^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\boldsymbol{\eta}}_0|^2\,d\mathbf{x}\right)^{1/2}.$$

Applying Lemma 3.6, (1.4), (1.10), and the analogs of estimates (10.8), (10.9) for $\widetilde{\boldsymbol{\eta}}_0$, we obtain

$$|\mathcal{J}_3^{(1)}(\varepsilon)| \le c(\varphi)^2\big(\breve{\nu}_5|\zeta|^{-1/2}\varepsilon + \breve{\nu}_6\varepsilon^2\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\mathbf{\Phi}\|_{L_2(\mathcal{O})}, \tag{11.45}$$

where

$$\breve{\nu}_5 = M_2(2d\beta_*\alpha_1 C_{\mathcal{O}}^{(1)} C_{\mathcal{O}}^{(2)}(\mathfrak{C}_0+1)c^\circ)^{1/2}(\nu_1+\nu_2+\nu_3),$$
$$\breve{\nu}_6 = M_2(2d\beta_*\alpha_1 C_{\mathcal{O}}^{(1)} C_{\mathcal{O}}^{(2)}(\mathfrak{C}_0+1)c^\circ)^{1/2}\nu_1.$$

Finally, it is easy to estimate the term (11.44) by using (11.25), (1.4), (1.19), and the analog of (10.9) for $\widetilde{\boldsymbol{\eta}}_0$:

$$|\mathcal{J}_3^{(2)}(\varepsilon)| \le c(\varphi)^2\big(\breve{\nu}_7|\zeta|^{-1/2}\varepsilon + \breve{\nu}_8\varepsilon^2\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\mathbf{\Phi}\|_{L_2(\mathcal{O})}, \tag{11.46}$$

where $\breve{\nu}_7 = 2M_1(d\alpha_1)^{1/2}C_{\mathcal{O}}^{(2)}c^\circ(\nu_2+\nu_3)$ and $\breve{\nu}_8 = 2M_1(d\alpha_1)^{1/2}C_{\mathcal{O}}^{(2)}c^\circ(\nu_1+\nu_2+\nu_3)$.

As a result, relations (11.37), (11.40)–(11.42), (11.45), and (11.46) imply that

$$|\widetilde{\mathcal{I}}_\varepsilon^{(2)}[\boldsymbol{\eta}_\varepsilon]| \le c(\varphi)^5\big(\breve{\nu}_9|\zeta|^{-1/2}\varepsilon + \breve{\nu}_{10}\varepsilon^2\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\mathbf{\Phi}\|_{L_2(\mathcal{O})}, \quad 0<\varepsilon\le\varepsilon_1,$$

where $\breve{\nu}_9 = \breve{\nu}_1+\breve{\nu}_3+\breve{\nu}_5+\breve{\nu}_7$ and $\breve{\nu}_{10} = \breve{\nu}_2+\breve{\nu}_4+\breve{\nu}_6+\breve{\nu}_8$. Together with (11.35), this yields

$$|\mathcal{I}_\varepsilon[\boldsymbol{\eta}_\varepsilon]| \le c(\varphi)^5\big(\mathfrak{C}_{16}|\zeta|^{-1/2}\varepsilon + \mathfrak{C}_{17}\varepsilon^2\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\mathbf{\Phi}\|_{L_2(\mathcal{O})}, \quad 0<\varepsilon\le\varepsilon_1, \tag{11.47}$$

where $\mathfrak{C}_{16} = \mathfrak{C}_0(\mathfrak{C}^{(1)}+\mathfrak{C}^{(2)}) + \breve{\nu}_9$ and $\mathfrak{C}_{17} = \breve{\nu}_{10}$.

From (11.34) and (11.47) it follows that

$$|(\mathbf{w}_\varepsilon, \mathbf{\Phi})_{L_2(\mathcal{O})}| \le c(\varphi)^5 \big(\mathfrak{C}_{16}|\zeta|^{-1/2}\varepsilon + \mathfrak{C}_{17}\varepsilon^2\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\mathbf{\Phi}\|_{L_2(\mathcal{O})}$$

for any $0 < \varepsilon \le \varepsilon_1$ and any $\mathbf{\Phi} \in L_2(\mathcal{O};\mathbb{C}^n)$, which implies the required estimate (11.33). □

Completion of the proof of Theorem 10.1. Estimate (10.25) and Lemma 11.3 show that (10.1) is true for $0 < \varepsilon \le \varepsilon_1$ with the constant $\mathfrak{C}_1 = \max\{\mathfrak{C}_9 + \mathfrak{C}_{16}, \mathfrak{C}_{17}\}$. □

§12. Results for the Neumann problem: the case where $\Lambda \in L_\infty$, special cases, estimates in a strictly interior subdomain

12.1. The case where $\Lambda \in L_\infty$. As for the problem in $\mathbb{R}^d$ (see Subsection 2.3) and for the Dirichlet problem (see Subsection 6.1), under Condition 2.8 the operator S_ε can be removed from the corrector. Now instead of the corrector (10.2) we shall use the operator

$$K_N^0(\varepsilon;\zeta) = [\Lambda^\varepsilon]b(\mathbf{D})(\mathcal{A}_N^0 - \zeta I)^{-1}. \tag{12.1}$$

Under Condition 2.8, the operator (12.1) is a continuous mapping of $L_2(\mathcal{O};\mathbb{C}^n)$ to the space $H^1(\mathcal{O};\mathbb{C}^n)$. Instead of (10.4) we use another approximation of the solution $\mathbf{u}_\varepsilon$ of problem (9.6):

$$\check{\mathbf{v}}_\varepsilon := (\mathcal{A}_N^0 - \zeta I)^{-1}\mathbf{F} + \varepsilon K_N^0(\varepsilon;\zeta)\mathbf{F} = \mathbf{u}_0 + \varepsilon\Lambda^\varepsilon b(\mathbf{D})\mathbf{u}_0. \tag{12.2}$$

Theorem 12.1. *Suppose that the assumptions of Theorem* 10.1 *and Condition* 2.8 *are satisfied. Let $\check{\mathbf{v}}_\varepsilon$ be defined by* (12.2). *Then for $0 < \varepsilon \le \varepsilon_1$ we have*

$$\|\mathbf{u}_\varepsilon - \check{\mathbf{v}}_\varepsilon\|_{H^1(\mathcal{O})} \le \big(\mathfrak{C}_2 c(\varphi)^2|\zeta|^{-1/4}\varepsilon^{1/2} + \mathfrak{C}_3^\circ c(\varphi)^4\varepsilon\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{12.3}$$

or, in operator terms,

$$\big\|(\mathcal{A}_{N,\varepsilon}-\zeta I)^{-1} - (\mathcal{A}_N^0-\zeta I)^{-1} - \varepsilon K_N^0(\varepsilon;\zeta)\big\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} \le \mathfrak{C}_2 c(\varphi)^2|\zeta|^{-1/4}\varepsilon^{1/2} + \mathfrak{C}_3^\circ c(\varphi)^4\varepsilon.$$

For the flux $\mathbf{p}_\varepsilon := g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$ we have

$$\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathcal{O})} \le \big(\widetilde{\mathfrak{C}}_2 c(\varphi)^2|\zeta|^{-1/4}\varepsilon^{1/2} + \widetilde{\mathfrak{C}}_3^\circ c(\varphi)^4\varepsilon\big)\|\mathbf{F}\|_{L_2(\mathcal{O})} \tag{12.4}$$

for $0 < \varepsilon \le \varepsilon_1$. The constants $\mathfrak{C}_2$ and $\widetilde{\mathfrak{C}}_2$ are the same as in Theorem 10.2. *The constants $\mathfrak{C}_3^\circ$ and $\widetilde{\mathfrak{C}}_3^\circ$ depend on d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the constants k_1, k_2 in* (9.2), *the parameters of the lattice Γ, the domain $\mathcal{O}$, and $\|\Lambda\|_{L_\infty}$.*

Proof. In order to deduce (12.3) from (10.5), it suffices to estimate the difference of the functions (10.4) and (12.2). Arguing as in (6.5)–(6.9), it is easy to obtain an analog of (6.9) with the same constant. Combining this with (10.5) and (10.9), we obtain (12.3) with the constant $\mathfrak{C}_3^\circ = \mathfrak{C}_3 + 2\mathcal{C}''' C_{\mathcal{O}}^{(2)} c^\circ$.

By analogy with (6.11)–(6.13), it is easy to check the corresponding analog of (6.13) with the same constant. Together with (9.14), (12.3), and (1.2), (1.5), this implies (12.4) with the constant $\widetilde{\mathfrak{C}}_3^\circ = \|g\|_{L_\infty}(d\alpha_1)^{1/2}\mathfrak{C}_3^\circ + 2\widetilde{\mathcal{C}}' c^\circ$. □

12.2. Special cases. The proof of the following two statements is similar to that of Propositions 6.2 and 6.3.

Proposition 12.2. *Under the assumptions of Theorem* 10.2, *if $g^0 = \overline{g}$, i.e.,* (1.13) *is true, then for $0 < \varepsilon \le \varepsilon_1$ we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{H^1(\mathcal{O})} \le \big(\mathfrak{C}_2 c(\varphi)^2|\zeta|^{-1/4}\varepsilon^{1/2} + \mathfrak{C}_3 c(\varphi)^4\varepsilon\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}.$$

Proposition 12.3. *Under the assumptions of Theorem* 10.2, *if $g^0 = \underline{g}$, i.e.,* (1.14) *is true, then for $0 < \varepsilon \le \varepsilon_1$ we have*

$$\|\mathbf{p}_\varepsilon - g^0 b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathcal{O})} \le \big(\widetilde{\mathfrak{C}}_2 c(\varphi)^2|\zeta|^{-1/4}\varepsilon^{1/2} + \widetilde{\mathfrak{C}}_3^\circ c(\varphi)^4\varepsilon\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}.$$

12.3. Approximation of the solutions of the Neumann problem in a strictly interior subdomain. By analogy with the proof of Theorem 7.1, the following theorem can be proved easily on the basis of Theorem 10.1 and the results for the homogenization problem in $\mathbb{R}^d$. We omit the proof.

Theorem 12.4. *Under the assumptions of Theorem* 10.2, *let $\mathcal{O}'$ be a strictly interior subdomain of the domain $\mathcal{O}$, and let $\delta := \operatorname{dist}\{\mathcal{O}'; \partial\mathcal{O}\}$. Then for $0 < \varepsilon \le \varepsilon_1$ we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon\|_{H^1(\mathcal{O}')} \le (\mathfrak{C}'_{18}\delta^{-1} + \mathfrak{C}''_{18}) c(\varphi)^6 \varepsilon \|\mathbf{F}\|_{L_2(\mathcal{O})},$$
$$\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathcal{O}')} \le (\widetilde{\mathfrak{C}}'_{18}\delta^{-1} + \widetilde{\mathfrak{C}}''_{18}) c(\varphi)^6 \varepsilon \|\mathbf{F}\|_{L_2(\mathcal{O})}.$$

The constants $\mathfrak{C}'_{18}$, $\mathfrak{C}''_{18}$, $\widetilde{\mathfrak{C}}'_{18}$, and $\widetilde{\mathfrak{C}}''_{18}$ depend on d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the constants k_1, k_2 in (9.2), *the parameters of the lattice Γ, and the domain $\mathcal{O}$.*

The proof of the next result is similar to the proof of Theorem 7.2.

Theorem 12.5. *Under the assumptions of Theorem* 12.1, *let $\mathcal{O}'$ be a strictly interior subdomain of $\mathcal{O}$, and let $\delta := \operatorname{dist}\{\mathcal{O}'; \partial\mathcal{O}\}$. Then for $0 < \varepsilon \le \varepsilon_1$ we have*

$$\|\mathbf{u}_\varepsilon - \check{\mathbf{v}}_\varepsilon\|_{H^1(\mathcal{O}')} \le (\mathfrak{C}'_{18}\delta^{-1} + \check{\mathfrak{C}}''_{18}) c(\varphi)^6 \varepsilon \|\mathbf{F}\|_{L_2(\mathcal{O})},$$
$$\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathcal{O}')} \le (\widetilde{\mathfrak{C}}'_{18}\delta^{-1} + \widehat{\mathfrak{C}}''_{18}) c(\varphi)^6 \varepsilon \|\mathbf{F}\|_{L_2(\mathcal{O})}.$$

The constants $\mathfrak{C}'_{18}$ and $\widetilde{\mathfrak{C}}'_{18}$ are the same as in Theorem 12.4. *The constants $\check{\mathfrak{C}}''_{18}$ and $\widehat{\mathfrak{C}}''_{18}$ depend on d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the constants k_1, k_2 in* (9.2), *the parameters of the lattice Γ, the domain $\mathcal{O}$, and $\|\Lambda\|_{L_\infty}$.*

§13. Another approximation of the resolvent $(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1}$

13.1. Approximation of the resolvent $(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1}$ for $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$. The next result is similar to Theorem 8.1.

Theorem 13.1. *Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class $C^{1,1}$, and let $\zeta = |\zeta|e^{i\varphi} \in \mathbb{C} \setminus \mathbb{R}_+$. Denote*

$$\rho_0(\zeta) = \begin{cases} c(\varphi)^2 |\zeta|^{-2} & \text{if } |\zeta| < 1, \\ c(\varphi)^2 & \text{if } |\zeta| \ge 1. \end{cases}$$

Let $\mathbf{u}_\varepsilon$ be the solution of problem (9.6) *and $\mathbf{u}_0$ the solution of problem* (9.13). *Let $\mathbf{v}_\varepsilon$ be defined by* (10.4). *Suppose that the number ε_1 is as in Condition* 3.4. *Then for $0 < \varepsilon \le \varepsilon_1$ we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O})} \le \mathfrak{C}_{19} \rho_0(\zeta) \varepsilon \|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{13.1}$$

$$\|\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon\|_{H^1(\mathcal{O})} \le \mathfrak{C}_{20} \rho_0(\zeta) \varepsilon^{1/2} \|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{13.2}$$

or, in operator terms,

$$\big\|(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1} - (\mathcal{A}_N^0 - \zeta I)^{-1}\big\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \le \mathfrak{C}_{19} \rho_0(\zeta) \varepsilon, \tag{13.3}$$

$$\big\|(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1} - (\mathcal{A}_N^0 - \zeta I)^{-1} - \varepsilon K_N(\varepsilon; \zeta)\big\|_{L_2(\mathcal{O}) \to H^1(\mathcal{O})} \le \mathfrak{C}_{20} \rho_0(\zeta) \varepsilon^{1/2}. \tag{13.4}$$

For the flux $\mathbf{p}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$ we have

$$\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathcal{O})} \le \mathfrak{C}_{21} \rho_0(\zeta) \varepsilon^{1/2} \|\mathbf{F}\|_{L_2(\mathcal{O})}, \quad 0 < \varepsilon \le \varepsilon_1. \tag{13.5}$$

The constants $\mathfrak{C}_{19}$, $\mathfrak{C}_{20}$, and $\mathfrak{C}_{21}$ depend on d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the constants k_1, k_2 in (9.2), *the parameters of the lattice Γ, and the domain $\mathcal{O}$.*

Proof. Applying Theorem 10.1 with $\zeta = -1$ and the corresponding analog of identity (2.10), for $0 < \varepsilon \leq \varepsilon_1$ we obtain

$$\big\|(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1} - (\mathcal{A}_N^0 - \zeta I)^{-1}\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leq 2\mathfrak{C}_1\varepsilon \sup_{x\geq 0}(x+1)^2|x-\zeta|^{-2}. \tag{13.6}$$

A direct calculation shows that

$$\sup_{x\geq 0}(x+1)^2|x-\zeta|^{-2} \leq 4\rho_0(\zeta). \tag{13.7}$$

Relations (13.6) and (13.7) imply (13.3) with $\mathfrak{C}_{19} = 8\mathfrak{C}_1$.

Now, applying Theorem 10.2 with $\zeta = -1$, for $0 < \varepsilon \leq \varepsilon_1$ we have

$$\big\|(\mathcal{A}_{N,\varepsilon} + I)^{-1} - (\mathcal{A}_N^0 + I)^{-1} - \varepsilon K_N(\varepsilon;-1)\big\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} \leq (\mathfrak{C}_2 + \mathfrak{C}_3)\varepsilon^{1/2}. \tag{13.8}$$

We put $\lambda := \|g^{-1}\|^{-1}_{L_\infty}(k_1 + k_2)$. Note that, by (9.5),

$$\|\mathbf{v}\|^2_{H^1(\mathcal{O})} \leq k_1^{-1}\|g^{-1}\|_{L_\infty}\|(\mathcal{A}_{N,\varepsilon} + \lambda I)^{1/2}\mathbf{v}\|^2_{L_2(\mathcal{O})}, \quad \mathbf{v} \in H^1(\mathcal{O};\mathbb{C}^n). \tag{13.9}$$

Using an analog of identity (2.34) and estimate (13.7), we obtain

$$\begin{aligned} &\|(\mathcal{A}_{N,\varepsilon} + \lambda I)^{1/2}\big((\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1} - (\mathcal{A}_N^0 - \zeta I)^{-1} - \varepsilon K_N(\varepsilon;\zeta)\big)\|_{L_2\to L_2} \\ &\leq 4\rho_0(\zeta)\big\|(\mathcal{A}_{N,\varepsilon}+\lambda I)^{1/2}\big((\mathcal{A}_{N,\varepsilon}+ I)^{-1} - (\mathcal{A}_N^0 + I)^{-1} - \varepsilon K_N(\varepsilon;-1)\big)\big\|_{L_2\to L_2} \\ &\quad + \varepsilon|\zeta+1|\sup_{x\geq 0}(x+\lambda)^{1/2}|x-\zeta|^{-1}\|K_N(\varepsilon;\zeta)\|_{L_2\to L_2}. \end{aligned} \tag{13.10}$$

Denote the summands on the right-hand side of (13.10) by $L_1(\varepsilon)$ and $L_2(\varepsilon)$. By (9.4) and (13.8), the first term satisfies

$$L_1(\varepsilon) \leq \mathfrak{C}_{22}\rho_0(\zeta)\varepsilon^{1/2}, \quad 0 < \varepsilon \leq \varepsilon_1, \tag{13.11}$$

where $\mathfrak{C}_{22} = 4(\max\{d\alpha_1\|g\|_{L_\infty}, \lambda\})^{1/2}(\mathfrak{C}_2 + \mathfrak{C}_3)$.

Next, the operator $K_N(\varepsilon;\zeta)$ can be written as

$$K_N(\varepsilon;\zeta) = R_{\mathcal{O}}[\Lambda^\varepsilon]S_\varepsilon b(\mathbf{D})P_{\mathcal{O}}(\mathcal{A}_N^0 + \lambda I)^{-1/2}(\mathcal{A}_N^0 + \lambda I)^{1/2}(\mathcal{A}_N^0 - \zeta I)^{-1}.$$

Combining this with (1.4), (1.19), and (4.3), we obtain

$$L_2(\varepsilon) \leq \varepsilon|\zeta+1|\big(\sup_{x\geq 0}(x+\lambda)|x-\zeta|^{-2}\big)M_1\alpha_1^{1/2}C_{\mathcal{O}}^{(1)}\|(\mathcal{A}_N^0 + \lambda I)^{-1/2}\|_{L_2\to H^1}. \tag{13.12}$$

A direct calculation shows that

$$|\zeta+1|\sup_{x\geq 0}(x+\lambda)|x-\zeta|^{-2} \leq 2(\lambda+1)\rho_0(\zeta). \tag{13.13}$$

Like in (13.9),

$$\|(\mathcal{A}_N^0 + \lambda I)^{-1/2}\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} \leq k_1^{-1/2}\|g^{-1}\|^{1/2}_{L_\infty}. \tag{13.14}$$

From (13.12)–(13.14) it follows that

$$L_2(\varepsilon) \leq \mathfrak{C}_{23}\rho_0(\zeta)\varepsilon, \tag{13.15}$$

where $\mathfrak{C}_{23} = 2(\lambda+1)M_1\alpha_1^{1/2}C_{\mathcal{O}}^{(1)}k_1^{-1/2}\|g^{-1}\|^{1/2}_{L_\infty}$. Now relations (13.10), (13.11), (13.15), and (13.9) imply (13.4) with $\mathfrak{C}_{20} = k_1^{-1/2}\|g^{-1}\|^{1/2}_{L_\infty}(\mathfrak{C}_{22} + \mathfrak{C}_{23})$.

We check (13.5). By (9.12) and (13.7),

$$\|(\mathcal{A}_N^0 - \zeta I)^{-1}\|_{L_2(\mathcal{O})\to H^2(\mathcal{O})} \leq 2c^\circ\rho_0(\zeta)^{1/2}.$$

Hence,

$$\|\mathbf{u}_0\|_{H^2(\mathcal{O})} \leq 2c^\circ\rho_0(\zeta)^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{13.16}$$

Arguing as in (5.19)–(5.22), it is easy to check an analog of (5.22) with the same constant. Together with (4.3), (13.16), (13.2), (1.2), and (1.5), this implies (13.5) with the constant $\mathfrak{C}_{21} = \mathfrak{C}_{20}\|g\|_{L_\infty}(d\alpha_1)^{1/2} + 2(\mathcal{C}' + \mathcal{C}'')C_{\mathcal{O}}^{(2)}c^\circ$. $\square$

The following result is an analog of Theorem 8.3.

Theorem 13.2. *Suppose that the assumptions of Theorem* 13.1 *and Condition* 2.8 *are satisfied. Let* $\check{\mathbf{v}}_\varepsilon$ *be defined by* (12.2)*. Then for* $0 < \varepsilon \le \varepsilon_1$ *we have*

$$\|\mathbf{u}_\varepsilon - \check{\mathbf{v}}_\varepsilon\|_{H^1(\mathcal{O})} \le \mathfrak{C}_{20}^\circ \rho_0(\zeta)\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{13.17}$$

or, in operator terms,

$$\left\|(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1} - (\mathcal{A}_N^0 - \zeta I)^{-1} - \varepsilon K_N^0(\varepsilon;\zeta)\right\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} \le \mathfrak{C}_{20}^\circ \rho_0(\zeta)\varepsilon^{1/2}.$$

For the flux $\mathbf{p}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$ *we have*

$$\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathcal{O})} \le \mathfrak{C}_{21}^\circ \rho_0(\zeta)\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \quad 0 < \varepsilon \le \varepsilon_1. \tag{13.18}$$

The constants $\mathfrak{C}_{20}^\circ$ *and* $\mathfrak{C}_{21}^\circ$ *depend on* d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the constants* k_1, k_2 *in* (9.2), *the parameters of the lattice* Γ, *the domain* $\mathcal{O}$, *and the norm* $\|\Lambda\|_{L_\infty}$.

Proof. Arguing as in (6.5)–(6.9), it is easy to check the corresponding analog of (6.9) with the same constant. Together with (4.3), (13.2), and (13.16), this implies (13.17) with the constant $\mathfrak{C}_{20}^\circ = \mathfrak{C}_{20} + 2\mathcal{C}'''C_{\mathcal{O}}^{(2)}c^\circ$.

Like in (6.11)–(6.13), we obtain an analog of (6.13) with the same constant. Combining this with (13.16), (13.17), (1.2), and (1.5), we arrive at (13.18) with the constant $\mathfrak{C}_{21}^\circ = \|g\|_{L_\infty}(d\alpha_1)^{1/2}\mathfrak{C}_{20}^\circ + 2c^\circ\widetilde{\mathcal{C}}'$. $\square$

13.2. Special cases. The following statements are similar to Propositions 8.4 and 8.5.

Proposition 13.3. *Under the assumptions of Theorem* 13.1*, if* $g^0 = \overline{g}$*, i.e.,* (1.13) *is true, then* $\Lambda = 0$, $\mathbf{v}_\varepsilon = \mathbf{u}_0$*, and for* $0 < \varepsilon \le \varepsilon_1$ *we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{H^1(\mathcal{O})} \le \mathfrak{C}_{20}\rho_0(\zeta)\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}.$$

Proposition 13.4. *Under the assumptions of Theorem* 13.1*, if* $g^0 = \underline{g}$*, i.e.,* (1.14) *is true, then for* $0 < \varepsilon \le \varepsilon_1$ *we have*

$$\|\mathbf{p}_\varepsilon - g^0 b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathcal{O})} \le \mathfrak{C}_{21}^\circ \rho_0(\zeta)\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}.$$

13.3. Approximation of the resolvent $(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1}$ for $\zeta \in \mathbb{C}\setminus\mathbb{R}_+$ in a strictly interior subdomain. The following result is an analog of Theorem 8.6.

Theorem 13.5. *Under the assumptions of Theorem* 13.1*, let* $\mathcal{O}'$ *be a strictly interior subdomain of the domain* $\mathcal{O}$*, and let* $\delta = \operatorname{dist}\{\mathcal{O}';\partial\mathcal{O}\}$*. Denote* $\widehat{\rho}_0(\zeta) := c(\varphi)\rho_0(\zeta) + c(\varphi)^{5/2}\rho_0(\zeta)^{3/4}$*. Then for* $0 < \varepsilon \le \varepsilon_1$ *we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon\|_{H^1(\mathcal{O}')} \le (\mathfrak{C}_{24}\delta^{-1} + \mathfrak{C}_{24}')\widehat{\rho}_0(\zeta)\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{13.19}$$

$$\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathcal{O}')} \le (\mathfrak{C}_{25}\delta^{-1} + \mathfrak{C}_{25}')\widehat{\rho}_0(\zeta)\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{13.20}$$

The constants $\mathfrak{C}_{24}$, $\mathfrak{C}_{24}'$, $\mathfrak{C}_{25}$, *and* $\mathfrak{C}_{25}'$ *depend on* d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the constants* k_1, k_2 *in* (9.2)*, the parameters of the lattice* Γ*, and the domain* $\mathcal{O}$.

Proof. The first part of the proof is similar to that of Theorem 7.1. Let $\chi(\mathbf{x})$ be a cut-off function satisfying (7.3). Let $\mathbf{u}_\varepsilon$ be the solution of problem (9.6) and $\widetilde{\mathbf{u}}_\varepsilon$ the solution of equation (10.12). Then $(\mathcal{A}_\varepsilon - \zeta)(\mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon) = 0$ in $\mathcal{O}$. Arguing as in (7.4)–(7.14), we deduce the estimate

$$\|\mathbf{D}(\chi(\mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon))\|_{L_2(\mathcal{O})} \le \mathcal{C}_{23}c(\varphi)\delta^{-1}\|\mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon\|_{L_2(\mathcal{O})}, \tag{13.21}$$

where $\mathcal{C}_{23}$ is the constant occurring in (7.14). Next, by Theorem 2.2,

$$\|\widetilde{\mathbf{u}}_\varepsilon - \widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} \le C_1 c(\varphi)^2 |\zeta|^{-1/2}\varepsilon \|\widetilde{\mathbf{F}}\|_{L_2(\mathbb{R}^d)}. \tag{13.22}$$

Relations (1.16) and (4.3) imply the following estimate for the function (10.10):

$$\begin{aligned}\|\widetilde{\mathbf{F}}\|_{L_2(\mathbb{R}^d)} &\le c_1 \|\widetilde{\mathbf{u}}_0\|_{H^2(\mathbb{R}^d)} + |\zeta| \|\widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} \\ &\le c_1 C_{\mathcal{O}}^{(2)} \|\mathbf{u}_0\|_{H^2(\mathcal{O})} + |\zeta| C_{\mathcal{O}}^{(0)} \|\mathbf{u}_0\|_{L_2(\mathcal{O})}.\end{aligned}$$

Combining this with (13.16) and the inequality $\|\mathbf{u}_0\|_{L_2(\mathcal{O})} \le |\zeta|^{-1} c(\varphi) \|\mathbf{F}\|_{L_2(\mathcal{O})}$, we obtain

$$\|\widetilde{\mathbf{F}}\|_{L_2(\mathbb{R}^d)} \le \mathfrak{C}_{26} \rho_0(\zeta)^{1/2} \|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{13.23}$$

where $\mathfrak{C}_{26} = 2c_1 C_{\mathcal{O}}^{(2)} c^\circ + C_{\mathcal{O}}^{(0)}$. By (13.22) and (13.23),

$$\|\widetilde{\mathbf{u}}_\varepsilon - \widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} \le C_1 \mathfrak{C}_{26} c(\varphi)^2 |\zeta|^{-1/2} \rho_0(\zeta)^{1/2} \varepsilon \|\mathbf{F}\|_{L_2(\mathcal{O})}.$$

Together with (13.1) this implies

$$\|\mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon\|_{L_2(\mathcal{O})} \le \big(\mathfrak{C}_{19}\rho_0(\zeta) + C_1 \mathfrak{C}_{26} c(\varphi)^{3/2} \rho_0(\zeta)^{3/4}\big)\varepsilon \|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{13.24}$$

Now, comparing (13.21) and (13.24), we get

$$\|\mathbf{D}(\chi(\mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon))\|_{L_2(\mathcal{O})} \le \mathcal{C}_{23}\delta^{-1} \mathfrak{C}_{27} \widehat{\rho}_0(\zeta) \varepsilon \|\mathbf{F}\|_{L_2(\mathcal{O})},$$

where $\mathfrak{C}_{27} = \max\{\mathfrak{C}_{19}, C_1 \mathfrak{C}_{26}\}$. Hence,

$$\|\mathbf{u}_\varepsilon - \widetilde{\mathbf{u}}_\varepsilon\|_{H^1(\mathcal{O}')} \le (\mathcal{C}_{23}\delta^{-1} + 1) \mathfrak{C}_{27} \widehat{\rho}_0(\zeta) \varepsilon \|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{13.25}$$

By Corollary 2.5 and (13.23),

$$\begin{aligned}\|\widetilde{\mathbf{u}}_\varepsilon - \widetilde{\mathbf{v}}_\varepsilon\|_{H^1(\mathbb{R}^d)} &\le (C_2 + C_3|\zeta|^{-1/2}) c(\varphi)^2 \varepsilon \|\widetilde{\mathbf{F}}\|_{L_2(\mathbb{R}^d)} \\ &\le (C_2 + C_3) \mathfrak{C}_{26} c(\varphi)^{3/2} \rho_0(\zeta)^{3/4} \varepsilon \|\mathbf{F}\|_{L_2(\mathcal{O})}.\end{aligned} \tag{13.26}$$

Relations (13.25), (13.26), and (10.3) imply (13.19) with the constants $\mathfrak{C}_{24} = \mathcal{C}_{23}\mathfrak{C}_{27}$ and $\mathfrak{C}'_{24} = \mathfrak{C}_{27} + \mathfrak{C}_{26}(C_2 + C_3)$.

Inequality (13.20) is deduced from (13.19) and an analog of (5.22). □

The following result is an analog of Theorem 8.7.

Theorem 13.6. *Under the assumptions of Theorem* 13.2, *let $\mathcal{O}'$ be a strictly interior subdomain of the domain $\mathcal{O}$, and let $\delta = \operatorname{dist}\{\mathcal{O}'; \partial\mathcal{O}\}$. Let $\widehat{\rho}_0(\zeta)$ be defined as in Theorem* 13.5. *Then for $0 < \varepsilon \le \varepsilon_1$ we have*

$$\|\mathbf{u}_\varepsilon - \check{\mathbf{v}}_\varepsilon\|_{H^1(\mathcal{O}')} \le (\mathfrak{C}_{24}\delta^{-1} + \widetilde{\mathfrak{C}}'_{24}) \widehat{\rho}_0(\zeta) \varepsilon \|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{13.27}$$

$$\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathcal{O}')} \le (\mathfrak{C}_{25}\delta^{-1} + \widetilde{\mathfrak{C}}'_{25}) \widehat{\rho}_0(\zeta) \varepsilon \|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{13.28}$$

The constants $\mathfrak{C}_{24}$ and $\mathfrak{C}_{25}$ are the same as in Theorem 13.5. *The constants $\widetilde{\mathfrak{C}}'_{24}$ and $\widetilde{\mathfrak{C}}'_{25}$ depend on d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the constants k_1, k_2 in* (9.2), *the parameters of the lattice Γ, the domain $\mathcal{O}$, and $\|\Lambda\|_{L_\infty}$.*

Proof. Using (4.3), an analog of (6.9), and (13.16), (13.19), we obtain (13.27) with the constant $\widetilde{\mathfrak{C}}'_{24} = \mathfrak{C}'_{24} + 2\mathcal{C}''' C_{\mathcal{O}}^{(2)} c^\circ$. Inequality (13.28) is deduced from an analog of (6.13), (13.16), and (13.27). □

§14. Approximation of the resolvent $(\mathcal{B}_{N,\varepsilon} - \zeta I)^{-1}$

14.1. The operator $\mathcal{B}_{N,\varepsilon}$. We denote

$$Z = \operatorname{Ker} b(\mathbf{D}) = \{\mathbf{z} \in H^1(\mathcal{O};\mathbb{C}^n) \,:\, b(\mathbf{D})\mathbf{z} = 0\}.$$

From (9.2) it follows that $\|\mathbf{D}\mathbf{z}\|^2_{L_2(\mathcal{O})} \le k_1^{-1} k_2 \|\mathbf{z}\|^2_{L_2(\mathcal{O})}$, $\mathbf{z} \in Z$. Due to the compactness of the embedding of $H^1(\mathcal{O};\mathbb{C}^n)$ in $L_2(\mathcal{O};\mathbb{C}^n)$, this shows that Z is finite-dimensional. Denote $\dim Z = p$. Obviously, Z contains the subspace of constant $\mathbb{C}^n$-valued functions in $\mathcal{O}$. We put $\mathcal{H}(\mathcal{O}) = L_2(\mathcal{O};\mathbb{C}^n) \ominus Z$, $H^1_\perp(\mathcal{O};\mathbb{C}^n) = H^1(\mathcal{O};\mathbb{C}^n) \cap \mathcal{H}(\mathcal{O})$. As was checked in [Su3, Proposition 9.1], the form $\|b(\mathbf{D})\mathbf{u}\|_{L_2(\mathcal{O})}$ determines a norm in $H^1_\perp(\mathcal{O};\mathbb{C}^n)$ equivalent to the standard H^1-norm: there exists a constant $\widetilde{k}_1$ such that

$$\widetilde{k}_1 \|\mathbf{u}\|^2_{H^1(\mathcal{O})} \le \|b(\mathbf{D})\mathbf{u}\|^2_{L_2(\mathcal{O})}, \quad \mathbf{u} \in H^1_\perp(\mathcal{O};\mathbb{C}^n). \tag{14.1}$$

Recall that $\mathcal{A}_{N,\varepsilon}$ is the operator in $L_2(\mathcal{O};\mathbb{C}^n)$ generated by the form (9.3), and $\mathcal{A}_N^0$ is generated by the form (9.11). Obviously, $\operatorname{Ker} \mathcal{A}_{N,\varepsilon} = \operatorname{Ker} \mathcal{A}_N^0 = Z$. The orthogonal decomposition $L_2(\mathcal{O};\mathbb{C}^n) = \mathcal{H}(\mathcal{O}) \oplus Z$ reduces the two operators $\mathcal{A}_{N,\varepsilon}$ and $\mathcal{A}_N^0$. Let $\mathcal{B}_{N,\varepsilon}$ (respectively, $\mathcal{B}_N^0$) be the part of $\mathcal{A}_{N,\varepsilon}$ (respectively, of $\mathcal{A}_N^0$) in the subspace $\mathcal{H}(\mathcal{O})$. In other words, $\mathcal{B}_{N,\varepsilon}$ is the selfadjoint operator in $\mathcal{H}(\mathcal{O})$ generated by the quadratic form

$$b_{N,\varepsilon}[\mathbf{u},\mathbf{u}] = (g^\varepsilon b(\mathbf{D})\mathbf{u}, b(\mathbf{D})\mathbf{u})_{L_2(\mathcal{O})}, \quad \mathbf{u} \in H^1_\perp(\mathcal{O};\mathbb{C}^n).$$

Similarly, $\mathcal{B}_N^0$ is the operator in $\mathcal{H}(\mathcal{O})$ generated by the quadratic form

$$b_N^0[\mathbf{u},\mathbf{u}] = (g^0 b(\mathbf{D})\mathbf{u}, b(\mathbf{D})\mathbf{u})_{L_2(\mathcal{O})}, \quad \mathbf{u} \in H^1_\perp(\mathcal{O};\mathbb{C}^n).$$

By (1.2), (1.5), (1.15), and (14.1), we have

$$\|g^{-1}\|^{-1}_{L_\infty} \widetilde{k}_1 \|\mathbf{u}\|^2_{H^1(\mathcal{O})} \le b_{N,\varepsilon}[\mathbf{u},\mathbf{u}] \le \|g\|_{L_\infty} d\alpha_1 \|\mathbf{D}\mathbf{u}\|^2_{L_2(\mathcal{O})}, \quad \mathbf{u} \in H^1_\perp(\mathcal{O};\mathbb{C}^n), \tag{14.2}$$

$$\|g^{-1}\|^{-1}_{L_\infty} \widetilde{k}_1 \|\mathbf{u}\|^2_{H^1(\mathcal{O})} \le b_N^0[\mathbf{u},\mathbf{u}] \le \|g\|_{L_\infty} d\alpha_1 \|\mathbf{D}\mathbf{u}\|^2_{L_2(\mathcal{O})}, \quad \mathbf{u} \in H^1_\perp(\mathcal{O};\mathbb{C}^n). \tag{14.3}$$

Let $\mathcal{P}$ denote the orthogonal projection of $L_2(\mathcal{O};\mathbb{C}^n)$ onto $\mathcal{H}(\mathcal{O})$, and let $\mathcal{P}_Z$ be the orthogonal projection onto Z; then $\mathcal{P} = I - \mathcal{P}_Z$.

Let $\zeta \in \mathbb{C} \setminus [c_\flat, \infty)$, where $c_\flat > 0$ is a common lower bound of the operators $\mathcal{B}_{N,\varepsilon}$ and $\mathcal{B}_N^0$. In other words, $0 < c_\flat \le \min\{\lambda_{2,\varepsilon}(N), \lambda_2^0(N)\}$, where $\lambda_{2,\varepsilon}(N)$ (respectively, $\lambda_2^0(N)$) is the first nonzero eigenvalue of the operator $\mathcal{A}_{N,\varepsilon}$ (respectively, $\mathcal{A}_N^0$). If the eigenvalues are enumerated in the nondecreasing order (counting multiplicities), $\lambda_{2,\varepsilon}(N)$ and $\lambda_2^0(N)$ are $(p+1)$-st eigenvalues.

Remark 14.1. 1) By (14.2) and (14.3), we can take $c_\flat$ equal to $\|g^{-1}\|^{-1}_{L_\infty} \widetilde{k}_1$.

2) Let $\nu > 0$ be arbitrarily small. If ε is sufficiently small, we can take $c_\flat$ equal to $c_\flat = \lambda_2^0(N) - \nu$.

3) It is easy to find an upper bound for $c_\flat$: from (14.2), (14.3), and the variational principle it follows that $c_\flat \le \|g\|_{L_\infty} d\alpha_1 \mu^0_{p+1}(N)$, where $\mu^0_{p+1}(N) > 0$ is the $(p+1)$st eigenvalue of the operator $-\Delta$ with the Neumann condition in $L_2(\mathcal{O};\mathbb{C}^n)$ (if the eigenvalues are enumerated in the nondecreasing order counting multiplicities). Thus, $c_\flat$ does not exceed a number depending only on d, n, p, α_1, $\|g\|_{L_\infty}$, and the domain $\mathcal{O}$.

Let $\boldsymbol{\varphi}_\varepsilon = (\mathcal{B}_{N,\varepsilon} - \zeta I)^{-1}\mathbf{F}$, where $\mathbf{F} \in \mathcal{H}(\mathcal{O})$, i.e., $\boldsymbol{\varphi}_\varepsilon$ is the generalized solution of the Neumann problem

$$\begin{aligned} &b(\mathbf{D})^* g^\varepsilon b(\mathbf{D})\boldsymbol{\varphi}_\varepsilon - \zeta \boldsymbol{\varphi}_\varepsilon = \mathbf{F} \ \text{ in } \ \mathcal{O}; \\ &\partial^\varepsilon_{\boldsymbol{\nu}} \boldsymbol{\varphi}_\varepsilon|_{\partial\mathcal{O}} = 0; \quad (\boldsymbol{\varphi}_\varepsilon, \mathbf{z})_{L_2(\mathcal{O})} = 0, \quad \mathbf{z} \in Z. \end{aligned} \tag{14.4}$$

Let $\boldsymbol{\varphi}_0 = (\mathcal{B}_N^0 - \zeta I)^{-1}\mathbf{F}$, $\mathbf{F} \in \mathcal{H}(\mathcal{O})$. Then $\boldsymbol{\varphi}_0$ is the generalized solution of the Neumann problem

$$\begin{aligned} &b(\mathbf{D})^* g^0 b(\mathbf{D})\boldsymbol{\varphi}_0 - \zeta\boldsymbol{\varphi}_0 = \mathbf{F} \ \text{ in } \ \mathcal{O}; \\ &\partial^0_{\boldsymbol{\nu}}\boldsymbol{\varphi}_0|_{\partial\mathcal{O}} = 0; \quad (\boldsymbol{\varphi}_0, \mathbf{z})_{L_2(\mathcal{O})} = 0, \quad \mathbf{z} \in Z. \end{aligned} \tag{14.5}$$

In (14.4) and (14.5), the condition that $\boldsymbol{\varphi}_\varepsilon$ and $\boldsymbol{\varphi}_0$ are orthogonal to Z is fulfilled automatically if $\zeta \neq 0$, and for $\zeta = 0$ this condition should be imposed additionally.

Denote

$$\mathcal{K}_N(\varepsilon;\zeta) := R_{\mathcal{O}}[\Lambda^\varepsilon] S_\varepsilon b(\mathbf{D}) P_{\mathcal{O}} (\mathcal{B}_N^0 - \zeta I)^{-1} \tag{14.6}$$

and put $\widetilde{\boldsymbol{\varphi}}_0 = P_{\mathcal{O}}\boldsymbol{\varphi}_0$,

$$\boldsymbol{\psi}_\varepsilon = \boldsymbol{\varphi}_0 + \varepsilon\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\boldsymbol{\varphi}}_0 = (\mathcal{B}_N^0 - \zeta I)^{-1}\mathbf{F} + \varepsilon\mathcal{K}_N(\varepsilon;\zeta)\mathbf{F}. \tag{14.7}$$

Theorem 14.2. *Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class $C^{1,1}$. Let $\zeta \in \mathbb{C} \setminus [c_\flat, \infty)$, where $c_\flat > 0$ is a common lower bound of the operators $\mathcal{B}_{N,\varepsilon}$ and $\mathcal{B}_N^0$. We put $\zeta - c_\flat = |\zeta - c_\flat| e^{i\vartheta}$ and denote*

$$\rho_\flat(\zeta) = \begin{cases} c(\vartheta)^2 |\zeta - c_\flat|^{-2} & \text{if } \ |\zeta - c_\flat| < 1, \\ c(\vartheta)^2 & \text{if } \ |\zeta - c_\flat| \geq 1. \end{cases} \tag{14.8}$$

Let $\boldsymbol{\varphi}_\varepsilon$ be the solution of problem (14.4) *and $\boldsymbol{\varphi}_0$ the solution of problem* (14.5) *with $\mathbf{F} \in \mathcal{H}(\mathcal{O})$. Let $\boldsymbol{\psi}_\varepsilon$ be defined by* (14.7). *Suppose that the number ε_1 satisfies Condition* 3.4. *Then for $0 < \varepsilon \leq \varepsilon_1$ we have*

$$\begin{aligned} \|\boldsymbol{\varphi}_\varepsilon - \boldsymbol{\varphi}_0\|_{L_2(\mathcal{O})} &\leq \mathfrak{C}_{28}\rho_\flat(\zeta)\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}, \\ \|\boldsymbol{\varphi}_\varepsilon - \boldsymbol{\psi}_\varepsilon\|_{H^1(\mathcal{O})} &\leq \mathfrak{C}_{29}\rho_\flat(\zeta)\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \end{aligned} \tag{14.9}$$

or, in operator terms,

$$\big\|(\mathcal{B}_{N,\varepsilon} - \zeta I)^{-1} - (\mathcal{B}_N^0 - \zeta I)^{-1}\big\|_{\mathcal{H}(\mathcal{O})\to\mathcal{H}(\mathcal{O})} \leq \mathfrak{C}_{28}\rho_\flat(\zeta)\varepsilon, \tag{14.10}$$

$$\big\|(\mathcal{B}_{N,\varepsilon} - \zeta I)^{-1} - (\mathcal{B}_N^0 - \zeta I)^{-1} - \varepsilon\mathcal{K}_N(\varepsilon;\zeta)\big\|_{\mathcal{H}(\mathcal{O})\to H^1(\mathcal{O})} \leq \mathfrak{C}_{29}\rho_\flat(\zeta)\varepsilon^{1/2}. \tag{14.11}$$

For the flux $g^\varepsilon b(\mathbf{D})\boldsymbol{\varphi}_\varepsilon$ we have

$$\|g^\varepsilon b(\mathbf{D})\boldsymbol{\varphi}_\varepsilon - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\boldsymbol{\varphi}}_0\|_{L_2(\mathcal{O})} \leq \mathfrak{C}_{30}\rho_\flat(\zeta)\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})} \tag{14.12}$$

for $0 < \varepsilon \leq \varepsilon_1$. The constants $\mathfrak{C}_{28}$, $\mathfrak{C}_{29}$, and $\mathfrak{C}_{30}$ depend on d, n, m, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the constants k_1, k_2 in (9.2), *the constant $\widetilde{k}_1$ in* (14.1), *the parameters of the lattice Γ, and the domain $\mathcal{O}$.*

Proof. Applying Theorem 10.1 with $\zeta = -1$, for $0 < \varepsilon \leq \varepsilon_1$ we obtain

$$\begin{aligned} &\big\|(\mathcal{B}_{N,\varepsilon} + I)^{-1} - (\mathcal{B}_N^0 + I)^{-1}\big\|_{\mathcal{H}(\mathcal{O})\to\mathcal{H}(\mathcal{O})} \\ &\quad = \big\|\big((\mathcal{A}_{N,\varepsilon} + I)^{-1} - (\mathcal{A}_N^0 + I)^{-1}\big)\mathcal{P}\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leq \mathfrak{C}_1(\varepsilon + \varepsilon^2) \leq 2\mathfrak{C}_1\varepsilon. \end{aligned}$$

Then, using an analog of identity (2.10) for the operators $\mathcal{B}_{N,\varepsilon}$ and $\mathcal{B}_N^0$, we arrive at the estimate

$$\big\|(\mathcal{B}_{N,\varepsilon} - \zeta I)^{-1} - (\mathcal{B}_N^0 - \zeta I)^{-1}\big\|_{\mathcal{H}(\mathcal{O})\to\mathcal{H}(\mathcal{O})} \leq 2\mathfrak{C}_1\varepsilon \sup_{x \geq c_\flat}(x+1)^2|x - \zeta|^{-2} \tag{14.13}$$

for $0 < \varepsilon \leq \varepsilon_1$. As in (8.8), we have

$$\sup_{x \geq c_\flat}(x+1)^2|x - \zeta|^{-2} \leq \check{c}_\flat\rho_\flat(\zeta), \quad \zeta \in \mathbb{C} \setminus [c_\flat, \infty), \tag{14.14}$$

where $\check{c}_\flat = (c_\flat + 2)^2$. By Remark 14.1(3), $\check{c}_\flat$ is bounded by a number depending only on d, n, p, α_1, $\|g\|_{L_\infty}$, and the domain $\mathcal{O}$. Relations (14.13) and (14.14) imply (14.10) with $\mathfrak{C}_{28} = 2\mathfrak{C}_1\check{c}_\flat$.

Now, we apply Theorem 10.2 with $\zeta = -1$. For $0 < \varepsilon \le \varepsilon_1$ we have

$$\big\|(\mathcal{A}_{N,\varepsilon} + I)^{-1} - (\mathcal{A}_N^0 + I)^{-1} - \varepsilon K_N(\varepsilon; -1)\big\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} \le (\mathfrak{C}_2 + \mathfrak{C}_3)\varepsilon^{1/2}. \tag{14.15}$$

Together with (9.4), this yields

$$\begin{aligned} &\big\|\mathcal{A}_{N,\varepsilon}^{1/2}\left((\mathcal{A}_{N,\varepsilon} + I)^{-1} - (\mathcal{A}_N^0 + I)^{-1} - \varepsilon K_N(\varepsilon; -1)\right)\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \\ &\qquad \le \|g\|_{L_\infty}^{1/2} (d\alpha_1)^{1/2} (\mathfrak{C}_2 + \mathfrak{C}_3)\varepsilon^{1/2}, \quad 0 < \varepsilon \le \varepsilon_1. \end{aligned} \tag{14.16}$$

Multiplying the operator under the norm sign in (14.16) by the projection $\mathcal{P}$ from the two sides, we obtain

$$\begin{aligned} &\big\|\mathcal{B}_{N,\varepsilon}^{1/2}\big((\mathcal{B}_{N,\varepsilon} + I)^{-1} - (\mathcal{B}_N^0 + I)^{-1} - \varepsilon \mathcal{P}\mathcal{K}_N(\varepsilon; -1)\big)\big\|_{\mathcal{H}(\mathcal{O})\to \mathcal{H}(\mathcal{O})} \\ &\qquad \le \|g\|_{L_\infty}^{1/2} (d\alpha_1)^{1/2} (\mathfrak{C}_2 + \mathfrak{C}_3)\varepsilon^{1/2}, \quad 0 < \varepsilon \le \varepsilon_1. \end{aligned} \tag{14.17}$$

We have

$$\begin{aligned} &(\mathcal{B}_{N,\varepsilon} - \zeta I)^{-1} - (\mathcal{B}_N^0 - \zeta I)^{-1} - \varepsilon \mathcal{P}\mathcal{K}_N(\varepsilon; \zeta) \\ &\quad = (\mathcal{B}_{N,\varepsilon} + I)(\mathcal{B}_{N,\varepsilon} - \zeta I)^{-1}\left((\mathcal{B}_{N,\varepsilon} + I)^{-1} - (\mathcal{B}_N^0 + I)^{-1} - \varepsilon \mathcal{P}\mathcal{K}_N(\varepsilon; -1)\right) \\ &\qquad \times (\mathcal{B}_N^0 + I)(\mathcal{B}_N^0 - \zeta I)^{-1} + (\zeta + 1)(\mathcal{B}_{N,\varepsilon} - \zeta I)^{-1}\varepsilon \mathcal{P}\mathcal{K}_N(\varepsilon; \zeta). \end{aligned} \tag{14.18}$$

Multiplying (14.18) by $\mathcal{B}_{N,\varepsilon}^{1/2}$ from the left and using (14.17) and (14.6), we arrive at

$$\begin{aligned} &\|\mathcal{B}_{N,\varepsilon}^{1/2}\left((\mathcal{B}_{N,\varepsilon} - \zeta I)^{-1} - (\mathcal{B}_N^0 - \zeta I)^{-1} - \varepsilon \mathcal{P}\mathcal{K}_N(\varepsilon; \zeta)\right)\|_{\mathcal{H}(\mathcal{O})\to \mathcal{H}(\mathcal{O})} \\ &\quad \le \|g\|_{L_\infty}^{1/2} (d\alpha_1)^{1/2} (\mathfrak{C}_2 + \mathfrak{C}_3)\varepsilon^{1/2} \sup_{x \ge c_\flat} (x+1)^2 |x - \zeta|^{-2} \\ &\qquad + |\zeta + 1|\varepsilon \sup_{x \ge c_\flat} x|x - \zeta|^{-2} \|\Lambda^\varepsilon S_\varepsilon b(\mathbf{D}) P_{\mathcal{O}} (\mathcal{B}_N^0)^{-1/2}\|_{\mathcal{H}(\mathcal{O})\to L_2(\mathbb{R}^d)}. \end{aligned} \tag{14.19}$$

We denote the summands on the right-hand side of (14.19) by $\mathcal{T}_1(\varepsilon)$ and $\mathcal{T}_2(\varepsilon)$. Taking (14.14) into account, we have

$$\mathcal{T}_1(\varepsilon) \le \mathfrak{C}_{31}\varepsilon^{1/2}\rho_\flat(\zeta), \quad 0 < \varepsilon \le \varepsilon_1, \tag{14.20}$$

where $\mathfrak{C}_{31} = \check{c}_\flat \|g\|_{L_\infty}^{1/2} (d\alpha_1)^{1/2} (\mathfrak{C}_2 + \mathfrak{C}_3)$. Next, from (14.3) it follows that

$$\|(\mathcal{B}_N^0)^{-1/2}\|_{\mathcal{H}(\mathcal{O})\to H^1(\mathcal{O})} \le \|g^{-1}\|_{L_\infty}^{1/2} \widetilde{k}_1^{-1/2}.$$

Combining this with (1.4), (1.19), and (4.3), we obtain

$$\|\Lambda^\varepsilon S_\varepsilon b(\mathbf{D}) P_{\mathcal{O}} (\mathcal{B}_N^0)^{-1/2}\|_{\mathcal{H}(\mathcal{O})\to L_2(\mathbb{R}^d)} \le M_1 \alpha_1^{1/2} C_{\mathcal{O}}^{(1)} \|g^{-1}\|_{L_\infty}^{1/2} \widetilde{k}_1^{-1/2}. \tag{14.21}$$

Like in (8.17), we have

$$|\zeta + 1| \sup_{x \ge c_\flat} x|x - \zeta|^{-2} \le (c_\flat + 2)(c_\flat + 1)\rho_\flat(\zeta), \quad \zeta \in \mathbb{C} \setminus [c_\flat, \infty). \tag{14.22}$$

Relations (14.21) and (14.22) imply the following estimate for the second summand in (14.19):

$$\mathcal{T}_2(\varepsilon) \le \mathfrak{C}_{32}\varepsilon\rho_\flat(\zeta), \tag{14.23}$$

where $\mathfrak{C}_{32} = (c_\flat + 2)(c_\flat + 1) M_1 \alpha_1^{1/2} C_{\mathcal{O}}^{(1)} \|g^{-1}\|_{L_\infty}^{1/2} \widetilde{k}_1^{-1/2}$.

As a result, using (14.19), (14.20), and (14.23), we get

$$\big\|\mathcal{B}_{N,\varepsilon}^{1/2}\big((\mathcal{B}_{N,\varepsilon} - \zeta I)^{-1} - (\mathcal{B}_N^0 - \zeta I)^{-1} - \varepsilon \mathcal{P}\mathcal{K}_N(\varepsilon; \zeta)\big)\big\|_{\mathcal{H}(\mathcal{O})\to \mathcal{H}(\mathcal{O})} \le (\mathfrak{C}_{31} + \mathfrak{C}_{32})\varepsilon^{1/2}\rho_\flat(\zeta),$$
$$0 < \varepsilon \le \varepsilon_1.$$

Together with (14.2) this yields

$$\big\|(\mathcal{B}_{N,\varepsilon} - \zeta I)^{-1} - (\mathcal{B}_N^0 - \zeta I)^{-1} - \varepsilon \mathcal{P}\mathcal{K}_N(\varepsilon;\zeta)\big\|_{\mathcal{H}(\mathcal{O})\to H^1(\mathcal{O})} \le \mathfrak{C}_{33}\varepsilon^{1/2}\rho_\flat(\zeta) \tag{14.24}$$

for $0 < \varepsilon \le \varepsilon_1$, where $\mathfrak{C}_{33} = \|g^{-1}\|^{1/2}_{L_\infty}\widetilde{k}_1^{-1/2}(\mathfrak{C}_{31} + \mathfrak{C}_{32})$.

Now we show that the last term under the norm sign in (14.24) can be replaced by $\varepsilon\mathcal{K}_N(\varepsilon;\zeta)$; this will only change a constant in the estimate. Multiplying the operator under the norm sign in (14.15) by $\mathcal{P}$ from the right, we obtain

$$\big\|(\mathcal{B}_{N,\varepsilon} + I)^{-1} - (\mathcal{B}_N^0 + I)^{-1} - \varepsilon \mathcal{K}_N(\varepsilon;-1)\big\|_{\mathcal{H}(\mathcal{O})\to H^1(\mathcal{O})} \le (\mathfrak{C}_2 + \mathfrak{C}_3)\varepsilon^{1/2} \tag{14.25}$$

for $0 < \varepsilon \le \varepsilon_1$. On the other hand, from (14.17) and (14.2) it follows that

$$\big\|(\mathcal{B}_{N,\varepsilon} + I)^{-1} - (\mathcal{B}_N^0 + I)^{-1} - \varepsilon \mathcal{P}\mathcal{K}_N(\varepsilon;-1)\big\|_{\mathcal{H}(\mathcal{O})\to H^1(\mathcal{O})} \le \mathfrak{C}_{34}\varepsilon^{1/2} \tag{14.26}$$

for $0 < \varepsilon \le \varepsilon_1$, where $\mathfrak{C}_{34} = \widetilde{k}_1^{-1/2}\|g^{-1}\|^{1/2}_{L_\infty}\|g\|^{1/2}_{L_\infty}(d\alpha_1)^{1/2}(\mathfrak{C}_2 + \mathfrak{C}_3)$. Comparing (14.25) and (14.26), we see that

$$\varepsilon\|\mathcal{P}_Z\mathcal{K}_N(\varepsilon;-1)\|_{\mathcal{H}(\mathcal{O})\to H^1(\mathcal{O})} \le (\mathfrak{C}_2 + \mathfrak{C}_3 + \mathfrak{C}_{34})\varepsilon^{1/2}, \quad 0 < \varepsilon \le \varepsilon_1.$$

Hence, by (14.14), we have

$$\begin{aligned} &\varepsilon\|\mathcal{P}_Z\mathcal{K}_N(\varepsilon;\zeta)\|_{\mathcal{H}(\mathcal{O})\to H^1(\mathcal{O})} \\ &\quad\le \varepsilon\|\mathcal{P}_Z\mathcal{K}_N(\varepsilon;-1)\|_{\mathcal{H}(\mathcal{O})\to H^1(\mathcal{O})}\|(\mathcal{B}_N^0 + I)(\mathcal{B}_N^0 - \zeta I)^{-1}\|_{\mathcal{H}(\mathcal{O})\to\mathcal{H}(\mathcal{O})} \\ &\quad\le (\mathfrak{C}_2 + \mathfrak{C}_3 + \mathfrak{C}_{34})\check{c}_\flat^{1/2}\varepsilon^{1/2}\rho_\flat(\zeta)^{1/2} \end{aligned} \tag{14.27}$$

for $0 < \varepsilon \le \varepsilon_1$. As a result, relations (14.24) and (14.27) together with the identity $\mathcal{P} + \mathcal{P}_Z = I$ imply (14.11) with the constant $\mathfrak{C}_{29} = \mathfrak{C}_{33} + (\mathfrak{C}_2 + \mathfrak{C}_3 + \mathfrak{C}_{34})\check{c}_\flat^{1/2}$.

It remains to check (14.12). From (14.9) and (1.2), (1.5) it follows that for $0 < \varepsilon \le \varepsilon_1$ we have

$$\|g^\varepsilon b(\mathbf{D})\boldsymbol{\varphi}_\varepsilon - g^\varepsilon b(\mathbf{D})\boldsymbol{\psi}_\varepsilon\|_{L_2(\mathcal{O})} \le \|g\|_{L_\infty}(d\alpha_1)^{1/2}\mathfrak{C}_{29}\rho_\flat(\zeta)\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{14.28}$$

As in (5.19)–(5.22),

$$\|g^\varepsilon b(\mathbf{D})\boldsymbol{\psi}_\varepsilon - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\boldsymbol{\varphi}}_0\|_{L_2(\mathcal{O})} \le (\mathcal{C}' + \mathcal{C}'')\varepsilon\|\widetilde{\boldsymbol{\varphi}}_0\|_{H^2(\mathbb{R}^d)}. \tag{14.29}$$

Now we estimate the $H^2(\mathcal{O})$-norm of the function $\boldsymbol{\varphi}_0 = (\mathcal{B}_N^0 - \zeta I)^{-1}\mathbf{F}$. By (9.12) and (14.14),

$$\begin{aligned} \|(\mathcal{B}_N^0 - \zeta I)^{-1}\|_{\mathcal{H}(\mathcal{O})\to H^2(\mathcal{O})} &\le \|(\mathcal{B}_N^0 + I)^{-1}\|_{\mathcal{H}(\mathcal{O})\to H^2(\mathcal{O})}\|(\mathcal{B}_N^0 + I)(\mathcal{B}_N^0 - \zeta I)^{-1}\|_{\mathcal{H}(\mathcal{O})\to\mathcal{H}(\mathcal{O})} \\ &\le c^\circ\check{c}_\flat^{1/2}\rho_\flat(\zeta)^{1/2}. \end{aligned}$$

Hence,

$$\|\boldsymbol{\varphi}_0\|_{H^2(\mathcal{O})} \le c^\circ\check{c}_\flat^{1/2}\rho_\flat(\zeta)^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{14.30}$$

Relations (14.28)–(14.30) and (4.3) imply inequality (14.12) with the constant $\mathfrak{C}_{30} = \|g\|_{L_\infty}(d\alpha_1)^{1/2}\mathfrak{C}_{29} + (\mathcal{C}' + \mathcal{C}'')C_{\mathcal{O}}^{(2)}c^\circ\check{c}_\flat^{1/2}$. □

Now we distinguish the case where $\Lambda \in L_\infty$. Denote

$$\mathcal{K}_N^0(\varepsilon;\zeta) := [\Lambda^\varepsilon]b(\mathbf{D})(\mathcal{B}_N^0 - \zeta I)^{-1} \tag{14.31}$$

and put

$$\check{\boldsymbol{\psi}}_\varepsilon = \boldsymbol{\varphi}_0 + \varepsilon\Lambda^\varepsilon b(\mathbf{D})\boldsymbol{\varphi}_0 = (\mathcal{B}_N^0 - \zeta I)^{-1}\mathbf{F} + \varepsilon\mathcal{K}_N^0(\varepsilon;\zeta)\mathbf{F}. \tag{14.32}$$

Theorem 14.3. *Suppose that the assumptions of Theorem* 14.2 *and Condition* 2.8 *are satisfied. Let* $\check{\boldsymbol{\psi}}_\varepsilon$ *be defined by* (14.32). *Then for* $0 < \varepsilon \le \varepsilon_1$ *we have*

$$\|\boldsymbol{\varphi}_\varepsilon - \check{\boldsymbol{\psi}}_\varepsilon\|_{H^1(\mathcal{O})} \le \mathfrak{C}_{29}^\circ \rho_\flat(\zeta)\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{14.33}$$

or, in operator terms,

$$\big\|(\mathcal{B}_{N,\varepsilon} - \zeta I)^{-1} - (\mathcal{B}_N^0 - \zeta I)^{-1} - \varepsilon \mathcal{K}_N^0(\varepsilon;\zeta)\big\|_{\mathcal{H}(\mathcal{O})\to H^1(\mathcal{O})} \le \mathfrak{C}_{29}^\circ \rho_\flat(\zeta)\varepsilon^{1/2}. \tag{14.34}$$

For the flux $g^\varepsilon b(\mathbf{D})\boldsymbol{\varphi}_\varepsilon$ *we have*

$$\|g^\varepsilon b(\mathbf{D})\boldsymbol{\varphi}_\varepsilon - \widetilde{g}^\varepsilon b(\mathbf{D})\boldsymbol{\varphi}_0\|_{L_2(\mathcal{O})} \le \mathfrak{C}_{30}^\circ \rho_\flat(\zeta)\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})} \tag{14.35}$$

for $0 < \varepsilon \le \varepsilon_1$. *The constants* $\mathfrak{C}_{29}^\circ$ *and* $\mathfrak{C}_{30}^\circ$ *depend on* d, n, m, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the constants* k_1, k_2 *in* (9.2), *the constant* $\widetilde{k}_1$ *in* (14.1), *the parameters of the lattice* Γ, *the domain* $\mathcal{O}$, *and the norm* $\|\Lambda\|_{L_\infty}$.

Proof. As in (6.5)–(6.9), we have

$$\|\boldsymbol{\psi}_\varepsilon - \check{\boldsymbol{\psi}}_\varepsilon\|_{H^1(\mathcal{O})} \le \mathcal{C}'''\varepsilon\|\widetilde{\boldsymbol{\varphi}}_0\|_{H^2(\mathbb{R}^d)}.$$

Together with (4.3) and (14.30), this yields

$$\|\boldsymbol{\psi}_\varepsilon - \check{\boldsymbol{\psi}}_\varepsilon\|_{H^1(\mathcal{O})} \le \mathcal{C}''' C_{\mathcal{O}}^{(2)} c^\circ \check{c}_\flat^{-1/2}\varepsilon\rho_\flat(\zeta)^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{14.36}$$

Now relations (14.9) and (14.36) imply the required estimate (14.33) with the constant $\mathfrak{C}_{29}^\circ = \mathfrak{C}_{29} + \mathcal{C}''' C_{\mathcal{O}}^{(2)} c^\circ \check{c}_\flat^{-1/2}$.

We check (14.35). From (14.33) and (1.2), (1.5) it follows that

$$\|g^\varepsilon b(\mathbf{D})\boldsymbol{\varphi}_\varepsilon - g^\varepsilon b(\mathbf{D})\check{\boldsymbol{\psi}}_\varepsilon\|_{L_2(\mathcal{O})} \le \|g\|_{L_\infty}(d\alpha_1)^{1/2}\mathfrak{C}_{29}^\circ \rho_\flat(\zeta)\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})} \tag{14.37}$$

for $0 < \varepsilon \le \varepsilon_1$. As in (6.11)–(6.13), we have

$$\|g^\varepsilon b(\mathbf{D})\check{\boldsymbol{\psi}}_\varepsilon - \widetilde{g}^\varepsilon b(\mathbf{D})\boldsymbol{\varphi}_0\|_{L_2(\mathcal{O})} \le \widetilde{\mathcal{C}}'\varepsilon\|\boldsymbol{\varphi}_0\|_{H^2(\mathcal{O})}. \tag{14.38}$$

Relations (14.30), (14.37), and (14.38) imply (14.35) with the following constant: $\mathfrak{C}_{30}^\circ = \|g\|_{L_\infty}(d\alpha_1)^{1/2}\mathfrak{C}_{29}^\circ + \widetilde{\mathcal{C}}' c^\circ \check{c}_\flat^{-1/2}$. $\square$

14.2. Special cases. The next statement concerns the case where the corrector is equal to zero.

Proposition 14.4. *Under the assumptions of Theorem* 14.2, *if* $g^0 = \overline{g}$, *i.e.,* (1.13) *is true, then* $\Lambda = 0$, $\boldsymbol{\psi}_\varepsilon = \boldsymbol{\varphi}_0$, *and for* $0 < \varepsilon \le \varepsilon_1$ *we have*

$$\|\boldsymbol{\varphi}_\varepsilon - \boldsymbol{\varphi}_0\|_{H^1(\mathcal{O})} \le \mathfrak{C}_{29}\rho_\flat(\zeta)\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}.$$

The next statement is similar to Proposition 8.5.

Proposition 14.5. *Under the assumptions of Theorem* 14.2, *if* $g^0 = \underline{g}$, *i.e.,* (1.14) *is true, then for* $0 < \varepsilon \le \varepsilon_1$ *we have*

$$\|g^\varepsilon b(\mathbf{D})\boldsymbol{\varphi}_\varepsilon - g^0 b(\mathbf{D})\boldsymbol{\varphi}_0\|_{L_2(\mathcal{O})} \le \mathfrak{C}_{30}^\circ \rho_\flat(\zeta)\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}.$$

14.3. Approximation of the resolvent of the operator $\mathcal{B}_{N,\varepsilon}$ in a strictly interior subdomain. Let $\mathcal{O}'$ be a strictly interior subdomain of the domain $\mathcal{O}$. Using Theorem 14.2 and the results for the problem in $\mathbb{R}^d$, we obtain approximation of the solution $\boldsymbol{\varphi}_\varepsilon$ in $H^1(\mathcal{O}')$ of sharp order with respect to ε.

Theorem 14.6. *Under the assumptions of Theorem* 14.2, *let $\mathcal{O}'$ be a strictly interior subdomain of $\mathcal{O}$, and let $\delta := \operatorname{dist}\{\mathcal{O}'; \partial\mathcal{O}\}$. Denote $\widehat{\rho}_\flat(\zeta) := c(\vartheta)\rho_\flat(\zeta) + c(\vartheta)^{5/2}\rho_\flat(\zeta)^{3/4}$. Then for $0 < \varepsilon \le \varepsilon_1$ we have*

$$\|\boldsymbol{\varphi}_\varepsilon - \boldsymbol{\psi}_\varepsilon\|_{H^1(\mathcal{O}')} \le \bigl(\mathfrak{C}'_\flat \delta^{-1}\widehat{\rho}_\flat(\zeta) + \mathfrak{C}''_\flat c(\vartheta)^{1/2}\rho_\flat(\zeta)^{5/4}\bigr)\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{14.39}$$

$$\|g^\varepsilon b(\mathbf{D})\boldsymbol{\varphi}_\varepsilon - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\boldsymbol{\varphi}}_0\|_{L_2(\mathcal{O}')} \le \bigl(\widetilde{\mathfrak{C}}'_\flat \delta^{-1}\widehat{\rho}_\flat(\zeta) + \widetilde{\mathfrak{C}}''_\flat c(\vartheta)^{1/2}\rho_\flat(\zeta)^{5/4}\bigr)\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{14.40}$$

The constants $\mathfrak{C}'_\flat$, $\mathfrak{C}''_\flat$, $\widetilde{\mathfrak{C}}'_\flat$, and $\widetilde{\mathfrak{C}}''_\flat$ depend on d, n, m, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the constants k_1, k_2 in (9.2), *the constant $\widetilde{k}_1$ in* (14.1), *the parameters of the lattice Γ, and the domain $\mathcal{O}$.*

The proof is similar to that of Theorem 8.6; we omit the details. □

Theorem 14.7. *Under the assumptions of Theorem* 14.3, *let $\mathcal{O}'$ be a strictly interior subdomain of $\mathcal{O}$, and let $\delta := \operatorname{dist}\{\mathcal{O}'; \partial\mathcal{O}\}$. Let $\widehat{\rho}_\flat(\zeta)$ be defined as in Theorem* 14.6. *Then for $0 < \varepsilon \le \varepsilon_1$ we have*

$$\|\boldsymbol{\varphi}_\varepsilon - \check{\boldsymbol{\psi}}_\varepsilon\|_{H^1(\mathcal{O}')} \le \bigl(\mathfrak{C}'_\flat \delta^{-1}\widehat{\rho}_\flat(\zeta) + \check{\mathfrak{C}}''_\flat c(\vartheta)^{1/2}\rho_\flat(\zeta)^{5/4}\bigr)\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{14.41}$$

$$\|g^\varepsilon b(\mathbf{D})\boldsymbol{\varphi}_\varepsilon - \widetilde{g}^\varepsilon b(\mathbf{D})\boldsymbol{\varphi}_0\|_{L_2(\mathcal{O}')} \le \bigl(\widetilde{\mathfrak{C}}'_\flat \delta^{-1}\widehat{\rho}_\flat(\zeta) + \widehat{\mathfrak{C}}''_\flat c(\vartheta)^{1/2}\rho_\flat(\zeta)^{5/4}\bigr)\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{14.42}$$

The constants $\mathfrak{C}'_\flat$ and $\widetilde{\mathfrak{C}}'_\flat$ are the same as in Theorem 14.6. *The constants $\check{\mathfrak{C}}''_\flat$ and $\widehat{\mathfrak{C}}''_\flat$ depend on d, n, m, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the constants k_1, k_2 in* (9.2), *the constant $\widetilde{k}_1$ in* (14.1), *the parameters of the lattice Γ, the domain $\mathcal{O}$, and $\|\Lambda\|_{L_\infty}$.*

Proof. This theorem can be proved easily by using Theorem 14.6 and relations (14.30), (14.36), and (14.38). □

14.4. Application of the results for $\mathcal{B}_{N,\varepsilon}$ to the operator $\mathcal{A}_{N,\varepsilon}$. Theorem 14.2 allows us to approximate the resolvent $(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1}$ at a regular point $\zeta \in \mathbb{C} \setminus [c_\flat, \infty)$, $\zeta \ne 0$.

Theorem 14.8. *Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class $C^{1,1}$. Let $\zeta \in \mathbb{C} \setminus [c_\flat, \infty)$, $\zeta \ne 0$. Let $\mathbf{u}_\varepsilon$ be the solution of problem* (9.6) *and $\mathbf{u}_0$ the solution of problem* (9.13) *with $\mathbf{F} \in L_2(\mathcal{O}; \mathbb{C}^n)$. Suppose that the number ε_1 satisfies Condition* 3.4. *Then for $0 < \varepsilon \le \varepsilon_1$ we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O})} \le \mathfrak{C}_{28}\rho_\flat(\zeta)\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})},$$

where $\rho_\flat(\zeta)$ is defined by (14.8). *In operator terms,*

$$\bigl\|(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1} - (\mathcal{A}_N^0 - \zeta I)^{-1}\bigr\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \le \mathfrak{C}_{28}\rho_\flat(\zeta)\varepsilon. \tag{14.43}$$

Denote $\widehat{\mathbf{v}}_\varepsilon = \mathbf{u}_0 + \varepsilon\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widehat{\mathbf{u}}_0$, where $\widehat{\mathbf{u}}_0 = P_{\mathcal{O}}(\mathcal{A}_N^0 - \zeta I)^{-1}\mathcal{P}\mathbf{F}$. Then for $0 < \varepsilon \le \varepsilon_1$ we have

$$\|\mathbf{u}_\varepsilon - \widehat{\mathbf{v}}_\varepsilon\|_{H^1(\mathcal{O})} \le \mathfrak{C}_{29}\rho_\flat(\zeta)\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})},$$

or, in operator terms,

$$\bigl\|(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1} - (\mathcal{A}_N^0 - \zeta I)^{-1} - \varepsilon\widehat{K}_N(\varepsilon;\zeta)\bigr\|_{L_2(\mathcal{O}) \to H^1(\mathcal{O})} \le \mathfrak{C}_{29}\rho_\flat(\zeta)\varepsilon^{1/2}, \tag{14.44}$$

where $\widehat{K}_N(\varepsilon;\zeta) := R_{\mathcal{O}}[\Lambda^\varepsilon]S_\varepsilon b(\mathbf{D})P_{\mathcal{O}}(\mathcal{A}_N^0 - \zeta I)^{-1}\mathcal{P}$. For the flux $\mathbf{p}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$ we have

$$\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widehat{\mathbf{u}}_0\|_{L_2(\mathcal{O})} \le \mathfrak{C}_{30}\rho_\flat(\zeta)\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})} \tag{14.45}$$

for $0 < \varepsilon \le \varepsilon_1$. The constants $\mathfrak{C}_{28}$, $\mathfrak{C}_{29}$, and $\mathfrak{C}_{30}$ are the same as in Theorem 14.2.

Proof. Note that for $\zeta \in \mathbb{C} \setminus [c_\flat, \infty)$, $\zeta \neq 0$, we have $(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1}\mathcal{P} = (\mathcal{B}_{N,\varepsilon} - \zeta I)^{-1}\mathcal{P}$ and $(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1}\mathcal{P}_Z = -\zeta^{-1}\mathcal{P}_Z$. Similarly, $(\mathcal{A}_N^0 - \zeta I)^{-1}\mathcal{P} = (\mathcal{B}_N^0 - \zeta I)^{-1}\mathcal{P}$ and $(\mathcal{A}_N^0 - \zeta I)^{-1}\mathcal{P}_Z = -\zeta^{-1}\mathcal{P}_Z$. Since $\mathcal{P} + \mathcal{P}_Z = I$, we have

$$(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1} - (\mathcal{A}_N^0 - \zeta I)^{-1} = \left((\mathcal{B}_{N,\varepsilon} - \zeta I)^{-1} - (\mathcal{B}_N^0 - \zeta I)^{-1}\right)\mathcal{P}. \tag{14.46}$$

Estimate (14.43) follows directly from (14.10) and (14.46). Note that $\widehat{K}_N(\varepsilon;\zeta) = \mathcal{K}_N(\varepsilon;\zeta)\mathcal{P}$. Together with (14.46), this implies

$$\begin{aligned}(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1} &- (\mathcal{A}_N^0 - \zeta I)^{-1} - \varepsilon\widehat{K}_N(\varepsilon;\zeta)\\ &= \left((\mathcal{B}_{N,\varepsilon} - \zeta I)^{-1} - (\mathcal{B}_N^0 - \zeta I)^{-1} - \varepsilon\mathcal{K}_N(\varepsilon;\zeta)\right)\mathcal{P}.\end{aligned} \tag{14.47}$$

Relations (14.47) and (14.11) yield (14.44).

Next, since $b(\mathbf{D})(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1}\mathcal{P}_Z = 0$, we have

$$\begin{aligned}g^\varepsilon b(\mathbf{D})(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1} &- \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D}) P_{\mathcal{O}}(\mathcal{A}_N^0 - \zeta I)^{-1}\mathcal{P}\\ &= \left(g^\varepsilon b(\mathbf{D})(\mathcal{B}_{N,\varepsilon} - \zeta I)^{-1} - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D}) P_{\mathcal{O}}(\mathcal{B}_N^0 - \zeta I)^{-1}\right)\mathcal{P}.\end{aligned} \tag{14.48}$$

Obviously, in the operator language (14.12) means that

$$\left\|g^\varepsilon b(\mathbf{D})(\mathcal{B}_{N,\varepsilon} - \zeta I)^{-1} - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D}) P_{\mathcal{O}}(\mathcal{B}_N^0 - \zeta I)^{-1}\right\|_{\mathcal{H}(\mathcal{O})\to L_2(\mathcal{O})} \leq \mathfrak{C}_{30}\rho_\flat(\zeta)\varepsilon^{1/2} \tag{14.49}$$

for $0 < \varepsilon \leq \varepsilon_1$. Now relations (14.48) and (14.49) imply

$$\left\|g^\varepsilon b(\mathbf{D})(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1} - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D}) P_{\mathcal{O}}(\mathcal{A}_N^0 - \zeta I)^{-1}\mathcal{P}\right\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leq \mathfrak{C}_{30}\rho_\flat(\zeta)\varepsilon^{1/2}, \quad 0 < \varepsilon \leq \varepsilon_1,$$

which is equivalent to (14.45). □

The following result is deduced from Theorem 14.3.

Theorem 14.9. *Suppose that that the assumptions of Theorem* 14.8 *and Condition* 2.8 *are satisfied. Let the corrector* $K_N^0(\varepsilon;\zeta)$ *be defined by* (12.1), *and let* $\check{\mathbf{v}}_\varepsilon$ *be given by* (12.2). *Then for* $0 < \varepsilon \leq \varepsilon_1$ *we have*

$$\|\mathbf{u}_\varepsilon - \check{\mathbf{v}}_\varepsilon\|_{H^1(\mathcal{O})} \leq \mathfrak{C}_{29}^\circ\rho_\flat(\zeta)\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})},$$

or, in operator terms,

$$\left\|(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1} - (\mathcal{A}_N^0 - \zeta I)^{-1} - \varepsilon K_N^0(\varepsilon;\zeta)\right\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} \leq \mathfrak{C}_{29}^\circ\rho_\flat(\zeta)\varepsilon^{1/2}. \tag{14.50}$$

For the flux $\mathbf{p}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$ *we have*

$$\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathcal{O})} \leq \mathfrak{C}_{30}^\circ\rho_\flat(\zeta)\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})} \tag{14.51}$$

for $0 < \varepsilon \leq \varepsilon_1$. *The constants* $\mathfrak{C}_{29}^\circ$ *and* $\mathfrak{C}_{30}^\circ$ *are the same as in Theorem* 14.3.

Proof. By (12.1), (14.31), and the identity $b(\mathbf{D})\mathcal{P}_Z = 0$, we have $K_N^0(\varepsilon;\zeta) = \mathcal{K}_N^0(\varepsilon;\zeta)\mathcal{P}$. Together with (14.46), this yields

$$\begin{aligned}(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1} &- (\mathcal{A}_N^0 - \zeta I)^{-1} - \varepsilon K_N^0(\varepsilon;\zeta)\\ &= \left((\mathcal{B}_{N,\varepsilon} - \zeta I)^{-1} - (\mathcal{B}_N^0 - \zeta I)^{-1} - \varepsilon\mathcal{K}_N^0(\varepsilon;\zeta)\right)\mathcal{P}.\end{aligned} \tag{14.52}$$

Relations (14.34) and (14.52) imply (14.50).

Next, since $b(\mathbf{D})\mathcal{P}_Z = 0$, we have

$$\begin{aligned}g^\varepsilon b(\mathbf{D})(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1} &- \widetilde{g}^\varepsilon b(\mathbf{D})(\mathcal{A}_N^0 - \zeta I)^{-1}\\ &= \left(g^\varepsilon b(\mathbf{D})(\mathcal{B}_{N,\varepsilon} - \zeta I)^{-1} - \widetilde{g}^\varepsilon b(\mathbf{D})(\mathcal{B}_N^0 - \zeta I)^{-1}\right)\mathcal{P}.\end{aligned} \tag{14.53}$$

Relations (14.35) and (14.53) yield (14.51). □

Now we deduce the following corollary to Theorem 14.6.

Theorem 14.10. *Under that the assumptions of Theorem* 14.8*, let $\mathcal{O}'$ be a strictly interior subdomain of the domain $\mathcal{O}$, and let $\delta := \operatorname{dist}\{\mathcal{O}';\partial\mathcal{O}\}$. Let $\widehat{\rho}_\flat(\zeta)$ be defined as in Theorem* 14.6*. Then for $0<\varepsilon\le\varepsilon_1$ we have*

$$\|\mathbf{u}_\varepsilon - \widehat{\mathbf{v}}_\varepsilon\|_{H^1(\mathcal{O}')} \le \big(\mathfrak{C}'_\flat \delta^{-1}\widehat{\rho}_\flat(\zeta) + \mathfrak{C}''_\flat c(\vartheta)^{1/2}\rho_\flat(\zeta)^{5/4}\big)\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{14.54}$$

$$\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widehat{\mathbf{u}}_0\|_{L_2(\mathcal{O}')} \le \big(\widetilde{\mathfrak{C}}'_\flat \delta^{-1}\widehat{\rho}_\flat(\zeta) + \widetilde{\mathfrak{C}}''_\flat c(\vartheta)^{1/2}\rho_\flat(\zeta)^{5/4}\big)\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{14.55}$$

The constants $\mathfrak{C}'_\flat$, $\mathfrak{C}''_\flat$, $\widetilde{\mathfrak{C}}'_\flat$, and $\widetilde{\mathfrak{C}}''_\flat$ are the same as in Theorem 14.6.

Proof. Estimate (14.54) follows from (14.39) and (14.47). Inequality (14.55) is a consequence of (14.40) and (14.48). ☐

The next result is deduced from Theorem 14.7.

Theorem 14.11. *Under the assumptions of Theorem* 14.9*, let $\mathcal{O}'$ be a strictly interior subdomain of $\mathcal{O}$, and let $\delta := \operatorname{dist}\{\mathcal{O}';\partial\mathcal{O}\}$. Let $\widehat{\rho}_\flat(\zeta)$ be defined as in Theorem* 14.6*. Then for $0<\varepsilon\le\varepsilon_1$ we have*

$$\|\mathbf{u}_\varepsilon - \check{\mathbf{v}}_\varepsilon\|_{H^1(\mathcal{O}')} \le \big(\mathfrak{C}'_\flat \delta^{-1}\widehat{\rho}_\flat(\zeta) + \check{\mathfrak{C}}''_\flat c(\vartheta)^{1/2}\rho_\flat(\zeta)^{5/4}\big)\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{14.56}$$

$$\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathcal{O}')} \le \big(\widetilde{\mathfrak{C}}'_\flat \delta^{-1}\widehat{\rho}_\flat(\zeta) + \widehat{\mathfrak{C}}''_\flat c(\vartheta)^{1/2}\rho_\flat(\zeta)^{5/4}\big)\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{14.57}$$

The constants $\mathfrak{C}'_\flat$, $\check{\mathfrak{C}}''_\flat$, $\widetilde{\mathfrak{C}}'_\flat$, and $\widehat{\mathfrak{C}}''_\flat$ are the same as in Theorem 14.7.

Proof. Estimate (14.56) follows from (14.41) and (14.52). Inequality (14.57) is a consequence of (14.42) and (14.53). ☐

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