Алгебра и анализ
Том 27 (2015), № 5

St. Petersburg Math. J.
Vol. 27 (2016), No. 5, Pages 709–730
http://dx.doi.org/10.1090/spmj/1413
Article electronically published on July 26, 2016

CONIC INJECTIVITY SETS FOR THE RADON TRANSFORMATION ON SPHERES

V. V. VOLCHKOV AND VIT. V. VOLCHKOV

ABSTRACT. The problem under study concerns description of nonzero functions that have zero integrals over all spheres with centers in a given set. For the corresponding integral transformation (Radon transformation on spheres), the kernel is described, and sharp uniqueness theorems are obtained. Applications of the main results to partial differential equations are considered: new uniqueness theorems are proved for the Darboux equation and the wave equation.

§1. INTRODUCTION

Let $\mathbb{R}^n$ be the real Euclidean space of dimension $n \geq 2$ with the Euclidean norm $|\cdot|$, let $d\omega_{n-1}$ be the surface measure on the sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ induced by Lebesgue measure in $\mathbb{R}^n$, and let ω_{n-1} be the area of $\mathbb{S}^{n-1}$. We denote by $\operatorname{dist}(x, \partial\mathcal{U})$ the distance from a point $x \in \mathbb{R}^n$ to the boundary $\partial\mathcal{U}$ of a domain $\mathcal{U} \subset \mathbb{R}^n$. Let $L^{1,\mathrm{loc}}(\mathcal{U})$ stand for the class of functions locally integrable in $\mathcal{U}$. For any $f \in L^{1,\mathrm{loc}}(\mathcal{U})$, and almost every $r \in (0; \operatorname{dist}(x, \partial\mathcal{U}))$ (with respect to Lebesgue measure), the following spherical mean operator is well defined:

$$\mathcal{R}f(x,r) = \frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} f(x + r\eta)\, d\omega_{n-1}(\eta). \tag{1}$$

It is called also the *Radon transformation on spheres* (see [2, Chapter 1, §4, item 1] and [3, Part 5, Chapter 1]).

Let $X(\mathcal{U})$ be a class of functions locally integrable in $\mathcal{U}$. By the *kernel* of the transformation $\mathcal{R}$ on the class $X(\mathcal{U})$ relative to a set $E \subset \mathcal{U}$ we mean the collection of all $f \in X(\mathcal{U})$ such that $\mathcal{R}f(x,r) = 0$ for all $x \in E$ and almost all $r \in (0; \operatorname{dist}(x, \partial\mathcal{U}))$. A set $E \subset \mathcal{U}$ is called an injectivity set for $\mathcal{R}$ if the kernel of $\mathcal{R}$ relative to E consists of the zero function only. Given $X(\mathcal{U})$ and $E \subset \mathcal{U}$, the following problems arise:

(i) to judge whether or not E is an injectivity set;
(ii) if not, to describe the kernel of $\mathcal{R}$ in $X(\mathcal{U})$;
(iii) if yes, to recover f by $\mathcal{R}f(x,r)$.

As a rule, the questions posed above are extremely difficult and studied little. Note that the domain $\mathcal{U}$ itself is an injectivity set for $C(\mathcal{U})$. The inversion formula is also obvious in this case:

$$f(x) = \lim_{r \to 0} \mathcal{R}f(x,r).$$

More generally, any set E dense in a ball $B \subset \mathcal{U}$ is an injectivity set for the class $L^{1,\mathrm{loc}}(\mathcal{U})$ (see [4, p. 103] and also [3, Part 5, Chapter 1, Theorem 1.1]).

As a simple example of a noninjectivity set for $L^{1,\mathrm{loc}}(\mathbb{R}^n)$ we mention a hyperplane. The kernel of $\mathcal{R}$ relative to a hyperplane E is formed by the $L^{1,\mathrm{loc}}(\mathbb{R}^n)$-functions that are odd with respect to E. A broad class of injectivity sets can be constructed with the

2010 *Mathematics Subject Classification.* Primary 44A12.
Key words and phrases. Conic sets, Radon transformation, Darboux equation, wave equation.

help of the mean value theorem for the eigenfunctions of the Laplace operator in $\mathbb{R}^n$ (see [5, Chapter 4, §3, item 4]). Namely, if f is a nonzero solution continuous in $\mathcal{U}$ for the equation

$$\Delta f + \lambda^2 f = 0, \quad \lambda \in \mathbb{C},$$

then the set $f^{-1}(\{0\})$ is an noninjectivity set of the Radon transformation for $C^\infty(\mathcal{U})$. For example, such is the set of zeros of a function harmonic in $\mathcal{U}$. As another example we can take

$$f(x) = J_{n/2-1}(\lambda|x|)/(\lambda|x|)^{n/2-1}, \quad \lambda > 0,$$

where $J_{n/2-1}$ is the Bessel function of the first kind. This example shows that a sphere in $\mathbb{R}^n$ is a noninjectivity set for the class $C^\infty(\mathbb{R}^n)$. The spherically symmetric injectivity sets of the transformation $\mathcal{R}$ for the class $L^{1,\mathrm{loc}}(\mathbb{R}^n)$ can be described as follows (see [6]): a spherically symmetric set $E \subset \mathbb{R}^n$ is an injectivity set of $\mathcal{R}$ for the class $L^{1,\mathrm{loc}}(\mathbb{R}^n)$ if and only if

$$E \subset \left\{x \in \mathbb{R}^n : |x|^{-n/2+1} J_{n/2+k-1}(\lambda|x|) = 0\right\}$$

for some $k \in \mathbb{Z}_+$, $\lambda > 0$. Also in [6], a series of final results were obtained about the spherical symmetric injectivity sets for the classes $L^{1,\mathrm{loc}}(B)$ (B is a ball in $\mathbb{R}^n$) and $L^{p,\mathrm{loc}}(\mathbb{R}^n)$. In particular, a sphere with fixed radius is an injectivity set of $\mathcal{R}$ for the class of functions f with

$$\liminf_{t\to+\infty} \frac{1}{\mu_p(t)} \int_{|x|<t} |f(x)|^p \, dx = 0, \quad 1 \le p \le \frac{2n}{n-1}, \tag{2}$$

where $\mu_p(t) = t^{n-p(n-1)/2}$ for $p < \frac{2n}{n-1}$ and $\mu_p(t) = \ln t$ for $p = \frac{2n}{n-1}$. Condition (2) cannot be relaxed substantially (see [6]). As for the classes $L^p(\mathbb{R}^n)$, the following result is known [7]: the boundary of an arbitrary bounded domain in $\mathbb{R}^n$ is an injectivity set of $\mathcal{R}$ for $L^p(\mathbb{R}^n)$ if and only if $1 \le p \le \frac{2n}{n-1}$.

There are practically no publications devoted to the description of arbitrary injectivity sets for a given class $X(\mathcal{U})$. The only result in this direction known presently was obtained in [8] for the class of compactly supported functions on the plane. This result looks like this: a set $E \subset \mathbb{R}^2$ is an injectivity set of $\mathcal{R}$ for the class of functions with finite support if and only if E cannot be included in the union of the set of zeros of a homogeneous harmonic polynomial and finitely many points in $\mathbb{R}^2$. In this connection, we mention the following conjecture stated by Zalcman, see [9]: a set E can be a noninjectivity set of the transformation $\mathcal{R}$ for the class of all continuous functions on $\mathbb{R}^n$ only if E is contained in the set of zeros for an eigenfunction of the Laplacian.

In this paper, we study the conic injectivity sets of the spherical Radon transformation. The main result known in this case is as follows [9]: if a nonempty subset E of the unit ball $B = \{x \in \mathbb{R}^n : |x| < 1\}$ satisfies $\alpha E \subset E$ for any $\alpha \in (0, 1)$, then E is an injectivity set of $\mathcal{R}$ for $L^{1,\mathrm{loc}}(B)$ if and only if E is not contained in the zero set of an arbitrary nonzero harmonic polynomial. Obviously, this implies the following description of the conic injectivity sets of $\mathcal{R}$ for $L^{1,\mathrm{loc}}(\mathbb{R}^n)$: a cone E in $\mathbb{R}^n$ is such a set if and only if E is not included is the zero set of a nonzero homogeneous harmonic polynomial. Some special cases of this statement and some modifications of it were considered in [10] and [11, Appendix 2].

In this paper, this problems stated above are studied in the case where $X(\mathcal{U}) = L^{1,\mathrm{loc}}(K_R)$ and $E = \partial K_r$ or $E = \partial K_{r_1} \cup \partial K_{r_2}$, where K_α denotes the circular cone of opening 2α (see §2 for the precise definitions). The main results of the paper are as follows (see Theorems 1–3 in §2):

1) a description of the kernel of $\mathcal{R}$ for the class $L^{1,\mathrm{loc}}(K_R)$ is found, and a sharp uniqueness theorem is proved for the functions belonging to that kernel;

2) necessary and sufficient conditions are obtained under which the set $\partial K_{r_1} \cup \partial K_{r_2}$ is an injectivity set of $\mathcal{R}$ for $L^{1,\mathrm{loc}}(K_R)$.

The methods used in the proof of Theorems 1–3 are bases on the study of the local spherical Radon transformation on a sphere and employs substantially the techniques elaborated in [12]. In the course of investigation, new facts about injectivity sets of $\mathcal{R}$ were found, which are of interest on their own (see Propositions 1 and 2 in §5). In the final part of the paper, we apply the main results to partial differential equations: new uniqueness theorems are proved for the Darboux equation and wave equation (see Theorems 4–6 in §7).

§2. Statement of main results

For convenience of presentation, in what follows we shall consider the transformation $\mathcal{R}$ on the space $\mathbb{R}^{n+1}$, $n \geq 2$. Put

$$K_\alpha = \Big\{x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \colon x_{n+1} > (\cot\alpha)\sqrt{x_1^2 + \cdots + x_n^2}\,\Big\}$$

if $0 < \alpha < \pi$, and $K_\pi = \mathbb{R}^{n+1} \setminus \big\{x \in \mathbb{R}^{n+1} \colon x_1 = \cdots = x_n = 0, x_{n+1} \leq 0\big\}$. Let $\bar{K}_\alpha$ stand for the closure of K_α and ∂K_α for the boundary of K_α.

We denote by $d(\cdot,\cdot)$ the inner metric in $\mathbb{S}^n$, by $B_R = \{\xi \in \mathbb{S}^n : d(o,\xi) < R\}$ the open geodesic ball (sheric hat) of radius $0 < R \leq \pi$ centered at the point $o = (0, \dots, 0, 1) \in \mathbb{S}^n$, and by $L^{1,\mathrm{loc}}(B_R)$ the set of all functions on B_R that are locally integrable with respect to the surface measure on $\mathbb{S}^n$. We put

$$\xi = (\xi_1, \dots, \xi_{n+1}) \in \mathbb{S}^n, \quad \xi' = (\xi_1, \dots, \xi_n) \neq 0, \quad \sigma = \xi'/|\xi'| \in \mathbb{S}^{n-1},$$

and let $\theta_1, \dots, \theta_n$ be the spherical coordinates of the point ξ ($0 \leq \theta_1 \leq 2\pi$, $0 \leq \theta_k \leq \pi$, $k \neq 1$, and $\xi_1 = \sin\theta_n \dots \sin\theta_1$, $\xi_2 = \sin\theta_n \dots \sin\theta_2 \cos\theta_1, \dots$, $\xi_{n+1} = \cos\theta_n$). Each function $f(\xi) = f(\sigma\sin\theta_n, \cos\theta_n) \in L^{1,\mathrm{loc}}(B_R)$ gives rise to its Fourier series

$$f(\xi) \sim \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} f_{k,l}(\theta_n) Y_l^{(k)}(\sigma), \quad \theta_n \in (0, R), \tag{3}$$

where $\big\{Y_l^{(k)}(\sigma)\big\}$, $1 \leq l \leq a_k$ is a fixed orthonormal basis in the space $\mathcal{H}^{n,k}$ of spherical harmonics of degree k on $\mathbb{S}^{n-1}$, and

$$f_{k,l}(\theta_n) = \int_{\mathbb{S}^{n-1}} f(\zeta\sin\theta_n, \cos\theta_n)\overline{Y_l^{(k)}(\zeta)}\, d\omega_{n-1}(\zeta) \tag{4}$$

(see [12, §3], [13, Chapter 4]).

The cone K_R splits into the spherical hats ρB_R: $K_R = \bigcup_{\rho>0} \rho B_R$. By (3) and (4), for any $f \in L^{1,\mathrm{loc}}(K_R)$ we have its Fourier series

$$f(x) \sim \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} F_{k,l}(x), \quad x \in K_R,$$

where

$$F_{k,l}(x) = \int_{\mathbb{S}^{n-1}} f(|x'|\zeta, x_{n+1})\overline{Y_l^{(k)}(\zeta)}\, d\omega_{n-1}(\zeta)\; Y_l^{(k)}\big(x'/|x'|\big), \tag{5}$$

with $x' = (x_1, \dots, x_n)$. Given $\nu \in \mathbb{C}$, we put

$$\psi_{\nu,k}(\theta) = (\sin\theta)^{1-\frac{n}{2}} P_{\nu+\frac{n}{2}-1}^{-\frac{n}{2}-k+1}(\cos\theta), \quad \theta \in (0;\pi),$$
$$\Psi_\nu^{k,l}(\xi) = \psi_{\nu,k}(\theta_n) Y_l^{(k)}(\sigma), \quad \xi \in B_\pi,$$

where $P^{-\frac{n}{2}-k+1}_{\nu+\frac{n}{2}-1}$ is the Legendre function of the 1st kind on $(-1,1)$ (see [14, 3.4 (6)]). For k and $r \in (0,\pi)$ fixed, the function $\psi_{\nu,k}(r)$ has infinitely many zeros ν. All these zeros are real and simple, they are located symmetrically relative to the point $(1-n)/2$, and $\psi_{\nu,k}(r) > 0$ for $-k-n+1 \le \nu \le k$ (see Lemma 8 in §4). Put

$$\mathcal{N}_k(r) = \{\nu > k\colon \psi_{\nu,k}(r) = 0\}.$$

Throughout in what follows, $0 < R \le \pi$ and $r \in (0,R)$ are fixed. The next result solves problems (i) and (ii) in the case where $X(\mathcal{U}) = L^{1,\mathrm{loc}}(K_R)$ and $E = \partial K_r$.

Theorem 1. *Let $E = \partial K_r$. A function $f \in L^{1,\mathrm{loc}}(K_R)$ belongs to the kernel of the transformation $\mathcal{R}$ relative to E if and only if the identity*

$$F_{k,l}(x) = \sum_{\nu\in\mathcal{N}_k(r)} c_{\nu,k,l}(|x|)\Psi_\nu^{k,l}(x/|x|), \tag{6}$$

is true for any $k \in \mathbb{Z}_+$, $1 \le l \le a_k$; here $c_{\nu,k,l} \in L^{1,\mathrm{loc}}(0;+\infty)$, and the series converges in the space of distributions $\mathcal{D}'(K_R)$.

Some results that specify the nature of the convergence of the series (6) in the case where f is smooth, and formulas for $c_{\nu,k,l}$, follow from Lemmas 12 and 13 below.

Theorem 2. (i) *Let $f \in L^{1,\mathrm{loc}}(K_R)$ be such that $f = 0$ in K_r and $\mathcal{R}f(x,t) = 0$ for all $x \in \partial K_r$ and almost all $t \in (0; \mathrm{dist}(x, \partial K_R))$. Then $f = 0$ in K_R.*

(ii) *For any $\varepsilon \in (0,r)$ there exists a nonzero function $f \in C^\infty(K_R)$ such that $f = 0$ in $K_{r-\varepsilon}$ and $\mathcal{R}f(x,t) = 0$ for all $x \in \partial K_r$ and almost all $t \in (0; \mathrm{dist}(x, \partial K_R))$.*

Corollary 1. *Suppose that $R < 2r$, $\mathcal{U} = K_R \setminus \bar{K}_{2r-R}$, $f \in L^{1,\mathrm{loc}}(\mathcal{U})$, and $\mathcal{R}f(x,t) = 0$ for all $x \in \partial K_r$ and almost all $t \in (0; \mathrm{dist}(x, \partial\mathcal{U}))$. If $f = 0$ in $K_r \setminus \bar{K}_{2r-R}$ or in $K_R \setminus \bar{K}_r$, then $f = 0$ in $\mathcal{U}$.*

Given $r_1, r_2 \in (0;\pi)$, $r_1 \ne r_2$, we put

$$\mathcal{N}_k(r_1,r_2) = \mathcal{N}_k(r_1) \cap \mathcal{N}_k(r_2), \quad \mathcal{N}(r_1,r_2) = \bigcup_{k=0}^{\infty} \mathcal{N}_k(r_1,r_2).$$

The following result solves problem (i) in the case where $X(\mathcal{U}) = L^{1,\mathrm{loc}}(K_R)$ and $E = \partial K_{r_1} \cup \partial K_{r_2}$.

Theorem 3. *Suppose* $\max\{r_1,r_2\} < R$. *Then the set $\partial K_{r_1} \cup \partial K_{r_2}$ is an injectivity set of the transformation $\mathcal{R}$ for the class $L^{1,\mathrm{loc}}(K_R)$ if and only if $r_1 + r_2 \le R$ and $\mathcal{N}(r_1,r_2) = \varnothing$.*

For other results related to the Radon transformation and its generalizations, we refer the reader to [2, 3, 15, 16] and the references theorem.

§3. Auxiliary constructions

Let $d\mathcal{T}_{n+1}$ be the normalized Haar measure on the orthogonal group $\mathrm{O}(n+1)$ in $\mathbb{R}^{n+1}$. For functions f, g on $\mathbb{S}^n$, their convolution $f * g$ is defined by

$$(f*g)(\xi) = \int_{\mathrm{O}(n+1)} f(\tau o) g(\tau^{-1}\xi)\, d\mathcal{T}_{n+1}(\tau), \tag{7}$$

whenever this integral exists (see [2, Introduction, §3, item 1]). If g is invariant under the action of the group $\mathrm{O}(n) = \{\tau \in \mathrm{O}(n+1) : \tau o = o\}$, then

$$(f*g)(\tau o) = \int_{\mathbb{S}^n} f(\xi) g(\tau^{-1}\xi)\, d\omega_n(\xi). \tag{8}$$

In a natural way, this definition extends to functions, and also to distributions, defined on open subsets of $\mathbb{S}^n$. For example, if $0 < r < R \le \pi$, $f, g \in L^{1,\mathrm{loc}}(B_R)$, and the support $\operatorname{supp} g$ of g lies in B_r, then $f * g \in L^{1,\mathrm{loc}}(B_{R-r})$. In the sequel, if not stated otherwise, we always assume that the conditions on r and R imposed above are satisfied.

Let φ_ε $(0 < \varepsilon < \pi)$ be a function with the following properties: 1) $\varphi_\varepsilon \in C^\infty(\mathbb{S}^n)$ and $\operatorname{supp} \varphi_\varepsilon \subset B_\varepsilon$; 2) φ_ε is $\mathrm{O}(n)$-invariant; 3) $\varphi_\varepsilon \ge 0$ and

$$\int_{\mathbb{S}^n} \varphi_\varepsilon(\xi)\, d\omega_n(\xi) = 1.$$

We shall need the following standard statement about smoothing via convolutions.

Lemma 1. *Let $f \in L^{1,\mathrm{loc}}(B_R)$. Then for any $r < R$ we have*

$$\lim_{\varepsilon \to 0} \int_{B_r} |(f * \varphi_\varepsilon)(\xi) - f(\xi)|\, d\omega_n(\xi) = 0.$$

The proof of Lemma 1 reduces to well-known arguments (see, e.g., [17, Chapter 1, §1]).

For $f \in L^{1,\mathrm{loc}}(B_R)$, $k \in \mathbb{Z}_+$, and $1 \le l, p \le a_k$, we put

$$f^{k,l,p}(\xi) = f_{k,l}(\theta_n) Y_p^{(k)}(\sigma).$$

Lemma 2. *Suppose $f, g \in L^{1,\mathrm{loc}}(B_R)$ and $\operatorname{supp} g \subset B_r$. Then*

$$(f * g)^{k,l,p} = f^{k,l,p} * g \text{ in } B_{R-r}.$$

Proof. Let $T^{n,k}(\tau)$ be the restriction of the quasiregular representation of the group $\mathrm{O}(n)$ to the space $\mathcal{H}^{n,k}$, and let $\{t_{l,p}^{n,k}(\tau)\}$ be the matrix of $T^{n,k}(\tau)$ in the basis $\{Y_l^{(k)}(\sigma)\}$, see [18, Chapter 1, §1]. Since $T^{n,k}(\tau)$ is irreducible, the orthogonality relations for $\{t_{l,p}^{n,k}(\tau)\}$ (see [18, Chapter 1, §4]) imply the formula

$$h^{k,l,p}(\xi) = a_k \int_{\mathrm{O}(n)} h(\tau^{-1}\xi) \overline{t_{l,p}^{n,k}(\tau)}\, d\mathcal{T}_n(\tau), \tag{9}$$

where h is an arbitrary function of class $L^{1,\mathrm{loc}}(B_R)$. Using (7) and relation (9) with $h = f * g$ and with $h = f$, in the ball B_{R-r} we get the identities

$$\begin{aligned}(f * g)^{k,l,p}(\xi) &= a_k \int_{\mathrm{O}(n)} \int_{\mathrm{O}(n+1)} f(uo) g(u^{-1}\tau^{-1}\xi)\, d\mathcal{T}_{n+1}(u) \overline{t_{l,p}^{n,k}(\tau)}\, d\mathcal{T}_n(\tau) \\ &= a_k \int_{\mathrm{O}(n)} \int_{\mathrm{O}(n+1)} f(\tau^{-1}vo) g(v^{-1}\xi)\, d\mathcal{T}_{n+1}(v) \overline{t_{l,p}^{n,k}(\tau)}\, d\mathcal{T}_n(\tau) \\ &= a_k \int_{\mathrm{O}(n+1)} \int_{\mathrm{O}(n)} f(\tau^{-1}vo) \overline{t_{l,p}^{n,k}(\tau)}\, d\mathcal{T}_n(\tau) g(v^{-1}\xi)\, d\mathcal{T}_{n+1}(v) \\ &= (f^{k,l,p} * g)(\xi),\end{aligned}$$

as required. □

Following [18, Chapter 9, §3], we introduce a canonical basis in the space $\mathcal{H}^{n+1,\ell}$. Let K denote a collection of integers $(k_1, \dots, k_{n-2}, \pm k_{n-1})$ such that $\ell \ge k_1 \ge \cdots \ge k_{n-1} \ge 0$. Put

$$A_K^\ell = \Big(\frac{1}{\Gamma\big(\frac{n+1}{2}\big)}\Big)^{1/2} \Big(\prod_{j=0}^{n-2} \frac{2^{2k_{j+1}+n-j-3}(k_j - k_{j+1})!(n-j+2k_j-1)\Gamma^2\big(\frac{n-j-1}{2} + k_{j+1}\big)}{\sqrt{\pi}\Gamma\big(k_j + k_{j+1} + n - j - 1\big)}\Big)^{1/2},$$

where $k_0 = \ell$ and Γ stands for the gamma-function. Note that

$$A_O^\ell = \Big(\frac{\ell!\, \Gamma(n-1)(n+2\ell-1)}{\Gamma(n+\ell-1)(n-1)}\Big)^{1/2}, \tag{10}$$

$$
\begin{aligned}
A^\ell_M &= 2^{m+n-3}\Gamma\Big(\frac{n-2}{2}\Big)\Gamma\Big(\frac{n-1}{2}+m\Big)\\
&= \Big(\frac{(\ell-m)!m!(n+2\ell-1)(n+2m-2)}{\pi\Gamma(\ell+m+n-1)\Gamma(m+n-2)(n-1)}\Big)^{1/2},
\end{aligned}
\tag{11}
$$

where $O=(0,\dots,0)$ and $M=(m,0,\dots,0)$ (see [18, Chapter 9, §3, item 6, formula (6) and §4, item 2]).

The functions

$$
\Xi^\ell_K(\xi) = A^\ell_K \prod_{j=0}^{n-2} C^{\frac{n-j-1}{2}+k_{j+1}}_{k_j-k_{j+1}}(\cos\theta_{n-j})(\sin\theta_{n-j})^{k_{j+1}} e^{\pm i k_{n-1}\theta_1},
\tag{12}
$$

where the C^p_m are the Gegenbauer polynomials, form an orthogonal basis in the space $\mathcal{H}^{n+1,\ell}\subset L^2(\mathbb{S}^n)$, and

$$
\int_{\mathbb{S}^n} |\Xi^\ell_K(\xi)|^2\, d\omega_n(\xi) = \omega_n
\tag{13}
$$

(see [18, Chapter 9, §3, item 6]). We have

$$
L(\Xi^\ell_K) = \ell(1-\ell-n)\Xi^\ell_K,
\tag{14}
$$

where L is the Laplace operator on $\mathbb{S}^n$. Also, we have

$$
\begin{aligned}
\max_{\xi\in\mathbb{S}^n} |\Xi^\ell_K(\xi)| &\le \sqrt{\dim\mathcal{H}^{n+1,\ell}}\\
&= \Big(\frac{(n+2\ell-1)(n+\ell-2)!}{\ell!(n-1)!}\Big)^{1/2} = O\big(\ell^{\frac{n-1}{2}}\big), \quad \ell\to+\infty,
\end{aligned}
\tag{15}
$$

(see [13, Chapter 4, Corollary 2.9 (b) and the proof of Theorem 2.10]). For $f\in L^1(\mathbb{S}^n)$, we put

$$
a^\ell_K(f) = \frac{1}{\omega_n}\int_{\mathbb{S}^n} f(\xi)\overline{\Xi^\ell_K(\xi)}\, d\omega_n(\xi).
$$

If $f\in C^m(\mathbb{S}^n)$, formula (14) and the symmetry of the operator L imply the relation

$$
a^\ell_K(f) = \frac{1}{\big(\ell(1-\ell-n)\big)^{[m/2]}} a^\ell_K(L^{[m/2]}f).
\tag{16}
$$

Using (13) and the Cauchy–Bunyakovskiĭ inequality, we obtain the estimate

$$
a^\ell_K(f) = O\big(\ell^{-2[m/2]}\big), \quad \ell\to+\infty.
\tag{17}
$$

The basis Ξ^ℓ_K will be employed for generalizing the classical multiplication formula for the Gegenbauer polynomials (see [18, Chapter 9, §4, item 3, formulas (2) and (5)]).

Lemma 3. *If t, r, and $t+r$ belong to $[0;\pi)$, then the following formulas hold true*:

$$
\begin{aligned}
&\int_0^\pi C_k^{\frac{n-2}{2}}\left(\frac{\sin t\cos r\cos\theta+\cos t\sin r}{\sqrt{1-(\cos t\cos r-\sin t\sin r\cos\theta)^2}}\right) C^{\frac{n-1}{2}+k}_{\ell-k}(\cos t\cos r-\sin t\sin r\cos\theta)\\
&\qquad\times\big(1-(\cos t\cos r-\sin t\sin r\cos\theta)^2\big)^{\frac{k}{2}}(\sin\theta)^{n-2}\,d\theta\\
&= \frac{\pi\Gamma(n-1)\Gamma(n+k-2)\ell!}{2^{n-3}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-2}{2}\right)k!\Gamma(\ell+n-1)}(\sin r)^k C^{\frac{n-1}{2}+k}_{\ell-k}(\cos r) C^{\frac{n-1}{2}}_{\ell}(\cos t),
\end{aligned}
\tag{18}
$$

$$\begin{aligned}&\int_{|t-r|}^{t+r} C_k^{\frac{n-2}{2}}\Big(\frac{\cos t-\cos\gamma\cos r}{\sin r\sin\gamma}\Big)\big((\cos\gamma-\cos(t+r))(\cos(t-r)-\cos\gamma)\big)^{\frac{n-3}{2}}\\ &\qquad\qquad\times C_{\ell-k}^{\frac{n-1}{2}+k}(\cos\gamma)(\sin\gamma)^{k+1}\,d\gamma\\ &=\frac{\pi\Gamma(n-1)\Gamma(n+k-2)\ell!}{2^{n-3}\Gamma\big(\frac{n}{2}\big)\Gamma\big(\frac{n-2}{2}\big)k!\Gamma(\ell+n-1)}(\sin r)^{n+k-2}C_{\ell-k}^{\frac{n-1}{2}+k}(\cos r)(\sin t)^{n-2}C_\ell^{\frac{n-1}{2}}(\cos t).\end{aligned}\tag{19}$$

Proof. Let a_r denote the rotation in the plane (x_{n+1},x_n) by the angle r, i.e.,

$$a_r x=(x_1,\dots,x_{n-1},x_n\cos r+x_{n+1}\sin r,-x_n\sin r+x_{n+1}\cos r).$$

For $K'=(k,0,\dots,0)$, we have

$$\Xi_{K'}^\ell(a_r\xi)=\sum_K t_{K\,K'}^{n+1,\ell}(a_r^{-1})\,\Xi_K^\ell(\xi)$$

(see [18, Chapter 9, §4, item 1, formula (1)]). We integrate this identity over the ball B_t. The definition of the canonical basis and the orthogonality of the Gegenbauer polynomials (see [18, Chapter 9, §3, item 4]) imply that

$$\int_{B_t}\Xi_K^\ell(\xi)d\omega_n(\xi)=\begin{cases}\omega_{n-1}A_O^\ell\displaystyle\int_0^t C_\ell^{\frac{n-1}{2}}(\cos\theta)(\sin\theta)^{n-1}d\theta & \text{if } K=O,\\ 0 & \text{if } K\neq O.\end{cases}$$

Therefore, as a result of integration we get

$$\int_{B_t}\Xi_{K'}^\ell(a_r\xi)\,d\omega_n(\xi)=\omega_{n-1}A_O^\ell\,t_{O\,K'}^{n+1,l}(a_r^{-1})\int_0^t C_\ell^{\frac{n-1}{2}}(\cos\theta)(\sin\theta)^{n-1}\,d\theta.\tag{20}$$

The expressions for $\Xi_{K'}^\ell(\xi)$ and $t_{O\,K'}^{n+1,\ell}(a_r^{-1})$ look like this:

$$\Xi_{K'}^\ell(\xi)=A_{K'}^\ell C_k^{\frac{n-2}{2}}\left(\frac{\xi_n}{\sqrt{1-\xi_{n+1}^2}}\right)C_{\ell-k}^{\frac{n-1}{2}+k}(\xi_{n+1})\big(1-\xi_{n+1}^2\big)^{k/2},\tag{21}$$

$$\begin{aligned}t_{O\,K'}^{n+1,\ell}(a_r^{-1})=\frac{2^k\Gamma\big(\frac{n-1}{2}+k\big)}{\Gamma\big(\frac{n-1}{2}\big)}\left(\frac{\ell!(\ell-k)!(n+2k-2)\Gamma(n-1)\Gamma(n+k-2)}{k!\Gamma(\ell+k+n-1)\Gamma(\ell+n-1)}\right)^{1/2}\\ \times(\sin r)^k C_{\ell-k}^{\frac{n-1}{2}+k}(\cos r)\end{aligned}\tag{22}$$

(see (12) and [18, Chapter 9, §4, item 1, formula (9)]). Relations (20)–(22) yield

$$\begin{aligned}&\int_0^t\int_0^{2\pi}\int_0^\pi\cdots\int_0^\pi A_{K'}^\ell C_k^{\frac{n-2}{2}}\left(\frac{\sin\theta_n\cos\theta_{n-1}\cos r+\cos\theta_n\sin r}{\sqrt{1-(\cos\theta_n\cos r-\sin\theta_n\sin r\cos\theta_{n-1})^2}}\right)\\ &\qquad\times C_{\ell-k}^{\frac{n-1}{2}+k}(\cos\theta_n\cos r-\sin\theta_n\sin r\cos\theta_{n-1})\\ &\qquad\times\big(1-(\cos\theta_n\cos r-\sin\theta_n\sin r\cos\theta_{n-1})^2\big)^{\frac{k}{2}}\\ &\qquad\times(\sin\theta_n)^{n-1}(\sin\theta_{n-1})^{n-2}\dots(\sin\theta_3)^2\sin\theta_2\,d\theta_1\dots\,d\theta_n\\ &=\omega_{n-1}A_O^\ell\,\frac{2^k\Gamma\left(\frac{n-1}{2}+k\right)}{\Gamma\left(\frac{n-1}{2}\right)}\\ &\qquad\times\left(\frac{\ell!(\ell-k)!(n+2k-2)\Gamma(n-1)\Gamma(n+k-2)}{k!\Gamma(\ell+k+n-1)\Gamma(\ell+n-1)}\right)^{1/2}(\sin r)^k C_{\ell-k}^{\frac{n-1}{2}+k}(\cos r)\\ &\qquad\times\int_0^t C_\ell^{\frac{n-1}{2}}(\cos\theta)(\sin\theta)^{n-1}d\theta.\end{aligned}\tag{23}$$

Differentiating (23) in t and using (10) and (11), we arrive at formula (18). Now, to get (19) it suffices to change the variable in the integral (18) by the rule $\cos t\cos r - \sin t\sin r\cos\theta = \cos\gamma$. □

Let a function H be defined as follows: $H(\theta)=0$ if $\theta\in[0;\pi]\setminus[\,|t-r|;t+r]$ and

$$H(\theta)=\frac{k!\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-2}{2}\right)}{2^{2k+1}\Gamma(n-1)\Gamma(n+k-2)\Gamma^2\left(\frac{n-1}{2}+k\right)} \times C_k^{\frac{n-2}{2}}\left(\frac{\cos t-\cos\theta\cos r}{\sin r\sin\theta}\right)\frac{\big((\cos\theta-\cos(t+r))(\cos(t-r)-\cos\theta)\big)^{\frac{n-3}{2}}}{(\sin\theta\sin r)^{n+k-2}(\sin t)^{n-2}}$$

if $|t-r|\le\theta\le t+r$.

We denote by $b_m(H)$ ($m\in\mathbb{Z}_+$) the Fourier coefficients of H with respect to the system $C_m^{\frac{n-1}{2}+k}(\cos\theta)$, i.e.,

$$b_m(H)=\frac{2^{n+2k-2}\left(m+k+\frac{n-1}{2}\right)m!\Gamma^2\left(\frac{n-1}{2}+k\right)}{\pi\Gamma(m+2k+n-1)}\int_0^\pi H(\theta)C_m^{\frac{n-1}{2}+k}(\cos\theta)(\sin\theta)^{2k+n-1}\,d\theta.$$

By Lemma 3, we have

$$b_m(H)=\frac{m!(m+k)!\left(m+k+\frac{n-1}{2}\right)}{\Gamma(m+2k+n-1)\Gamma(m+k+n-1)}C_m^{\frac{n-1}{2}+k}(\cos r)C_{m+k}^{\frac{n-1}{2}}(\cos t). \tag{24}$$

Lemma 4. *Let $p=(8k+4n)/(4k+2n+1)$. Then*

$$\lim_{N\to+\infty}\int_0^\pi\left|H(\theta)-\sum_{\ell=k}^{N+k}b_{\ell-k}(H)C_{\ell-k}^{\frac{n-1}{2}+k}(\cos\theta)\right|^p(\sin\theta)^{2k+n-1}\,d\theta=0.$$

Proof. Since the number p satisfies the inequality

$$\frac{2\lambda+1}{\lambda+1}<p<\frac{2\lambda+1}{\lambda},$$

where $\lambda=k+\frac{n-1}{2}$, the claim follows from [14, vol. 2, p. 209 (7)]. □

Lemma 5. *Suppose that $0<r<R<\pi$, $\Phi\in C^1[0;R]$, $\Phi=0$ on $[0;r]$, and*

$$\int_{|t-r|}^{t+r}\Phi(\theta)H(\theta)(\sin\theta)^{k+n-2}\,d\theta=0,\quad 0<t<R-r. \tag{25}$$

Then $\Phi=0$ on $[0;R]$.

Proof. We start with the case where $R\le 2r$. Then $|t-r|=r-t<r$ for any $t\in(0;R-r)$, and relation (25) can be written in the form

$$\int_r^{t+r}\Phi(\theta)C_k^{\frac{n-2}{2}}\left(\frac{\cos t-\cos\theta\cos r}{\sin r\sin\theta}\right)\big((\cos\theta-\cos(t+r))(\cos(t-r)-\cos\theta)\big)^{\frac{n-3}{2}}d\theta=0,\quad 0<t<R-r, \tag{26}$$

whence

$$\int_r^{t+r}\Phi(\theta)C_k^{\frac{n-2}{2}}\left(\frac{\cos t-\cos\theta\cos r}{\sin r\sin\theta}\right)\big((\cos t-\cos(\theta+r))(\cos(\theta-r)-\cos t)\big)^{\frac{n-3}{2}}d\theta=0,\quad 0<t<R-r, \tag{27}$$

because

$$(\cos\theta-\cos(t+r))(\cos(t-r)-\cos\theta)=(\cos t-\cos(\theta+r))(\cos(\theta-r)-\cos t).$$

The substitution $\theta = r + \arccos s$ in (27) yields

$$\int_u^1 \phi(s)(s-u)^{\frac{n-3}{2}}\varphi(s,u)\,ds = 0, \quad \cos(R-r) < u < 1, \tag{28}$$

where

$$\phi(s) = \frac{\Phi(r+\arccos s)}{\sqrt{1-s^2}},$$
$$\varphi(s,u) = C_k^{\frac{n-2}{2}}\left(1 + \frac{u-s}{s\,\sin^2 r + \sqrt{1-s^2}\sin r\cos r}\right)\left(u + \sqrt{1-s^2}\sin 2r - s\cos 2r\right)^{\frac{n-3}{2}}.$$

Let $\cos(R-r) < t < 1$. We multiply (28) by $(u-t)^{\frac{n-3}{2}}$ and integrate in u from t to 1. After interchanging the order of integration, we get

$$\int_t^1 \phi(s)\int_t^s \big((s-u)(u-t)\big)^{\frac{n-3}{2}}\varphi(s,u)\,du\,ds = 0, \quad \cos(R-r) < t < 1.$$

Putting $(s-t)v = s+t-2u$ in the inner integral, we find

$$\int_t^1 \phi(s)(s-t)^{n-2}\Psi(s,t)\,ds = 0, \quad \cos(R-r) < t < 1, \tag{29}$$

where

$$\Psi(s,t) = \int_{-1}^1 (1-v^2)^{\frac{n-3}{2}}\varphi\Big(s, \frac{s+t-(s-t)v}{2}\Big)\,dv.$$

It is easy to check that for any $N \in \mathbb{N}$ the function $\frac{\partial^N}{\partial t^N}\Psi(s,t)$ is continuous on the triangle $\cos(R-r) \le t \le s \le 1$. We have

$$\begin{aligned}\Psi(t,t) &= \int_{-1}^1 (1-v^2)^{\frac{n-3}{2}}\,dv\;\varphi(t,t)\\ &= C_k^{\frac{n-2}{2}}(1)\int_{-1}^1 (1-v^2)^{\frac{n-3}{2}}\,dv\big(t + \sqrt{1-t^2}\sin 2r - t\cos 2r\big)^{\frac{n-3}{2}} \ne 0.\end{aligned}$$

We differentiate (29) $n-1$ times in t, obtaining

$$(-1)^{n-1}(n-2)!\phi(t)\Psi(t,t) + \int_t^1 \phi(s)\frac{\partial^{n-1}}{\partial t^{n-1}}\big((s-t)^{n-2}\Psi(s,t)\big)\,ds = 0.$$

Thus, the function ϕ solves a homogeneous second kind integral Volterra equation with a bounded kernel, which implies that $\phi = 0$ on $(\cos(R-r); 1)$ (see, e.g., [19, Chapter 9, §2, item 5]). This proves the claim of Lemma 5 in the case where $R \le 2r$.

Suppose that Lemma 5 is proved whenever $R \le mr$, where $m \ge 2$ is a fixed integer. In this case, if $mr < R \le (m+1)r$, then by the proved above, again (25) implies (26). As above, we conclude that $\Phi = 0$, which completes the proof of Lemma 5. □

§4. Properties of the functions $\psi_{\nu,k}$ and $\Psi_\nu^{k,l}$

A series of properties of the functions $\psi_{\nu,k}$ and $\Psi_\nu^{k,l}$ was proved in [12, §3–5]. Here we recall some of them and present some new ones.

For $m \in \mathbb{Z}$, let d_m denote the differential operator given on $C^1(0;\pi)$ by the formula

$$(d_m u)(\theta) = (\sin\theta)^m \frac{d}{d\theta}\left(u(\theta)(\sin\theta)^{-m}\right), \quad u \in C^1(0;\pi). \tag{30}$$

Observe that

$$d_{1-k-n}d_k - k(n+k-1)\,\mathrm{Id} = d_{k-1}d_{2-n-k} - (k-1)(n+k-2)\,\mathrm{Id} = l_k, \tag{31}$$

where Id is the identity operator, and

$$l_k = \frac{d^2}{d\theta^2} + (n-1)\cot\theta \frac{d}{d\theta} - \frac{k(n+k-2)}{\sin^2\theta}\,\mathrm{Id}\,.$$

Lemma 6. (i) *We have*

$$d_k \psi_{\nu,k} = (k-\nu)(k+\nu+n-1)\psi_{\nu,k+1}, \quad d_{1-n-k}\psi_{\nu,k+1} = \psi_{\nu,k}, \tag{32}$$

$$(L - c_\nu\,\mathrm{Id})\,\Psi_\nu^{k,l} = 0, \tag{33}$$

where $c_\nu = \nu(1-n-\nu)$.

(ii) *Let* $h(\xi) = u(\theta_n) Y_l^{(k)}(\sigma) \in C^\infty(B_R)$, *and let*

$$(L - c_{\lambda_1}\,\mathrm{Id}) \dots (L - c_{\lambda_m}\,\mathrm{Id}) h = 0 \quad in \quad B_R,$$

where $\lambda_1, \dots, \lambda_m$ *are arbitrary complex numbers that satisfy (for* $m \geq 2$*) the condition* $(\lambda_i - \lambda_j)(\lambda_i + \lambda_j + n - 1) \neq 0$ *for* $i \neq j$. *Then*

$$h = \sum_{j=1}^m \gamma_j \Psi_{\lambda_j}^{k,l}, \quad where \quad \gamma_j \in \mathbb{C}.$$

(iii) *Let* $\mu, \nu \in \mathbb{C}$, $a, b \in [0;\pi)$. *Then*

$$\begin{aligned}(\mu-\nu)(\mu+\nu+n-1) &\int_a^b \psi_{\mu,k}(\theta)\psi_{\nu,k}(\theta)(\sin\theta)^{n-1}\,d\theta \\ &= (\sin\theta)^{n-1}\left(\psi_{\mu,k}(\theta)\frac{d}{d\theta}\psi_{\nu,k}(\theta) - \psi_{\nu,k}(\theta)\frac{d}{d\theta}\psi_{\mu,k}(\theta)\right)\Bigg|_a^b.\end{aligned} \tag{34}$$

Proof. Formulas (32) and (33) can be found in [12, §3]. We prove (ii). The operator L acts on h by the rule

$$(Lh)(\xi) = (l_k u)(\theta_n) Y_l^{(k)}(\sigma) \tag{35}$$

(see [12, (3.5)]). Next, the identity

$$c_\nu - c_\mu = (\nu - \mu)(1 - n - \nu - \mu) \tag{36}$$

and the conditions imposed on the λ_j show that all numbers c_{λ_j} are distinct. We apply the theorem on the structure of the general solution of a linear nonhomogeneous differential equation. Then, using the smoothness of h and formula (33), we arrive at the required assertion via induction on m.

Statement (iii) with $k = 1$ was obtained in [12, §4]. In the case of an arbitrary $k \in \mathbb{Z}_+$ the proof is similar (see (35) and (31)). □

We denote by $\mathcal{D}(B_R)$ the space of compactly supported and infinitely differentiable functions on B_R.

Lemma 7. (i) *If* $\xi \in B_{\pi/2}$, *then*

$$\int_{\mathbb{S}^{n-1}} \left(\xi_{n+1} + i(\zeta,\xi')\right)^\nu Y_l^{(k)}(\zeta)\,d\omega_n(\zeta) = (2\pi)^{\frac{n}{2}} i^k \frac{\Gamma(\nu+1)}{\Gamma(\nu-k+1)} \Psi_\nu^{k,l}(\xi), \tag{37}$$

where (ζ,ξ') *is the Euclidean scalar product of the vectors* $\zeta, \xi' \in \mathbb{R}^n$.

(ii) *Suppose* $0 < r < \pi$, $t \in (0;\pi - r)$, $\eta \in S_r = \{\xi \in \mathbb{S}^n : d(o,\xi) = r\}$. *Then*

$$\int_{S_t(\eta)} \Psi_\nu^{k,l}(\xi)\,d\omega_{n-1}(\xi) = 2^{n/2-1}\Gamma\Big(\frac{n}{2}\Big)\omega_{n-1}(\sin t)^{n-1}\psi_{\nu,0}(t)\Psi_\nu^{k,l}(\eta), \tag{38}$$

where

$$S_t(\eta) = \{\xi \in \mathbb{S}^n : d(\xi,\eta) = t\}.$$

(iii) *For any* $\psi \in \mathcal{D}(B_\pi)$ *and any* $m > 0$ *we have*

$$\int_{\mathbb{S}^n} \Psi_\nu^{k,l}(\xi)\psi(\xi)\, d\omega_n(\xi) = O\Big(\frac{1}{\nu^m}\Big) \tag{39}$$

as $\nu \to +\infty$.

(iv) *Let* $r \in (0;\pi)$. *Then as* $\nu \to +\infty$ *we have the estimates*

$$\psi_{\nu,k}(r) = \sqrt{\frac{2}{\pi}}\,(\sin r)^{\frac{1-n}{2}}\,\frac{\cos\big(\big(\nu+\frac{n-1}{2}\big)\,r - \frac{\pi}{4}(n+2k-1)\big)}{\big(\nu+\frac{n-1}{2}\big)^{k+\frac{n-1}{2}}} + O\left(\frac{1}{\nu^{k+\frac{n+1}{2}}}\right), \tag{40}$$

$$\max_{t\in[0;r]}\left|\frac{d^s\psi_{\nu,k}(t)}{dt^s}\right| = O\big(\nu^{s-k}\big). \tag{41}$$

Proof. Statement (i) is contained in [12, (4.5)]. Using (33) and [12, proof of Lemma 7.1], we obtain (ii). Statement (iii) is a consequence of (i) and the identity

$$\int_{\mathbb{S}^n} \Psi_\nu^{k,l}(\xi)\psi(\xi)\, d\omega_n(\xi) = \frac{1}{\big(\nu(1-n-\nu)\big)^m}\int_{\mathbb{S}^n} \Psi_\nu^{k,l}(\xi)(L^m\psi)(\xi)\, d\omega_n(\xi) \tag{42}$$

(see (33) and the proof of (16)). Finally, (iv) follows from [12, (3.8), (4.2), and (4.7)]. □

Lemma 8. (i) *If* $r \in (0;\pi)$ *and* $k \in \mathbb{Z}_+$, *then the function* $\psi_{\nu,k}(r)$ *has infinitely many zeros* ν. *All these zeros are real and simple, and are located symmetrically with respect to the point* $(1-n)/2$. *Also, we have*

$$\psi_{\nu,k}(r) > 0 \quad \textit{for} \quad -k-n+1 \le \nu \le k \tag{43}$$

and

$$\sum_{\nu\in\mathcal{N}_k(r)} \frac{1}{\nu^{1+\varepsilon}} < +\infty \quad \textit{for any} \quad \varepsilon > 0. \tag{44}$$

(ii) *For all sufficiently large zeros* $\nu \in \mathcal{N}_k(r)$ *we have*

$$\big|\psi_{\nu,k+1}(r)\big| \ge \frac{c}{\nu^{k+\frac{n+1}{2}}}, \tag{45}$$

where $c > 0$ *is independent of* ν.

(iii) *Suppose* $u \in L^1[0;r]$ *and*

$$v(z) = \int_0^r u(\theta)\psi_{z,k}(\theta)(\sin\theta)^{n-1}\, d\theta.$$

If $v(\nu) = 0$ *for all* $\nu \in \mathcal{N}_k(r)$, *then* $u = 0$.

(iv) *Let*

$$\delta(\mu,\nu) = \int_0^r \psi_{\mu,k}(\theta)\psi_{\nu,k}(\theta)(\sin\theta)^{n-1}\, d\theta$$

and let $\mu,\nu \in \mathcal{N}_k(r)$. *Then* $\delta(\mu,\nu) = 0$ *for* $\mu \ne \nu$ *and we have*

$$\delta(\nu,\nu) > \frac{c}{\nu^{\,n+2k}}, \tag{46}$$

where $c > 0$ *is independent of* ν.

Proof. For $k = 1$, statements (i), (iii), and (iv) were proved in [12, §5]. The general case requires only minor modifications of those arguments. Statement (ii) is an easy consequence of Lemma 7 (iv). □

§5. Properties of functions of class $\mathfrak{S}_r(B_R)$

We denote by $\mathfrak{S}_r(B_R)$ the class of functions $f \in L^{1,\mathrm{loc}}(B_R)$ such that

$$\int_{S_t(\eta)} f(\xi)\, d\omega_{n-1}(\xi) = 0 \quad \text{for every} \quad \eta \in S_r \tag{47}$$

and almost every $t \in (0; R-r)$. Since

$$\int_{B_t(\eta)} f(\xi)\, d\omega_n(\xi) = \int_0^t \int_{S_\varrho(\eta)} f(\xi)\, d\omega_{n-1}(\xi)\, d\varrho,$$

where

$$B_t(\eta) = \{\xi \in \mathbb{S}^n \,:\, d(\xi, \eta) < t\},$$

a function $f \in L^{1,\mathrm{loc}}(B_R)$ belongs to $\mathfrak{S}_r(B_R)$ if and only if

$$\int_{B_t(\eta)} f(\xi)\, d\omega_n(\xi) = 0 \quad \text{on} \quad S_r \tag{48}$$

for all $t \in (0; R-r)$.

Our aim in this section is to obtain a description the smooth functions of class $\mathfrak{S}_r(B_R)$ and to prove some uniqueness results about functions satisfying (47).

We shall need some auxiliary statements. Let $\mathcal{D}_\natural(B_R)$ denote the set of $\mathrm{O}(n)$-invariant functions in $\mathcal{D}(B_R)$.

Lemma 9. *Let $f \in L^{1,\mathrm{loc}}(B_R)$. Then $f \in \mathfrak{S}_r(B_R)$ if and only if $f * g = 0$ on S_r for every $g \in \mathcal{D}_\natural(B_{R-r})$.*

Proof. If $f \in \mathfrak{S}_r(B_R)$ and $g(\xi) = h(\theta_n)$, then by (8) for $\tau o \in S_r$ we have

$$\begin{aligned}(f * g)(\tau o) &= \int_{B_{R-r}} f(\tau\xi) g(\xi)\, d\omega_n(\xi) = \int_0^{R-r} \int_{S_\varrho} f(\tau\xi) g(\xi) d\omega_{n-1}(\xi)\, d\varrho \\ &= \int_0^{R-r} h(\varrho) \int_{d(\tau o, \xi) = \varrho} f(\xi)\, d\omega_{n-1}(\xi)\, d\varrho = 0.\end{aligned}$$

To prove the converse, let χ_t denote the indicator function of the ball B_t. Approximating χ_t by functions $g \in \mathcal{D}_\natural(B_{R-r})$, we see that $f * \chi_t = 0$ on S_r for all $t \in (0; R-r)$. Hence, $f \in \mathfrak{S}_r(B_R)$ (see (48)). □

Put $\mathfrak{S}_r^\infty(B_R) = (\mathfrak{S}_r \cap C^\infty)(B_R)$.

Lemma 10. *Let $f \in \mathfrak{S}_r^\infty(B_R)$. Then:*

(i) *$f(\xi) = 0$ for every $\xi \in S_r$;*
(ii) *$f^{k,l,p} \in \mathfrak{S}_r^\infty(B_R)$ for all $k \in \mathbb{Z}_+$ and $1 \le l, p \le a_k$ (a similar statement is valid also for $f \in \mathfrak{S}_r(B_R)$);*
(iii) *$Lf \in \mathfrak{S}_r^\infty(B_R)$.*

Proof. For any $\eta \in S_r$ we have

$$|f(\eta)| = \frac{1}{\omega_{n-1}(\sin t)^{n-1}} \left| \int_{S_t(\eta)} \big(f(\xi) - f(\eta)\big)\, d\omega_{n-1}(\xi) \right| \le \max_{\xi \in S_t(\eta)} \big|f(\xi) - f(\eta)\big|.$$

Passing to the limit as $t \to +0$, we get (i). Statement (ii) follows from (9) and the definition of the classes $\mathfrak{S}_r^\infty(B_R)$ and $\mathfrak{S}_r(B_R)$. Finally, (iii) is a consequence of Lemma 9, because the operator L is convolution invariant and takes $\mathcal{D}_\natural(B_{R-r})$ to $\mathcal{D}_\natural(B_{R-r})$. □

Lemma 11. *If $f \in \mathfrak{S}_r^\infty(B_R)$ and $f = 0$ in B_r, then $f = 0$ in B_R.*

Proof. We fix arbitrary $k \in \mathbb{Z}_+$ and $l \in \{1, \dots, a_k\}$. By Lemma 10 (ii), the function $f_{k,l}(\theta_n) C_k^{\frac{n-2}{2}}(\cos\theta_{n-1})$ belongs to $\mathfrak{S}_r^\infty(B_R)$ and $f_{k,l} = 0$ on $[0; r]$ (see (9)). We extend $f_{k,l}$ by zero to $[R; \pi]$. For $\theta \in [0; \pi]$, we put $\Phi = f_{k,l}\, w_\varepsilon$, where w_ε ($\varepsilon \in (0; R - r)$) is a function such that 1) $w_\varepsilon \in C^\infty[0; \pi]$; 2) $w_\varepsilon = 1$ on $[0; R - \varepsilon]$; and 3) $w_\varepsilon = 0$ on $[-\varepsilon/2 + R; \pi]$. The function

$$F(\xi) = \Phi(\theta_n) C_k^{\frac{n-2}{2}}(\cos\theta_{n-1}) \tag{49}$$

expands in a series of the form

$$F(\xi) = \sum_{\ell=0}^{\infty} \sum_K a_K^\ell(F) \Xi_K^\ell(\xi),$$

which converges in the mean; here

$$a_K^\ell(F) = \frac{1}{\omega_n} \int_{\mathbb{S}^n} F(\xi) \overline{\Xi_K^\ell(\xi)}\, d\omega_n(\xi)$$

(see [18, Chapter 9, §3, item 7, (2), (3)]). Using (49), (12), and the orthogonality relations for the Gegenbauer polynomials, we see that $a_K^\ell(F) = 0$ for $\ell < k$ or $\ell \geq k$ and $K \neq K' = (k, 0, \dots, 0)$. Consequently,

$$F(\xi) = \sum_{\ell=k}^{\infty} a_{K'}^\ell(F) \Xi_{K'}^\ell(\xi), \tag{50}$$

where

$$\begin{aligned} a_{K'}^\ell(F) = \frac{2^k \Gamma\left(\frac{n-1}{2} + k\right)}{\Gamma\left(\frac{n-2}{2}\right)} \left(\frac{(n-1)(\ell-k)!(n+2\ell-1)\Gamma(n+k-2)}{\pi(n+2k-2)k!\Gamma(\ell+k+n-1)} \right)^{1/2} \\ \times \int_0^\pi \Phi(\theta) C_{\ell-k}^{\frac{n-1}{2}+k}(\cos\theta)(\sin\theta)^{k+n-1}\, d\theta. \end{aligned} \tag{51}$$

Since $F \in C^\infty(\mathbb{S}^n)$, the series (50) converges absolutely and uniformly on $\mathbb{S}^n$ (see (15) and (17)).

Next, let a_r be the mapping defined in Lemma 3. By (50) and the assumptions of Lemma 11, we have

$$\sum_{\ell=k}^{\infty} a_{K'}^\ell(F) \int_{B_t} \Xi_{K'}^\ell(a_r\xi)\, d\omega_n(\xi) = \int_{B_t} F(a_r\xi)\, d\omega_n(\xi) = 0, \quad 0 < t < R - r - \varepsilon. \tag{52}$$

After differentiation in t, identity (52) yields

$$\begin{aligned} \sum_{\ell=k}^{\infty} \frac{\ell!(\ell-k)!(\ell+\frac{n-1}{2})}{\Gamma(\ell+k+n-1)\Gamma(\ell+n-1)} \int_0^\pi \Phi(\theta) C_{\ell-k}^{\frac{n-1}{2}+k}(\cos\theta)(\sin\theta)^{k+n-1}\, d\theta \\ \times C_{\ell-k}^{\frac{n-1}{2}+k}(\cos r) C_\ell^{\frac{n-1}{2}}(\cos t) = 0, \quad 0 < t < R - r - \varepsilon \end{aligned} \tag{53}$$

(see the proof of Lemma 3 and (51)). Combining this and (24), we see that

$$\sum_{\ell=k}^{\infty} \int_0^\pi \Phi(\theta) C_{\ell-k}^{\frac{n-1}{2}+k}(\cos\theta)(\sin\theta)^{k+n-1}\, d\theta, \quad b_{\ell-k}(H) = 0, \quad 0 < t < R - r - \varepsilon.$$

By using Lemma 4 and the Hölder inequality, it is not hard to show that this identity is equivalent to the equation

$$\int_{|t-r|}^{t+r} \Phi(\theta) H(\theta)(\sin\theta)^{k+n-1}\, d\theta = 0, \quad 0 < t < R - r - \varepsilon.$$

Also, we have $\Phi = 0$ on $[0; r]$ by construction. Then, by Lemma 5, $\Phi = 0$ on $[0; R - \varepsilon]$. Since $\varepsilon \in (0; R - r)$, $k \in \mathbb{Z}_+$, and $l \in \{1, \dots, a_k\}$ are arbitrary, we conclude that $f = 0$ in B_R. □

Lemma 12. *Let $f \in C^\infty(B_R)$. Then f belongs to $\mathfrak{S}_r(B_R)$ if and only if for any $k \in \mathbb{Z}_+$, and any $1 \le l \le a_k$ we have*

$$f^{k,l,l} = \sum_{\nu \in \mathcal{N}_k(r)} c_{\nu,k,l} \Psi_\nu^{k,l}, \tag{54}$$

where

$$c_{\nu,k,l} = \frac{1}{\delta(\nu,\nu)} \int_0^r f_{k,l}(\theta) \psi_{\nu,k}(\theta) (\sin\theta)^{n-1}\, d\theta, \tag{55}$$

$$c_{\nu,k,l} = O\big(\nu^{-m}\big) \quad \textit{for} \quad \nu \to +\infty, \;\; m > 0. \tag{56}$$

The series (54) *converges uniformly on the compact subsets of B_R.*

Proof. Let $f \in \mathfrak{S}_r^\infty(B_R)$, and k, l be fixed. By Lemma 10 and (35), the function $f^{k,l,l}$ belongs to $\mathfrak{S}_r^\infty(B_R)$ and we have

$$(l_k^m f_{k,l})(r) = 0, \quad m \in \mathbb{Z}_+. \tag{57}$$

For $\nu \in \mathcal{N}_k(r)$, let $c_{\nu,k,l}$ be defined by (55). Let $s \in \mathbb{Z}_+$. In (55), we integrate by parts $2s$ times, using (30)–(32) and the boundary conditions (57). As a result, we arrive at

$$c_{\nu,k,l} = \frac{1}{\delta(\nu,\nu)\big(\nu(1-n-\nu)\big)^s} \int_0^r \big(l_k^s f_{k,l}\big)(\theta) \psi_{\nu,k}(\theta) (\sin\theta)^{n-1}\, d\theta. \tag{58}$$

This representation and estimates (41), (44) show that inequality (56) is true and that the function φ defined by

$$\varphi(\theta) = \sum_{\nu \in \mathcal{N}_k(r)} c_{\nu,k,l}\, \psi_{\nu,k}(\theta), \tag{59}$$

belongs to $C^\infty[0, R)$. Formula (59) and Lemma 8 (iv) yield

$$c_{\nu,k,l} = \frac{1}{\delta(\nu,\nu)} \int_0^r \varphi(\theta) \psi_{\nu,k}(\theta) (\sin\theta)^{n-1}\, d\theta. \tag{60}$$

Comparing (60) and (55) and applying Lemma 8 (iii), we conclude that $f_{k,l} = \varphi$ on $[0, r]$. Also, we have $\big(f_{k,l}(\theta_n) - \varphi(\theta_n)\big) Y_l^{(k)}(\sigma) \in \mathfrak{S}_r^\infty(B_R)$ (see (59) and (38)). Then the expansion (54) is valid by Lemma 11.

Conversely, if we have (54) with the coefficient estimate (56), then, as above, the series in (54) converges uniformly on the compact subsets of B_R, and $f^{k,l,l} \in \mathfrak{S}_r(B_R)$. Therefore, $f \in \mathfrak{S}_r(B_R)$. □

Next, let $R \in (0; \pi]$, and let $r_1, r_2 \in (0; R)$, $r_1 \ne r_2$. Put

$$\mathfrak{S}_{r_1,r_2}(B_R) = \big(\mathfrak{S}_{r_1} \cap \mathfrak{S}_{r_2}\big)(B_R).$$

Proposition 1. *Let $\mathfrak{S}_{r_1,r_2}(B_R)$, where $r_1 + r_2 \le R$ and let $\mathcal{N}(r_1, r_2) = \varnothing$. Then $f = 0$ in B_R.*

Proof. It suffices to check that $f^{k,l,l} = 0$ for every k, l. Put $F = f^{k,l,l}$ and $F_\varepsilon = F * \varphi_\varepsilon$, where $\varepsilon \in (0; R - \max\{r_1, r_2\})$ (see §3). Lemmas 2, 9, and 10 (ii) show that $F_\varepsilon \in (\mathfrak{S}_{r_1,r_2} \cap C^\infty)(B_{R-\varepsilon})$ and that $(F_\varepsilon)^{k,l,l} = F_\varepsilon$. Then, by Lemma 12,

$$F_\varepsilon = \sum_{\nu \in \mathcal{N}_k(r_1)} a_{\nu,k,l}(F_\varepsilon) \Psi_\nu^{k,l} = \sum_{\nu \in \mathcal{N}_k(r_2)} b_{\nu,k,l}(F_\varepsilon) \Psi_\nu^{k,l} \quad \text{in} \quad B_{R-\varepsilon}, \tag{61}$$

where, for any $m > 0$,

$$a_{\nu,k,l}(F_\varepsilon) = \frac{1}{\delta(\nu,\nu)} \int_{B_{r_1}} F_\varepsilon(\xi) \overline{\Psi_\nu^{k,l}(\xi)}\, d\omega_n(\xi) = O(\nu^{-m}) \quad \text{as} \quad \nu \to +\infty,$$
$$b_{\nu,k,l}(F_\varepsilon) = \frac{1}{\delta(\nu,\nu)} \int_{B_{r_2}} F_\varepsilon(\xi) \overline{\Psi_\nu^{k,l}(\xi)}\, d\omega_n(\xi) = O(\nu^{-m}) \quad \text{as} \quad \nu \to +\infty.$$

Passing in (61) to the limit as $\varepsilon \to +0$ in the space $\mathcal{D}'(B_R)$ and using Lemma 1, (39), and (41), we put

$$f^{k,l,l} = \sum_{\nu \in \mathcal{N}_k(r_1)} \alpha_{\nu,k,l} \Psi_\nu^{k,l} = \sum_{\nu \in \mathcal{N}_k(r_2)} \beta_{\nu,k,l} \Psi_\nu^{k,l},$$

where the series converge in $\mathcal{D}'(B_R)$, and

$$|\alpha_{\nu,k,l}| + |\beta_{\nu,k,l}| = O\Big(\frac{1}{\nu^k \delta(\nu,\nu)}\Big) = O(\nu^{n+k}), \quad \nu \to +\infty. \tag{62}$$

Given an integer $m \geq n + k + 2$, we choose numbers $\lambda_1, \dots, \lambda_m$ with the following properties:

(a) $\lambda_1, \dots, \lambda_m \in [-k - n + 1; k]$;

(b) $(\lambda_i - \lambda_j)(\lambda_i + \lambda_j + n - 1) \neq 0$ for $i \neq j$;

(c) if $k \geq 1$, then $\lambda_i = i - 1$ for $1 \leq i \leq k$.

We put

$$h = \sum_{\nu \in \mathcal{N}_k(r_1)} \frac{\alpha_{\nu,k,l} \Psi_\nu^{k,l}}{(c_\nu - c_{\lambda_1}) \dots (c_\nu - c_{\lambda_m})} - \sum_{\nu \in \mathcal{N}_k(r_2)} \frac{\beta_{\nu,k,l} \Psi_\nu^{k,l}}{(c_\nu - c_{\lambda_1}) \dots (c_\nu - c_{\lambda_m})}, \tag{63}$$

where $c_\nu = \nu(1 - n - \nu)$ (see (a) and (36), (43)). Relations (62), (37), (41), (44), and (33) show that the series on the left-hand side in (63) converges uniformly on the compact subsets of B_R, $h \in C(B_R)$, and

$$(L - c_{\lambda_1} \operatorname{Id}) \dots (L - c_{\lambda_m} \operatorname{Id}) h = 0 \tag{64}$$

in the space $\mathcal{D}'(B_R)$. Since the operator L is elliptic, the function h is of class $C^\infty(B_R)$, and identity (64) is valid in the usual sense (see [20, Chapter 8.6]). Since h is smooth at the point zero, and the c_{λ_j} are distinct (see (b) and (36)), identities (64) and (35) yield the representation

$$h = \sum_{j=1}^m \gamma_j \Psi_{\lambda_j}^{k,l}, \quad \gamma_j \in \mathbb{C}. \tag{65}$$

We equate the right-hand sides of the expansions (63) and (65) and integrate them over the sphere $S_t(\eta)$, where $\eta \in S_{r_2}$, $0 < t < r_1$. Using (38), we obtain

$$\sum_{j=1}^m \gamma_j \Psi_{\lambda_j}^{k,l}(\eta) \psi_{\lambda_j,0}(t) = \sum_{\nu \in \mathcal{N}_k(r_1)} \frac{\alpha_{\nu,k,l} \Psi_\nu^{k,l}(\eta) \psi_{\nu,0}(t)}{(c_\nu - c_{\lambda_1}) \dots (c_\nu - c_{\lambda_m})}. \tag{66}$$

If $k \geq 1$, then we apply the operator $d_{k-1} \dots d_0$ to the two sides of (66) and use (32), (62), and (41) to get the identity

$$\sum_{j=1}^m \delta_j \psi_{\lambda_j,k}(t) = \sum_{\nu \in \mathcal{N}_k(r_1)} \frac{\alpha_{\nu,k,l} \Psi_\nu^{k,l}(\eta) \psi_{\nu,k}(t)}{(c_\nu - c_{\lambda_1}) \dots (c_\nu - c_{\lambda_m})} \prod_{q=0}^{k-1} (q - \nu)(q + \nu + n - 1),$$

where

$$\delta_j = \gamma_j \Psi_{\lambda_j}^{k,l}(\eta) \prod_{q=0}^{k-1} (q - \nu)(q + \nu + n - 1).$$

(The product over q is assumed to be equal to one if $k=0$.) Referring to Lemma 8 (iv) (see also (c)), we see that

$$\begin{aligned}&\frac{\delta(\nu,\nu)\,\alpha_{\nu,k,l}\,\Psi_\nu^{k,l}(\eta)}{(c_\nu-c_{\lambda_1})\dots(c_\nu-c_{\lambda_m})}\prod_{q=0}^{k-1}(q-\nu)(q+\nu+n-1)\\ &\quad=(\sin r_1)^{n-1}(k-\nu)(k+\nu+n-1)\psi_{\nu,k+1}(r_1)\sum_{j=k+1}^{m}\frac{\delta_j\,\psi_{\lambda_j,k}(r_1)}{(\lambda_j-\nu)(\lambda_j+\nu+n-1)}.\end{aligned}\tag{67}$$

Observe that, by (a) and (43), $\psi_{\lambda_j,k}(r_1)\neq 0$ for any $j=1,\dots,m$.

Suppose that not all among the δ_j are equal to zero. Then (45) shows that, for all sufficiently large $\nu\in\mathcal{N}_k(r_1)$, the absolute value of the right-hand side in (67) is at least $c\,\nu^{-2m+k-\frac{n-3}{2}}$ with $c>0$. On the other hand, in accordance with (62) and (40), the absolute value of the left-hand side in (67) is $O\big(\nu^{-m-\frac{n-1}{2}}\big)$. Comparing these estimates, we arrive at the inconsistent relation $\nu^{k+1}=O(1)$, $\nu\in\mathcal{N}_k(r_1)$. Thus, $\delta_j=0$ for all $j=1,\dots,m$. Since $\eta\in S_{r_2}$ in (67) is arbitrary, we see that $\alpha_{\nu,k,l}\psi_{\nu,k}(r_2)=0$ for $\nu\in\mathcal{N}_k(r_1)$. Recalling that $\mathcal{N}_k(r_1,r_2)=\varnothing$, we conclude that this is possible only if $\alpha_{\nu,k,l}=0$, $\nu\in\mathcal{N}_k(r_1)$. Therefore, $f^{k,l,l}=0$ in B_R, as claimed. □

Proposition 2. *If $\mathcal{N}(r_1,r_2)\neq\varnothing$ or $r_1+r_2>R$, then there exists a nonzero function $f\in\big(\mathfrak{S}_{r_1,r_2}\cap C^\infty\big)(B_R)$.*

Proof. If $\mathcal{N}(r_1,r_2)\neq\varnothing$, then $\mathcal{N}_k(r_1,r_2)\neq\varnothing$ for some $k\in\mathbb{Z}_+$. In this case, $\Psi_\nu^{k,l}\in(\mathfrak{S}_{r_1,r_2}\cap C^\infty)(B_R)$ for $\nu\in\mathcal{N}_k(r_1,r_2)$. Now, suppose that $\mathcal{N}(r_1,r_2)=\varnothing$ and $r_1+r_2>R$. Then we can find a nonzero function $f(\xi)=\varphi(\theta_n)\in C^\infty(B_R)$ such that $f*\chi_{r_i}=0$ in B_{R-r_i} $(i=1,2)$ (see [12, the proof of Theorem 2.1(5)]). The function φ is not constant, and for any $m>0$ we have

$$\varphi(\theta_n)=\sum_{\nu\in\mathcal{N}_1(r_1)}\alpha_\nu\psi_{\nu,0}(\theta_n)=\sum_{\nu\in\mathcal{N}_1(r_2)}\beta_\nu\psi_{\nu,0}(\theta_n),$$

where $|\alpha_\nu|+|\beta_\nu|=O(\nu^{-m})$ as $\nu\to+\infty$ (see [12, Lemma 8.3]). Using (32) and Lemma 12 with $k=1$, we conclude that the nonzero function $\varphi'(\theta_n)Y_1^{(1)}(\sigma)$ is of class $(\mathfrak{S}_{r_1,r_2}\cap C^\infty)(B_R)$, which completes the proof. □

§6. Proof of main results

For $\alpha>0$ and $x\in\mathbb{R}^{n+1}$, put

$$\mathcal{B}_\alpha(x)=\{y\in\mathbb{R}^{n+1}\,:\,|x-y|<\alpha\},\quad \mathcal{B}_\alpha=\mathcal{B}_\alpha(0),$$
$$\mathcal{S}_\alpha(x)=\{y\in\mathbb{R}^{n+1}\,:\,|x-y|=\alpha\},\quad \mathcal{S}_\alpha=\mathcal{S}_\alpha(0).$$

The identity

$$\int_{\mathcal{B}_\alpha(x)}f(y)\,dy=\int_0^\alpha\int_{\mathcal{S}_t(x)}f(u)d\omega_n(u)\,dt\tag{68}$$

shows that a function $f\in L^{1,\mathrm{loc}}(K_R)$ belongs to the kernel of $\mathcal{R}$ relative to the set ∂K_r if and only if

$$\int_{\mathcal{B}_\alpha(x)}f(y)\,dy=0$$

for all $x\in\partial K_r$, $\alpha\in(0;\mathrm{dist}(x,\partial K_R))$. The next statement gives a characterization of the functions that belong to this kernel in terms of the class $\mathfrak{S}_r(B_R)$.

Lemma 13. *Suppose that $f\in L^{1,\mathrm{loc}}(K_R)$ and $r\in(0;R)$. The following statements are equivalent*:

(i) $\mathcal{R}f(x,t)=0$ *for every* $x\in\partial K_r$ *and almost every* $t\in(0;\operatorname{dist}(x,\partial K_R))$;
(ii) *for a.e.* $\rho\in(0;+\infty)$, *the function* $f(\rho\xi)$, *defined on* B_R, *belongs to* $\mathfrak{S}_r(B_R)$.

Proof. Let a nonnegative and $\mathrm{O}(n+1)$-invariant function ψ_ε with support in $\mathcal{B}_\varepsilon$ be such that

$$\int_{\mathcal{B}_\varepsilon}\psi_\varepsilon(x)\,dx=1.$$

The function

$$\mathcal{F}_\varepsilon(x)=\int_{\mathcal{B}_\varepsilon}f(x-y)\psi_\varepsilon(y)\,dy \tag{69}$$

possesses the following properties: (a) $\mathcal{F}_\varepsilon\in C^\infty(K_R^\varepsilon)$, where

$$K_R^\varepsilon=\{x\in\mathbb{R}^{n+1}\,:\,x-\mathcal{B}_\varepsilon\subset K_R\};$$

(b) for any $a,b>0$, any $\eta\in S_r$, and any $t\in(0;R-r)$ we have

$$\lim_{\varepsilon\to0}\int_a^b\rho^n\int_{B_t(\eta)}|\mathcal{F}_\varepsilon(\rho\xi)-f(\rho\xi)|d\omega_n(\xi)\,d\rho=0; \tag{70}$$

(c)

$$\int_{\mathcal{B}_\alpha(w)}\mathcal{F}_\varepsilon(x)\,dx=\int_{\mathcal{S}_\alpha(w)}\mathcal{F}_\varepsilon(u)\,d\omega_n(u)=0, \tag{71}$$

where $\mathcal{B}_\alpha(w)$ is an arbitrary ball with center on ∂K_r whose closure lies in K_R^ε (see (68), (69)). For any sufficiently small number h, from (71) it follows that

$$\int_{\mathcal{B}_\alpha}\mathcal{F}_\varepsilon(w(1+h)+x)\,dx=\int_{\mathcal{B}_\alpha(w(1+h))}\mathcal{F}_\varepsilon(x)\,dx=0,$$

$$\int_{\mathcal{B}_\alpha}\frac{\mathcal{F}_\varepsilon(w(1+h)+x)-\mathcal{F}_\varepsilon(w+x)}{h|w|}\,dx=0. \tag{72}$$

The limit passage as $h\to0$ yields

$$\int_{\mathcal{B}_\alpha}\left(\frac{\partial\mathcal{F}_\varepsilon}{\partial x_1}(w+x)w_1+\cdots+\frac{\partial\mathcal{F}_\varepsilon}{\partial x_{n+1}}(w+x)w_{n+1}\right)\,dx=0.$$

Consequently, by the Stokes formula,

$$\int_{\mathcal{S}_\alpha}\mathcal{F}_\varepsilon(w+u)(w,u)\,d\omega_n(u)=0,$$

whence

$$\int_{\mathcal{S}_\alpha(w)}\mathcal{F}_\varepsilon(u)(w,u)\,d\omega_n(u)=0$$

(see (71)). Since the scalar product is homogeneous, this shows that the limit passage described above admits iteration. As a result, we have

$$\int_{\mathcal{S}_\alpha(w)}\mathcal{F}_\varepsilon(u)p\left((w,u)\right)d\omega_n(u)=0 \tag{73}$$

for any algebraic polynomial p. Relation (73) implies the identity

$$\int_{\mathbb{S}^n}\mathcal{F}_\varepsilon(w+\alpha\xi)p\left((w/|w|,\xi)\right)\,d\omega_n(\xi)=0,$$

which can be rewritten in the form

$$\int_0^\pi p(\cos t)\int_{S_t(w/|w|)}\mathcal{F}_\varepsilon(w+\alpha\xi)\,d\omega_{n-1}(\xi)\,dt=0.$$

Since p is arbitrary, we obtain

$$\int_{\mathcal{M}} \mathcal{F}_\varepsilon(u)\,d\omega_{n-1}(u) = 0, \quad t \in (0;\pi), \tag{74}$$

where $\mathcal{M} = \{u \in \mathcal{S}_\alpha(w) : (w,u) = |w|^2 + \alpha|w|\cos t\}$. The set $\mathcal{M}$ is the sphere on $\mathcal{S}_\alpha(w)$ with center at $(1+\alpha|w|^{-1})w$ and of radius αt. Therefore, from (74) it follows that for any $a, b > 0$, $\eta \in S_r$, and $t \in (0; R-r)$ we have

$$\int_a^b \rho^n \int_{B_t(\eta)} \mathcal{F}_\varepsilon(\rho\xi)\,d\omega_n(\xi)\,d\rho = 0$$

for all sufficiently small $\varepsilon > 0$. Letting $\varepsilon \to 0$ and using (70), we arrive at a similar relation for the function f, proving the implication i) $\Rightarrow$ ii).

The reverse implication follows from the formula

$$\int_{\mathcal{B}_\alpha(w)} f(x)\,dx = \int_{|w|-\alpha}^{|w|+\alpha} \int_{\mathcal{B}_\alpha(w)\cap\mathcal{S}_\rho} f(u)\,d\omega_n(u)\,d\rho.$$

□

Proof of Theorem 1. We prove the "only if" part. Given $0 < a < b$ and $0 < R' < R$, put $K_{R'}^{a,b} = \bigcup_{\rho\in(a;b)} \rho B_{R'}$. Let ψ be an infinitely differentiable function with support in $K_{R'}^{a,b}$. From (4), (5), and Lemma 13 it follows that (see the proof of Proposition 1)

$$\int_{K_{R'}^{a,b}} F_{k,l}(x)\psi(x)\,dx = \int_a^b \sum_{\nu\in\mathcal{N}_k(r)} \rho^n c_{\nu,k,l}(\rho) \int_{B_{R'}} \Psi_\nu^{k,l}(\xi)\psi(\rho\xi)\,d\omega_n(\xi)\,d\rho, \tag{75}$$

where

$$c_{\nu,k,l}(\rho) = \frac{1}{\delta(\nu,\nu)} \int_{B_r} f(\rho\xi)\overline{\Psi_\nu^{k,l}(\xi)}\,d\omega_n(\xi).$$

We have

$$\int_a^b \rho^n |c_{\nu,k,l}(\rho)|\,d\rho = O(\nu^{n+k}), \quad \nu \to +\infty \tag{76}$$

(see (37), (41), and (46)). Next, the identity

$$L(\psi(\rho\xi)) = \frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1}\left(\left(x_i\frac{\partial}{\partial x_j} - x_i\frac{\partial}{\partial x_j}\right)^2\psi\right)(\rho\xi)$$

(see, e.g., [21, Chapter. 6, §4]) and relation (42) show that for any $m > 0$ we have

$$\sup_{\varrho\in[a;b]} \left| \int_{B_{R'}} \Psi_\nu^{k,l}(\xi)\psi(\rho\xi)\,d\omega_n(\xi)\right| = O\Big(\frac{1}{\nu^m}\Big), \quad \nu \to +\infty. \tag{77}$$

Using estimates (76), (77) and the B. Levi theorem, we see that the function

$$g(\rho) = \sum_{\nu\in\mathcal{N}_k(r)} \rho^n |c_{\nu,k,l}(\rho)| \left|\int_{B_{R'}} \Psi_\nu^{k,l}(\xi)\psi(\rho\xi)\,d\omega_n(\xi)\right|$$

is integrable on $[a;b]$. Then the dominated convergence theorems applies, and from (75) we obtain

$$\int_{K_{R'}^{a,b}} F_{k,l}(x)\psi(x)\,dx = \sum_{\nu\in\mathcal{N}_k(r)} \int_{K_{R'}^{a,b}} c_{\nu,k,l}(|x|)\Psi_\nu^{k,l}(x/|x|)\psi(x)\,dx.$$

This proves the "only if" part in Theorem 1. The "if" part follows from (38), (5), and Lemma 13. □

Proof of Theorem 2. We start with proving the first claim. By Lemma 13, there exists a set $E \subset (0,+\infty)$ of zero Lebesgue measure such that for any $\rho \in (0,+\infty) \setminus E$ the function $f(\rho\xi)$ belongs to $\mathfrak{S}_r(B_R)$ and vanishes in B_r. For such ρ and fixed k,l, we put $F = u^{k,l,l}$, where $u(\xi) = f(\rho\xi)$ in B_R. Also, let $F_\varepsilon = F * \varphi_\varepsilon$, $\varepsilon \in (0; R-r)$ (see §3). As in the proof of Proposition 1, we have

$$F_\varepsilon = \sum_{\nu \in \mathcal{N}_k(r)} c_{\nu,k,l}(F_\varepsilon)\Psi_\nu^{k,l} \text{ in } B_{R-\varepsilon}, \tag{78}$$

where, for any $m > 0$,

$$c_{\nu,k,l}(F_\varepsilon) = \frac{1}{\delta(\nu,\nu)} \int_{B_r} F_\varepsilon(\xi)\overline{\Psi_\nu^{k,l}(\xi)}\, d\omega_n(\xi) = O(\nu^{-m}) \text{ as } \nu \to +\infty.$$

Passing in (78) to the limit as $\varepsilon \to +0$ in the space $\mathcal{D}'(B_R)$ and using Lemma 1 and relations (39), (41) we obtain

$$F = \sum_{\nu \in \mathcal{N}_k(r)} \gamma_{\nu,k,l}\Psi_\nu^{k,l} \tag{79}$$

with

$$\gamma_{\nu,k,l} = \frac{1}{\delta(\nu,\nu)} \int_0^r u_{k,l}(\theta)\psi_{\nu,k}(\theta)(\sin\theta)^{n-1}\, d\theta \tag{80}$$

and the series (79) converges locally uniformly in B_R. Since $u = 0$ in B_r, (80) implies that $\gamma_{\nu,k,l} = 0$. By (79), then $F = 0$ in B_R. Since ρ, k, l were chosen arbitrarily, we arrive at statement (i).

In oreder to prove the second statement, we consider a nonzero $O(n)$-invariant function $g \in C^\infty(\overline{B_r})$ such that $g = 0$ in $B_{r-\varepsilon} \cup (\overline{B_r} \setminus B_{r-\varepsilon/2})$. Then $g = g^{0,1,1}$, and identity (57) is fulfilled for $k = 0$, $l = 1$, and all $m \in \mathbb{Z}_+$. The proof of Lemma 12 shows that in $\overline{B_r}$ we have an expansion of the form

$$g = \sum_{\nu \in \mathcal{N}_0(r)} c_\nu \Psi_\nu^{0,1}, \tag{81}$$

where $c_\nu = O(\nu^{-m})$ as $\nu \to +\infty$ for any fixed $m > 0$. The series in (81) converges in the space $C^\infty(B_R)$. Extending g to B_R with the help of (81), we get $g \in (\mathfrak{S}_r \cap C^\infty)(B_R)$. Next, take $h \in C^\infty(0,+\infty)$ with $\operatorname{supp} h \subset [r-\varepsilon, r-\varepsilon/2]$, and let the function $f(x) = h(|x|)g(x/|x|)$ be nonzero. Then f obeys all requirements of statement (ii). □

Proof of Corollary 1. First, we treat the case where $f = 0$ in $K_r \setminus \bar{K}_{2r-R}$. Then, to deduce the required claim from Theorem 2, it suffices to extend f by zero to K_{2r-R}.

Now, suppose that $f = 0$ in $K_R \setminus \bar{K}_r$. The set $\mathcal{U}_1 = \{\xi \in \mathbb{S}^n : r < d(o,\xi) < R\}$ can be presented in the form

$$\mathcal{U}_1 = \big\{\xi \in \mathbb{S}^n : 2r_1 - R_1 < d(\operatorname{Ant}\{o\}, \xi) < r_1\big\},$$

where $r_1 = \pi - r$, $R_1 = \pi - 2r + R$, $\operatorname{Ant}\{o\} = (0,\dots,0,-1) \in \mathbb{S}^n$. Therefore, by using what has been proved in the first case, once again we conclude that $f = 0$ in $\mathcal{U}$. This completes the proof of Corollary 1. □

Proof of Theorem 3. Suppose that the set $\partial K_{r_1} \cup \partial K_{r_2}$ is an injectivity set of the Radon transformation $\mathcal{R}$ for $L^{1,\mathrm{loc}}(K_R)$. If $r_1 + r_2 > R$ or $\mathcal{N}(r_1,r_2) \neq \varnothing$, then, by Proposition 2, the class $(\mathfrak{S}_{r_1,r_2} \cap C^\infty)(B_R)$ contains a nonzero function f. In this case, by Lemma 13, the function $f(x/|x|)$ belongs to the kernel of $\mathcal{R}$ relative to $\partial K_{r_1} \cup \partial K_{r_2}$, which contradicts the definition of an injectivity set. Thus, $r_1 + r_2 \le R$ and $\mathcal{N}(r_1,r_2) = \varnothing$. The converse statement can be proved similarly, with the use of Lemma 13 and Proposition 2. □

§7. Applications to partial differential equations

Let
$$G_R = \{(x,\rho) \in \mathbb{R}^{n+1} \times [0;+\infty) \colon x \in K_R,\, 0 \le \rho < \operatorname{dist}(x, \partial K_R)\}.$$
Consider the Darboux equation

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{n}{\rho}\frac{\partial u}{\partial \rho} = \Delta_x u, \quad u = u(x,\rho), \quad (x,\rho) \in G_R, \tag{82}$$

where Δ_x is the Laplace operator in the variables $x_1, \dots, x_{n+1}$.

Theorem 4. *Let $u \in C^2(G_R)$ be a solution of equation* (82). *Suppose that $r_1, r_2 \in (0;\pi)$ are fixed, $\max\{r_1, r_2\} < R$, and that $u(x,\rho) = 0$ for $x \in \partial K_{r_1} \cup \partial K_{r_2}$, $0 < \rho < \operatorname{dist}(x, \partial K_R)$. If $r_1 + r_2 \le R$ and $\mathcal{N}(r_1, r_2) = \varnothing$, then $u = 0$ in G_R. If $r_1 + r_2 > R$ or $\mathcal{N}(r_1, r_2) \ne \varnothing$, then this claim may fail.*

Proof. Let $r_1 + r_2 \le R$, and let $\mathcal{N}(r_1, r_2) = \varnothing$. Let $x \in K_R$ and $y \in \mathbb{R}^{n+1}$ be such that $|y| < \operatorname{dist}(x, \partial K_R)$; we put $U(x,y) = u(x,|y|)$. The function U satisfies the equation
$$\triangle_y U(x,y) = \frac{\partial^2 u}{\partial \rho^2}(x,|y|) + \frac{n}{|y|}\frac{\partial u}{\partial \rho}(x,|y|) = \triangle_x U(x,y).$$
By the Asgeirsson theorem (see [2, Chapter 2, item 5.6]), we have
$$\int_{\mathcal{S}_\rho(x)} U(\eta, 0) d\omega_n(\eta) = \int_{\mathcal{S}_\rho} U(x,y)\, d\omega_n(y).$$
Using this identity and the assumptions, we see that
$$u(x,\rho) = \frac{1}{\omega_n \rho^n} \int_{\mathcal{S}_\rho(x)} u(\eta, 0)\, d\omega_n(\eta) = 0$$
for $x \in \partial K_{r_1} \cup \partial K_{r_2}$, $0 < \rho < \operatorname{dist}(x, \partial K_R)$. Now Theorem 3 shows that $u = 0$ in G_R.

Next, we suppose that $r_1 + r_2 > R$ or $\mathcal{N}(r_1, r_2) \ne \varnothing$. The proof of Theorem 3 shows that the kernel of $\mathcal{R}$ relative to the set $\partial K_{r_1} \cup \partial K_{r_2}$ contains a nonzero function $f \in C^\infty(K_R)$. Then the function $\mathcal{R}f(x,\rho)$ is a nonzero solution of equation (82) and $\mathcal{R}f(x,\rho) = 0$ for $x \in \partial K_{r_1} \cup \partial K_{r_2}$, $0 < \rho < \operatorname{dist}(x, \partial K_R)$ (see [2, Chapter 1, item 2.3] and §1 in the present paper). Thus, Theorem 4 is proved completely. □

Similarly, from Corollary 1 we deduce the following result.

Theorem 5. *Let $u \in C^2(G_R)$ be a solution of equation* (82). *Suppose that $R < 2r$, $u(x,0) = 0$ in the domain $K_r \setminus \bar{K}_{2r-R}$, and $u(x,\rho) = 0$ for $x \in \partial K_r$, $0 < \rho < \operatorname{dist}(x, \partial K_R)$. Then $u = 0$ in the domain*
$$\{(x,\rho) \in \mathbb{R}^{n+1} \times [0;+\infty) : x \in K_R \setminus \bar{K}_{2r-R},\, 0 < \rho < \min\{\operatorname{dist}(x, \partial K_R), \operatorname{dist}(x, \partial K_{2r-R})\}\}.$$

Consider the Cauchy problem for the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta_x u, \quad u = u(x,t), \quad (x,t) \in G_R \tag{83}$$

with the initial data

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = f(x), \quad x \in K_R. \tag{84}$$

Theorem 6. *Let u be a solution of problem* (83), (84) *with $f \in C^m(K_R)$, $m = \left[\frac{n+3}{2}\right]$. Suppose that $r_1, r_2 \in (0;\pi)$, $\max\{r_1, r_2\} < R$, and that $u(x,t) = 0$ for $x \in \partial K_{r_1} \cup \partial K_{r_2}$, $0 < t < \operatorname{dist}(x, \partial K_R)$. If $r_1 + r_2 \le R$ and $\mathcal{N}(r_1, r_2) = \varnothing$, then $u = 0$ in G_R. If $r_1 + r_2 > R$ or $\mathcal{N}(r_1, r_2) \ne \varnothing$, then this claim may fail.*

For the proof of Theorem 6 we need an auxiliary statement.

Lemma 14. *Let* $f \in C^m(K_R)$, $m = \left[\frac{n+3}{2}\right]$. *Then*

$$\left\{x \in K_R \,:\, \mathcal{R}f(x,\rho) = 0,\, \rho \in (0;\operatorname{dist}(x,\partial K_R))\right\} = \left\{x \in K_R \,:\, u(x,t) = 0,\, t \in (0;\operatorname{dist}(x,\partial K_R))\right\},$$

where u *is a solution of problem* (83), (84).

Proof. For $f \in C^m(K_R)$, the solution of (83), (84) is given by the Poisson–Kirchhoff formula

$$u(x,t) = \frac{1}{(n-1)!}\frac{\partial^{n-1}}{\partial t^{n-1}}\left(F(x,t)\right), \tag{85}$$

where

$$F(x,t) = \int_0^t (t^2-\rho^2)^{(n-2)/2}\rho\,\mathcal{R}f(x,\rho)\,d\rho \tag{86}$$

(see [2, Chapter 1, §2, item 7]). Therefore, for $x \in K_R$ fixed, the identity $u(x,t) = 0$ is equivalent to the fact that $F(x,t)$ is a polynomial in t of degree at most $n-2$. The change $\rho = tv$ in the integral (86) yields the representation

$$F(x,t) = t^n \int_0^1 (1-v^2)^{(n-2)/2} v\,\mathcal{R}f(x,vt)\,dv,$$

whence $F(x,t) = O(t^n)$, $t \to 0$. This is possible only if $F(x,t) = 0$ for any $t \in (0;\operatorname{dist}(x,\partial K_R))$. Thus, $\mathcal{R}f$ satisfies the integral equation

$$\int_0^t (t^2-\rho^2)^{(n-2)/2}\rho\,\mathcal{R}f(x,\rho)\,d\rho = 0,$$

which implies that $\mathcal{R}f(x,\rho) = 0$ for $\rho \in (0;\operatorname{dist}(y,\partial K_R))$ (see the proof of Lemma 5). □

Proof of Theorem 6. If $r_1+r_2 \le R$ and $\mathcal{N}(r_1,r_2) = \varnothing$, then Lemma 14 and Proposition 1 show that $f = 0$ in K_R. Consequently, $u = 0$ in G_R. Suppose that $r_1 + r_2 > R$ or $\mathcal{N}(r_1,r_2) \neq \varnothing$. Now we define u by the identity (85) in which f is a nonzero function of class $C^\infty(K_R)$ that belongs to the kernel of $\mathcal{R}$ relative to the set $\partial K_{r_1} \cup \partial K_{r_2}$ (see the proof of Theorem 3). Then u is a nonzero solution of problem (83), (84), and $u(x,t) = 0$ for $x \in \partial K_{r_1} \cup \partial K_{r_2}$, $0 < t < \operatorname{dist}(x,\partial K_R)$. This proves Theorem 6. □

The results of this paper were reported by the first author at L. Zaltsman's seminar in Bar-Ilan university, Israel. The authors are grateful to L. Zaltsman and the members of his seminar for numerous helpful discussions, and also to S. Helgason for a copy of the book [15].

References

[1] A. B. Aleksandrov, *A-integrability of boundary values of harmonic functions*, Mat. Zametki **30** (1981), no. 1, 59–72; English transl., Math. Notes **30** (1981), no. 1, 515–523. MR627941

[2] S. Helgason, *Groups and geometric analysis. Integral geometry, invariant differential operators and special functions*, Math. Surveys Monogr., vol. 83, Amer. Math. Soc., Providence, RI, 2000. MR1790156

[3] V. V. Volchkov, *Integral geometry and convolution equations*, Kluwer. Acad. Publ., Dordrecht, 2003. MR2016409

[4] F. John, *Plane waves and spherical means applied to partial differential equations*, Intersci. Publ., New York-London, 1955. MR0075429

[5] R. Courant, D. Hilbert, *Methods of mathematical physics*, Vol. II. *Partial differential equations*, Intersci. Publ., New York-London, 1962. MR0140802

[6] V. V. Volchkov, *Injectivity sets for the Radon transform on spheres*, Izv. Ross. Akad. Nauk Ser. Mat. **63** (1999), no. 3, 63–76; English transl., Izv. Math. **63** (1999), no. 3, 481–493. MR1712132

[7] M. L. Agranovsky, C. A. Berenstein, and P. Kuchment, *Approximation by spherical waves in L^p spaces*, J. Geom. Anal. **6** (1996), no. 3, 365–383. MR1471897
[8] M. L. Agranovsky and E. T. Quinto, *Injectivity sets for the Radon transform over circles and complete systems of radial functions*, J. Funct. Anal. **139** (1996), no. 2, 383–414. MR1402770
[9] M L. Agranovsky, V. V. Volchkov, and L. A. Zalcman, *Conical injectivity sets for the spherical Radon transform*, Bull. London Math. Soc. **31** (1999), no. 2, 231–236. MR1664137
[10] D. H. Armitage, *Cones on which entire harmonic functions can vanish*, Proc. Roy. Irish Acad. Sect. A **92** (1992), no. 1, 107–110. MR1173388 (93h:31003)
[11] V. P. Burskiĭ, *Investigation methods of boundary value problems for general differential equations*, Kiev, Naukova Dumka, 2002. (Russian)
[12] Vit. V. Volchkov, *A local two-radius theorem on the sphere*, Algebra i Analiz **16** (2004), no. 3, 24–55; English transl., St. Petersburg Math. J. **16** (2005), no. 3, 453–475. MR2083565
[13] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Math. Ser., vol. 32, Princeton Univ. Press, Princeton, NS, 1971. MR0304972
[14] A. Erdelyi, F. Oberheftinger, and F. Triconi, *Higher transcendental functions.* Vol. I, II, *Based in part on notes left by H. Bateman*, McGraw-Hell Book Co., New-York, 1953. MR0058756
[15] S. Helgason, *Integral geometry and radon transforms*, Springer, New York, 2011. MR2743116
[16] V. V. Volchkov and Vit. V. Volchkov, *Harmonic analysis of mean periodic functions on symmetric spaces and the Heisenberg group*, Springer Monogr. Math., Springer-Verlag, London, 2009. MR2527108 (2011f:43021)
[17] S. Lang, $SL_2(R)$, Addison-Wesley Publ. Co., Reading, Mass, 1975. MR0430163
[18] N. Ya. Vilenkin, *Special functions and the theory of group representations*, Nauka, Moscow, 1991; English transl. first. edition, Transl. Math. Monogr., vol. 22, Amer. Math. Soc., Providence, RI, 1968. MR0229863 (37:5429), MR1177172 (93d:33013)
[19] A. N. Kolmogorov and S. V. Fomin, *Elements of the theory of functions and functional analysis*, Nauka, Moscow, 1989; English transl. first edition, Graylock Press, Rochester, N.Y., 1957. MR0085462 (19:44d); MR1025126 (90k:46001)
[20] L. Hörmander, *The analysis of linear partial differential operators.* I. *Distribution theory and Fourier analysis*, 2nd ed., Grundlehren Math. Wiss., Bd. 256, Springer-Verlag, Berlin, 1990. MR1065993 (91m:35001a)
[21] Ya. B. Lopatinskiĭ, *Introduction to the modern theory of partial differential equations*, Kiev, Nakova Dumka, 1980. (Russian). MR591676

Donetsk national university, 24 Universitetskaya str., Donetsk 83001, Ukraine
E-mail address: valeriyvolchkov@gmail.com

Received 16/DEC/2014

Translated by A. PLOTKIN