Алгебра и анализ
Том 27 (2015), № 5

St. Petersburg Math. J.
Vol. 27 (2016), No. 5, Pages 731–756
http://dx.doi.org/10.1090/spmj/1414
Article electronically published on July 26, 2016

HAAR NEGLIGIBILITY OF POSITIVE CONES IN BANACH SPACES

J. ESTERLE, É. MATHERON, AND P. MOREAU

ABSTRACT. The Haar negligibility of the positive cone associated with a basic sequence is discussed in the case of a separable Banach space. In particular, it is shown that, up to equivalence, the canonical basis of c_0 is the only normalized subsymmetric unconditional basic sequence whose positive cone is not Haar null, and the only normalized unconditional basic sequence whose positive cone contains a translate of every compact set. It is also proved that an unconditional basic sequence with a non-Haar null positive cone must be c_0-saturated in a very strong sense, and that every quotient of the space generated by such a sequence is c_0-saturated.

§1. INTRODUCTION

The main topic of this paper is *Haar negligibility*, a quite well-known but still rather mysterious notion of smallness introduced by J. P. R. Christensen [C] in the 1970's (and rediscovered much later in [HSY]). A Borel subset A of a Polish Abelian group G is said to be *Haar null* if there exists a (Borel) probability measure μ on G such that $\mu(A + x) = 0$ for every $x \in G$. Any such measure μ is called a *test measure* for A.

If the group G is locally compact, Haar negligibility turns out to be equivalent to negligibility with respect to the Haar measure of G. Haar null sets always have empty interior, and they form a σ-ideal, i.e., any countable union of Haar null sets is again Haar null. Also, "Pettis' lemma" is available: if a Borel set A is not Haar null, then $A - A$ is a neighborhood of 0; in particular, every compact subset of G is Haar null if G is not locally compact. Finally, from the inner regularity of measures it follows that if a Borel set $A \subseteq G$ contains a translate of every compact subset of G, then A is not Haar null. (This is essentially the only known way of showing that a given set is not Haar null.) We refer to [BL, Chapter 6] for more information.

In the present paper, we study Haar negligibility in the framework of separable Banach spaces.

Our work is motivated by beautiful results obtained by Matoušková and Stegall, which provide the following surprising link between the geometry of a Banach space and the Haar negligibility of its closed convex subsets with empty interior: *A separable Banach space X is reflexive if and only if every closed convex subset of X with empty interior is Haar null.* More precisely: *If X is reflexive, then every closed convex subset of X with empty interior is Haar null* ([M3]); *and if X is not reflexive, then there exists a closed convex subset of X with empty interior that contains a translate of every compact subset of X* ([MS]).

Thus, we see that as far as Haar negligibility is concerned, closed convex sets in reflexive Banach spaces are "uninteresting". However, this does not rule out the possibility that closed convex sets with empty interior taken from an interesting specific class turn

2010 *Mathematics Subject Classification.* Primary 46B25.
Key words and phrases. Haar negligibility, positive cone, Schauder base, Gauss negligibility.

out to be Haar null in some nonreflexive Banach spaces, and non-Haar null in some others.

With this in mind, a quite natural class of examples to consider is that of *positive cones* associated with basic sequences. If $\mathbf{e} = (e_i)_{i\geq 1}$ is a basic sequence in a Banach space X, the positive cone associated with $\mathbf{e}$ is the set of all $x \in X$ that can be written as $x = \sum_1^\infty x_i e_i$ with *nonnegative* coefficients x_i. This positive cone will be denoted by $Q^+(\mathbf{e})$. (The fact that $Q^+(\mathbf{e})$ has indeed empty interior is easy to check.)

In some sense, the real starting point of our study is the following well-known fact: the positive cone of c_0 is not Haar-null, whereas the positive cone of ℓ_1 *is* Haar null. More generally, positive cones show-up naturally in the study of Haar negligibility, and play a crucial role in the proofs of the Matoušková and Stegall results. They are the main objects of study in the present paper.

Throughout the paper, we shall say that a positive cone $Q^+(\mathbf{e})$ in a Banach space X *is Haar null* if it is Haar null in $[\mathbf{e}]$, the closed subspace of X generated by $\mathbf{e}$.

Ideally, one would like to characterize the basic sequences $\mathbf{e}$ whose positive cone is Haar null. (Note that since Haar negligibility is preserved under linear isomorphisms, this can be done only up to equivalence). However, this seems rather too ambitious, and we mostly concentrate on *unconditional* basic sequences.

Note that if $\mathbf{e}$ is unconditional, then $Q^+(\mathbf{e}) - Q^+(\mathbf{e}) = [\mathbf{e}]$; so one would rather expect positive cones to be non-Haar null. Nevertheless, all our results suggest that the following "conjecture" might be true: *up to equivalence, the only unconditional normalized basic sequence whose positive cone is* not *Haar null is the canonical basis of* c_0.

We have been unable to prove this in full generality, but we believe that some of our results do support this conjecture rather convincingly. The most "important" ones seem to be the following.

- The conjecture holds true for *subsymmetric* unconditional basic sequences, and also for all unconditional basic sequences if "non-Haar negligibility" is replaced by the stronger property "to contain a translate of every compact set".
- A normalized unconditional basic sequence $\mathbf{e}$ whose positive cone is not Haar null has to be "c_0-saturated" in quite a strong sense: every normalized block-sequence of $\mathbf{e}$ has a subsequence equivalent to the canonical basis of c_0.
- If a Banach space X has an unconditional basis whose positive cone is not Haar null, then every quotient of X is c_0-saturated.

The paper is organized as follows.

In §2, we introduce the notions and tools that will be used throughout the paper, and we recall some known results concerning the Haar negligibility of positive cones.

In §3 we give a characterization of those basic sequences whose positive cone contains a translate of every compact set, and we use it to show that the canonical basis of c_0 is (up to equivalence) the only normalized unconditional basic sequence with this property. We also observe that unconditionality cannot be dispensed with, one counterexample being the positive cone of the classical James space.

In §4 we obtain several results concerning unconditional basic sequences whose positive cone is not Haar null. We show in Subsection 4.1 that any block-sequence of such a sequence $\mathbf{e}$ also has a non-Haar null positive cone, from which it follows that all block-sequences of $\mathbf{e}$ are both shrinking and non-boundedly complete. In Subsection 4.2, we isolate a simple necessary condition for the non-Haar negligibility of a positive cone, which turns out to be crucial in our study. We use this condition to prove in a very elementary way that the positive cone of the Schreier space is Haar null, and that the canonical basis of c_0 is the only *symmetric* sequence with a non-Haar null positive cone. The c_0-saturation property stated above is established in Subsection 4.3, and the result concerning subsymmetric sequences follows immediately. In Subsection 4.4, we prove a stronger c_0-saturation result. However, in Subsection 4.5 we give an example showing that this kind of c_0-saturation is not even sufficient to ensure the non-Haar negligibility of the positive cone.

In §5 we show that the non-Haar negligibility of the positive cone associated with an unconditional basis entails c_0-saturation of the quotients.

§6 is independent of the rest of the paper. We give simple examples of Haar null positive cones admitting a *Gaussian* test measure. On the other hand, we observe that the positive cone associated with an arbitrary Schauder basis is never *Gauss null*.

Finally, we list a few natural questions in §7.

This work is based on Chapter 3 of the third author's PhD thesis [Mo].

§2. Background

2.1. Notation and terminology. We start by fixing some notation and recalling a few quite well-known notions from the Banach space theory. Our references are [LT] and [AK].

Throughout the paper, the symbol X will denote a Banach space with norm $\|\cdot\|$. As a rule, all Banach spaces are real, separable, and infinite-dimensional.

For any family of vectors $\mathbf{e} = (e_i)_{i\in I}$ in X, we denote by $[e_i;\ i \in I]$, or simply by $[\mathbf{e}]$, the closed linear span of $\{e_i;\ i \in I\}$.

A sequence $\mathbf{e} = (e_i)_{i\geq 1} \subseteq X$ is a *basic sequence* if it is a Schauder basis of its closed linear span $[\mathbf{e}]$, i.e., every $x \in [\mathbf{e}]$ can be written in a unique way as $x = \sum_{i\geq 1} x_i e_i$, where $x_i \in \mathbb{R}$ and the series is norm-convergent. If, further, every (convergent) series $\sum x_i e_i$ is unconditionally convergent, the basic sequence $\mathbf{e}$ is said to be *unconditional.* A basic sequence (e_i) is said to be *normalized* if $\|e_i\| = 1$ for all i, and *seminormalized* if it is bounded and $\inf_i \|e_i\| > 0$.

If $\mathbf{e} = (e_i)$ is a basic sequence, we denote by (e_i^*) the associated sequence of coordinate functionals on $[\mathbf{e}]$. Note that if $\inf_i \|e_i\| > 0$ then (e_i^*) is w^*-null, i.e., $\langle e_i^*, x\rangle \to 0$ for every $x \in [\mathbf{e}]$.

The *basis constant* of $\mathbf{e} = (e_i)_{i\geq 1}$ is the number $K(\mathbf{e}) := \sup\{\|P_n\|, n \geq 1\}$, where P_n is the canonical projection of $[\mathbf{e}]$ onto $[e_i;\ 1 \leq i \leq n]$. This depends heavily on the given norm $\|\cdot\|$, but the formula $|\!|\!|x|\!|\!| := \sup\{\|P_n(x)\|,\ n \geq 1\}$ defines an equivalent norm on $[\mathbf{e}]$ with respect to which the basis constant of $\mathbf{e}$ is equal to 1. Since Haar negligibility is preserved under linear isomorphisms, it follows that, for our purpose, we can safely restrict ourselves to basic sequences with basis constant equal to 1.

Likewise, if $\mathbf{e}$ is unconditional, the *unconditional basis constant* of $\mathbf{e}$ is the number $K_u(\mathbf{e}) := \sup\big\{\|P_I\|;\ I \subseteq \mathbb{N}\big\}$, where P_I is the canonical projection of $[\mathbf{e}]$ onto $[e_i;\ i \in I]$; and this can be made equal to 1 by replacing the original norm with the equivalent norm $|\!|\!|x|\!|\!| := \sup\{\|P_I(x)\|;\ I \subseteq \mathbb{N}\}$. When $K_u(\mathbf{e}) = 1$, the basic sequence $\mathbf{e}$ is said to be 1-*unconditional.*

If $\mathbf{e}$ is a basic sequence in X, we denote by $\|\cdot\|^{\mathbf{e}}$ the restriction of $\|\cdot\|$ to the closed subspace $[\mathbf{e}] \subseteq X$. If $\mathbf{e}$ is normalized, we denote by $\|\cdot\|^{\mathbf{e}}_\infty$ the "ℓ_∞ norm built on $\mathbf{e}$", i.e. the norm on $[\mathbf{e}]$ defined by

$$\Big\|\sum_{i\geq 1} x_i e_i\Big\|^{\mathbf{e}}_\infty := \sup_{i\geq 1} |x_i|.$$

When $\mathbf{e}$ is a basis of X, we write $\|\cdot\|_\infty$ instead of $\|\cdot\|^{\mathbf{e}}_\infty$.

It is important to keep in mind that the original norm always dominates the ℓ_∞ norm built on a normalized basic sequence $\mathbf{e}$; more precisely, we have

$$\|\cdot\|^{\mathbf{e}}_\infty \leq 2K(\mathbf{e})\|\cdot\|^{\mathbf{e}}.$$

Two basic sequences $(e_i)_{i\geq 1}$ and $(f_i)_{i\geq 1}$ (living on possibly different Banach spaces) are said to be *equivalent* if, for every sequence of scalars $(x_i)_{i\geq 1}$, the convergence of the series $\sum x_i e_i$ is equivalent to the convergence of the series $\sum x_i f_i$; in other words, if there is an isomorphism from $[e_i;\ i\geq 1]$ onto $[f_i;\ i\geq 1]$ sending e_i to f_i. Note that the property "to have a Haar null positive cone" is invariant under equivalence.

If $\mathbf{e}=(e_i)_{i\geq 1}$ is a basic sequence, a *block-sequence* built on $\mathbf{e}$ is a sequence $\mathbf{f}=(f_j)_{j\geq 1}$ of the form

$$f_j=\sum_{i=p_{j-1}}^{p_j-1} a_i e_i,$$

where $(p_j)_{j\geq 0}$ is a monotone increasing sequence of integers and (a_i) is a sequence of scalars. Any such sequence $\mathbf{f}$ is basic, with $K(\mathbf{f})\leq K(\mathbf{e})$; and if $\mathbf{e}$ is unconditional then so is $\mathbf{f}$, with $K_u(\mathbf{f})\leq K_u(\mathbf{e})$.

Finally, recall that a basic sequence $\mathbf{e}=(e_i)$ is said to be *shrinking* if the linear span of (e_i^*) is dense in the dual space of $[\mathbf{e}]$, and *boundedly complete* if the convergence of a series $\sum x_i e_i$ is equivalent to the boundedness of its partial sums.

By a famous result due to R. C. James, a basic sequence generates a reflexive Banach space if and only if it is both shrinking and boundedly complete. Moreover, an *unconditional* basic sequence is nonshrinking if and only if it has a subsequence equivalent to the canonical basis of ℓ_1, and non-boundedly complete if and only if it has a subsequence equivalent to the canonical basis of c_0.

2.2. Three useful facts. For the sake of future reference, we now recall three extremely useful results concerning basic sequences.

The first is merely an observation.

Fact 2.1. If $\mathbf{e}=(e_i)_{i\geq 1}$ is an unconditional basic sequence, then every bounded sequence of scalars $\mathbf{a}=(a_i)_{i\geq 1}$ determines a "bounded multiplier" on $[\mathbf{e}]$; more precisely, for any convergent series $\sum x_i e_i$, the series $\sum a_i x_i e_i$ is convergent with

$$\Big\|\sum_{i\geq 1} a_i x_i e_i\Big\|\leq K_u(\mathbf{e})\,\|\mathbf{a}\|_\infty\,\Big\|\sum_{i\geq 1} x_i e_i\Big\|.$$

In particular, any fixed change of signs on the e_i's determines an isomorphism of $[\mathbf{e}]$. Since Haar negligibility is preserved under linear isomorphisms, it follows that the property "to have a Haar null positive cone" is invariant under changes of signs: if an unconditional basic sequence (e_i) has a Haar null positive cone, then so does any sequence of the form $(\pm e_i)$.

The second result we want to state explicitly is the "principle of small perturbations", which is a standard tool for establishing the equivalence of two basic sequences.

Lemma 2.2. *Let $\mathbf{e}=(e_i)_{i\geq 1}$ be a normalized basic sequence in X. If $\mathbf{f}=(f_i)_{i\geq 1}$ is a sequence in X such that $\eta:=\sum_1^\infty \|e_i-f_i\|<\frac{1}{2K(\mathbf{e})}$, then $\mathbf{f}$ is basic and equivalent to $\mathbf{e}$. Moreover, for every convergent series $\sum x_i f_i$, we have the following estimate*:

$$(1-2K(\mathbf{e})\eta)\Big\|\sum_{i=1}^\infty x_i e_i\Big\|\leq\Big\|\sum_{i=1}^\infty x_i f_i\Big\|\leq 2K(\mathbf{e})\Big\|\sum_{i=1}^\infty x_i e_i\Big\|.$$

The above estimate is not explicitly stated in [AK] or [LT], but follows easily from the proofs of the principle of small perturbations given therein.

Finally, we recall the so-called "Bessaga–Pelczynski selection principle".

Lemma 2.3. *If $\mathbf{e}$ is a basic sequence, then any normalized sequence $(y_n)\subseteq[\mathbf{e}]$ such that $\langle e_i^*, y_n\rangle \xrightarrow[n\to\infty]{} 0$ for all $i\in\mathbb{N}$ admits a basic subsequence equivalent to a (normalized) block-sequence of $\mathbf{e}$.*

This is often stated with a *weakly null* sequence (y_n), but the greater generality is useful (see [AK]).

2.3. Some Haar null and non-Haar null positive cones. To conclude this background section, we now state some known results concerning the Haar negligibility of positive cones.

Proposition 2.4. *Let $\mathbf{e} = (e_i)_{i\geq 1}$ be a normalized basic sequence.*

(1) *If $\mathbf{e}$ is equivalent to the canonical basis of c_0, then $Q^+(\mathbf{e})$ contains a translate of every compact subset of $[\mathbf{e}]$, and so $Q^+(\mathbf{e})$ is not Haar null.*
(2) *If $\mathbf{e}$ is equivalent to the canonical basis of ℓ_p, $1 \leq p < \infty$, then $Q^+(\mathbf{e})$ is Haar null. More generally, $Q^+(\mathbf{e})$ is Haar null as soon as $\mathbf{e}$ dominates the canonical basis of ℓ_p, i.e., there exists a finite constant M such that*

$$\sum_i |x_i|^p \leq M \Big\| \sum_i x_i e_i \Big\|^p$$

for every finite sequence of scalars (x_i).
(3) *If $[\mathbf{e}]$ is reflexive, then $Q^+(\mathbf{e})$ is Haar null.*
(4) *If $\mathbf{e}$ is unconditional and satisfies*

$$\lim_{k\to\infty} \inf_{n_1<\ldots<n_k} \Big\| \sum_{i=1}^{k} e_{n_i} \Big\| = \infty,$$

then $Q^+(\mathbf{e})$ is Haar null.

Part (1) is almost obvious (see [BL, p. 132]). Part (2) can be found in [BN] and [M] (see also [BL]). Part (3) follows from the main result of [M3], and (4) can be found in [M2].

Concerning (1), in §3 it will be shown that the canonical basis of c_0 is in fact the only unconditional basic sequence for which the positive cone contains a translate of every compact set. We shall also see that (2), (3), and (4) can be obtained as applications of Proposition 4.6 and Theorem 4.9 below. (However, the direct proof of (3) given in [BL, Proposition 6.8] provides an explicit test measure for the positive cone.)

§3. Positive cones containing a translate of every compact set

In this section, we study the positive cones $Q^+(\mathbf{e})$ *containing a translate of every compact set.* By this, we mean of course that $Q^+(\mathbf{e})$ contains a translate of every compact subset of $[\mathbf{e}]$. As already mentioned, this is a strong way of being non-Haar null.

The main result of this section is the following.

Theorem 3.1. *Let $\mathbf{e} = (e_i)_{i\geq 1}$ be a normalized basic sequence. The positive cone $Q^+(\mathbf{e})$ contains a translate of every compact set if and only if there exists a sequence of real numbers $(\lambda_i)_{i\geq 1}$, with $\lambda_i \geq 1$ for all i, such that*

$$\sup_{n\geq 1} \Big\| \sum_{i=1}^{n} \lambda_i e_i \Big\| < \infty.$$

From this, it is easy to deduce the following statement.

Corollary 3.2. *Up to equivalence, the only normalized* unconditional *basic sequence whose positive cone contains a translate of every compact set is the canonical basis of c_0.*

Proof. Let $\mathbf{e}$ be a normalized unconditional basic sequence such that $Q^+(\mathbf{e})$ contains a translate of every compact set, and set $X := [\mathbf{e}]$. Without loss of generality, we may assume that $\mathbf{e}$ is 1-unconditional.

We already know that the original norm $\|\cdot\|$ of X dominates the norm $\|\cdot\|_\infty$, so we only need to show that $\|\cdot\|_\infty$ also dominates $\|\cdot\|$. Let $(\lambda_i)_{i\geq 1}$ be the sequence given by Theorem 3.1, and set

$$M := \sup_{n\geq 1} \left\| \sum_{i=1}^n \lambda_i e_i \right\| < \infty.$$

For every $x = \sum_{i\geq 1} x_i e_i \in X \setminus \{0\}$ and all $n \geq 1$, we have

$$P_n(x) := \sum_{i=1}^n x_i e_i = \sum_{i=1}^n \frac{x_i}{\lambda_i} \lambda_i e_i.$$

If we set $a_i := x_i/\lambda_i$, then $|a_i| \leq \|x\|_\infty$ and so, by Fact 2.1:

$$\|P_n(x)\| \leq \|x\|_\infty \left\| \sum_{i=1}^n \lambda_i e_i \right\| \leq M\|x\|_\infty.$$

Hence, we obtain $\|x\| = \lim_{n\to\infty} \|P_n(x)\| \leq M\|x\|_\infty$. □

The proof of Theorem 3.1 relies on the following lemma. Here and afterwards, for any positive number r and any set $Q \subseteq X$, we set

$$Q_r := Q \cap \bar{B}(0, r).$$

Lemma 3.3. *Let Q be a closed convex cone in a Banach space X, and let D be any dense subset of $\bar{B}(0,1)$. The following statements are equivalent.*

(i) *The cone Q contains a translate of every compact set.*
(ii) *There exists $R > 0$ such that Q_R contains a translate of every compact subset of $\bar{B}(0,1)$.*
(iii) *There exists $M > 0$ such that Q_M contains a translate of every finite subset of $\bar{B}(0,1) \cap D$.*

Moreover if condition (iii) *is satisfied, then* (ii) *is fulfilled with $R = 2M$.*

Proof. The proof of this lemma is essentially the same as that of [MS, Theorem 3]. However, we give some details for the convenience of the reader.

First, it is obvious that (ii) implies (iii), and rather clear that (ii) implies (i) because Q is stable under dilations. Moreover, arguing by contradiction, it is not hard to show that (i) implies (ii).

Let us now concentrate on the subtler implication (iii) $\Longrightarrow$ (ii). This relies on the following claim.

Fact. For any compact set $L \subseteq \bar{B}(0, 2^{-1})$, one can find a sequence $(F_i)_{i\geq 1}$ of finite subsets of D such that

$$L \subseteq \sum_{i\geq 1} 2^{-i} F_i.$$

Proof of Fact. The meaning of the notation $\sum_{i\geq 1} 2^{-i} F_i$ is obvious, and this set is easily seen to be compact, for any sequence (F_i) of finite subsets of $\bar{B}(0,1)$.

Set $B := \bar{B}(0,1)$. Then the set $2^{-1}D$ is dense in $2^{-1}B = \bar{B}\big(0, \frac{1}{2}\big)$, so one can find a finite set $F_1 \subseteq D$ such that

$$L \subseteq 2^{-1}F_1 + 2^{-2}B.$$

Then $L_1 := 2(L - 2^{-1}F_1)$ is a compact set contained in $2^{-1}B$; so one can find a finite set $F_2 \subseteq D$ such that $L_1 \subseteq 2^{-1}F_2 + 2^{-2}B$, i.e.,

$$L \subseteq 2^{-1}F_1 + 2^{-2}F_2 + 2^{-3}B.$$

The process should be clear: one constructs inductively a sequence (F_i) of finite subsets of D such that for all $n \geq 1$ we have

$$L \subseteq \sum_{i=1}^{n} 2^{-i}F_i + 2^{-(n+1)}B \subseteq \sum_{i\geq 1} 2^{-i}F_i + 2^{-n}B + 2^{-(n+1)}B.$$

Since $\sum_{i\geq 1} 2^{-i}F_i$ is a closed set, it follows that $L \subseteq \sum_{i\geq 1} 2^{-i}F_i$. □

Now, assume that (iii) holds true, and let K be any compact subset of $\bar{B}(0,1)$. Applying the Fact to $L := 2^{-1}K$, we get a sequence (F_i) of finite subsets of D such that

$$K \subseteq 2\sum_{i\geq 1} 2^{-i}F_i.$$

By (ii), one can find points $z_i \in X$ such that $z_i + F_i \subseteq Q_M$ for all $i \geq 1$. Note that the sequence (z_i) is bounded, because $F_i \subseteq \bar{B}(0,1)$; so we may set $z := 2\sum_{i\geq 1} 2^{-i}z_i$, and we get

$$z + K \subseteq 2\sum_{i\geq 1} 2^{-i}(z_i + F_i) \subseteq 2Q_M = Q_{2M},$$

where we have used the fact that the cone Q is closed. □

Proof of Theorem 3.1. Without loss of generality, we may assume that the basis constant of $\mathbf{e}$ is equal to 1.

Assume first that $Q^+(\mathbf{e})$ contains a translate of every compact subset of $X := [\mathbf{e}]$. By Lemma 3.3, there exists $M > 0$ such that $Q^+_M(\mathbf{e})$ contains a translate of every finite subset of $D := \bar{B}(0,1) \cap \operatorname{span}(e_i;\ i \geq 1)$.

For every $k \geq 1$, set $F_k := \big\{ -e_i; 1 \leq i \leq k\big\}$. This is a finite subset of D, so there exists $z_k = \sum_{i=1}^{\infty} z_{i,k}e_i \in X$ such that $z_k + F_k \subseteq Q^+_M(\mathbf{e})$. Then $\|z_k\| \leq M+1$, and so $|z_{i,k}| \leq 2(M+1)$ because $K(\mathbf{e}) = 1$. Moreover, since $z_k + F_k \subseteq Q^+(\mathbf{e})$, we have $z_{i,k} \geq 1$ whenever $k \geq i$; hence $1 \leq z_{i,k} \leq 2(M+1)$ for $i \geq 1$ and $k \geq i$.

By a diagonal process, one can find a monotone increasing sequence of integers $(p_k)_{k\geq 1}$ such that for every $i \geq 1$, the sequence $(z_{i,p_k})_{k\geq 1}$ converges to some real number $\lambda_i \geq 1$ as $k \to \infty$. Then, since $K(\mathbf{e}) = 1$, for any $n \geq 1$ we have

$$\Big\| \sum_{i=1}^{n} \lambda_i e_i \Big\| = \lim_{k\to\infty} \Big\| \sum_{i=1}^{n} z_{i,p_k} e_i \Big\| \leq \limsup_{k\to\infty} \|z_{p_k}\| \leq M + 1.$$

Conversely, assume that there exists $M > 0$ and a sequence of real numbers $(\lambda_i)_{i\geq 1}$ with $\lambda_i \geq 1$ such that $\big\| \sum_{i=1}^{n} \lambda_i e_i \big\| \leq M$ for every $n \geq 1$.

Set again $D := \bar{B}(0,1) \cap \operatorname{span}(e_i;\ i \geq 1)$. Then $R := 1 + 2M$ satisfies property (iii) of Lemma 3.3. Indeed let F be any finite subset of D. One can find $m \geq 1$ such that $\langle e_n^*, x\rangle = 0$ for every $x \in F$ if $n \geq m$. Since $\big|\langle e_n^*, x\rangle\big| \leq 2$ for all $n \geq 1$ if $x \in F$ (because $\|x\| \leq 1$), we see that $F + 2\sum_{i=1}^{m} \lambda_i e_i \subseteq Q^+_{1+2M}(\mathbf{e})$. □

We conclude this section with a simple example showing that Corollary 3.2 breaks down if we drop the unconditionality assumption.

Example 3.4. The positive cone associated with the canonical basis of the *James space* $\mathcal{J}$ contains a translate of every compact set.

Proof. Recall that $\mathcal{J}$ is the space of all sequences of real numbers $x = (x_i)_{i\geq 1}$ that converge to 0 and satisfy the following condition:

$$\|x\|_{\mathcal{J}} := \sup_{m\in\mathbb{N}} \sup\bigl\{ \bigl[(x_{p_2} - x_{p_1})^2 + \cdots + (x_{p_m} - x_{p_{m-1}})^2\bigr]^{1/2} ;\ 1 \leq p_1 \cdots < p_m \bigr\} < \infty.$$

We refer, e.g., to [LT, p. 25] for more details on the James space.

The canonical basis $\mathbf{e} = (e_i)_{i\geq 0}$ is a normalized Schauder basis of $\mathcal{J}$, and for every $n \geq 1$ we have

$$\Bigl\| \sum_{i=1}^{n} e_i \Bigr\|_{\mathcal{J}} = 1.$$

Then Theorem 3.1 shows that $Q^+(\mathbf{e})$ contains a translate of every compact set. □

§4. Positive cones that are not Haar null

Now we address the question of characterizing the basic sequences whose positive cone is not Haar null. We obtain a complete answer in the case of subsymmetric unconditional sequences: within this class of sequences, the canonical basis of c_0 is the only one whose positive cone is not Haar null. We also show that an unconditional basic sequence whose positive cone is not Haar null must be "very close" to the canonical basis of c_0, in a sense that will be made precise.

4.1. Haar negligibility and block-sequences. The following result shows that for unconditional basic sequences, the Haar negligibility of the positive cone can be detected by looking at any block-sequence.

Proposition 4.1. *Let $\mathbf{e} = (e_i)_{i\geq 1}$ be an unconditional basic sequence. If $\mathbf{e}$ admits a block-sequence $\mathbf{f}$ whose positive cone is Haar null, then $Q^+(\mathbf{e})$ is Haar null.*

Proof. Set $X := [\mathbf{e}]$, and let Y be the closed subspace of X generated by the block-sequence $\mathbf{f} = (f_j)_{j\geq 1}$. There exists a monotone increasing sequence of integers $(p_j)_{j\geq 0}$, with $p_0 = 1$ such that $f_j = \sum_{i=p_{j-1}}^{p_j - 1} a_i e_i$; since Haar negligibility is invariant under any change of signs on the e_i's, we may assume that $a_i \geq 0$ for all i.

By assumption, one can find a probability measure μ on Y such that $\mu(Q^+(\mathbf{f}) + y) = 0$ for every $y \in Y$. Denote by ν the measure μ viewed as a probability measure on X; that is, $\nu(A) = \mu(A \cap Y)$ for every Borel set $A \subseteq X$. We show that ν is a test measure for $Q^+(\mathbf{e})$.

Since $\mathbf{e}$ is unconditional, we can write any $x \in X$ as $x = x^+ + x^-$, where $x^+ \in Q^+(\mathbf{e})$ and $x^- \in Q^-(\mathbf{e})$. Then $x + Q^+(\mathbf{e}) \subseteq x^- + Q^+(\mathbf{e})$, and so

$$\nu(x + Q^+) \leq \nu(x^- + Q^+).$$

Hence, it suffices to check that $\nu(x + Q^+(\mathbf{e})) = 0$ for every $x \in Q^-(\mathbf{e})$. This follows immediately from the following statement. □

Fact. Given $x \in Q^-(\mathbf{e})$, one can find $w \in Y$ such that $\bigl(x + Q^+(\mathbf{e})\bigr) \cap Y \subseteq w + Q^+(\mathbf{f})$.

Indeed, once this is known, we only need to write

$$\nu(x + Q^+(\mathbf{e})) = \mu((x + Q^+(\mathbf{e})) \cap Y) \leq \mu(w + Q^+(\mathbf{f})) = 0.$$

Proof of Fact. Fix $x \in Q^-(\mathbf{e})$, and let $z \in (x + Q^+(\mathbf{e})) \cap Y$ be arbitrary. Write

$$z = \sum_{i=1}^{\infty} (x_i + \lambda_i) e_i = \sum_{j=1}^{\infty} z_j f_j = \sum_{j=1}^{\infty} z_j \Bigl(\sum_{i=p_{j-1}}^{p_j - 1} a_i e_i \Bigr),$$

where $\lambda_i \geq 0$ for all i. Then $z_j a_i - x_i = \lambda_i \geq 0$ for every $j \geq 1$ and all $i \in [p_{j-1}, p_j)$. So, for $j \geq 1$, we get

$$w_j := \sup\left\{\frac{x_i}{a_i};\ p_{j-1} \leq i < p_j,\ a_i \neq 0\right\} \leq z_j. \tag{4.1}$$

Since $x_i \leq 0$ for all i, the series $\sum |x_i| e_i$ is convergent (with $\sum_{i=1}^\infty |x_i| e_i = -x$). By Fact 2.1 and the definition of the w_j, it follows that the series $\sum_{j\geq 1}\big(\sum_{i=p_{j-1}}^{p_j-1} |w_j| a_i e_i\big) = \sum |w_j| f_j$ is convergent; and so the series $\sum w_j f_j$ is also convergent in Y. Set $w := \sum_{j=1}^\infty w_j f_j$. Note that w depends only on x and $\mathbf{f}$ (not on the point z we are considering).

Using (4.1), we see that $z = w + \sum_{j=1}^\infty (z_j - w_j) f_j \in w + Q^+(\mathbf{f})$, which concludes the proof. □

We know that the positive cone of ℓ_1 is Haar null, and that positive cones in reflexive spaces are also Haar null. By James' characterization of reflexivity, we deduce the following statement.

Corollary 4.2. *If $\mathbf{e}$ is an unconditional basic sequence such that $Q^+(\mathbf{e})$ is not Haar null, then every block-sequence of $\mathbf{e}$ is both shrinking and non-boundedly complete.*

Since an unconditional basic sequence is non-boundedly complete if and only if it has a block-sequence equivalent to the canonical basis of c_0, it follows that if $\mathbf{e}$ is unconditional and $Q^+(\mathbf{e})$ is not Haar null, then every block-sequence of $\mathbf{e}$ admits a block-sequence equivalent to the canonical basis of c_0. We shall see in Subsection 4.4 that $\mathbf{e}$ has in fact a much stronger "c_0-saturation" property.

4.2. The (δ, R)-condition. Matoušková observed in [M2] that a Borel subset A of a Polish Abelian group G is Haar null if and only if the following is true: *for every $\delta > 0$ and every $r > 0$, there exists a probability measure μ supported on the open ball $B(0, r)$ such that $\mu(A + g) \leq \delta$ for every $g \in G$.* From this, it is easy to deduce the following.

Lemma 4.3. *Let $\mathbf{e}$ be a normalized basic sequence. The cone $Q^+(\mathbf{e})$ is not Haar null if and only if there exist $\delta > 0$ and $R > 0$ with the following property.*

$(*)$ For every probability measure μ on X supported on $\bar{B}(0,1)$, there exists $x \in X$ such that $\mu(x + Q^+_R(\mathbf{e})) \geq \delta$.

In this situation we have $\|x\| \leq 1 + R$.

Proof. Since a countable union of Haar null sets is Haar null, $Q^+(\mathbf{e})$ is not Haar null if and only if $Q^+_n(\mathbf{e})$ is not Haar null for some integer $n \geq 1$.

If $Q^+_n(\mathbf{e})$ is not Haar null, from Matoušková's result mentioned above it follows that there exist $\delta > 0$ and $r > 0$ such that for every probability measure μ on X of support contained in $\bar{B}(0, r)$, there exists $x \in X$ with $\mu(x + Q^+_n) \geq \delta$. This gives the required condition with $R := n/r$. Conversely, if $(*)$ holds then Matoušková's result implies that $Q^+_R(\mathbf{e})$ is not Haar null, and so $Q^+(\mathbf{e})$ is not Haar null either.

The inequality $\|x\| \leq 1 + R$ follows from the triangle inequality. □

In what follows we shall consider a weaker version of condition $(*)$ above, where we only allow a special type of measures.

Definition 4.4. Let $\delta > 0$ and $R > 0$. We say that a normalized basic sequence $\mathbf{e} = (e_i)_{i\geq 1}$ satisfies the (δ, R)-*condition* if for every probability measure μ having finite support contained in $\{-e_i;\ i \geq 1\}$, there exists $x \in X$ (necessarily satisfying $\|x\| \leq 1+R$) such that $\mu(x + Q^+_R(\mathbf{e})) \geq \delta$.

The following observation shows the relevance of this condition to our matters.

Observation 4.5. *Let $\mathbf{e} = (e_i)$ be a normalized basic sequence.*

(i) *If $Q^+(\mathbf{e})$ is not Haar null, then $\mathbf{e}$ satisfies the (δ, R)-condition for some $\delta, R > 0$.*
(ii) *If $\mathbf{e}$ is unconditional and if $Q^+(\mathbf{e})$ is not Haar null, then there exist $\delta, R > 0$ such that all normalized block-sequences of $\mathbf{e}$ satisfy the (δ, R)-condition.*

Proof. Part (i) follows immediately from Lemma 4.3. To prove (ii), argue by contradiction and use Proposition 4.1. □

When $\mathbf{e}$ is unconditional, this implies a condition on the norm of all finite sums of the form $\sum_{i\in J} e_i$.

Proposition 4.6. *Let $\mathbf{e} = (e_i)$ be a normalized, 1-unconditional basic sequence, and assume that $\mathbf{e}$ satisfies the (δ, R)-condition for some $\delta, R > 0$. Then the following facts hold.*

(a) *For every finite set $I \subseteq \mathbb{N}$, there exists $J \subseteq I$ such that*

$$|J| \geq \delta|I| \quad \textit{and} \quad \Big\|\sum_{i\in J} e_i\Big\| \leq 1 + R.$$

(b) *If $\delta = 1$, then $\mathbf{e}$ is equivalent to the canonical basis of c_0.*
(c) *If $\delta < 1$ then, for any finite set $I \subseteq \mathbb{N}$, we have the estimate*

$$\Big\|\sum_{i\in I} e_i\Big\| \leq \frac{1+R}{-\log(1-\delta)}\log(|I|) + 2(1+R).$$

Proof. (a) Set $\lambda_i = \frac{1}{|I|}$ for $i \in I$. Then $\mu = \sum_{i\in I}\lambda_i\delta_{-e_i}$ is a probability measure with finite support contained in $\big\{-e_i, i \geq 1\big\}$. So there exists $x \in X$ such that $\mu(x + Q_R^+) \geq \delta$. This implies that there exists $J \subseteq I$ such that $|J| \geq \delta|I|$ and such that $-e_j \in x + Q_R^+(\mathbf{e})$ for all $j \in J$. So $\langle e_j^*, x\rangle \leq -1$ for all $j \in J$ (and hence $|\langle e_j^*, x\rangle| \geq 1$). Since $\mathbf{e}$ is 1-unconditional and $\|x\| \leq 1 + R$, from Fact 2.1 it follows that $\big\|\sum_{i\in J} e_i\big\| \leq 1 + R$.

(b) If $\delta = 1$, then for every finite set I we have $\big\|\sum_{i\in I} e_i\big\| \leq 1 + R$. By 1-unconditionality, this implies that $\|\cdot\| \leq (1+R)\|\cdot\|_\infty$, and hence $\mathbf{e}$ is equivalent to the canonical basis of c_0.

(c) If $\delta < 1$, we construct a family of disjoint subsets of I in the following way: set $I_0 := I$ and choose $J_0 \subseteq I_0$ in accordance with (a). Then we set $I_1 = I_0 - J_0$ and choose $J_1 \subseteq I_1$ in the same way, and so on. At the step k, we thus have $|I_k| \leq (1-\delta)|I_{k-1}|$, $|I_k| \leq (1-\delta)^k|I|$ and $\big\|\sum_{i\in J_k} e_i\big\| \leq 1 + R$.

We stop when I_{k+1} is empty, so at the latest when $(1-\delta)^k|I| \leq 1$, which gives $k \geq \frac{\log(|I|)}{-\log(1-\delta)}$. So we can choose $k \leq \frac{\log(|I|)}{-\log(1-\delta)} + 1$. This gives

$$\Big\|\sum_{i\in I} e_i\Big\| \leq \sum_{j=0}^{k}\Big\|\sum_{i\in J_j} e_i\Big\| \leq (k+1)(1+R) \leq \frac{1+R}{-\log(1-\delta)}\log(|I|) + 2(1+R). \qquad \square$$

Remark 1. Part (4) of Proposition 2.4 follows immediately from (a).

Remark 2. If $\mathbf{e} = (e_i)$ satisfies the assumption of Proposition 4.6, then (c) shows that

$$\lim_{|I|\to\infty}\Big\|\frac{1}{|I|}\sum_{i\in I} e_i\Big\| = 0.$$

The sequences $\mathbf{e}$ satisfying this property are called the *Blum–Hanson sequences* in [LMP]. They are quite interesting objects of study.

Remark 3. For every normalized basic sequence $\mathbf{e} = (e_i)$ and any $M \in \mathbb{R}^+$, let $\mathcal{I}_M(\mathbf{e})$ denote the family of all subsets σ of $\mathbb{N}$ such that

$$\Big\|\sum_{i\in I} e_i\Big\| \leq M \text{ for every finite set } I \subseteq \sigma.$$

Note that if $\mathbf{e}$ is unconditional and $\sigma \in \mathcal{I}_M(\mathbf{e})$ is infinite, then $(e_i)_{i\in\sigma}$ is equivalent to the canonical basis of c_0. So our "main conjecture" would be proved if one could show the following: *if* $\mathbf{e}$ *is a normalized basic sequence such that* $Q^+(\mathbf{e})$ *is not Haar null, then one can partition* $\mathbb{N}$ *into finitely many sets from* $\bigcup_{M>0}\mathcal{I}_M(\mathbf{e})$. Even though this is of course very unlikely to be relevant, it is hard not to notice the formal analogy with the "paving" formulation of the famous and recently solved Kadison–Singer Problem. Note that, by standard arguments, it would suffice to show that any ultrafilter $\mathcal{U}$ on $\mathbb{N}$ contains a set from $\bigcup_{M>0}\mathcal{I}_M(\mathbf{e})$.

Assuming that $\mathbf{e}$ is unconditional and $Q^+(\mathbf{e})$ is not Haar null (so that $\mathbf{e}$ satisfies some (δ, R)-condition), we deduce from Proposition 4.6 (a) the existence of a constant M such that $\mathcal{I}_M(\mathbf{e})$ has an interesting "largeness" property, namely that inside any finite set $I \subseteq \mathbb{N}$, one can find a set $J \in \mathcal{I}_M(\mathbf{e})$ with comparable cardinality; but this is definitely not sufficient to get a partition result.

In the same spirit (and under the same assumptions on $\mathbf{e}$), the argument of [BT, Corollary 1.4] shows the following: there exists a constant M, a probability measure $\mathfrak{m}$ on the compact set $\mathcal{I}_M(\mathbf{e}) \subseteq 2^{\mathbb{N}}$, and a constant $\alpha > 0$ such that

$$\mathfrak{m}(\{\sigma \in \mathcal{I}_M(\mathbf{e});\ i \in \sigma\}) \geq \alpha \text{ for every } i \in \mathbb{N}.$$

This is another "largeness" property of the family $\mathcal{I}_M(\mathbf{e})$ that makes perhaps plausible the existence of a suitable partition.

Now we show that Proposition 4.6 can be used to characterize the *symmetric* basic sequences whose positive cone is Haar null. Recall that a basic sequence (e_i) is said to be symmetric if, for any permutation π of the integers, the sequence $(e_{\pi(i)})$ is equivalent to (e_i). In this case (see [LT, Chapter 3]), there exists a constant $C > 0$ such that for every convergent series $\sum_{i\geq 1} x_i e_i$ and every permutation $\pi : \mathbb{N} \to \mathbb{N}$ we have

$$\frac{1}{C}\Big\|\sum_{i\geq 1} x_i e_{\pi(i)}\Big\| \leq \Big\|\sum_{i\geq 1} x_i e_i\Big\| \leq C\Big\|\sum_{i\geq 1} x_i e_{\pi(i)}\Big\|.$$

Every symmetric sequence is unconditional, but not conversely (consider any enumeration of the "canonical" basis of $l_1 \oplus c_0$).

Corollary 4.7. *Up to equivalence, the only normalized and symmetric basic sequence whose positive cone is not Haar null is the canonical basis of* c_0.

Proof. Let $\mathbf{e} = (e_i)$ be a normalized, symmetric sequence such that $Q^+(\mathbf{e})$ is not Haar null. Assume without loss of generality that $\mathbf{e}$ is 1-unconditional. Choose $\delta, R > 0$ such that $\mathbf{e}$ satisfies the (δ, R)-condition. By Proposition 4.6 and since $K_u(\mathbf{e}) = 1$, for every integer $p \geq 1$ one can find a finite set $I_p \subseteq \mathbb{N}$ of cardinality p such that $\big\|\sum_{i\in I_p} e_i\big\| \leq 1+R$. Since $\mathbf{e}$ symmetric, there exists a finite constant C such that, for *every* finite set I of integers of cardinality p, we have

$$\Big\|\sum_{i\in I} e_i\Big\| \leq C\Big\|\sum_{i\in I_p} e_i\Big\|.$$

So the finite sums $\sum_{i\in I} e_i$ are uniformly bounded, which implies (by unconditionality) that the given norm $\|\cdot\|^{\mathbf{e}}$ on $[\mathbf{e}]$ is dominated by the norm $\|\cdot\|^{\mathbf{e}}_\infty$. Hence, these two norms are equivalent, and $\mathbf{e}$ is equivalent to the canonical basis of c_0. $\square$

As another illustration of Proposition 4.6, now we show that the positive cone associated with the canonical basis of the *Schreier space* $\mathcal{S}$ is Haar null.

First, we recall the definition of $\mathcal{S}$. Let c_{00} be the linear space of all real sequences with finite support. For $x = (x_i)_{i\geq 1} \in c_{00}$, set

$$\|x\|_{\mathcal{S}} := \sup\left\{ \sum_{i=1}^{p} |x_{k_i}| \,:\, p \geq 1,\, p \leq k_1 < \cdots < k_p \right\}.$$

Then $\|\cdot\|_{\mathcal{S}}$ is a norm on c_{00}, and the Schreier space $\mathcal{S}$ is the completion of c_{00} with respect to this norm. The canonical basis (e_i) of c_{00} is a normalized, 1-unconditional basis of $\mathcal{S}$, and it is also shrinking.

Corollary 4.8. *The positive cone of the Schreier space is Haar null.*

Proof. Let $n \geq 1$, and let $I \subseteq \mathbb{N}$ be a set of cardinality $2n$. Then the set $I \cap [n, \infty)$ contains at least n elements of I. The definition of $\|\cdot\|_{\mathcal{S}}$ implies

$$\left\| \sum_{i\in I} e_i \right\|_{\mathcal{S}} \geq n.$$

Since n is arbitrary, the result now follows from Proposition 4.6. □

4.3. Extracting c_0-subsequences. In this section, we show that a normalized unconditional basic sequence whose positive cone is not Haar null has a wealth of subsequences equivalent to the canonical basis of c_0. This will follow easily from the next result.

Theorem 4.9. *Let* $\mathbf{e}$ *be a normalized basic sequence, and assume that* $\mathbf{e}$ *satisfies the (δ, R)-condition for some $\delta, R > 0$. Then there exists a monotone increasing sequence of integers $(m_j)_{j\geq 1}$ and a sequence $(x_n)_{n\geq 1} \subseteq \bar{B}(0, 1+R)$ such that*

$$\left\langle e^*_{m_j}, x_n \right\rangle \geq 1 \ \textit{for every } n \geq 1 \textit{ and all } j \in \{1, \dots, n\}.$$

Remark. This result implies immediately that a normalized Schauder basis of a reflexive Banach space cannot satisfy a (δ, R)-condition, which gives part (3) of Proposition 2.4. Indeed let X be a Banach with a normalized Schauder basis $\mathbf{e} = (e_i)_{i\geq 1}$, and assume that $\mathbf{e}$ satisfies the (δ, R)-condition for some $\delta, R > 0$. Should the space X be reflexive, the sequence $(x_n)_{n\geq 1}$ given by Theorem 4.9 would admit a subsequence converging weakly to some $x \in X$. Then we would have $\left\langle e^*_{m_j}, x \right\rangle \geq 1$ for every $j \geq 1$, a contradiction because $\left\langle e^*_{m_j}, x \right\rangle \to 0$ (recall that $\mathbf{e}$ is normalized).

Part (2) of Proposition 2.4 is also a consequence of Theorem 4.9. Indeed, let $\mathbf{e} = (e_i)$ be a normalized basic sequence that dominates the canonical basis of ℓ_p. Should $\mathbf{e}$ satisfy some (δ, R)-condition, the sequence $(x_n)_{n\geq 1}$ given by Theorem 4.9 would satisfy $\sum_{i\geq 1} |\langle e^*_i, x_n\rangle|^p \geq n$ for every $n \geq 1$, which is impossible because (x_n) is bounded. Hence, $\mathbf{e}$ does not satisfy any (δ, R)-condition, so $Q^+(\mathbf{e})$ is Haar null.

Theorem 4.9 is in fact a simple consequence of a classical combinatorial lemma due to V. Ptak, which was already an essential tool in [M3].

Lemma 4.10. *Let W be an infinite set, and let $\mathfrak{V}$ be a family of subsets of W. Denote by $\mathbf{P}(W)$ the set of all finitely supported probability measures on W. Assume that*

$$\inf_{\lambda\in\mathbf{P}(W)} \sup_{V\in\mathfrak{V}} \lambda(V) > 0.$$

Then, there exists a sequence $(V_j)_{j\geq 1}$ of elements of $\mathfrak{V}$ and a sequence $(w_j)_{j\geq 1}$ of pairwise distinct elements of W such that $\{w_1, \dots, w_n\} \subseteq V_n$ for every $n \geq 1$.

For a proof of Ptak's lemma, see [P] or [To].

Proof of Theorem 4.9. Let $W := \{-e_i;\ i \in \mathbb{N}\}$, and denote by $\mathfrak{V}$ the family of all subsets V of W such that $x + V \subseteq Q_R^+(\mathbf{e})$ for some $x \in \bar{B}(0, 1+R)$.

Let $\lambda \in \mathbf{P}(W)$, and write $\lambda(w)$ instead of $\lambda(\{w\})$, $w \in W$. Set

$$I_\lambda := \{i \in \mathbb{N};\ \lambda(-e_i) > 0\}.$$

Then I_λ is a finite set and $\sum_{i \in I_\lambda} \lambda(-e_i) = 1$.

Since $\mathbf{e}$ satisfies the (δ, R)-condition, there exist $J_\lambda \subseteq I_\lambda$ and $x \in \bar{B}(0, 1+R)$ such that $\sum_{i \in J_\lambda} \lambda(-e_i) \geq \delta > 0$ and $x + V \subseteq Q_R^+(\mathbf{e})$, where $V := \{ -e_i;\ i \in J_\lambda \}$. Then $V \in \mathfrak{V}$ and $\lambda(J) \geq \delta$.

By Ptak's lemma, there exists an increasing sequence of integers $(m_j)_{j \geq 1}$ and a sequence (V_j) of elements of $\mathfrak{V}$ such that $\{-e_{m_1}, \ldots, -e_{m_n}\} \subseteq V_n$ for every $n \geq 1$. So there exists a sequence $(x_n) \subseteq \bar{B}(0, 1+R)$ such that $x_n + \{-e_{m_1}, \ldots, -e_{m_n}\} \subseteq Q_R^+$ for all $n \geq 1$. Hence, for every $n \geq 1$ and all $j \in \{1, \ldots, n\}$, we have $\langle e_{m_j}^*, x_n \rangle \geq 1$. □

From Theorem 4.9, we easily deduce the following.

Corollary 4.11. *If* $\mathbf{e}$ *is a normalized,* 1*-unconditional basic sequence satisfying some* (δ, R)*-condition, then* $\mathbf{e}$ *has a subsequence* $(1+R)$*-equivalent to the canonical basis of* c_0*.*

Proof. Let (m_j) and (x_n) be given by Theorem 4.9, and set $\mathbf{f} := (e_{m_j})_{j \geq 1}$. By Fact 2.1, we have

$$\left\| \sum_{j=1}^{n} e_{m_j} \right\| \leq \|x_n\| \leq 1 + R$$

for all $n \geq 1$, which implies that $\| \cdot \|^{\mathbf{f}} \leq (1+R) \| \cdot \|_\infty^{\mathbf{f}}$. On the other hand, since $\mathbf{f}$ is normalized, we also have $\| \cdot \|_\infty^{\mathbf{f}} \leq \| \cdot \|^{\mathbf{f}}$, so the subsequence $\mathbf{f}$ is $(1+R)$-equivalent to the canonical basis of c_0. □

Remark. Of course, the same result holds provided $\mathbf{e}$ is only assumed to be semi-normalized, if we replace $1+R$ by some constant M depending only on R and on $\inf_{i \geq 1} \|e_i\|$.

If $\mathbf{e}$ is an unconditional basic sequence with a non-Haar null positive cone, then, after the usual renorming to make it 1-unconditional and keeping in mind Proposition 4.1, one can apply Corollary 4.11 to any normalized block-sequence of $\mathbf{e}$. This yields the next statement.

Corollary 4.12. *If* $\mathbf{e}$ *is an unconditional basic sequence whose positive cone is not Haar null, then every normalized block-sequence* $\mathbf{f}$ *of* $\mathbf{e}$ *is* c_0-saturated; *that is, any subsequence of* $\mathbf{f}$ *has a further subsequence equivalent to the canonical basis of* c_0*, with uniform bounds on the isomorphism constants.*

Another immediate consequence of Corollary 4.11 is an extension of Corollary 4.7 to the case of *subsymmetric* unconditional basic sequences. Recall that a basic sequence $\mathbf{e}$ is said to be *subsymmetric* if it is equivalent to all its subsequences. Every symmetric sequence is subsymmetric, but not conversely (see [LT]).

Corollary 4.13. *Up to equivalence, the canonical basis of* c_0 *is the only normalized and subsymmetric unconditional basic sequence whose positive cone is not Haar null.*

Proof. Let $\mathbf{e}$ be such a sequence. Since $\mathbf{e}$ is unconditional, Corollary 4.11 implies that $\mathbf{e}$ has a subsequence equivalent to the canonical basis of c_0. Since $\mathbf{e}$ is equivalent to all its subsequences, the result follows. □

Finally, we quote the following "decomposition" result for Banach spaces with a non-Haar null positive cone.

Corollary 4.14. *Let $\mathbf{e} = (e_i)_{i\geq 1}$ be a normalized unconditional basic sequence, and set $X := [\mathbf{e}]$. Assume that $Q^+(\mathbf{e})$ is not Haar null. Then one can find some constant M and a partition $(I_\lambda)_{\lambda\in\Lambda}$ of $\mathbb{N}$ such that each space $X_\lambda := [e_i;\ i \in I_\lambda]$ is M-isomorphic to c_0 and $X = \oplus_{\lambda\in\Lambda} X_\lambda$, where $\oplus$ denotes an unconditional Schauder decomposition.*

Proof. By Corollary 4.12, one can find a constant C such that every subsequence of $\mathbf{e}$ has a further subsequence that is C-equivalent to the canonical basis of c_0. By an obvious transfinite induction argument, it follows that one can partition $\mathbb{N}$ as $\mathbb{N} = I_0 \cup \bigcup_{\lambda\in\Lambda} I_\lambda$, where I_0 is finite (possibly empty) and each space X_λ is M-isomorphic to c_0. Replacing one set I_λ by $I_\lambda \cup I_0$ and using the unconditionality of $\mathbf{e}$, we get the result. ☐

4.4. More on c_0-saturation. We saw in Corollary 4.12 that unconditional basic sequences with a non-Haar null positive cone have a strong "c_0-saturation" property. In this section, we elaborate a little bit more on this.

For the sake of brevity, we adopt the following *ad hoc* terminology: given $M \geq 1$, a normalized basic sequence $\mathbf{e}$ is said to be *of type* Mc_0 if $\|\cdot\|^{\mathbf{e}} \leq M\|\cdot\|^{\mathbf{e}}_\infty$. This, of course, implies that $\mathbf{e}$ is equivalent to the canonical basis of c_0.

Definition 4.15. Let $\mathbf{e} = (e_i)_{i\geq 1}$ be a normalized basic sequence. We say that

- $\mathbf{e}$ is Mc_0*-saturated* (for some $M \geq 1$) if every subsequence of $\mathbf{e}$ has a further subsequence of type Mc_0;
- $\mathbf{e}$ is *strongly* Mc_0*-saturated* (for some $M \geq 1$) if the following is true: from every sequence $(I_j)_{j\geq 1}$ of pairwise disjoint subsets of $\mathbb{N}$ for which there exists $N \geq 1$ such that all sequence $(e_i)_{i\in I_j}$ are of type Nc_0, it is possible to extract a subsequence $(I'_j)_{j\geq 1}$ such that $(e_i)_{i\in\bigcup_{j\geq 1} I'_j}$ is of type NMc_0;
- $\mathbf{e}$ is (*strongly*) c_0*-saturated* if it is (strongly) Mc_0-saturated for some constant M.

Note that taking each I_i to be a singleton in the above definition, we see that strong Mc_0-saturation indeed implies Mc_0-saturation.

Recall also that a *space* X is said to be c_0-saturated if every (infinite-dimensional) closed subspace of X admits a further subspace isomorphic to c_0. Using the Bessaga–Pelczynski selection principle (Lemma 2.3), it is fairly easy to check that if a Banach space X admits a c_0-saturated (normalized) Schauder basis, then X is c_0-saturated.

Our aim is to prove the following strengthening of Corollary 4.12.

Theorem 4.16. *Let $\mathbf{e}$ be a normalized, 1-unconditional basic sequence. Assume that there exists $\delta, R > 0$ such that all normalized block-sequences satisfy the (δ, R)-condition. Then all these block-sequences are strongly $(1 + R)c_0$-saturated.*

This immediately yields the following.

Corollary 4.17. *If $\mathbf{e}$ is an unconditional basic sequence whose positive cone is not Haar null, then all normalized block-sequences of $\mathbf{e}$ are strongly c_0-saturated.*

The proof of Theorem 4.16 relies on the following lemma.

Lemma 4.18. *Let $\mathbf{e} = (e_i)_{i\geq 1}$ be a normalized, 1-unconditional basic sequence, and let $\delta, R > 0$. Assume that all normalized block-sequences of $\mathbf{e}$ satisfy the (δ, R)-condition. Let also $\mathbf{y} = (y_i)_{i\geq 1}$ be a normalized block-sequence of $\mathbf{e}$. Finally, let n be a positive integer, let $(\lambda_j)_{1\leq j\leq n}$ be a family of n nonnegative real numbers satisfying $\sum_j \lambda_j = 1$, and let $(I_j)_{1\leq j\leq n}$ be a family of n pairwise disjoint infinite subsets of $\mathbb{N}$ such that each sequence $(y_i)_{i\in I_j}$ is of type Nc_0 for some $N \geq 1$. Then one can find a set $J \subseteq \{1, \dots, n\}$ such that $\sum_{j\in J} \lambda_j \geq \delta$ and the sequence $(y_i)_{i\in\cup_{j\in J} I_j}$ is of type $N(1 + R)c_0$.*

Proof. For $m \geq 1$ and $1 \leq j \leq n$, let I_j^m be the set consisting of the first m elements of I_j, with the convention $I_j^m := I_j$ if $m \geq |I_j|$. Set $z_{m,j} := \sum_{i \in I_j^m} y_i$ (so that $\|z_{m,j}\| \leq N$) and $\widehat{z}_{m,j} = \frac{z_{m,j}}{\|z_{m,j}\|}$.

For any fixed m, the sets I_j^m are pairwise disjoint. Hence $(\widehat{z}_{m,j})_{1 \leq j \leq n}$ is a (finite) block-sequence of $(y_i)_{i \geq 1}$ for every $m \geq 1$. So it is also a block-sequence of $\mathbf{e}$, and it is normalized; hence it is 1-unconditional and satisfies the (finite version of) the (δ, R)-condition. Forgetting the dependence on m, we denote $\widehat{Q}^+$ its positive cone in $Z := \operatorname{span}(\widehat{z}_{m,j};\ 1 \leq j \leq n)$.

For $j \in \{1, \dots, n\}$, denote by Δ_j the Dirac measure at $-\widehat{z}_{m,j}$. Then $\mu = \sum_{j=1}^n \lambda_j \Delta_j$ is a probability measure with finite support contained in $\{-\widehat{z}_{m,j};\ 1 \leq j \leq n\}$. So there exists $z \in Z$ such that $\mu\big(z + \widehat{Q}_R^+\big) \geq \delta$, and hence there exists $J^m \subseteq \{1, \dots, n\}$ such that $\big\langle \widehat{z}_{m,j}^*, z \big\rangle \leq -1$ for all $j \in J^m$ and $\sum_{j \in J^m} \lambda_j \geq \delta$. Since $\big(\widehat{z}_{m,j}\big)_{1 \leq j \leq n}$ is 1-unconditional and $\|z\| \leq 1 + R$, from Fact 2.1 we deduce that $\big\|\sum_{j \in J^m} \widehat{z}_{m,j}\big\| \leq 1 + R$; and, again from Fact 2.1, it follows that $\big\|\sum_{j \in J^m} z_{m,j}\big\| \leq N(1 + R)$.

Since there are finitely many sets J^m, one can find $J \subseteq \{1, \dots, n\}$ such that $J = J^m$ for infinitely many m. Moreover, since the basis constant of $\mathbf{e}$ is equal to 1, we have

$$\Big\|\sum_{j \in J} z_{m,j}\Big\| \leq \Big\|\sum_{j \in J} z_{m',j}\Big\|$$

for every $m \geq 1$ and every $m' \geq m$.

It follows that $\big\|\sum_{j \in J} z_{m,j}\big\| \leq N(1 + R)$ for every $m \geq 1$. Hence, all finite sums built on $(y_j)_{j \in \cup_{i \in J} I_i}$ are bounded in norm by $N(1 + R)$, which implies that $(y_j)_{j \in \cup_{i \in J} I_i}$ is of type $N(1 + R)c_0$. □

Proof of Theorem 4.16. Since every normalized block-sequence of $\mathbf{e}$ satisfies the same assumptions as $\mathbf{e}$, it suffices to show that $\mathbf{e}$ is strongly $(1 + R)c_0$-saturated.

Let $\big(I_j\big)_{j \geq 1}$ be a sequence of pairwise disjoint subsets of $\mathbb{N}$ such that each sequence $(e_i)_{i \in I_j}$ is of type Nc_0, for some fixed $N \geq 1$.

Denote by $\mathfrak{V}$ the family of all subsets V of $\mathbb{N}$ such that the basic sequence $(e_i)_{i \in \cup_{j \in V} I_j}$ is of type $N(1 + R)c_0$. Lemma 4.18 ensures that

$$\inf_{\lambda \in \mathbf{P}(\mathbb{N})} \sup_{V \in \mathfrak{V}} \lambda(V) \geq \delta > 0.$$

It follows then from Ptak's lemma that there exists a sequence $(V_j)_{i \geq 1}$ of elements of $\mathfrak{V}$, and a sequence $(w_j)_{j \geq 1}$ of pairwise distinct positive integers such that $\{w_1, \dots, w_n\} \subseteq V_n$ for every $n \geq 1$. Then $(e_i)_{i \in \cup_{j \geq 1} I_{w_j}}$ is of type $N(1 + R)c_0$. □

4.5. An example. So far, we have shown that for a normalized unconditional basic sequence $\mathbf{e}$, the following implications hold true:

$Q^+(\mathbf{e})$ not Haar null $\Longrightarrow$ there exist $\delta, R > 0$ such that every normalized block-sequence of $(e_i)_{i \geq 1}$ satisfies the (δ, R)-condition $\Longrightarrow$ every normalized block-sequence of $\mathbf{e}$ is strongly c_0-saturated $\Longrightarrow$ the space $X := [\mathbf{e}]$ is c_0-saturated.

With our "main conjecture" in mind, it is rather tempting to believe that the strong saturation of all block-sequences suffices already to ensure that $\mathbf{e}$ is equivalent to the canonical basis of c_0. The following result shows that this is not so.

Example 4.19. There exists a normalized, 1-unconditional basic sequence $\mathbf{e}$ that is not isomorphic to the canonical basis of c_0 but has the property that all its block-sequences are strongly Mc_0-saturated for any $M \geq 1$.

Proof. Not surprisingly, the sequence $\mathbf{e}$ will be defined as the "canonical basis" of a c_0-direct sum of suitably chosen finite-dimensional ℓ_p spaces.

Let (p_n) be an increasing sequence of real numbers to be chosen later, with $p_n \geq 1$ and $\lim_{n\to\infty} p_n = \infty$. We set $X_n := \ell_{p_n}(n)$, and we denote by $(e_i^n)_{1\leq i\leq n}$ the canonical basis of X_n. Finally, we denote by X the c_0-direct sum of the spaces X_n,

$$X = \bigoplus_{c_0} \{X_n, n\in\mathbb{N}\}\,.$$

Denote by $(e_i)_{i\geq 1}$ the "canonical" enumeration of the vectors e_i^n; that is, $e_1 = e_1^1$, $e_2 = e_1^2$, $e_3 = e_2^2$, $e_4 = e_1^3$ an so on. Then (e_i) is a Schauder basis of X. This basis is normalized and 1-unconditional.

For $n \geq 1$, denote by $P_n\colon X \to X_n$ the canonical projection of X onto X_n. Since $K(\mathbf{e}) = 1$, we have $\|x\| = \sup\big\{\|P_n(x)\|, n\in\mathbb{N}\big\}$ for every $x\in X$.

We want (e_i) *not* to be equivalent to the canonical basis of c_0. For that it suffices to ensure that

$$\limsup_{m\to\infty} \Big\| \sum_{i=1}^m e_i \Big\| = \infty;$$

and this will be true if the sequence (p_n) satisfies

$$\lim_{n\to\infty} n^{1/p_n} = \infty.$$

Having fixed (p_n) in this way, we now show that every normalized block-sequence of $(e_i)_{i\geq 1}$ is strongly Rc_0-saturated for every $R\geq 1$. So, let us fix $R\geq 1$, let $(y_i)_{i\geq 1}$ be a normalized block-sequence of (e_i), and let $(I_i)_{i\geq 1}$ be a sequence of pairwise disjoint subsets of $\mathbb{N}$ such that, for some $N\geq 1$, all sequences $(y_j)_{j\in I_i}$ are of type Nc_0.

We claim that for any $k\geq 1$, one can find a positive integer n_k such that, for every $n\geq n_k$ and every family $(x_i)_{1\leq i\leq k}$ of disjointly supported elements of $X_n\cap \bar{B}(0,N)$, we have

$$\Big\| \sum_{i=1}^k x_i \Big\| \leq NR. \tag{4.2}$$

Indeed, denoting by J_i the support of x_i and writing $x_i = \sum_{q\in J_i} x_{i,q} e_q^n$, we have

$$\Big\| \sum_{i=1}^k x_i \Big\| = \Big[\sum_{i=1}^k \Big(\sum_{q\in J_i} |x_{i,q}^{p_n}| \Big) \Big]^{1/p_n} \leq \Big(\sum_{i=1}^k N^{p_n} \Big)^{1/p_n} \leq N\, k^{1/p_n}.$$

So inequality (4.2) is satisfied as soon as $k^{1/p_n} \leq R$, which holds true when n is sufficiently large because $p_n\to\infty$ as $n\to\infty$.

Since (y_i) is a block-sequence of (e_i), for every $k\geq 1$ there exists an integer $j_k\geq 1$ such that, for every $i\geq j_k$,

$$P_n(y_i) = 0 \ \text{ if } \ n < n_k.$$

Hence, for every $k\geq 1$, one can find an integer $i_k\geq 1$ such that

$$P_n(y_j) = 0 \ \text{ for } \ j\in I_{i_k} \ \text{ if } \ n < n_k.$$

To conclude the proof, now it suffices to show that the sequence $(y_j)_{j\in\cup_{k\geq 1} I_{i_k}}$ is of type NRc_0; and since this sequence is normalized and 1-unconditional, we only need to show that the norm of any finite sum built on it does not exceed NR.

Every finite sum w built on $(y_j)_{j\in\cup_{k\geq 1} I_{i_k}}$ can be written as

$$w = \sum_{1\leq k\leq m} w_k,$$

where $m \geq 1$ is an integer and w_k is a finite sum built on $(y_j)_{j \in I_{i_k}}$. Then, from the definition of the norm on X it follows that

$$\|w\| = \Big\| \sum_{1 \leq k \leq m} w_k \Big\| = \sup_{n \in \mathbb{N}} \Big\| P_n\Big(\sum_{1 \leq k \leq m} w_k \Big) \Big\|.$$

The definition of n_m shows that

$$\Big\| P_n\Big(\sum_{1 \leq k \leq m} w_k \Big) \Big\| \leq NR$$

for $n \geq n_m$. For $n_{i-1} \leq n < n_i$ with $1 \leq i \leq m$, we have

$$P_n\Big(\sum_{1 \leq k \leq m} w_k \Big) = P_n\Big(\sum_{1 \leq k \leq i-1} w_k \Big),$$

and so, by the definition of n_{i-1},

$$\Big\| P_n\Big(\sum_{1 \leq k \leq m} w_k \Big) \Big\| \leq NR.$$

Hence, we do get $\|w\| \leq NR$ for any finite sum w built on $(y_j)_{j \in \cup_{k \geq 1} I_{i_k}}$, which concludes the proof. □

Remark. As it turns out, the above sequence $\mathbf{e}$ does not satisfy any (δ, R)-condition, and hence $Q^+(\mathbf{e})$ is Haar null. Indeed if $\delta > 0$ is given, then for any set $I \subseteq \{1, \ldots, n\}$ of cardinality δn we have $\big\| \sum_{i \in I} e_i^n \big\| \geq (\delta n)^{1/p_n}$, which tends to ∞ by the choice of the sequence (p_n). By Proposition 4.6, it follows that $\mathbf{e}$ cannot satisfy any (δ, R)-condition. Therefore, this example leaves our "main conjecture" open.

§5. Quotients

We saw in the previous section that if a Banach space X admits an unconditional basis X whose positive cone is not Haar null, then X is c_0-saturated. One of our main objectives in the present section is to prove the following more precise result.

Theorem 5.1. *Let X be a Banach space, and assume that X admits an unconditional basis whose positive cone is not Haar null. Then X* has c_0-saturated quotients; *that is, every quotient space of X is c_0-saturated.*

To put this result into perspective, it is worth mentioning that the class of Banach spaces with c_0-saturated quotients seems far from being completely understood.

Odell showed in [O] that the Schreier space $\mathcal{S}$ has this property, and asked whether this holds true for every c_0-saturated Banach space having an unconditional and shrinking Schauder basis. Leung [L] first gave a negative answer to this question, by constructing a c_0-saturated space with a shrinking unconditional basis admitting a quotient isomorphic to ℓ_2. This was generalized by Gasparis [G], who obtained the same result for any ℓ_p, $1 < p < \infty$.

Theorem 5.1 gives one class of examples, possibly consisting of the only space c_0 (!). In any event, it shows that the c_0-saturation of quotients is a property strictly weaker than the non-Haar negligibility of the positive cone (by Odell's result and Corollary 4.8), and that the positive cones of the spaces constructed in [L] and [G] are Haar null.

As a matter of fact, we shall deduce Theorem 5.1 from a more general result having little to do with Haar negligibility. First, we introduce the following terminology.

Definition 5.2. A Banach space X will be said to have *property* (P) if every weakly null seminormalized sequence in X admits a subsequence equivalent to the canonical basis of c_0.

It is clear that in this definition, one can replace "seminormalized" by "normalized".

Note that if X admits a Schauder basis $\mathbf{e}$, then, by Bessaga–Pelczynki's selection principle, property (P) can be tested by looking only at those weakly null normalized sequences that are block-sequence of $\mathbf{e}$. By Theorem 4.16 (or Corollary 4.12) it follows that if the basis $\mathbf{e}$ is unconditional and has a non-Haar null positive cone, then X has property (P).

Note also that if X has separable dual, then every quotient space E of X with property (P) is c_0-saturated: indeed, any infinite-dimensional subspace of E contains a weakly null normalized sequence (because it has separable dual), and hence a subspace isomorphic to c_0 by property (P).

From these two remarks, it is clear that the following result implies Theorem 5.1.

Theorem 5.3. *Let X be a Banach space admitting an unconditional shrinking Schauder basis. If X has property* (P)*, then all quotients of X have property* (P)*.*

Indeed, assume that X is a Banach space admitting an unconditional basis $\mathbf{e}$ whose positive cone is not Haar null. Then X has property (P) by Corollary 4.12. Moreover, the basis $\mathbf{e}$ is shrinking by Corollary 4.2. By Theorem 5.3, all quotients of X have property (P), and hence all these quotients are c_0-saturated because X has separable dual.

The proof of Theorem 5.3 relies on the following result from [O].

Proposition 5.4. *Let X be a Banach space admitting a normalized, shrinking, and 1-unconditional basis $(e_i)_{i\geq 1}$. Let T be a continuous surjective linear operator from X onto another Banach space Y, and let C be any finite constant such that $\bar{B}_Y(0,1)\subseteq C\,T(\bar{B}_X(0,1))$. Let also $(\varepsilon_i)_{i\geq 1}$ be a monotone decreasing sequence of positive real numbers tending to 0.*

Then, given any weakly null sequence $(y_i)_{i\geq 1}$ in the unit ball of Y, one can find a subsequence (y_i') of (y_i) and a monotone increasing sequence of integers $(p_i)_{i\geq 0}$ with $p_0=0$ satisfying the following property.

For every sequence of real numbers $(a_i)_{i\geq 1}$ such that $\big\|\sum_{i=1}^\infty a_iy_i'\big\|\leq 2$, there exists a monotone increasing sequence of integers $(r_i)_{i\geq 0}$ with $r_0=0$ and a sequence $(x_i)_{i\geq 1}\subseteq X$ such that

1. $p_i<r_i<p_{i+1}$ *for every* $i\geq 1$;
2. $x_i\in \operatorname{span}\big(e_{r_{i-1}},\dots,e_{r_i}\big)$;
3. $\|T(x_i)-a_iy_i'\|\leq\varepsilon_i$;
4. *the series* $\sum x_i$ *is convergent and* $\big\|\sum_{i=1}^\infty x_i\big\|\leq 2C$.

Proof of Theorem 5.3. Fix a normalized shrinking unconditional basis $\mathbf{e}=(e_i)_{i\geq 1}$ for X. Since property (P) is invariant under renormings, we may assume that $\mathbf{e}$ is 1-unconditional.

Let Y be any quotient of X, and let $T\colon X\to Y$ be the canonical quotient map. Note that $\bar{B}_Y(0,1)\subseteq T\big(\bar{B}_X(0,1)\big)$.

Let $(y_i)_{i\geq 1}$ be a weakly null normalized sequence in Y. Replacing this sequence by a subsequence, we may suppose that (y_i) is a basic sequence, with basis constant K.

We apply Proposition 5.4 to (y_i), with $\varepsilon_i=\frac{1}{2K}\,2^{-i}$; this gives a subsequence (y_i') of (y_i) and a monotone increasing sequence of integers $(p_i)_{i\geq 1}$.

For $i,n\geq 1$ denote by $\delta_{i,n}$ the usual "Kronecker symbol". Then $\big\|\sum_{i=1}^\infty \delta_{i,n}y_i'\big\|=1\leq 2$ for every $n\geq 1$. Hence, for each $n\geq 1$, one can find a sequence $(x_{i,n})_{i\geq 1}\subseteq X$ such that

1. $x_{i,n}\in[e_r;\ p_{i-1}<r<p_{i+1}]$;
2. $\big\|\sum_{i=1}^\infty x_{i,n}\big\|\leq 2$, and in particular $\|x_{n,n}\|\leq 2$;
3. $\|T(x_{n,n})-y_n'\|\leq\varepsilon_n$.

Set $u_i := x_{2i,2i}$ and $v_i := y'_{2i}$. Note that by (3), we have $\|u_i\| \geq \frac{1-\varepsilon_{2i}}{\|T\|} \geq \frac{3}{4\|T\|}$, so that (u_i) is a seminormalized block-sequence of (e_i).

By the definition of (u_i) and (v_i), we have

$$\sum_{i=1}^{\infty} \|T(u_i) - v_i\| \leq \sum_{i=1}^{\infty} \varepsilon_{2i} < \frac{1}{2K}.$$

Since $(v_i)_{i\geq 1}$ is a normalized basic sequence, the principle of small perturbations, (Lemma 2.2) shows that $(T(u_i))$ is a basic sequence equivalent to (v_i) and that for every sequence of real numbers (a_i) such that the series $\sum a_i v_i$ is convergent we have

$$\left(1 - 2K \sum_{i=1}^{\infty} \|v_i - T(u_i)\|\right) \left\| \sum_{i=1}^{\infty} a_i v_i \right\| \leq \left\| \sum_{i=1}^{\infty} a_i T(u_i) \right\| \leq 2K \left\| \sum_{i=1}^{\infty} a_i v_i \right\|.$$

In particular, there exists a constant $c > 0$ such that

$$c \left\| \sum_{i=1}^{\infty} a_i v_i \right\| \leq \|T\| \cdot \left\| \sum_{i=1}^{\infty} a_i u_i \right\|$$

for any such sequence (a_i).

Now, observe that the sequence (u_i) is weakly null, being a bounded block-sequence of the shrinking Schauder basis $\mathbf{e}$. Since (u_i) is also seminormalized and X has property (P), it follows that (u_i) has a subsequence equivalent to the canonical basis of c_0. The above inequality then implies that the corresponding subsequence of (v_i) is a subsequence of (y_i) equivalent to the canonical basis of c_0. $\square$

To conclude this section (and although this has nothing to do with Haar negligibility), now we show that property (P) is "almost" a three spaces property.

Proposition 5.5. *Let X be a Banach space admitting a shrinking unconditional basis, and let F be a closed subspace of X. If F and X/F have property* (P)*, then X has property* (P).

Proof. Let $\mathbf{e} = (e_i)_{i\geq 1}$ be a normalized, shrinking unconditional basis for X, and assume without loss of generality that $\mathbf{e}$ is 1-unconditional. Also, set $Y := X/F$ and denote by $T\colon X \to Y$ the canonical quotient map.

Let $(x_i)_{i\geq 1}$ be a weakly null normalized sequence in X. To show that (x_i) admits a subsequence equivalent to the canonical basis of c_0, we distinguish two cases.

Case 1. $\inf_i \|T(x_i)\| = 0$.

We may assume (upon extracting a subsequence) that $\lim_{i\to\infty} \|T(x_i)\| = 0$. Then there exists a sequence $(f_i) \subseteq F$ such that $\|x_i - f_i\| \to 0$. The sequence (f_i) is weakly null, and since (x_i) is normalized we may assume that it is also normalized. Since F has property (P), (f_i) has a subsequence (f'_i) equivalent to the canonical basis of c_0. By the principle of small perturbations, (x_i) has a subsequence equivalent to some subsequence of (f'_i), which gives the required result.

Case 2. $\inf_i \|T(x_i)\| > 0$.

In this case, $(T(x_i))$ is a a weakly null seminormalized sequence in Y. Since Y has property (P), we may assume that this sequence is equivalent to the canonical basis of c_0. Choose a constant $M > 0$ such that for every finite set $I \subseteq \mathbb{N}$, we have

$$\left\| \sum_{i\in I} \frac{1}{M} T(x_i) \right\| \leq 2.$$

Since $\bar{B}_Y(0,1) \subseteq T(\bar{B}_X(0,1))$, we may apply Proposition 5.4 with $C := 1$ and $y_i := \frac{1}{M}\,T(x_i)$, taking, e.g., $\varepsilon_i := 2^{-i}$. This gives a subsequence $(y_i')_{i\geq 1}$ of $\big(\frac{1}{M}\,T(x_i)\big)_{i\geq 1}$ and a monotone increasing sequence of positive integers $(p_i)_{i\geq 0}$. We denote by $(x_i')_{i\geq 1}$ the subsequence of (x_i) such that

$$y_i' = \frac{1}{M}\,T(x_i').$$

For each $n \geq 1$, we have $\big\|\sum_{i=1}^n y_i'\big\| \leq 2$. So there exists a sequence $(x_{i,n})_{i\geq 1} \subseteq X$ such that

- $x_{i,n} \in E_i := [e_r;\ p_{2i-1} < r < p_{2i+1}]$;
- $\|T(x_{i,n}) - y_i'\| \leq \varepsilon_i$ if $i \leq n$;
- $\big\|\sum_{i=1}^\infty x_{i,n}\big\| \leq 2$, and hence $\big\|\sum_{i\in I} x_{i,n}\big\| \leq 2$ for every finite set $I \subseteq \mathbb{N}$.

For each fixed $i \geq 1$, the sequence $(x_{i,n})_{n\geq 1}$ is a bounded sequence in the finite-dimensional space E_i; in fact $\|x_{i,n}\| \leq 2$. By a diagonal argument we may assume that $(x_{i,n})_{n\geq 1}$ converges to some $\bar{x}_i \in E_i$ for every $i \geq 1$. Then $\|\bar{x}_i\| \leq 2$ and $\|T(\bar{x}_i) - y_i'\| \leq \varepsilon_i$. Since (y_i') is seminormalized, the latter inequality implies that $\liminf_i \|\bar{x}_i\| > 0$; hence, we may assume that $(\bar{x}_i)$ is seminormalized.

Now set $u_i := \bar{x}_{2i}$ and $v_i := \frac{1}{M}x_{2i}'$. Observe that (u_i) is a seminormalized block-sequence of $\mathbf{e}$. For every finite set $I \subseteq \mathbb{N}$, we have

$$\Big\|\sum_{i\in I} u_i\Big\| = \lim_{n\to\infty}\Big\|\sum_{i\in I} x_{2i,n}\Big\| \leq 2.$$

Since (u_i) is unconditional (being a block-sequence of $\mathbf{e}$) and seminormalized, it follows that (u_i) is equivalent to the canonical basis of c_0. Moreover, since $\|T(\bar{x}_i) - y_i'\| \leq \varepsilon_i$ for all i, we have

$$\lim_{i\to\infty} T(u_i - v_i) = 0.$$

Set $z_i := u_i - v_i$. If there exists a subsequence of $(z_i)_{i\geq 1}$ that converges to 0, the principle of small perturbations tells us that some subsequence of (v_i) is equivalent to (u_i), i.e., to the canonical basis of c_0, and the conclusion follows because $v_i = \frac{1}{M}x_i$. Otherwise (discarding finitely many terms) the sequence (z_i) is seminormalized, and hence we may apply Case 1 to it; so there exists a subsequence of (z_i) equivalent to the canonical basis of c_0. It is then easily checked that the same property holds true for (v_i), and hence for (x_i). □

§6. Positive cones and Gaussian measures

6.1. Gaussian test measures. In this section we intend to produce "explicit" test measures for Haar null positive cones. More precisely, we shall see that under reasonable assumptions, it is possible to construct test measures that are *Gaussian* measures.

Gaussian measures on Banach spaces may be defined as follows: a Borel probability measure μ on a separable Banach space X is Gaussian if and only if it is the distribution of an X-valued random variable of the form

$$\xi(\omega) = x_0 + \sum_{i=1}^\infty g_i(\omega)\,x_i,$$

where $x_i \in X$, the g_i are independent real-valued random variables (defined on the same probability space $(\Omega, \mathfrak{A}, \mathbb{P})$) having standard Gaussian distribution $\gamma = \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\,dt$, and the series is almost surely convergent. This means that for any Borel set $A \subseteq X$, we have

$$\mu(A) = \mathbb{P}(\xi \in A).$$

A Gaussian measure μ is said to be *centered* if $x_0 = 0$ in the above representation, which means that $\int_X x^*\, d\mu = 0$ for every continuous linear functional $x^* \in X^*$. The measure μ is *nondegenerate* if $\mu(V) > 0$ for every open set $V \neq \varnothing$; in the above representation, this means that the linear span of the $x_i, i \geq 1$, is dense in X. For more information on Gaussian measures, we refer to [Bo] or [CTV].

In the remaining part of this section, we fix a normalized basic sequence $\mathbf{e}$ whose positive cone is Haar null, and we denote by X the Banach space generated by $\mathbf{e}$. For the sake of notational simplicity, we write Q^+ instead of $Q^+(\mathbf{e})$. For technical reasons, we will in fact mostly deal with the *negative* cone $Q^- := -Q^+$.

We shall consider Gaussian measures of a very special type. Let us fix once and for all a sequence of independent real-valued random variables (defined on the same probability space $(\Omega, \mathfrak{A}, \mathbb{P})$) having standard Gaussian distribution. We denote by $\mathfrak{R}(\mathbf{e})$ the family of all sequences of positive real numbers $\mathfrak{r} = (r_i)_{i\geq 1}$ such that the random series $\sum r_i g_i e_i$ is almost surely convergent. If $\mathfrak{r} \in \mathfrak{R}(\mathbf{e})$, the corresponding Gaussian measure is denoted by $\mu_\mathfrak{r}$:

$$\mu_\mathfrak{r} \sim \sum r_i g_i e_i.$$

Any such measure will be called a *diagonal* Gaussian measure. Note that $\mu_\mathfrak{r}$ is centered, and also nondegenerate because $r_i \neq 0$ for all i.

We restrict ourselves to diagonal measures because this allows us to compute very easily the measure of a translate of the negative cone. Indeed, if $x = \sum_1^\infty x_i e_i \in X$ then, by independence, we have

$$\mu(x + Q^-) = \prod_{i\geq 1} \mathbb{P}\Big(g_i \leq \frac{x_i}{r_i}\Big). \tag{6.1}$$

Having set this notation, we are going to prove the following result.

Theorem 6.1. *Assume that the basic sequence* $\mathbf{e}$ *is unconditional. Let* $\mathfrak{r} \in \mathfrak{R}(\mathbf{e})$*, and assume that there exists a permutation* π_0 *of the integers and a constant* $\alpha > 0$ *such that* $r_{\pi_0(i)} = O(i^{-\alpha})$*. Then* $\mu_\mathfrak{r}$ *is a test measure for* $Q^- = Q^-(\mathbf{e})$ *if and only if the series* $\sum \log(i)^{1/2} r_{\pi_0(i)} e_{\pi_0(i)}$ *is divergent.*

When $\mathbf{e}$ is the canonical basis of ℓ_p, it is well known that a Gaussian series $\sum r_i g_i e_i$ is almost surely convergent in ℓ_p if and only if $\sum_1^\infty |r_i|^p < \infty$ (see [CTV, Chapter V]). Applying Theorem 6.1, we get the following statement.

Corollary 6.2. *Assume that* $\mathbf{e}$ *is the canonical basis of* ℓ_p*, and let* $\mathfrak{r} = (r_i)$ *be a sequence of positive real numbers such that* $\sum_1^\infty r_i^p < \infty$*. Denote by* $(\bar{r}_i)$ *the decreasing rearrangement of* (r_i)*. Then, the measure* $\mu_\mathfrak{r}$ *is a test measure for* Q^- *if and only if*

$$\sum_{i\geq 1} \log(i)^{p/2}\, \bar{r}_i^p = \infty.$$

Proof. This follows immediately from Theorem 6.1 because $\bar{r}_i = o\big(i^{-1/p}\big)$. $\square$

To prove Theorem 6.1, we basically need to determine whether an expression like (6.1) above can be positive or not. The following remark shows that it suffices to consider vectors $x \in X$ having *positive* coefficients on the basis $\mathbf{e}$. The set of all such vectors x will be denoted by Q^{++}:

$$Q^{++} = \Big\{x = \sum_{i=1}^\infty x_i e_i \in X;\ x_i > 0 \text{ for all } i \geq 1\Big\}.$$

Remark 6.3. Let $\mathfrak{r} \in \mathfrak{R}(\mathbf{e})$. For any $a \in X$, one can find $x \in Q^{++}$ such that $\mu_\mathfrak{r}(x+Q^-) \geq \mu_\mathfrak{r}(a+Q^-)$. In particular, if $\mu_\mathfrak{r}$ is not a test measure for Q^-, then there exists $x \in Q^{++}$ such that $\mu_\mathfrak{r}(x+Q^-) > 0$.

Proof. Write $a = \sum_1^\infty a_i e_i$ and assume without loss of generality that $\mu_{\mathfrak{r}}(a+Q^-) > 0$. Set $I_a^- = \{i \in \mathbb{N};\ , a_i \le 0\}$ and $I_a^+ = \{i \in \mathbb{N};\ a_i > 0\}$; then

$$\mu_{\mathfrak{r}}(a+Q^-) = \prod_{i \in I_a^-} \mathbb{P}\Big(g_i \le \frac{a_i}{r_i}\Big) \times \prod_{i \in I_a^+} \mathbb{P}\Big(g_i \le \frac{a_i}{r_i}\Big),$$

and hence

$$\mu_{\mathfrak{r}}(a+Q^-) \le \prod_{i \in I_a^-} \mathbb{P}\Big(g_i \le \frac{a_i}{r_i}\Big) \le 2^{-|I_a^-|}.$$

Hence, the set I_a^- is finite. So, setting $x_i := 1$ if $i \in I_a^-$ and $x_i := a_i$ if $i \in I_a^+$, we see that $x := \sum_{i\ge 1} x_i e_i$ is a well-defined element of Q^{++} such that $\mu_{\mathfrak{r}}(x+Q^-) \ge \mu_{\mathfrak{r}}(a+Q^-)$. □

In view of this remark, in what follows we shall only consider translates of the negative cone Q^- by elements of Q^{++}.

Lemma 6.4. *Let $\mathfrak{r} = (r_i) \in \mathfrak{R}(\mathbf{e})$, and let $x = \sum_{i\ge 1} x_i e_i \in Q^{++}$. Then*

$$\mu_{\mathfrak{r}}(x+Q^-) > 0 \ \textit{if and only if} \ \sum_{i\ge 1} \frac{e^{-\frac{1}{2}(x_i/r_i)^2}}{(x_i/r_i)} < \infty.$$

Proof. We have $\mu(x+Q^-) = \prod_{i\ge 1} \mathbb{P}\big(g_i \le \frac{x_i}{r_i}\big) = \prod_{i \ge 1}\big(1 - \mathbb{P}\big(g_i \ge \frac{x_i}{r_i}\big)\big)$. Therefore, $\mu(x+Q^-) > 0$ if and only if

$$\sum_{i\ge 1} \mathbb{P}\Big(g_i \ge \frac{x_i}{r_i}\Big) < \infty.$$

Set $g(t) := \frac{e^{-t^2/2}}{\sqrt{2\pi}}$ and $G(a) = \int_a^\infty g(t)\,dt$, $a > 0$. Then $\mathbb{P}\big(g_i \ge \frac{x_i}{r_i}\big) = G\big(\frac{x_i}{r_i}\big)$. Now, it is well known that

$$G(a) \sim \frac{e^{-a^2/2}}{\sqrt{2\pi} a} \text{ as } a \to \infty,$$

and the lemma follows. □

This lemma allows us now to state the following "comparison criterion".

Lemma 6.5. *Let $\mathfrak{r} \in \mathfrak{R}(\mathbf{e})$, and let $x \in Q^{++}$. Let also π be a permutation of the integers.*

- *If* $\limsup_{i\to\infty} \big(\frac{x_{\pi(i)}}{r_{\pi(i)}} \log(i)^{-1/2}\big) < \sqrt{2}$, *then* $\mu_{\mathfrak{r}}(x+Q^-) = 0$.
- *If* $\liminf_{i\to\infty} \big(\frac{x_{\pi(i)}}{r_{\pi(i)}} \log(i)^{-1/2}\big) > \sqrt{2}$, *then* $\mu_{\mathfrak{r}}(x+Q^-) > 0$.

Proof. Set $b_i := \frac{x_i}{r_i}$ and $a_i = \frac{x_{\pi(i)}}{r_{\pi(i)}}$. Then $\mu(x+Q^-) > 0$ if and only if $\sum_1^\infty \frac{e^{(b_i)^2/2}}{b_i} < \infty$, which is equivalent to the convergence of the series $\sum \frac{e^{-(a_i)^2/2}}{a_i}\cdot$ We need the following statement.

Fact. For any $M > 0$, set $h_M(t) := g(M \log(t)^{1/2})$, where $g(t) = \frac{e^{-t^2/2}}{t}\cdot$ Then the series $\sum_{i\ge 1} h_M(i)$ converges if $M > \sqrt{2}$, and diverges if $M < \sqrt{2}$.

Proof of Fact. The function h_M is positive and monotone decreasing on $(0, \infty)$, and so the convergence of the series $\sum_{i \ge 1} h_M(i)$ is equivalent to the convergence of the integral $\int_1^\infty h_M(t)\,dt$. A simple computation gives $h_M'(t) = -h_M(t)\big(\frac{M^2}{2t} + \frac{1}{2t\log(t)}\big)$, and so $\frac{h_M'(t)}{h_M(t)} \sim -\frac{M^2}{2t}$ as $t \to \infty$. It follows that the integral $\int_1^\infty h_M(t)$ is converges if $M > \sqrt{2}$ and diverges if $M < \sqrt{2}$. □

If $\liminf_{i\to\infty}\big(a_i\log(i)^{-1/2}\big) > \sqrt{2}$, take $M > \sqrt{2}$ such that $a_i \geq M\log(i)^{1/2}$ for i sufficiently large, say $i \geq i_M$. Since $g(t) = \frac{e^{-t^2/2}}{t}$ is monotone decreasing on $(0,\infty)$, we obtain

$$\sum_{i\geq i_M}\frac{e^{-(a_i)^2/2}}{a_i} = \sum_{i\geq i_M} g(a_i) \leq \sum_{i\geq i_M} g\big(M\,\log(i)^{1/2}\big) = \sum_{i\geq i_M} h_M(i),$$

and hence $\mu_{\mathfrak{r}}(x+Q^-) > 0$ by the above Fact and Lemma 6.4.

If $\limsup_{i\to\infty}(a_i\log(i)^{-1/2}) < \sqrt{2}$, then in the same way we get $\mu_{\mathfrak{r}}(x+Q^-) = 0$. $\square$

Lemma 6.5 immediately implies the following.

Corollary 6.6. *Let $\mathfrak{r} \in \mathfrak{R}(\mathbf{e})$. If there exists a permutation π of the positive integers such that the series $\sum_{i\geq 1}\log(i)^{1/2}r_{\pi(i)}e_{\pi(i)}$ converges in X, then $\mu_{\mathfrak{r}}$ is not a test measure for Q^-.*

Proof. If we set $x_M := M\sum_{i\geq 1}\log(i)^{1/2}r_{\pi(i)}e_{\pi(i)}$, then $\mu_{\mathfrak{r}}(x_M+Q^-) > 0$ for any $M > \sqrt{2}$. $\square$

The next lemma gives a kind of converse to Corollary 6.6.

Lemma 6.7. *Assume that the basic sequence $\mathbf{e}$ is unconditional, and let $\mathfrak{r} \in \mathfrak{R}(\mathbf{e})$. If $\mu_{\mathfrak{r}}$ is not a test measure for Q^-, then, for every permutation π of the positive integers and every $\alpha > 0$, there exists a set $I = I_{\pi,\alpha} \subseteq \mathbb{N}$ such that $\sum_{i\in I}\frac{1}{i^\alpha\log(i)^{1/2}} < \infty$ and the series $\sum_{i\notin I}\log(i)^{1/2}r_{\pi(i)}e_{\pi(i)}$ is convergent.*

Proof. Assume that μ is not a test measure Q^-, and choose a point $x \in Q^{++}$ such that $\mu(x+Q^-) > 0$. Write $x = \sum_{i\geq 1}a_ir_ie_i$, so that $a_i > 0$ for all i. By Lemma 6.4, we know that the series $\sum\frac{e^{-(a_i)^2/2}}{a_i}$ is convergent.

Now, fix $\alpha > 0$ and a permutation π of the integers. Set

$$I := \big\{i\in\mathbb{N};\ a_{\pi(i)} \leq \sqrt{2\alpha}\log(i)^{1/2}\big\}.$$

By unconditionality and the definition of I, the series $\sum_{i\notin I}\log(i)^{1/2}r_{\pi(i)}e_{\pi(i)}$ is convergent. Moreover, we have

$$\sum_{i\in I}\frac{1}{i^\alpha\log(i)^{1/2}} = \sum_{i\in I}\frac{e^{-\alpha\log(i)}}{\log(i)^{1/2}} \leq \sum_{i\in I}\frac{e^{-(a_{\pi(i)})^2/2}}{a_{\pi(i)}} < \infty.$$

$\square$

Proof of Theorem 6.1. Taking α smaller if necessary, we may assume that in fact

$$r_{\pi_0(i)} = O\Big(\frac{1}{i^\alpha\log(i)}\Big).$$

By Corollary 6.6, we know that if $\mu_{\mathfrak{r}}$ is a test measure for Q^-, then the series $\sum\log(i)^{1/2}r_{\pi_0(i)}e_{\pi_0(i)}$ is divergent.

Conversely, assume that $\mu_{\mathfrak{r}}$ is not a test measure for Q^-. Then, by Lemma 6.7, there exists a set $I\subseteq\mathbb{N}$ such that $\sum_{i\in I}\frac{1}{i^\alpha\log(i)^{1/2}} < \infty$ and the series $\sum_{i\notin I}\log(i)^{1/2}r_{\pi_0(i)}e_{\pi_0(i)}$ is convergent.

Since $r_{\pi_0(i)} = O\big(\frac{1}{i^\alpha\log(i)}\big)$, we have

$$\sum_{i\in I}\|\log(i)^{1/2}r_{\pi_0(i)}e_{\pi_0(i)}\| = \sum_{i\in I}\log(i)^{1/2}r_{\pi_0(i)} < \infty.$$

So the series $\sum_{i\in I}\log(i)^{1/2}r_{\pi_0(i)}e_{\pi_0(i)}$ is convergent, and altogether the entire series $\sum\log(i)^{1/2}r_{\pi_0(i)}e_{\pi_0(i)}$ is convergent. $\square$

6.2. Positive cones are not Gauss null. Besides Haar negligibility, there are many natural notions of smallness for subsets of Banach spaces. A most useful one is *Gauss negligibility*, which was introduced by R. R. Phelps in [Ph]. A Borel set A in a separable Banach space X is said to be *Gauss null* if $\mu(A) = 0$ for every nondegenerate Gaussian measure μ on X.

Since the family of all Gaussian measures on X is translation invariant, any Gauss null set is obviously Haar null; more precisely, a Borel set $A \subseteq X$ is Gauss null exactly when every nondegenerate Gaussian measure on X is a test measure for A.

Gauss negligibility is in fact a *much* stronger notion of smallness than Haar negligibility. For example, a compact set need not be Gauss null (see [BL]). More in the spirit of the present paper, we have the following result.

Proposition 6.8. *Positive cones determined by basic sequences are never Gauss null.*

Proof. Let $\mathbf{f} = (f_i)_{i\geq 1}$ be any normalized basic sequence, and set $X := [\mathbf{f}]$.

Let also $(e_i)_{i\geq 1}$ be the canonical basis of ℓ_2, and fix any vector $x = \sum_{i\geq 1} x_i e_i \in \ell_2$ such that $x_i \neq 0$ for all $i \geq 1$. Let (α_i) be a sequence of positive numbers such that the formula

$$T\left(\sum_{i\geq 1} y_i e_i\right) := \sum_{i\geq 1} \alpha_i \,\mathrm{sgn}(x_i)\, y_i f_i$$

defines a bounded operator $T\colon \ell_2 \to X$. (For example, we can take any square-summable sequence (α_i).) Note that the operator T has dense range because $\alpha_i x_i \neq 0$ for all i.

Now, choose a sequence $(a_i)_{i\geq 1} \subseteq (0,1)$ such that $\prod_{i\geq 1}(1-a_i) > 0$, and a sequence of real numbers (r_i) such that $0 < \frac{r_i}{x_i} \leq \sqrt{a_i}$ for all $i \geq 1$. The series $\sum_{i\geq 1} r_i e_i$ is convergent in ℓ_2, and so the Gaussian series $\sum_{i\geq 1} r_i g_i e_i$ is almost surely convergent. Let ν be the associated Gaussian measure on ℓ_2:

$$\nu \sim \sum_{i=1}^{\infty} r_i g_i e_i.$$

Since the operator $T\colon \ell_2 \to X$ has dense range, the measure

$$\mu := \nu \circ T^{-1} \sim \sum_{i=1}^{\infty} r_i g_i T e_i$$

is a nondegenerate Gaussian measure on X.

Set

$$H := \left\{ y = \sum_{i\geq 1} y_i e_i \in \ell_2;\ \frac{y_i}{r_i} \geq 0 \text{ for all } i \right\}.$$

By the independence of the random variables g_i, we have

$$\nu(-x+H) = \prod_{i\geq 1} \mathbb{P}\big(g_i \geq -x_i/r_i\big) \geq \prod_{i\geq 1} \mathbb{P}\big(|g_i| \leq x_i/r_i\big) \geq \prod_{i\geq 1}\big(1 - \mathbb{P}(|g_i| > x_i/r_i)\big).$$

Since $\mathbb{P}(|g_i| > x_i/r_i) \leq r_i^2/x_i^2$ by Chebyshev's inequality, it follows that

$$\nu(-x+H) \geq \prod_{i\geq 1}\left(1 - \frac{r_i^2}{x_i^2}\right) \geq \prod_{i\geq 1}(1-a_i) > 0.$$

On the other hand, since $T(H) \subseteq Q^+(\mathbf{f})$ by the definition of T and since T is one-to-one, we have

$$\mu(T(-x) + Q^+(\mathbf{f})) \geq \mu\big(T(-x+H)\big) = \nu(-x+H).$$

So $\mu(T(-x) + Q^+(\mathbf{f})) > 0$, and hence $Q^+(\mathbf{f})$ is not Gauss null. $\square$

Remark. Another way of proving Proposition 6.8 is by using a deep result of M. Csörnyei, according to which Gauss negligibility is equivalent to *cube negligibility* (see [Cs] or [BL]). Indeed, it is not difficult to check directly that the positive cone of a Schauder basis is never cube-null (see [Mo, Proposition 3.5.2]).

§7. Some questions

(1) Let us say that a Borel set A in a Polish Abelian group G is *compactivorous* if for every compact set $K \subseteq G$, one can find $x \in G$ and an open set V such that $V \cap K \neq \varnothing$ and $x + (V \cap K) \subseteq A$. Does there exist a non-Haar null positive cone (in some Banach space) that is not compactivorous?

(1′) Does there exist *any* closed convex non-Haar null set that is not compactivorous? Note that for a quite large class of Polish Abelian groups G, one can find a G_δ set in G that is non-Haar null and non-compactivorous (see [M, Proposition 1.2.2]), but it does not seem clear that there always exists a *closed* set with these properties.

(2) What can be said of a Banach lattice whose positive cone is not Haar null?

(3) Let $\mathbf{e}$ be a normalized unconditional basic sequence. Assume that there exist $\delta, R > 0$ such that all block-sequences of $\mathbf{e}$ satisfy the (δ, R)-condition. Does this imply that $\mathbf{e}$ must be equivalent to the canonical basis of c_0?

(3′) In the situation of (3), is it at least true that $Q^+(\mathbf{e})$ is not Haar null?

(4) Regarding our "main conjecture", one may ask if the following weaker result holds true: if $\mathbf{e}$ is an unconditional basic sequence whose positive cone is not Haar null, then the Banach space $[\mathbf{e}]$ is isomorphic to a *subspace* of c_0. The methods of [GKL] might be relevant here.

(5) Is property (P) a three spaces property?

(6) Let $\mathbf{e}$ be a basic sequence with a Haar null positive cone. Assume that $Q^+(\mathbf{e})$ admits a Gaussian test measure. Does it admit a *diagonal* measure of the same sort?

(7) Does there exist a Haar null positive cone that does not admit any Gaussian test measure?

(8) It is known that if G is a non-locally compact Polish Abelian group, then the family of all closed Haar null sets of G is extremely complicated from a descriptive point of view (see [S2] and [SR]). What is the descriptive complexity of the family of all basic sequences having a Haar null positive cone?

References

[AK] F. Albiac and N. J. Kalton, *Topics in Banach space theory*, Grad. Texts in Math., vol. 233, Springer, New York, 2006. MR2192298

[BL] Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis*, Amer. Math. Soc. Colloq. Publ., vol. 48, Amer. Math. Soc., Providence, RI, 2000. MR1727673

[Bo] V. I. Bogachev, *Gaussian mesures*, Math. Surveys Monogr., vol. 62, Amer. Math. Soc., Providence, RI, 1998. MR1642391

[BN] J. Borwein and D. Noll, *Second order differentiability of convex functions in Banach spaces*, Trans. Amer. Math. Soc. **342** (1994), no. 1, 43–81. MR1145959

[BT] J. Bourgain and L. Tzafriri, *On a problem of Kadison and Singer*, J. Reine Angew. Math. **420** (1991), 1–43. MR1124564

[CTV] S. A. Chobangan, V. I. Tarieladze, and N. N. Vakhaniya, *Probabilités distributions in Banach spaces*, Nauka, Moscow, 1985; English transl., Math. Appl. (Soviet Ser.), vol. 14, D. Reidel Publ. Co., Dordrecht, 1987. MR1435288 (97k:60007)

[C] J. P. R. Christensen, *On sets of Haar measure zero in abelian Polish groups*, Israel J. Math. **13** (1972), 255–260. MR0326293

[C2] ———, *Topology and Borel structures*, North-Holland Math. Stud., vol. 10, North-Holland Publ. Co., Amsterdam–London, 1974. MR0348724 (50:1221)

[Cs] M. Csörnyei, *Aronszajn null and Gaussian null sets coincide*, Israel J. Math. **111** (1999), 191–201. MR1710738

[G] I. Gasparis, *New examples of c_0-saturated Banach spaces.* II, J. Funct. Anal. **256** (2009), no. 11, 3830–3840. MR2514062

[GKL] G. Godefroy, N. J. Kalton, and G. Lancien, *Subspaces of $c_0(\mathbb{N})$ and Lipschitz isomorphisms*, Geom. Funct. Anal. **10** (2000), no. 4, 798–820. MR1791140

[HSY] B. R. Hunt, T. Sauer, and J. A. Yorke, *Prevalence: a translation invariant "almost every" on infinite-dimensional spaces*, Bull. Amer. Math. Soc. (N.S.) **27** (1992), no. 2, 217–238. MR1161274

[LMP] P. Lefèvre, É. Matheron, and A. Primot, *Smoothness, asymptotic smoothness and the Blum–Hanson property*, Israel J. Math. **211** (2016), no. 1, 271–309. MR3474965

[L] D. H. Leung, *On c_0-saturated Banach spaces*, Illinois J. Math. **39** (1995), no. 1, 15–29. MR1299646

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces.* I. Secuence spaces, Ergeb. Math. Grenzgeb., vol. 92, Springer-Verlag, Berlin, 1977. MR0500056

[M] E. Matoušková, *Convexity and Haar null sets*, Proc. Amer. Math. Soc. **125** (1997), no. 6, 1793–1799. MR1372040

[M2] ———, *The Banach–Saks property and Haar null sets*, Comment. Math. Univ. Carolin. **39** (1998), no. 1, 71–80. MR1622974

[M3] ———, *Translating finite sets into convex sets*, Bull. London Math. Soc. **33** (2001), no. 6, 711–714. MR1853783

[MS] E. Matoušková and C. Stegall, *A characterization of reflexive Banach spaces*, Proc. Amer. Math. Soc. **124** (1996), no. 4, 1083–1090. MR1301517

[Mo] P. Moreau, *Notions de petitesse, géométrie des espaces de Banach et hypercyclicité*, Thèse, Univ. Bordeaux, 2009.

[O] E. Odell, *On quotients of Banach spaces having shrinking unconditional bases*, Illinois J. Math. **36** (1992), no. 4, 681–695. MR1215801

[Ph] R. R. Phelps, *Gaussian null sets and differentiability of Lipschitz maps on Banach spaces*, Pacific J. Math. **77** (1978), no. 2, 523–531. MR510938

[P] V. Ptak, *A combinatorial lemma on the existence of convex means and its application to weak compactness*, Proc. Symp. Pure Math., vol. 7, Amer. Math. Soc., Providence, RI, 1963, pp. 437–450. MR0161128

[SR] J. Saint-Raymond, *Quasi-bounded trees and analytic inductions*, Fund. Math. **191** (2006), no. 2, 175–185. MR2231061

[S2] S. Solecki, *Haar null and non-dominating sets*, Fund. Math. **170** (2001), no. 1–2, 197–217. MR1881376

[To] S. Todorcevic, *Topics in topology*, Lecture Notes in Math., vol. 1652, Springer-Verlag, Berlin, 1997. MR1442262

IMB, UMR 5251, Université de Bordeaux, 351 cours de la Libération, 33405 Talence, France

E-mail address: `esterle@math.u-bordeaux1.fr`

Laboratoire de Mathématiques de Lens, Université d'Artois, 18 Rue Jean Souvraz S. P., 62307 Lens, France

E-mail address: `etienne.matheron@euler.univ-artois.fr`

Lycée La Pérouse-Kerichen, Rue Prince de Joinville - BP 82517, 29225 Brest cedex 2, France

E-mail address: `pierre.moreau@ac-rennes.fr`

Received 6/APR/2015

Originally published in English