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CORONA THEOREM AND INTERPOLATION

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Abstract. Let E be a Banach ideal space of sequences and E' its order dual. By definition, E verifies the corona theorem if for arbitrary bounded functions f_j analytic in the unit disk $\mathbb{D}$ and such that $0 < \delta \le \|\{f_j(z)\}\|_E \le 1$, there is a sequence $\{g_j\}$ of bounded analytic functions with $\sum_j f_j(z)g_j(z) \equiv 1$ and $\|\{g_j(z)\}\|_{E'} \le C(\delta)$, $z \in \mathbb{D}$. It is shown that the spaces ℓ^p, $1 \le p < \infty$, and some more general Banach lattices verify the corona theorem.

In this paper, we shall be interested in quantitative versions of the corona theorem. Let E be a Banach sequence space satisfying the condition

$$|y_n| \le x_n, \quad \{x_n\} \in E \Rightarrow \{y_n\} \in E \quad \text{and} \quad \|\{y_n\}\| \le \|\{x_n\}\|.$$

Denote by E' the order dual space:

$$E' = \Big\{\{y_n\}_{n\in\mathbb{N}} : \sum |y_n|\,|x_n| < \infty \text{ for all } \{x_n\} \in E\Big\}$$

with natural norm. Next, let $\{f_j\}$ be a sequence of bounded analytic functions on the unit disk $\mathbb{D}$ of the complex plane satisfying the condition

$$0 < \delta \le \|\{f_j(z)\}_{j\in\mathbb{N}}\|_E \le 1 \tag{1}$$

for all $z \in \mathbb{D}$. Do there exist functions $\{g_j\}$ bounded and analytic on $\mathbb{D}$ and such that

$$\sum_{j=1}^{\infty} f_j(z)g_j(z) \equiv 1 \quad \text{and} \quad \|\{g_j(z)\}_{j\in\mathbb{N}}\|_{E'} \le C, \quad z \in \mathbb{D}, \tag{2}$$

where the constant $C = C_E(\delta)$ depends on E and δ?

If the answer to this question is in the positive, E is said to verify the corona theorem. For E finite-dimensional, the validity of the corona theorem was first proved by Carleson at the end of the 1950s. A different and simpler proof that employs the Euclidean metric on (again, a finite-dimensional space) E was suggested by Wolff in 1979. Soon after, M. Rosenblum, Tolokonnikov, and Uchiyama observed independently that Wolff's arguments can be adjusted to the infinite-dimensional case of $E = \ell^2$. See [1, Appendix 3] for bibliographic references and other details, and for the proof of the estimate $C_{\ell^2}(\delta) = O\big(\delta^{-2}\log\frac{1}{\delta}\big)$, $\delta \to 0$, by Wolff's method.

Thus, ℓ^2 verifies the corona theorem. Uchiyama showed in [2] that so does ℓ^∞. His constructions are based on Carleson's original ideas and are fairly complicated.

It is natural to wonder for which p, $1 \le p \le \infty$ (except the known cases of $p = 2, \infty$), the space ℓ^p verifies the corona theorem. This question was mentioned in [1], for instance. In [3], a positive answer was given for $p \in (2, +\infty)$, moreover, this can be deduced from the case of $p = 2$ at the difficulty level of the Hölder inequality.

In the present paper, the following result will be proved.

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Theorem 1. *The spaces ℓ^p verify the corona theorem for $1 \le p < \infty$.*

The most difficult case here is that of $p = 1$. This case is obtained from the Wolff theorem ($p = 2$) by an interpolation argument. After that, the case of an arbitrary $p > 1$ follows easily (much as the case of $p \in (2, \infty)$ was deduced in [3] from the case of $p = 2$).

We mention an interesting consequence of Theorem 1.

Corollary. *Let $\{f_j\}_{j\in\mathbb{N}}$ be a sequence of bounded analytic functions on $\mathbb{D}$ with*

$$0 < \delta \le \Big(\sum |f_j(z)|^2\Big)^{1/2} \le 1, \quad z \in \mathbb{D}.$$

Then there exist bounded analytic functions $\{\varphi_j\}_{j=1}^{\infty}$ on $\mathbb{D}$ such that $\sum_j f_j\varphi_j \equiv 1$ and $|\varphi_j| \le C(\delta)|f_j|$ in $\mathbb{D}$.

Proof. We apply Theorem 1 to the sequence $\{f_j^2\}$ with $p = 1$. This yields a sequence g_j of functions analytic in the disk such that $\sup\{|g_j(z)| : z \in \mathbb{D},\ j \in \mathbb{N}\} \le C(\delta)$ and $\sum_{j\in\mathbb{N}} f_j^2 g_j \equiv 1$. It remains to put $\varphi_j = f_j g_j$, $j \in \mathbb{N}$. □

The proof of Theorem 1 can be extended to more general ideal sequence spaces. For example, we shall show that the corona theorem can be verified for an arbitrary p-concave lattice E with the Fatou property under some additional assumption of interpolation nature (the definitions of the notions used here will be given later). Note that, quite recently, D. V. Rutsky proved an even stronger statement: he lifted nearly all restrictions on E.[1] However, his argument is based on the corona theorem for ℓ^∞ (this is Uchiyama's *difficult* result mentioned above) and fairly complicated fixed point theorems. We emphasize that in the present paper all results are deduced by interpolation from the fairly simple case of $p = 2$ in Theorem 1. Sometimes, this yields better estimates for $C_E(\delta)$ than those obtained with the help of Uchiyama's result. It would be tempting to deduce the Uchiyama theorem from the case of $p = 2$ at roughly the same complexity level, but this has not yet been done.

We shall start with the proof of Theorem 1, which will be prefaced by minimal information about interpolation of Hardy type spaces. Generalization to abstract sequence spaces is done within the same methods, but requires more prerequisites about lattices. This generalization will be presented in the last part of the paper.

1. Interpolation by the real method can be viewed, in a broad sense, as merely an art of decomposing functions into pieces that are estimated in different norms. As has already been said, we need some information about interpolation of Hardy type subspaces in lattices of measurable functions. By a lattice of measurable functions, we mean an arbitrary linear space of measurable functions on a measure space (S, μ) (for simplicity, the measure μ is assumed to be σ-finite) endowed with a complete quasinorm $\|\cdot\|$ and possessing the following property: if $g \in X$, f is measurable, and $|f| \le |g|$, then $f \in X$ and $\|f\| \le \|g\|$. We have already considered certain lattices of measurable functions on the set $\mathbb{N}$ of natural numbers with the counting measure ν. In what follows, the principal measure space will be $(\mathbb{T}, m) \times (\mathbb{N}, \nu) = (\mathbb{T} \times \mathbb{N}, m \otimes \nu)$, where $\mathbb{T}$ is the unit circle on the complex plane, and m is the normalized Lebesgue measure on $\mathbb{T}$. A sequence $\{f_j\}$ of functions on $\mathbb{T}$ or $\mathbb{D}$ will be viewed as a function on $\mathbb{T} \times \mathbb{N}$ or $\mathbb{D} \times \mathbb{N}$; accordingly, we write $f_j(z) = f(z, j)$. Then, in a natural way, formula (1) can be rewritten as $\delta \le \|f(z, \cdot)\|_E \le 1$, and formulas (2) can be rewritten as $\langle f(z, \cdot), g(z, \cdot)\rangle = 1$ and $\|g(z, \cdot)\|_{E'} \le C$.

[1] See [11] (added in translation).

A quasi-Banach lattice X of measurable functions on $\mathbb{T}\times\mathbb{N}$ is said to be BMO-regular if for every nonzero $f \in X$ there exists $g \in X$ with $|f| \le g$, $\|g\|_X \le C\|f\|_X$, and $\sup_{j\in\mathbb{N}} \|g(\,\cdot\,,j)\|_{\mathrm{BMO}} \le C$, where C does not depend on f and g.

For every lattice X of measurable functions on $\mathbb{T}\times\mathbb{N}$, we introduce its *analytic part* (or *Hardy-type subspace*) X_A by setting $X_A = \{f \in X : f(\,\cdot\,,j) \in N_+, j = 1,2,\dots\}$, where N_+ is the (boundary) Smirnov class. We shall need two statements, whose highly more general versions can be found in the survey [4].

Proposition 1. *If X and Y are* BMO*-regular lattices of measurable functions on $\mathbb{T}\times\mathbb{N}$, then the couple (X,Y) possesses the following property. Suppose that $f \in (X+Y)_A$ and $f = g+h$, where $g \in X$ and $h \in Y$. Then also $f = \widetilde{g}+\widetilde{h}$, where $\widetilde{g} \in X_A$, $\widetilde{h} \in Y_A$ and $\|\widetilde{g}\|_X \le d\|g\|_X$, $\|\widetilde{h}\|_Y \le d\|h\|_Y$; here d only depends on the constant in the definition of* BMO*-regularity.*

Proposition 2. *The lattice $L^\infty(\ell^p)$ is* BMO*-regular for $0 < p \le \infty$.*

The property formulated in Proposition 1 is called the (strong) analytic K-stability. In fact, Proposition 1 is the principal interpolation result to be used below. Recently, Rutsky proved that Proposition 1 can "almost" be converted, see [5]. See also the paper [6] (this issue) about a refinement of Proposition 2.

Let (S,μ) be an arbitrary measure space, and let w be a weight, i.e., a measurable function on S that is positive a.e. If X is a lattice of measurable functions on S, we can introduce the weighted lattice $X(w)$ by putting

$$X(w) = \{f : fw^{-1} \in X\}, \quad \|f\|_{X(w)} = \|fw^{-1}\|_X.$$

A weight w on $\mathbb{T}\times\mathbb{N}$ is said to have logarithm belonging to BMO uniformly if

$$\sup_j \|\log w(\,\cdot\,,j)\|_{\mathrm{BMO}} < \infty.$$

The following statement is a direct consequence of the definitions.

Proposition 3. *If X is a* BMO*-regular lattice of measurable functions on $\mathbb{T}\times\mathbb{N}$, and w is a weight whose logarithm belongs to* BMO *uniformly, then $X(w)$ is a* BMO*-regular lattice.*

Corollary 1. *The lattices $L^\infty(\ell^\infty,w) = \{f : |f(\zeta,j)| \le Cw(\zeta,j),\ \zeta\in\mathbb{T},\ j = 1,2,\dots\}$ and $L^\infty(\ell^1,w^{-1}) = \{f : \sum_{j\in\mathbb{N}} |f(\zeta,j)|w(\zeta,j) \le C,\ \zeta \in \mathbb{T}\}$ are* BMO*-regular if $\log w$ belongs to* BMO *uniformly. Consequently, the couple $(L^\infty(\ell^\infty,w), L^\infty(\ell^1,w^{-1}))$ is strongly analytically K-stable.*

2. Proof of Theorem 1. First, we treat the main case of $p = 1$, from which the claim for $p \in (1,\infty)$ will follow easily. In the sequel, we usually write $f(\zeta,\,\cdot\,)$ when we mean boundary values and $f(z,\,\cdot\,)$ when we mean values of $f(\,\cdot\,,\,\cdot\,)$ in the disk for a fixed value of the second variable. Suppose that the functions $f(\,\cdot\,,j)$ are analytic in the disk and satisfy the condition

$$0 < \delta \le \|f(z,\,\cdot\,)\|_{l_1} \le 1 \quad \text{for} \quad z \in \mathbb{D}.$$

Let $\alpha(\zeta,j) = |f(\zeta,j)|^{1/2}$, $\zeta\in\mathbb{T}$, $j \in \mathbb{Z}$. Then $\|\alpha\|_{L^\infty(\ell^2)} \le 1$ and by Proposition 2 there exists a function $v \in L^\infty(\ell^2)$ with $v \ge \alpha$, $\|v\|_{L^\infty(\ell^2)} \le C$, and $\sup_j \|\log v(\,\cdot\,,j)\|_{\mathrm{BMO}} \le C$, where C is a universal constant. Denote by ψ the outer function (in the first variable) constructed by v:

$$\psi(\,\cdot\,,j) = \exp[\log v(\,\cdot\,,j) + i\mathcal{H}(\log v(\,\cdot\,,j))], \quad j \in \mathbb{N},$$

where $\mathcal{H}$ is the harmonic conjugation operator. The function $\varphi = f/\psi$ is analytic in the disk, and we have $|\varphi(\zeta,\,\cdot\,)| \le \alpha(\zeta,\,\cdot\,) \le v(\zeta,\,\cdot\,)$ a.e. on $\mathbb{T}$. In particular, $\varphi \in L^\infty(\ell^2)_A$.

Next, by the Cauchy inequality, for $z \in \mathbb{D}$ we obtain

$$\delta \leq \|f(z,\cdot)\|_{\ell^1} = \|\varphi(z,\cdot)\psi(z,\cdot)\|_{\ell^1} \leq \|\psi(z,\cdot)\|_{\ell^2}\|\psi(z,\cdot)\|_{\ell^2} \leq C\|\varphi(z,\cdot)\|_{\ell^2} \leq C.$$

Thus, φ satisfies the assumptions of Theorem 1 for $p = 2$ (with δ replaced by $C^{-1}\delta$). As has already been said, Theorem 1 is well known for $p = 2$, so there exists $h \in L^\infty(\ell^2)_A$ with

$$\langle \varphi(z,\cdot), h(z,\cdot)\rangle \equiv 1 \quad \text{in} \quad \mathbb{D} \quad \text{and} \quad \|h\|_{L^\infty(\ell^2)} \leq C_{\ell^2}(\delta C^{-1}).$$

We split h into two measurable summands in the following way: $h = h\chi_e + h\chi_{e^c}$, where

$$e = \{(\zeta, j) \in \mathbb{T}\times\mathbb{N} : |h(\zeta,j)| \leq A|\psi(\zeta,j)| = Av(\zeta,j)\},$$

and e^c stands for the complement of e in $\mathbb{T}\times\mathbb{N}$. The constant A will be chosen later. Clearly,

$$\begin{aligned}
\|h\chi_e\|_{L^\infty(\ell^\infty,v)} &\leq A,\\
\|h\chi_{e^c}\|_{L^\infty(\ell^1,v^{-1})} &= \operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} \sum_{j\in\mathbb{N}} |h(\zeta,j)|v(\zeta,j)\chi_{e^c}(\zeta,j)\\
&\leq \frac{1}{A}\operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} \sum_{j\in\mathbb{N}} |h(\zeta,j)|^2 \leq A^{-1}[C_{\ell^2}(\delta C^{-1})]^2.
\end{aligned}$$

By Corollary 1, there exists a splitting $h = h_1 + h_2$ of h with summands analytic in the first variable ($h_1 \in L^\infty(\ell^\infty, v)_A$, $h_2 \in L^\infty(\ell^1, v^{-1})_A$) and similar estimates:

$$\|h_1\|_{L^\infty(\ell^\infty,v)} \leq DA,\ \|h_2\|_{L^\infty(\ell^1,v^{-1})} \leq DA^{-1}C_{\ell^2}(\delta C^{-1})^2, \tag{3}$$

where D is a universal constant. For $z \in \mathbb{D}$, we have

$$1 = \big\langle \varphi(z,\cdot), h(z,\cdot)\big\rangle = \left\langle f(z,\cdot), \frac{h_1(z,\cdot)}{\psi(z,\cdot)}\right\rangle + \big\langle \varphi(z,\cdot), h_2(z,\cdot)\big\rangle. \tag{4}$$

In the first summand in the last expression, the function h_1/ψ is analytic in the first variable and its modulus is dominated by DA. We shall show that the modulus of the second summand does not exceed $1/2$ for all $z \in \mathbb{D}$ provided A is sufficiently large. If this is done, put $U(z) = 1 - \langle\varphi(z,\cdot), h_2(z,\cdot)\rangle$, then $1/U$ is analytic in the disk and its modulus does not exceed 2. Therefore,

$$1 = \left\langle f(z,\cdot), \frac{h_1(z,\cdot)}{U(z)\psi(z,\cdot)}\right\rangle, \quad z \in \mathbb{D},$$

and $\left\|\frac{h_1(z,\cdot)}{U(z)\psi(z,\cdot)}\right\|_{\ell^\infty} \leq 2DA$, as required in Theorem 1 for $p = 1$.

Thus, it remains to estimate the second summand in the last expression in (4) from above; moreover, it suffices to do this on the unit circle. But $|\varphi| \leq v$, consequently,

$$|\langle\varphi(z,\cdot), h_2(z,\cdot)\rangle| \leq \sum_{j\in\mathbb{N}} |h_2(\cdot,j)|v(z,j) \leq DA^{-1}C_{\ell^2}(\delta C^{-1})^2,$$

by (3). Therefore, it suffices to take $A = 2DC_{\ell^2}(\delta C^{-1})^2$.

Remark. Incidentally, we have proved that $C_{\ell^1}(\delta) \leq C'C_{\ell^2}(\delta/C)^2$.

Now, we deduce the case of arbitrary finite p in Theorem 1 from the case of $p = 1$. The argument is similar to that in [3, §3].

Denote by q the exponent conjugate to $p \in (1,\infty)$. Let f be a function in $L^\infty(\ell^p)_A$ satisfying the condition

$$0 < \delta \leq \|f(z,\cdot)\|_{\ell^p} \leq 1, \quad z \in \mathbb{D}.$$

Consider the canonical factorization $f = \theta F$ (i.e., $f(\,\cdot\,, j) = \theta(\,\cdot\,, j)F(\,\cdot\,, j)$), where the functions $F(\,\cdot\,, j)$ are outer and the $\theta(\,\cdot\,, j)$ are inner. Clearly, on the circle we have $\|F(\zeta, \,\cdot\,)\|_{\ell^p} = \|f(\zeta, \,\cdot\,)\|_{\ell^p} \le 1$, whence $\|F(z, \,\cdot\,)\|_{\ell^p} \le 1$ for $z \in \mathbb{D}$.

Next, we put $\varphi = \theta F^p$. Then φ is analytic in the first variable, and the inequalities

$$\delta^p \le \sum_{j\in\mathbb{N}} |F(z,j)|^p |\theta(z,j)|^p \le \sum_{j\in\mathbb{N}} |F(z,j)|^p |\theta(z,j)| \le \sum_{j\in\mathbb{N}} |F(z,j)|^p \le 1$$

show that

$$\delta^p \le \|\varphi(z, \,\cdot\,)\|_{\ell^1} \le 1 \quad \text{for} \quad z \in \mathbb{D}.$$

By the facts already proved, there exists $h \in L^\infty(\ell^\infty)_A$ with

$$\|h\|_{L^\infty(\ell^\infty)} \le C_{\ell^1}(\delta^p) \quad \text{and} \quad \langle \varphi(z, \,\cdot\,), h(z, \,\cdot\,)\rangle = 1, \ z \in \mathbb{D}.$$

The second relation can be rewritten in the form $\langle f(z, \,\cdot\,), H(z, \,\cdot\,)\rangle = 1$, where $H = F^{p-1}h$. It is easily seen that H belongs to $L^\infty(\ell^q)_A$: analyticity is obvious and, at the same time,

$$\sum_{j\in\mathbb{N}} |H(\zeta,j)|^q \le C_{\ell^1}(\delta^p)^q \sum_{j\in\mathbb{N}} |F(\zeta,j)|^{(p-1)\cdot q} = C_{\ell^1}(\delta^p)^q \|F(\zeta, \,\cdot\,)\|_{\ell^p} \le C_{\ell^1}(\delta^p)^q$$

for a.e. $\zeta \in \mathbb{T}$. Thus, the required claim follows with the estimate $C_{\ell^p}(\delta) \le C_{\ell^1}(\delta^p)$. □

3. Some generalizations. We shall require standard operations with lattices of measurable functions. Let X and Y be two such lattices on an arbitrary measure space (S, μ), and let $\alpha > 0$. We put

$$XY = \{h = fg \,:\, f \in X, g \in Y\}, \quad \|h\|_{XY} = \inf \|f\|_X \|g\|_Y,$$

where the infimum is taken over all representations $h = fg$. Next, let

$$X^\alpha = \{g \,:\, |g|^{1/\alpha} \in X\}, \quad \|g\|_{X^\alpha} = \||g|^{1/\alpha}\|_X^\alpha.$$

Generally speaking, the lattices XY and Y^α are only quasinormed, but sometimes these are Banach lattices. For instance, $X^{1-\gamma}Y^\gamma$ is such for $0 < \gamma < 1$ if X and Y are Banach. In particular, $Y^\gamma = L^\infty(\mu)^{1-\gamma}Y^\gamma$ is Banach if so is Y. If the lattice $Y^{1/\gamma}$ is Banach (i.e., admits a norm equivalent to its quasinorm), then Y itself is said to be p-convex, $p = 1/\gamma$. See [7] for the details.

For a Banach lattice X of measurable functions, its order dual X' is defined to be the lattice of all measurable functions g with

$$\|g\|_{X'} = \sup\left\{ \int |fg| \,:\, f \in X, \|f\| \le 1 \right\} < +\infty.$$

We shall tacitly assume that all lattices of measurable functions that will occur in this paper satisfy the condition $X'' = X$. This condition is equivalent to the so-called *Fatou property* (see [8]), whose precise definition will not be required here. If E is a lattice of sequences (i.e., of measurable functions on the measure space $(\mathbb{N}, \nu)$; we recall that the condition $E = E''$ is assumed without special mention), we define the lattice $L^\infty(E)$ on $(\mathbb{T} \times \mathbb{N}, m \times \nu)$ as follows:

$$f \in L^\infty(E) \iff \operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} \sup_{N\in\mathbb{N}} \|f(\zeta, \,\cdot\,)\chi_{\mathbb{T}\times\{1,\dots,N\}}(\zeta, \,\cdot\,)\|_E.$$

It should be noted that $L^\infty(E)$ may fail to coincide with the space of strongly measurable essentially bounded E-valued functions (consider the case of $E = \ell^\infty$).

It is convenient to formulate the announced generalization of Theorem 1 in the form of two separate statements.

Theorem 2. *Let E and F be Banach lattices of sequences such that the product EF is also Banach and $E = XF^a$ for yet another Banach lattice X and some $a > 0$. If E verifies the corona theorem, then so does EF provided $L^\infty(F)$ is* BMO*-regular.*

Theorem 3. *If a Banach lattice E of sequences verifies the corona theorem, then so does E^α for $0 < \alpha < 1$.*

Taking $E = F = \ell^2$ (and $X = \ell^\infty$, $a = 1$) in Theorem 2, we recapture case of $p = 1$ in Theorem 1. Theorem 3 formalizes the passage from $p = 1$ to $p > 1$ in Theorem 1.

3.1. Preparations and a corollary. We refer, e.g., to [9] for the proof of the following two propositions.

Proposition 4 (Lozanovskiĭ)**.** *For every Banach lattice X of measurable functions on a measure space (S, μ), we have $XX' = L^1(\mu)$.*

Proposition 5. *Let X and Y be Banach lattices of measurable functions on (S, μ). If $0 < \alpha < 1$, then $(X^{1-\alpha}Y^\alpha)' = (X')^{1-\alpha}(Y')^\alpha$.*

We remind the reader that the conditions $X'' = X$ and $Y'' = Y$ are assumed tacitly. Proposition 5 shows that, in particular,

$$(Y^\alpha)' = \big((L^\infty)^{1-\alpha}Y^\alpha\big)' = (L^1)^{1-\alpha}(Y')^\alpha.$$

Lemma 1. *Assume that the three lattices X, Y, and XY are Banach, and put $V = (XY)'$. Then $VX = Y'$ and $VY = X'$.*

Proof. We only verify the second identity (the proof of the first is similar). By Lozanovskiĭ's factorization,

$$(XY)V = L^1(\mu) = XX'. \tag{5}$$

Setting $E = (YV)^{1/2} = Y^{1/2}V^{1/2}$, $F = (X')^{1/2}$, we deduce from (5) that $ZE = ZF$, where $Z = X^{1/2}$. Thus, we have reduced the claim to the following statement. *If Z, E and F are arbitrary Banach lattices of measurable functions and $ZE = ZF$, then $E = F$.*

We prove it. Multiplying by Z', we obtain $L^1E = L^1F$, whence

$$(L^1)^{1/2}E^{1/2} = (L^1)^{1/2}F^{1/2}.$$

Taking order duals, we arrive at $(E')^{1/2} = (F')^{1/2}$, i.e., $E' = F'$. But then $E = (E')' = (F')' = F$. ☐

The condition for $L^\infty(F)$ in Theorem 2 to be BMO-regular is equivalent to the boundedness of the harmonic conjugation operator (in the first variable) on the space $L^p((F')^\beta)$ for some $p \in (1, \infty)$ and $\beta \in (0, 1)$. Usually, it is verified precisely in this form. See [4, 6] and the references therein.

We mention a consequence of Theorem 2.

Corollary 2. *If U is a Banach lattice of sequences such that U' is p-convex for some $p > 1$ and $L^\infty(U)$ is* BMO*-regular, then U verifies the corona theorem.*

The lattices U with p-convex U' are said to be q-concave ($p^{-1} + q^{-1} = 1$). This property can easily be described in terms of U itself, see [7].

Proof of the corollary. By assumption, $U' = Z^{1/p}$ for some Banach lattice Z, and then $U = (\ell^1)^{1/q}(Z')^{1/q} = \ell^q F$, where $F = (Z')^{1/q}$. Since the spaces $L^\infty(U)$ and $L^\infty(\ell^q)$ are BMO-regular and $L^\infty(U) = L^\infty(\ell^q)L^\infty(F)$, it follows that $L^\infty(F)$ is also BMO-regular, see [10] (we signalize that this "division" property is not easy to prove). In order to apply Theorem 2 to $E = \ell^q$ and the space F introduced above, it suffices to observe that $E = (\ell^1)^{1/q} = (F')^{1/q}F^{1/q}$ (so, the assumptions of Theorem 2 are fulfilled with $X = (F')^{1/q}$ and $a = 1/q$). ☐

3.2. Proof of Theorems 2 and 3. First, we establish Theorem 2. Let a function f in $L^\infty(EF)$ satisfy the condition

$$0 < \delta \le \|f(z, \cdot)\|_{EF} \le 1, \quad z \in \mathbb{D}.$$

Observe that $EF = XF^{a+1}$. Therefore, for the boundary function for f we have $|f(\zeta, \cdot)| = \sigma(\zeta, \cdot)\rho(\zeta, \cdot)^{a+1}$, where σ and ρ are nonnegative measurable functions on $\mathbb{T} \times \mathbb{N}$ with $\|\sigma(\zeta, \cdot)\|_X \le 1$ and $\|\rho(\zeta, \cdot)\|_F \le 2$ for a.e. $\zeta \in \mathbb{T}$. Since $L^\infty(F)$ is BMO-regular, there exists $v \in L^\infty(F)$ such that $v \ge \rho$, $\|v\|_{L^\infty(F)} \le C$, and $\sup_j \|\log v(\cdot, j)\|_{\mathrm{BMO}} \le C$. Let ψ be the other function constructed by v (with respect to the first variable). The function $\varphi = f/\psi^{a+1}$ is analytic in the disk and $|\varphi(\zeta, \cdot)| \le \sigma(\zeta, \cdot)$, so that $\|\varphi(\zeta, \cdot)\psi^a(\zeta, \cdot)\|_E \le \|\sigma(\zeta, \cdot)\|_X \|\psi^a(\zeta, \cdot)\|_{E^a} \le F^a$. Next, for $z \in \mathbb{D}$ we have

$$\delta \le \|f(z, \cdot)\|_{EF} \le \|\varphi(z, \cdot)\psi^a(z, \cdot)\|_E \|\psi(z, \cdot)\|_F \le C\|\varphi(z, \cdot)\psi^a(z, \cdot)\|_E.$$

Since E verifies the corona theorem, there exists $h \in L^\infty(E')_A$ with

$$\langle \varphi(z, \cdot)\psi^a(z, \cdot), h(z, \cdot)\rangle \equiv 1 \quad \text{and} \quad \|h\|_{L^\infty(E')} \le d_1 C_E(d_2\delta).$$

Applying Lemma 1 and the trick with outer functions used above, we can find a factorization $h = \alpha \cdot \beta$, where $\alpha \in L^\infty((EF)')_A$, $\beta \in L^\infty(F)_A$, and $\|\alpha\|_{L^\infty((EF)')} \le d_1 C_E(d_2\delta)$, $\|\beta\|_{L^\infty(F)} \le 2$. We split the function β into two measurable functions: $\beta = \beta\chi_e + \beta\chi_{e^c}$, where

$$e = \{(\zeta, j) \in \mathbb{T} \times \mathbb{N} : |\beta(\zeta, j)| \le A|\psi(\zeta, j)| = Av(\zeta, j)\}.$$

The constant A will be chosen later. Obviously,

$$\|\beta\chi_e\|_{L^\infty(\ell^\infty, v)} \le A,$$
$$\|\beta\chi_{e^c}\|_{L^\infty(F^{a+1}, v^{-a})} = \|\beta v^a \chi_{e^c}\|_{L^\infty(F^{a+1})} \le \frac{1}{A^a}\||\beta|^{a+1}\|_{L^\infty(F^{a+1})} \le \frac{d_3}{A^a}.$$

It is clear from the definitions that the spaces $L^\infty(\ell^\infty, v)$ and $L^\infty(F^{a+1}, v^{-a})$ are BMO-regular, whence there exists a decomposition $\beta = \beta_1 + \beta_2$, where $\beta_1 \in L^\infty(\ell^\infty, v)_A$, $\beta_2 \in L^\infty(F^{a+1}, v^{-a})_A$ and

$$\|\beta_1\|_{L^\infty(\ell^\infty, v)} \le DA, \quad \|\beta_2\|_{L^\infty(F^{a+1}, v^{-a})} \le \frac{D}{A^a}.$$

Next, for $z \in \mathbb{D}$ we have

$$\begin{aligned} 1 &= \big\langle \varphi(z, \cdot)\psi^a(z, \cdot), \alpha(z, \cdot)\beta(z, \cdot)\big\rangle \\ &= \left\langle f(z, \cdot), \frac{\alpha(z, \cdot)\beta_1(z, \cdot)}{\psi(z, \cdot)}\right\rangle + \big\langle \varphi(z, \cdot)\psi^a(z, \cdot), \alpha(z, \cdot)\beta_2(z, \cdot)\big\rangle. \end{aligned}$$

In the first summand, the function $\alpha\beta\psi^{-1}$ is analytic in the first variable and its modulus does not exceed $DA|\alpha|$, whence we see that

$$\|(\alpha\beta\psi^{-1})(z, \cdot)\|_{(EF)'} \le d_4 C_E(d_2\delta).$$

We estimate the second summand from above (it suffices to do this on the boundary):

$$\begin{aligned} &\operatorname*{ess\,sup}_{\zeta \in \mathbb{T}} |\langle \varphi(\zeta, \cdot)\psi^a(\zeta, \cdot), \alpha(\zeta, \cdot)\beta_2(\zeta, \cdot)\rangle| \le \|\alpha\|_{L^\infty((EF)')}\|\varphi\psi^a\beta_2\|_{L^\infty(EF)} \\ &\quad = \|\alpha\|_{L^\infty((EF)')}\|\varphi\psi^a\beta_2\|_{L^\infty(XF^{a+1})} \le \|\alpha\|_{L^\infty((EF)')}\|\varphi\|_{L^\infty(X)}\|\psi^a\beta_2\|_{L^\infty(F^{a+1})} \\ &\quad = \|\alpha\|_{L^\infty((EF)')}\|\varphi\|_{L^\infty(X)}\|\beta_2\|_{L^\infty(F^{a+1}, v^{-a})} \le \frac{d_5}{A^a} C_E(d_2\delta). \end{aligned}$$

The last quantity is dominated by $1/2$ if A is large, and the argument is finished precisely as the proof of the case of $p = 1$ in Theorem 1.

Now we establish Theorem 2. Assume that

$$0 < \delta \le \|f(z, \cdot)\|_{E^\alpha} \le 1, \quad z \in \mathbb{D},$$

for some $\alpha \in (0,1)$. Consider the inner-outer factorization $f = \theta F$ (the functions $F(\cdot, j)$ being outer and the $\theta(\cdot, j)$ being inner) and put $\varphi = \theta F^p$, where $p = \frac{1}{\alpha}$. Then

$$\delta^p \le \|\varphi(z, \cdot)\|_E \le 1, \quad z \in \mathbb{D}.$$

Since E verifies the corona theorem, there exists $h \in L^\infty(E')_A$ with $\langle \varphi(z, \cdot), h(z, \cdot)\rangle = 1$ and $\|h(z, \cdot)\|_{E'} \le C_E(\delta^p)$. Then, surely, $\langle f(z, \cdot), (F^{p-1}h)(z, \cdot)\rangle \equiv 1$, so it remains to prove that $F^{p-1}h \in L^\infty((E^\alpha)')$ (the analyticity of this function in the first variable is obvious). But in the identity $E = E^\alpha E^{1-\alpha}$ all lattices are Banach, whence we see that $E'E^{1-\alpha} = (E^\alpha)'$ by Lemma 1. It remains to observe that $(F(\zeta, \cdot))^{p-1} = (F(\zeta, \cdot)^{\frac{1}{\alpha}})^{1-\alpha} \in E^{1-\alpha}$ uniformly in $\zeta \in \mathbb{T}$ with an appropriate estimate for the norm.

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