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PRIME ENDS AND ORLICZ–SOBOLEV CLASSES

D. A. KOVTONYUK AND V. I. RYAZANOV

Abstract. A canonical representation of prime ends is obtained in the case of regular spatial domains, and the boundary behavior is studied for the so-called lower Q-homeomorphisms, which generalize the quasiconformal mappings in a natural way. In particular, a series of efficient conditions on a function Q are found for continuous and homeomorphic extendibility to the boundary along prime ends. On that basis, a theory is developed that describes the boundary behavior of mappings in the Sobolev and Orlicz–Sobolev classes and also of finitely bi-Lipschitz mappings, which are a far-reaching generalization of the well-known classes of isometries and quasiisometries.

§1. Introduction

The problem of boundary behavior is among the basic subjects in the theory of quasiconformal mappings and their generalizations. In the last years, various classes of mappings with finite distortion were studied, which naturally generalize conformal, quasiconformal, and quasiregular mappings, see the references in [13, 15] and [36]. As before, the main geometric method in the modern mapping theory is the modulus method, see, e.g., the monographs [13, 36, 43, 51, 68, 69] and [72].

From the viewpoint of the theory of conformal mappings, it was not quite satisfactory to view points of the boundary of a simply connected domain as its elementary constituents. Indeed, by the Riemann theorem, any such domain can be mapped onto the unit disk by a conformal mapping. Under such a mapping, the points of the unit circle correspond to the so-called *prime ends* of the domain.

The term "prime end" goes back to Carathéodory [5] who initiated the systematic study of the structure of the boundary of a simply connected domain in the plane. His approach was topological and was based on the notions of subdomains, cuts, etc. in the given domain. The conformal invariance of the prime ends was proved in one of Carathéodory's fundamental theorems.

Lindelöf [32] avoided these difficulties by defining the prime ends via conformal mappings of the unit disk onto a domain, namely, in terms of indeterminacy or cluster sets. However, his method does not involve any explicit analysis of the topological structure of the domain.

Two other approaches to defining prime ends also deserve attention. Mazurkiewicz [39] introduced a certain metric $\rho_\pi(z_1, z_2)$, which is equivalent to the Euclidean metric in the sense that $\rho_\pi(z_j, z_0) \to 0$ if and only if $|z_j - z_0| \to 0$ for each sequence of points $\{z_j\}$ of the domain in question. The boundary of the domain corresponding to ρ_π, i.e., the completion of the domain in the metric ρ_π, is a space that can be identified with the set of prime ends in the sense of Carathéodory.

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Finally, Ursell and Young [67] introduced the prime ends of a plane domain by applying the notion of the classes of equivalent ways that converge to the boundary of the domain.

In the space, the theory of prime ends was also developed intensively, see, e.g., [9, 20, 40, 42, 66] and [71]. Zorich was the first to apply the theory of prime ends to spatial quasiconformal mappings, see [77, 78, 79]. We define the prime ends via cross-sections and apply yet another condition in terms of moduli, following Näkki (see [42] and also a capacity approach in [71]), and develop the theory of more general spatial mappings. The history of the problem and further references can be found also in [1, 7] and [42].

Here we often write I, $\overline{\mathrm{I}}$, $\mathbb{R}$, $\overline{\mathbb{R}}$, $\mathbb{R}^+$, $\overline{\mathbb{R}^+}$, and $\overline{\mathbb{R}^n}$ for $[1,\infty)$, $[1,\infty]$, $(-\infty,\infty)$, $[-\infty,\infty]$, $[0,\infty)$, $[0,\infty]$ and $\mathbb{R}^n \cup \{\infty\}$, respectively, and D for a domain in $\mathbb{R}^n$. We also use the *spherical* (*chordal*) *metric* $h(x,y) = |\pi(x) - \pi(y)|$ in $\overline{\mathbb{R}^n}$, where π is the stereographic projection of $\overline{\mathbb{R}^n}$ onto the sphere $S^n\left(\frac{1}{2}e_{n+1}, \frac{1}{2}\right)$ in $\mathbb{R}^{n+1}$, i.e.,

$$h(x,y) = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}, \quad x \neq \infty \neq y, \qquad h(x,\infty) = \frac{1}{\sqrt{1+|x|^2}},$$

and *spherical* (*chordal*) *diameter* of a E in $\overline{\mathbb{R}^n}$:

$$h(E) = \sup_{x,y\in E} h(x,y).$$

Let ω be an open set in $\mathbb{R}^k$, $k = 1,\dots,n-1$. A continuous mapping $\sigma\colon \omega \to \overline{\mathbb{R}^n}$ is called a *k-dimensional surface* in $\overline{\mathbb{R}^n}$. An $(n-1)$-dimensional surface σ in $\overline{\mathbb{R}^n}$ is called simply a surface. A surface $\sigma\colon \omega \to D$ is called a *Jordan surface* in D if $\sigma(z_1) \neq \sigma(z_2)$ for $z_1 \neq z_2$. In what follows, sometimes we use σ to denote the image $\sigma(\omega) \subseteq \overline{\mathbb{R}^n}$ under the mapping σ, and write $\bar{\sigma}$ instead of $\overline{\sigma(\omega)}$ in $\overline{\mathbb{R}^n}$, and $\partial\sigma$ instead of $\overline{\sigma(\omega)} \setminus \sigma(\omega)$. A Jordan surface σ in D is called a *cross-cut* of the domain D if σ splits D, i.e., $D \setminus \sigma$ has more than one component, $\partial\sigma \cap D = \varnothing$, and $\partial\sigma \cap \partial D \neq \varnothing$.

A sequence $\sigma_1, \sigma_2, \dots, \sigma_m, \dots$ of cross-cuts of a domain D is called a *chain* if:

(i) $\bar{\sigma}_i \cap \bar{\sigma}_j = \varnothing$ for all $i \neq j$, $i,j = 1,2,\dots$;

(ii) σ_{m-1} and σ_{m+1} are included in different components of $D \setminus \sigma_m$ for all $m > 1$;

(iii) $\cap\, d_m = \varnothing$, where d_m is the component of $D \setminus \sigma_m$ containing σ_{m+1}.

Finally, we say that a chain of cross-cuts $\{\sigma_m\}$ is *regular* if

(iv) $h(\sigma_m) \to 0$ as $m \to \infty$.

In accordance with that definition, a chain of cross-cuts $\{\sigma_m\}$ determines a chain of domains $d_m \subset D$ such that $\partial\, d_m \cap D \subseteq \sigma_m$ and $d_1 \supset d_2 \supset \cdots \supset d_m \supset \dots$. Two chains of cross-cuts $\{\sigma_m\}$ and $\{\sigma'_k\}$ are said to be *equivalent* if for every $m = 1,2,\dots$ the domain d_m contains all domains d'_k except finitely many of them, and for every $k = 1,2,\dots$ the domain d'_k also contains all domains d_m except finitely many. An *end* of a domain D is an equivalence class of chains of cross-cuts of D.

Let K be an end of D in $\overline{\mathbb{R}^n}$, $\{\sigma_m\}$ and $\{\sigma'_m\}$ two chains in K, and d_m and d'_m the domains corresponding to σ_m and σ'_m. Then

$$\bigcap_{m=1}^{\infty} \overline{d_m} \subseteq \bigcap_{m=1}^{\infty} \overline{d'_m} \subset \bigcap_{m=1}^{\infty} \overline{d_m},$$

and, thus,

$$\bigcap_{m=1}^{\infty} \overline{d_m} = \bigcap_{m=1}^{\infty} \overline{d'_m},$$

i.e., the set

$$I(K) := \bigcap_{m=1}^{\infty} \overline{d_m}$$

depends only on K but not on the choice of a chain of cross-cuts $\{\sigma_m\}$ in K. The set $I(K)$ is called the *body of the end* K.

As is well known, the set $I(K)$ is a continuum, i.e., it is a connected compact set, see, e.g., I(9.12) in [73]. Moreover, conditions (ii) and (iii) imply that

$$I(K) = \bigcap_{m=1}^{\infty} (\partial d_m \cap \partial D) = \partial D \cap \bigcap_{m=1}^{\infty} \partial d_m.$$

Thus, we obtain the following statement.

Proposition 1. *For every end K of a domain D in $\overline{\mathbb{R}^n}$, we have*

$$I(K) \subseteq \partial D. \tag{1}$$

In what follows, as usual, for sets A, B and C in $\overline{\mathbb{R}^n}$, we denote by $\Delta(A, B; C)$ the family of all paths connecting A and B in C.

As in [42], we say that an end K is a *prime end* if K contains a chain of cross-cuts $\{\sigma_m\}$ such that

$$\lim_{m\to\infty} M(\Delta(C, \sigma_m; D)) = 0 \tag{2}$$

for some continuum C in D, where M is the modulus of the family $\Delta(C, \sigma_m; D)$, see the following section.

If an end K contains at least one regular chain, then we say that K is *regular*. As it is easy to see from Lemma 1 below, every regular end is prime.

§2. Lower and ring Q-homeomorphisms

The class of lower Q-homeomorphisms was introduced in the paper [25], see also [29] and the monograph [36]; it was motivated by the ring definition of quasiconformal mappings by Gehring, see [10]. The theory of lower Q-homeomorphisms has found interesting applications to the theory of the Beltrami equations in the plane and to the theory of the Sobolev and Orlicz–Sobolev classes in the space, see e.g. [23, 29, 30, 22, 21, 28, 36] and [53].

In our paper, key part will be played by the moduli of families of surfaces. Let ω be an open set in $\overline{\mathbb{R}^k}$, $k = 1, \dots, n-1$. Recall that a continuous mapping $S\colon \omega \to \mathbb{R}^n$ is called a *k-dimensional surface* in $\mathbb{R}^n$. The number of preimages

$$N(S, y) = \operatorname{card} S^{-1}(y) = \operatorname{card}\{x \in \omega : S(x) = y\}, \quad y \in \mathbb{R}^n \tag{3}$$

is called the *multiplicity function* of the surface S. As is well known, it is lower semicontinuous, i.e.,

$$N(S, y) \geq \liminf_{m\to\infty} N(S, y_m)$$

for every sequence $y_m \in \mathbb{R}^n$, $m = 1, 2, \dots$, such that $y_m \to y \in \mathbb{R}^n$ as $m \to \infty$, see e.g., [47, p. 160]. Thus, the function $N(S, y)$ is Borel and, consequently, measurable with respect to each Hausdorff measure H^k, see, e.g., Theorems II(3.5) and II(7.4) in [61].

Recall that the k-dimensional Hausdorff area in $\mathbb{R}^n$ (or simply *area*) associated with a surface $S\colon \omega \to \mathbb{R}^n$ is defined by the formula

$$\mathcal{A}_S(B) = \mathcal{A}_S^k(B) := \int_B N(S, y)\, dH^k y \tag{4}$$

for every Borel $B \subseteq \mathbb{R}^n$ and, more generally, for arbitrary set measurable with respect to H^k in $\mathbb{R}^n$, see [8, 3.2.1] and [36, 9.2].

If $\varrho\colon \mathbb{R}^n \to \overline{\mathbb{R}^+}$ is a Borel function, then its *integral over* S is defined as

$$\int_S \varrho\, d\mathcal{A} := \int_{\mathbb{R}^n} \varrho(y)\, N(S,y)\, dH^k y. \tag{5}$$

Given a family Γ of k-dimensional surfaces S, we say that a Borel function $\varrho\colon \mathbb{R}^n \to [0,\infty]$ is *admissible* for Γ and write $\varrho \in \operatorname{adm}\Gamma$ if

$$\int_S \varrho^k\, d\mathcal{A} \geq 1 \tag{6}$$

for all $S \in \Gamma$. The *modulus* of the family Γ is the following quantity:

$$M(\Gamma) = \inf_{\varrho \in \operatorname{adm}\Gamma} \int_{\mathbb{R}^n} \varrho^n(x)\, dm(x), \tag{7}$$

where m denotes Lebesgue measure in $\mathbb{R}^n$. We also say that a Lebesgue measurable function $\varrho\colon \mathbb{R}^n \to [0,\infty]$ is *extensively admissible* for a family Γ of k-dimensional surfaces S in $\mathbb{R}^n$ and write $\varrho \in \operatorname{ext\,adm}\Gamma$ if the subfamily of all surfaces S in Γ for which (6) fails has modulus zero.

Given domains D and D' in $\overline{\mathbb{R}^n}$, $n \geq 2$, $x_0 \in \bar{D} \setminus \{\infty\}$, and a measurable function $Q\colon \mathbb{R}^n \to (0,\infty)$, we say that a homeomorphism $f\colon D \to D'$ is a *lower Q-homeomorphism at the point* x_0 if

$$M(f\Sigma_\varepsilon) \geq \inf_{\varrho \in \operatorname{ext\,adm}\Sigma_\varepsilon} \int_{D \cap R_\varepsilon} \frac{\varrho^n(x)}{Q(x)}\, dm(x) \tag{8}$$

for every ring $R_\varepsilon = \{x \in \mathbb{R}^n : \varepsilon < |x - x_0| < \varepsilon_0\}$, $\varepsilon \in (0,\varepsilon_0)$, $\varepsilon_0 \in (0,d_0)$, where $d_0 = \sup_{x\in D} |x - x_0|$, and Σ_ε denotes the family of all intersections of the spheres $S(x_0,r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}$, $r \in (\varepsilon,\varepsilon_0)$, with D. This notion can be extended to the case of $x_0 = \infty \in \bar{D}$ via inversion T with respect to the unit sphere in $\overline{\mathbb{R}^n}$, $T(x) = x/|x|^2$, $T(\infty) = 0$, $T(0) = \infty$. Namely, a homeomorphism $f\colon D \to D'$ is called a *lower Q-homeomorphism at* $\infty \in \bar{D}$ if $F = f \circ T$ is a lower Q_*-homeomorphism with $Q_* = Q \circ T$ at 0.

We also say that a homeomorphism $f\colon D \to \overline{\mathbb{R}^n}$ is a *lower Q-homeomorphism in the domain* D if f is a lower Q-homeomorphism at every point $x_0 \in \bar{D}$.

We cite a criterion for being a lower Q-homeomorphism in $\mathbb{R}^n$, see Theorem 2.1 in [25] or Theorem 9.2 in [36].

Proposition 2. *Suppose that D and D' are domains in $\overline{\mathbb{R}^n}$, $n \geq 2$, $x_0 \in \bar{D} \setminus \{\infty\}$, and $Q\colon D \to (0,\infty)$ is a measurable function. A homeomorphism $f\colon D \to D'$ is a lower Q-homeomorphism at the point x_0 if and only if*

$$M(f\Sigma_\varepsilon) \geq \int_\varepsilon^{\varepsilon_0} \frac{dr}{\|Q\|_{n-1}(x_0,r)} \quad \textit{for all } \varepsilon \in (0,\varepsilon_0),\ \ \varepsilon_0 \in (0, d(x_0)), \tag{9}$$

where $d(x_0) = \sup_{x\in D} |x - x_0|$, and

$$\|Q\|_{n-1}(x_0,r) = \left(\int_{D(x_0,r)} Q^{n-1}(x)\, d\mathcal{A}\right)^{\frac{1}{n-1}} \tag{10}$$

is the L_{n-1}-norm of the function Q over $D(x_0,r) = \{x \in D : |x - x_0| = r\} = D \cap S(x_0,r)$.

Let domains D and D' in $\overline{\mathbb{R}^n}$, $n \geq 2$, and a measurable function $Q\colon \mathbb{R}^n \to (0,\infty)$ be given. Set $S_i := S(x_0,r_i)$. We say that a homeomorphism $f\colon D \to D'$ is *ring Q-homeomorphism at a point* $x_0 \in \bar{D} \setminus \{\infty\}$ if

$$M\big(f(\Delta(S_1,S_2;D))\big) \leq \int_{A\cap D} Q(x) \cdot \eta^n(|x - x_0|)\, dm(x) \tag{11}$$

for every ring $A = A(x_0, r_1, r_2)$, $0 < r_1 < r_2 < d(x_0) = \sup_{x\in D} |x - x_0|$, and every measurable function $\eta\colon (r_1, r_2) \to [0, \infty]$ such that

$$\int_{r_1}^{r_2} \eta(r)\, dr \geq 1. \tag{12}$$

The notion of a ring Q-homeomorphism can be extended to the case of $x_0 = \infty$ in a standard way, as in the case of lower Q-homeomorphisms.

For the first time, the notion of a ring Q-homeomorphism was introduced for interior points of a domain in [56] in connection with the study of the Beltrami equations in the plane and was extended to the spatial case in [54], see also the monograph [36]. Then this notion was extended to boundary points in the papers [33] and [57, 58, 59], see also [13]. By Corollary 5 in [29], we obtain the following fact.

Proposition 3. *In $\mathbb{R}^n$, $n \geq 2$, any lower Q-homeomorphism $f\colon D \to D'$ at a point $x_0 \in \bar{D}$ with Q integrable in the power $n-1$ in a neighborhood of x_0, is a ring Q_*-homeomorphism at x_0 with $Q_* = Q^{n-1}$.*

Remark 1. By Remark 8 in [29], the conclusion of Proposition 3 remains valid if the function Q is integrable in the power $n-1$ on almost all spheres centered at x_0 with sufficiently small radius.

Note also that, in the definitions of the ring and lower Q-homeomorphisms, it suffices to define the function Q only in the domain D, and to extend it by zero outside of D.

§3. Canonical representation of ends of spatial domains

Lemma 1. *Every regular end K of a domain D in $\overline{\mathbb{R}^n}$ contains a chain of cross-cuts σ_m lying on spheres S_m and centered at a point $x_0 \in \partial D$, with chordal radii ρ_m tending to zero as $m \to \infty$. Every regular end K of a bounded domain D in $\mathbb{R}^n$ contains a chain of cross-cuts σ_m lying on spheres S_m and centered at a point $x_0 \in \partial D$, with Euclidean radii r_m tending to zero as $m \to \infty$.*

Proof. We restrict ourselves to the case of domains D in $\overline{\mathbb{R}^n}$ with the chordal metric. The second case is similar.

Let $\{\sigma_m\}$ be a chain of cross-cuts in the end K, and let x_m be a sequence of points in σ_m. Without loss of generality we may assume that $x_m \to x_0 \in \partial D$ as $m \to \infty$, because $\overline{\mathbb{R}^n}$ is a compact metric space. Then $\rho_m^- := h(x_0, \sigma_m) \to 0$ because $h(\sigma_m) \to 0$ as $m \to \infty$. Moreover,

$$\rho_m^+ := H(x_0, \sigma_m) = \sup_{x\in\sigma_m} h(x, x_0) = \sup_{x\in\overline{\sigma_m}} h(x, x_0)$$

is the Hausdorff distance between the compact sets $\{x_0\}$ and $\overline{\sigma_m}$ in $\overline{\mathbb{R}^n}$. Condition (i) in the definition of ends allows us to assume with no loss of generality that $\rho_m^- > 0$ and $\rho_{m+1}^+ < \rho_m^-$ for all $m = 1, 2, \ldots$.

Set

$$\delta_m = \Delta_m \setminus d_{m+1},$$

where $\Delta_m = S_m \cap d_m$ and

$$S_m = \Big\{x \in \overline{\mathbb{R}^n} \,:\, h(x_0, x) = \frac{1}{2}\big(\rho_m^- + \rho_{m+1}^+\big)\Big\}.$$

Clearly, Δ_m and δ_m are closed relative to d_m.

Note that d_{m+1} is contained in one of the connected components of the open set $d_m \setminus \delta_m$. Indeed, suppose that there exist two points x_1 and x_2 in d_{m+1} that belong to different components Ω_1 and Ω_2 of the set $d_m \setminus \delta_m$. Then x_1 and x_2 can be joined by a continuous curve $\gamma\colon [0,1] \to d_{m+1}$. However, by construction, d_{m+1} (and hence also γ)

does not intersect δ_m; consequently, $[0,1] = \bigcup_{k=1}^{\infty} \omega_k$, where $\omega_k = \gamma^{-1}(\Omega_k)$, Ω_k being some enumeration of the components of $d_m \setminus \delta_m$. But ω_k is open in $[0,1]$ because Ω_k is open and γ is continuous. This contradicts the connectedness of $[0,1]$ because $\omega_1 \neq \varnothing$, $\omega_2 \neq \varnothing$, and, moreover, ω_i and ω_j are mutually disjoint for $i \neq j$.

Let d_m^* be a component of $d_m \setminus \delta_m$ containing d_{m+1}. Then by construction we have $d_{m+1} \subseteq d_m^* \subseteq d_m$. It remains to show that $\partial d_m^* \setminus \partial D \subseteq \delta_m$. Indeed, it is obvious that $\partial d_m^* \setminus \partial D \subseteq \delta_m \cup \sigma_m$, because every point in $d_m \setminus \delta_m$ belongs either to d_m^* or to another component of $d_m \setminus \delta_m$, and, hence, it does not belong to the boundary of d_m^* in view of the relative closeness of δ_m in d_m. Thus, it suffices to prove that $\sigma_m \cap \partial d_m^* \setminus \partial D \neq \varnothing$.

Suppose that there is a point $x_* \in \sigma_m$ in $d_m^* \setminus \partial D$. Then there is a point $y_* \in d_m^*$ that is sufficiently close to σ_m and

$$h(x_0, y_*) > \frac{1}{2}\left(\rho_m^- + \rho_{m+1}^+\right),$$

because $h(x_0, y_*) \geq \rho_m^-$ and $\rho_{m+1}^+ < \rho_m^-$. On the other hand, there is a point $z_* \in d_{m+1}$ that is sufficiently close to σ_{m+1} and

$$h(x_0, z_*) < \frac{1}{2}\left(\rho_m^- + \rho_{m+1}^+\right).$$

Moreover, the points z_* and y_* can be joined by a continuous curve $\gamma\colon [0,1] \to d_{m+1}^*$. Note that the set $\gamma^{-1}(d_m^* \setminus \overline{d_{m+1}})$ consists of a countable collection of open mutually disjoint intervals in $[0,1]$ and the interval $(t_0, 1]$ with $t_0 \in (0,1)$ and $z_0 = \gamma(t_0) \in \sigma_{m+1}$. Thus,

$$h(x_0, z_0) < \frac{1}{2}\left(\rho_m^- + \rho_{m+1}^+\right)$$

because $h(x_0, z_0) \leq \rho_{m+1}^+$ and $\rho_{m+1}^+ < \rho_m^-$. Now, by the continuity of the function $\varphi(t) = h(x_0, \gamma(t))$, there is a point $\tau_0 \in (t_0, 1)$ such that

$$h(x_0, y_0) = \frac{1}{2}\left(\rho_m^- + \rho_{m+1}^+\right),$$

where $y_0 = \gamma(\tau_0) \in d_m^*$ by the choice of γ. We arrive at a contradiction, which completes the proof. □

In the sequel, given a domain D in $\mathbb{R}^n$, $n \geq 2$, we say that a sequence of points $x_k \in D$, $k = 1, 2, \ldots$, *converges to its end* K if, for every chain of cross-cuts $\{\sigma_m\}$ in K and every domain d_m, all points x_k, except maybe finitely many of them belong to d_m.

§4. On regular domains

First, we recall a few topological notions. A domain $D \subset \mathbb{R}^n$, $n \geq 2$, is said to be *locally connected at a point* $x_0 \in \partial D$ if for every neighborhood U of x_0 there is a neighborhood $V \subseteq U$ of x_0 such that $V \cap D$ is connected. Note that a Jordan domain D in $\mathbb{R}^n$ is locally connected at every point ∂D, see, e.g., [74, p. 66].

As in [24] and [25] (see also [36] and [52]), we say that the boundary ∂D is *weakly flat at a point* $x_0 \in \partial D$ if for every neighborhood U of x_0 and every number $P > 0$, there is a neighborhood $V \subset U$ of x_0 such that

$$M(\Delta(E, F; D)) \geq P \tag{13}$$

for every continua E and F in D that intersect ∂U and ∂V. We also say that the boundary ∂D is *weakly flat* if it is weakly flat at every point of ∂D.

Next, we say that a point $x_0 \in \partial D$ is *strictly accessible* if for every neighborhood U of x_0 there is a compact set E in D, a neighborhood $V \subset U$ of x_0, and a number $\delta > 0$

such that

$$M\big(\Delta(E,F;D)\big) \geq \delta \tag{14}$$

for all continua F in D that intersect ∂U and ∂V. Finally, we say that the boundary ∂D is *strictly accessible* if all points $x_0 \in \partial D$ are strictly accessible.

Remark 2. In the definition of strictly accessible and weakly flat boundaries, we may assume that the neighborhoods U and V of x_0 are balls (closed or open) centered at x_0 or that they are neighborhoods of the point x_0 belonging to another fundamental system of its neighborhoods. These concepts extend in a natural way to the case of $\overline{\mathbb{R}^n}$ and $x_0 = \infty$. Then we should use the corresponding neighborhoods of ∞.

It is easily seen that if a domain D in $\mathbb{R}^n$ is weakly flat at a point $x_0 \in \partial D$, then it is strictly accessible at x_0. Furthermore, we have proved that if a domain D in $\mathbb{R}^n$ is weakly flat at a point $x_0 \in \partial D$, then D is locally connected at x_0, see, e.g., Lemma 5.1 in [25] or Lemma 3.15 in [36].

In accordance with the classic geometric definition of Väisälä, see, e.g., [69, 13.1], a homeomorphism f between domains D and D' in $\mathbb{R}^n$, $n \geq 2$, is said to be K*-quasiconformal* if

$$\frac{1}{K}\, M(\Gamma) \leq M(f\Gamma) \leq K\, M(\Gamma)$$

for every family of curves Γ in D. A homeomorphism $f\colon D \to D'$ is *quasiconformal* if f is K-quasiconformal for some $K \in [1,\infty)$, i.e., the distortion of the moduli of curves under the mapping f is bounded.

We say that the boundary of a domain D in $\mathbb{R}^n$ is *locally quasiconformal* if every point $x_0 \in \partial D$ has a neighborhood U that admit a conformal mapping φ onto the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ such that $\varphi(\partial D \cap U)$ is the intersection of $\mathbb{B}^n$ and a coordinate hyperplane. Note that the definition shows immediately that the locally quasiconformal boundaries are weakly flat.

In the theory of mappings and in the theory of differential equations, we often meet the so-called Lipschitz domains whose boundaries are locally quasiconformal and, consequently, weakly flat.

Recall that a mapping $\varphi\colon X \to Y$ between metric spaces X and Y is said to be *Lipschitz* if the inequality $\operatorname{dist}(\varphi(x_1), \varphi(x_2)) \leq M \operatorname{dist}(x_1, x_2)$ is fulfilled for some $M < \infty$ and all x_1 and $x_2 \in X$. A mapping φ is *bi-Lipschitz* if, moreover, we have $M^* \operatorname{dist}(x_1, x_2) \leq \operatorname{dist}(\varphi(x_1), \varphi(x_2))$ for some $M^* > 0$ and all x_1 and $x_2 \in X$. In what follows, X and Y are subsets of $\mathbb{R}^n$ with the Euclidean distance.

We say that a domain D in $\mathbb{R}^n$ is *Lipschitz* if every point $x_0 \in \partial D$ has a neighborhood U that admits a Lipschitz homeomorphism φ onto the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ such that $\varphi(\partial D \cap U)$ is the intersection of $\mathbb{B}^n$ and a coordinate hyperplane, and $f(x_0) = 0$, see, e.g., [43]. Note that the bi-Lipschitz homeomorphisms are quasiconformal, so that the Lipschitz domains have locally quasiconformal boundaries.

We say that a bounded domain D in $\mathbb{R}^n$ is *regular* if D can be mapped quasiconformally onto a domain with a locally quasiconformal boundary.

Obviously, any regular domain is finitely connected, because for every homeomorphism between domains D and D' in $\overline{\mathbb{R}^n}$, $n \geq 2$, there is a natural one-to-one correspondence between the components of boundaries ∂D and $\partial D'$, see, e.g., Lemma 5.3 in [16] or Lemma 6.5 in [36]. Note also that every finitely connected plane domain whose boundary has no boundary components consisting of a single point can be conformally mapped onto a domain bounded by a finitely many mutually disjoint circles; hence, it is a regular domain, see, e.g., Theorem V.6.2 in [12]. From [42, Theorem 5.1] it follows that every prime end of a regular domain in $\overline{\mathbb{R}^n}$, $n \geq 2$, is regular. Combining this and Lemma 1, we obtain the following statement.

Lemma 2. *Every prime end P of a regular domain D in $\mathbb{R}^n$, $n \geq 2$, contains a chain of cross-cuts σ_m that lie on spheres S_m centered at $x_0 \in \partial D$ with Euclidean radii r_m tending to* 0 *as $m \to \infty$.*

Remark 3. By [42, Theorem 4.1], for any quasiconformal mapping g of a domain D_0 with locally quasiconformal boundary onto a bounded domain D in $\mathbb{R}^n$, $n \geq 2$, there is a natural one-to-one correspondence between the points of ∂D_0 and the prime ends of D and, moreover, the cluster sets $C(g, b)$, $b \in \partial D_0$, coincide with the body $I(P)$ of the corresponding prime end P of D.

If $\bar{D}_P$ is the completion of a regular domain D by its prime ends and g_0 is a quasiconformal mapping of a domain D_0 with locally quasiconformal boundary onto D, then this mapping naturally determines the metric $\rho_0(p_1, p_2) = \left|\widetilde{g_0}^{-1}(p_1) - \widetilde{g_0}^{-1}(p_2)\right|$ in $\bar{D}_p$, where $\widetilde{g_0}$ is the extension of g_0 onto $\overline{D_0}$.

If g_* is another quasiconformal mapping of a domain D_* with locally quasiconformal boundary onto the domain D, then the corresponding metric

$$\rho_*(p_1, p_2) = \left|\widetilde{g_*}^{-1}(p_1) - \widetilde{g_*}^{-1}(p_2)\right|$$

generates the same convergence and, consequently, the same topology in $\bar{D}_P$ as the metric ρ_0, because $g_0 \circ g_*^{-1}$ is a quasiconformal mapping between the domains D_* and D_0, which extends, by Theorem 4.1 in [42], up to a homeomorphism between $\overline{D_*}$ and $\overline{D_0}$.

In what follows, we shall call this topology in the space $\bar{D}_P$ the *topology of prime ends* and by the continuity of mappings $F\colon \bar{D}_P \to \overline{D'}_P$ we shall mean continuity with respect to this topology.

§5. On extension of direct mappings

Lemma 3. *Let D and D' be regular domains in $\mathbb{R}^n$, $n \geq 2$, and let $f\colon D \to D'$ be a lower Q-homeomorphism. If*

$$\int_0^{\delta(x_0)} \frac{dr}{\|Q\|_{n-1}(x_0, r)} = \infty \quad \text{for all} \quad x_0 \in \partial D \tag{15}$$

for some $\delta(x_0) < d(x_0) = \sup_{x \in D} |x - x_0|$, where

$$\|Q\|_{n-1}(x_0, r) = \left(\int_{D \cap S(x_0, r)} Q^{n-1}\, d\mathcal{A}\right)^{\frac{1}{n-1}}, \tag{16}$$

then f extends up to a continuous mapping of $\bar{D}_P$ onto $\overline{D'}_P$.

Proof. By Remark 3, there is no loss of generality in assuming that the domain D' has a locally quasiconformal boundary and $\overline{D'}_P = \overline{D'}$. Again in accordance with Remark 3, by the metrizability of the spaces $\bar{D}_P$ and $\overline{D'}_P$, it suffices to prove that, for every prime end P of the domain D, the cluster set

$$L = C(P, f) := \left\{y \in \mathbb{R}^n : y = \lim_{k \to \infty} f(x_k), x_k \to P, x_k \in D\right\}$$

consists of a single point $y_0 \in \partial D'$.

Note that $L \neq \varnothing$ because the set $\overline{D'}$ is compact, and L is a subset of $\partial D'$, see, e.g., Proposition 2.5 in [52] or Proposition 13.5 in [36]. Suppose that L contains at least two points y_0 and z_0. Set $U = B(y_0, r_0)$, where $0 < r_0 < |y_0 - z_0|$.

Let $x_0 \in I(P) \subseteq \partial D$, and let σ_k, $k = 1, 2, \ldots$, be a chain of cross-cuts in D lying on spheres $S_k = S(x_0, r_k)$ as in Lemma 2 with the associated domains D_k, $k = 1, 2, \ldots$.

Then there exist points y_k and z_k in the domains $D'_k = f(D_k)$ such that $|y_0 - y_k| < r_0$ and $|y_0 - z_k| > r_0$, and, moreover, $y_k \to y_0$ and $z_k \to z_0$ as $k \to \infty$. Let C_k be continuous curves joining y_k and z_k in D'_k. Note that $\partial U \cap C_k \neq \varnothing$ by construction.

Since the point y_0 is strictly accessible, see Remark 2, there exists a continuum $E \subset D'$ and a number $\delta > 0$ such that

$$M\big(\Delta(E, C_k; D')\big) \geq \delta$$

for all sufficiently large k.

With no loss of generality, we may assume that the last condition is satisfied for all $k = 1, 2, \dots$. Note that $C = f^{-1}(E)$ is a compact subset of the domain D, so that $\varepsilon_0 = \operatorname{dist}(x_0, C) > 0$. Again with no loss of generality, we may assume that $r_k < \varepsilon_0$ for all $k = 1, 2, \dots$.

Let Γ_m be the family of all continuous curves in $D \setminus D_m$ that join the sphere $S_0 = S(x_0, \varepsilon_0)$ with $\overline{\sigma_m}$, $m = 1, 2, \dots$. Note that, by construction, $C_k \subset D'_k \subset D'_m$ for all $m \leq k$, and thus, by the minorization principle, $M(f(\Gamma_m)) \geq \delta$ for all $m = 1, 2, \dots$.

On the other hand, the quantity $M(f(\Gamma_m))$ is equal to the capacity of the condenser in D' with the facings $\overline{D'_m}$ and $f(\overline{D \setminus B_0})$, where $B_0 = B(x_0, \varepsilon_0)$ (see, e.g., [63]). Thus, using the minorization principle and Theorem 3.13 in [76], we get

$$M(f(\Gamma_m)) \leq \frac{1}{M^{n-1}(f(\Sigma_m))},$$

where Σ_m is the collection of all intersections of the domain D and the spheres $S(x_0, \rho)$, $\rho \in (r_m, \varepsilon_0)$, because $f(\Sigma_m) \subset \Sigma(f(S_m), f(S_0))$, where $\Sigma(f(S_m), f(S_0))$ consists of all relatively closed subsets of the domain D' that separate $f(S_m)$ and $f(S_0)$. Finally, condition (15) shows that $M(f(\Gamma_m)) \to 0$ as $m \to \infty$.

The contradiction we obtained disproves the assumption that the cluster set $C(P, f)$ consists of more than one point. □

§6. On extension of inverse mappings

Lemma 4. *Let D and D' be regular domains in $\mathbb{R}^n$, $n \geq 2$, P_1 and P_2 different prime ends of the domain D, and f a lower Q-homeomorphism of D onto the domain D'. Let σ_m, $m = 1, 2, \dots$, be a chain of cross-cuts of the end P_1 as in Lemma 2 lying on the spheres $S(z_1, r_m)$, $z_1 \in I(P_1)$, with the associated domains D_m. Suppose that the function Q is integrable in the power $n - 1$ over the spheres*

$$D(r) = \big\{x \in D : |x - z_1| = r\big\} = D \cap S(z_1, r) \tag{17}$$

for a set E of numbers $r \in (0, d)$ of positive linear measure, where $d = r_{m_0}$, and m_0 is the smallest among all numbers such that the domain D_{m_0} contains no sequences converging to P_2. If $\partial D'$ is weakly flat, then

$$C(P_1, f) \cap C(P_2, f) = \varnothing. \tag{18}$$

Note that, since the completion $\bar{D}_P$ of the domain D by its prime ends is metrizable, see Remark 3, the number m_0 in Lemma 4 always exists.

Proof. Choose $\varepsilon \in (0, d)$ such that $E_0 := \{r \in E : r \in (\varepsilon, d)\}$ has positive linear measure. Such a choice is possible by the countable subadditivity of the linear measure and the exhaustion $E = \cup E_m$, where $E_m = \{r \in E : r \in (1/m, d)\}$, $m = 1, 2, \dots$. Note that, by Proposition 2, we have

$$M(f(\Sigma_\varepsilon)) > 0, \tag{19}$$

where Σ_ε is the family of all surfaces $D(r)$, $r \in (\varepsilon, d)$, as in (17).

Assume that $C_1 \cap C_2 \neq \varnothing$, where $C_i = C(P_i, f)$, $i = 1, 2$. By construction, there is $m_1 > m_0$ such that σ_{m_1} lies on the sphere $S(z_1, r_{m_1})$ with $r_{m_1} < \varepsilon$. Let $D_0 = D_{m_1}$, and let $D_* \subseteq D \setminus D_{m_0}$ be the domain associated with the chain of cross-cuts of the prime end P_2. Let $y_0 \in C_1 \cap C_2$. Choose $r_0 > 0$ such that $S(y_0, r_0) \cap f(D_0) \neq \varnothing$ and $S(y_0, r_0) \cap f(D_*) \neq \varnothing$.

Set $\Gamma = \Gamma(\overline{D_0}, \overline{D_*}; D)$. Then by the minorization principle and Theorem 3.13 in [76], from (19) it follows that

$$M(f(\Gamma)) \leq \frac{1}{M^{n-1}(f(\Sigma_\varepsilon))} < \infty. \tag{20}$$

Let $M_0 > M(f(\Gamma))$ be a finite number. Since $\partial D'$ is weakly flat, there is $r_* \in (0, r_0)$ with

$$M\big(\Delta(E, F; D')\big) \geq M_0$$

for all continua E and F in D' that intersect the spheres $S(y_0, r_0)$ and $S(y_0, r_*)$. However, these spheres can be joined by continuous curves c_1 and c_2 in the domains $f(D_0)$ and $f(D_*)$, respectively, and, in particular, for these curves we have

$$M_0 \leq M(\Delta(c_1, c_2; D')) \leq M(f(\Gamma)). \tag{21}$$

The contradiction obtained disproves the assumption that $C_1 \cap C_2 \neq \varnothing$. □

Theorem 1. *Let D and D' be regular domains in $\mathbb{R}^n$, $n \geq 2$. If f is a lower Q-homeomorphism of D onto D' with $Q \in L^{n-1}(D)$, then f^{-1} extends up to a continuous mapping of $\overline{D'}_P$ onto $\bar{D}_P$.*

Proof. By the Fubini theorem (see, e.g., [61]), the set

$$E(x_0) = \big\{r \in (0, d(x_0)) \,:\, Q\big|_{D(x_0,r)} \in L^{n-1}(D(x_0, r))\big\}, \quad x_0 \in \partial D,$$

where $d(x_0) = \sup_{x \in D} |x - x_0|$ and $D(x_0, r) = D \cap S(x_0, r)$, has positive linear measure because $Q \in L^{n-1}(D)$. By Remark 3, there is no loss of generality in assuming that the domain D' has a weakly flat boundary. Thus, arguing by contradiction and taking into account the metrizability of the spaces $\overline{D'}_P$ and $\bar{D}_P$ (see Remark 3), we deduce the desired conclusion from Lemma 4. □

Similarly, combining Lemma 4 and [25, Lemma 9.2], see also Lemma 9.6 in [36], we obtain the following statement.

Theorem 2. *Let D and D' be regular domains in $\mathbb{R}^n$, $n \geq 2$. If $f\colon D \to D'$ is a lower Q-homeomorphism with condition (15) for the function Q, then f^{-1} extends up to a continuous mapping of $\overline{D'}_P$ onto $\bar{D}_P$.*

§7. On a homeomorphic extension to the boundary

Combining Lemma 3 and Theorem 2, we obtain the next statement.

Theorem 3. *Let D and D' be regular domains in $\mathbb{R}^n$, $n \geq 2$, and let $f\colon D \to D'$ be a lower Q-homeomorphism with*

$$\int_0^{\delta(x_0)} \frac{dr}{\|Q\|_{n-1}(x_0, r)} = \infty \quad \textit{for all} \quad x_0 \in \partial D \tag{22}$$

with some $\delta(x_0) \in (0, d(x_0))$, where $d(x_0) = \sup_{x \in D} |x - x_0|$ and

$$\|Q\|_{n-1}(x_0, r) = \left(\int_{D \cap S(x_0, r)} Q^{n-1}(x)\, d\mathcal{A}\right)^{\frac{1}{n-1}}.$$

Then f extends up to a homeomorphism of $\bar{D}_P$ onto $\overline{D'}_P$.

Corollary 1. *In particular, the conclusion of Theorem* 3 *holds true if*

$$q_{x_0}(r) = O\left(\left[\log\frac{1}{r}\right]^{n-1}\right) \quad \text{for all} \quad x_0 \in \partial D \tag{23}$$

as $r \to 0$*, where* $q_{x_0}(r)$ *is the average of* Q^{n-1} *over the sphere* $|x - x_0| = r$*.*

Applying [54, Lemma 2.2], see also [36, Lemma 7.4], from Theorem 3 we obtain the following general lemma, which in its turn makes it possible to obtain a great number of new criteria.

Lemma 5. *Let* D *and* D' *be regular domains in* $\mathbb{R}^n$*,* $n \geq 2$*, and let* $f\colon D \to D'$ *be a lower* Q*-homeomorphism. Suppose that*

$$\int_{D(x_0,\varepsilon)} Q^{n-1}(x) \cdot \psi^n_{x_0,\varepsilon}(|x - x_0|)\, dm(x) = o\big(I^n_{x_0}(\varepsilon)\big) \quad \text{for all} \quad x_0 \in \partial D \tag{24}$$

as $\varepsilon \to 0$*, where* $D(x_0, \varepsilon) = \{x \in D : \varepsilon < |x - x_0| < \varepsilon_0\}$ *for some* $\varepsilon_0 = \varepsilon(x_0) > 0$*,* $\varepsilon(x_0) < d(x_0) = \sup_{x\in D} |x - x_0|$*, and the* $\psi_{x_0,\varepsilon}(t)\colon (0,\infty) \to [0,\infty]$*,* $\varepsilon \in (0, \varepsilon_0)$*, form a family of measurable functions such that*

$$0 < I_{x_0}(\varepsilon) := \int_{\varepsilon}^{\varepsilon_0} \psi_{x_0,\varepsilon}(t)\, dt < \infty \qquad \forall \varepsilon \in (0, \varepsilon_0).$$

Then f *extends up to a homeomorphism of* $\bar{D}_P$ *onto* $\overline{D'}_P$*.*

Remark 4. Note that (24) is true, in particular, if

$$\int_{B(x_0,\varepsilon_0)} Q^{n-1}(x) \cdot \psi^n(|x - x_0|)\, dm(x) < \infty \quad \text{for all} \quad x_0 \in \partial D, \tag{25}$$

where $B(x_0, \varepsilon_0) = \{x \in \mathbb{R}^n : |x - x_0| < \varepsilon_0\}$ for some $\varepsilon_0 = \varepsilon(x_0) > 0$ and $\psi(t)\colon (0,\infty) \to [0,\infty]$ is a measurable function such that $I_{x_0}(\varepsilon) \to \infty$ as $\varepsilon \to 0$. In other words, for the extendibility of f up to a homeomorphism of $\bar{D}_P$ onto $\overline{D'}_P$, it suffices to have the convergence of the integrals in (25) for a nonnegative function $\psi(t)$ that is locally integrable over $(0, \varepsilon_0]$ but has a nonintegrable singularity at the origin. Here the role of the common ε_0 can be played by any number in the interval $(0, d/2)$, where d is the diameter of the domain D.

Let D be a domain in $\mathbb{R}^n$, $n \geq 1$. Recall that a real-valued function $\varphi \in L^1_{\text{loc}}(D)$ has *bounded mean oscillation* in D, written $\varphi \in \mathrm{BMO}(D)$, or simply $\varphi \in \mathrm{BMO}$, if

$$\|\varphi\|_* = \sup_{B\subset D} ⨍_B |\varphi(z) - \varphi_B|\, dm(z) < \infty, \tag{26}$$

where the supremum is taken over all balls B in D, and

$$\varphi_B = ⨍_B \varphi(z)\, dm(z) = \frac{1}{|B|}\int_B \varphi(z)\, dm(z) \tag{27}$$

is the average of the function φ over B. Note that $L^\infty(D) \subset \mathrm{BMO}(D) \subset L^p_{\text{loc}}(D)$ for all $1 \leq p < \infty$ (see, e.g., [49]).

A function φ in BMO has *vanishing mean oscillation*, written $\varphi \in \mathrm{VMO}$, if the supremum in (26) over all balls B in D with the radii $r < \varepsilon$ converges to 0 as $\varepsilon \to 0$. The class VMO was introduced by Sarason in the paper [62]. Note that there is a series of papers on PDE devoted to the study of equations with coefficients in the class VMO, see, e.g., [6, 18, 37, 46, 48].

Following [16], we say that a function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$, $n \geq 2$, has *finite mean oscillation* at a point x_0, written $\varphi \in \mathrm{FMO}(x_0)$, if $\varphi \in L^1_{\mathrm{loc}}$ and

$$\limsup_{\varepsilon \to 0} \fint_{B(x_0,\varepsilon)} |\varphi(x) - \widetilde{\varphi}_\varepsilon| \, dm(x) < \infty, \tag{28}$$

where $\widetilde{\varphi}_\varepsilon$ denotes the average of the function φ over the ball $B(x_0, \varepsilon)$. We also write $\varphi \in \mathrm{FMO}(D)$ or simply $\varphi \in \mathrm{FMO}$ if this property is fulfilled at every point $x_0 \in D$. Obviously, $\mathrm{BMO} \subset \mathrm{FMO}$. By the definition, $\mathrm{FMO} \subset L^1_{\mathrm{loc}}$, but FMO is not a subset of L^p_{loc} for any $p > 1$, see [36]. Thus, the class FMO is essentially wider than $\mathrm{BMO}_{\mathrm{loc}}$.

Choosing $\psi(t) = \frac{1}{t \log 1/t}$ in Lemma 5 and applying [16, Corollary 2.3], see also [36, Corollary 6.3], we obtain the following result.

Theorem 4. *Let D and D' be regular domains in $\mathbb{R}^n$, $n \geq 2$, and let $f\colon D \to D'$ be a lower Q-homeomorphism. If the function Q^{n-1} has finite mean oscillation at every point of ∂D, then f extends up to a homeomorphism of $\bar{D}_P$ onto $\overline{D'}_P$.*

Using this theorem and [16, Proposition 2.1], see also [36, Proposition 6.1], we easily deduce the following statements.

Corollary 2. *In particular, the conclusion of Theorem 4 holds true if*

$$\limsup_{\varepsilon \to 0} \fint_{B(x_0,\varepsilon)} Q^{n-1}(x) \, dm(x) < \infty \quad \textit{for all} \quad x_0 \in \partial D \tag{29}$$

Recall that a point x_0 is called a *Lebesgue point* of a function $\varphi\colon D \to \mathbb{R}$ if φ is integrable over a neighborhood of x_0 and

$$\lim_{\varepsilon \to 0} \fint_{B(x_0,\varepsilon)} |\varphi(x) - \varphi(x_0)| \, dm(x) = 0. \tag{30}$$

Corollary 3. *The conclusion of Theorem 4 holds true if every point $x_0 \in \partial D$ is a Lebesgue point of the function Q^{n-1}.*

The next statement follows also from Lemma 5 under the choice $\psi(t) = 1/t$.

Theorem 5. *Let D and D' be regular domains in $\mathbb{R}^n$, $n \geq 2$, and let $f\colon D \to D'$ be a lower Q-homeomorphism. If for some $\varepsilon_0 = \varepsilon(x_0) \in (0, d_0)$, where $d_0 = d(x_0) = \sup_{x \in D} |x - x_0|$, we have*

$$\int_{\varepsilon < |x - x_0| < \varepsilon_0} Q(x) \, \frac{dm(x)}{|x - x_0|^n} = o\left(\left[\log \frac{1}{\varepsilon}\right]^n\right) \quad \textit{for all} \quad x_0 \in \partial D \tag{31}$$

as $\varepsilon \to 0$, then f extends up to a homeomorphism of $\bar{D}_P$ onto $\overline{D'}_P$.

Remark 5. If we choose the function $\psi(t) = 1/(t \log 1/t)$ instead of $\psi(t) = 1/t$ in Lemma 5, then condition (31) can be replaced by the weaker condition

$$\int_{\varepsilon < |x - x_0| < \varepsilon_0} \frac{Q(x) \, dm(x)}{|x - x_0| \, \log \frac{1}{|x - x_0|}} = o\left(\left[\log \log \frac{1}{\varepsilon}\right]^n\right) \tag{32}$$

and (23) by the condition

$$q_{x_0}(r) = o\left(\left[\log \frac{1}{r} \log \log \frac{1}{r}\right]^{n-1}\right). \tag{33}$$

Of course, a series of similar logarithmic type conditions could be given by applying suitable functions $\psi(t)$.

Theorem 3 has many other consequences; for example, we mention the following.

Theorem 6. *Let D and D' be regular domains in $\mathbb{R}^n$, $n \geq 2$, and let $f\colon D \to D'$ be a lower Q-homeomorphism with*

$$\int_D \Phi\big(Q^{n-1}(x)\big)\, dm(x) < \infty \tag{34}$$

for a monotone nonincreasing convex function $\Phi\colon [0,\infty] \to [0,\infty]$ such that

$$\int_\delta^\infty \frac{d\tau}{\tau\big[\Phi^{-1}(\tau)\big]^{\frac{1}{n-1}}} = \infty \tag{35}$$

with $\delta > \Phi(0)$. Then f extends up to a homeomorphism of $\bar{D}_P$ onto $\overline{D'}_P$.

Indeed, by Theorem 3.1 and Corollary 3.2 in [60], (34) and (35) imply (22) and, thus, Theorem 6 is a direct consequence of Theorem 3.

Corollary 4. *In particular, the conclusion of Theorem* 6 *holds true if, for some $\alpha > 0$, we have*

$$\int_D e^{\alpha Q^{n-1}(x)}\, dm(x) < \infty. \tag{36}$$

Remark 6. Note that condition (35) is not only sufficient but also necessary for the continuous extendibility to the boundary of the mappings f with integral restrictions of the form (34), see, e.g., Theorem 5.1 and Remark 5.1 in [27].

Moreover, by [60, Theorem 2.1], see also [55, Proposition 2.3], condition (35) is equivalent to each of the following conditions:

$$\int_\delta^\infty H'_{n-1}(t)\, \frac{dt}{t} = \infty, \qquad \delta > 0, \tag{37}$$

$$\int_\delta^\infty \frac{dH_{n-1}(t)}{t} = \infty, \qquad \delta > 0, \tag{38}$$

$$\int_\delta^\infty H_{n-1}(t)\, \frac{dt}{t^2} = \infty, \qquad \delta > 0, \tag{39}$$

$$\int_0^\Delta H_{n-1}\left(\frac{1}{t}\right) dt = \infty, \qquad \Delta > 0, \tag{40}$$

$$\int_{\delta_*}^\infty \frac{d\eta}{H^{-1}_{n-1}(\eta)} = \infty, \qquad \delta_* > H_{n-1}(+0), \tag{41}$$

$$\int_{\delta_*}^\infty \frac{d\tau}{\tau \Phi^{-1}_{n-1}(\tau)} = \infty, \qquad \delta_* > \Phi(+0), \tag{42}$$

where

$$H_{n-1}(t) = \log \Phi_{n-1}(t), \qquad \Phi_{n-1}(t) = \Phi(t^{n-1}). \tag{43}$$

Here, in (37) and (38) the integrals are assumed to be equal to ∞ if $\Phi_{n-1}(t) = \infty$, respectively, $H_{n-1}(t) = \infty$, for all $t \geq T \in \mathbb{R}^+$. The integral in (38) is understood as a Lebesgue–Stiltjes integral, and the integrals in (37) and (39)–(42) are the usual Lebesgue integrals.

Some more explanations are in order. On the right-hand sides of conditions (37)–(42) we mean $+\infty$. If $\Phi_{n-1}(t) = 0$ for $t \in [0, t_*]$, then $H_{n-1}(t) = -\infty$ for $t \in [0, t_*]$ and we set $H'_{n-1}(t) = 0$ for $t \in [0, t_*]$. Note that conditions (38) and (39) exclude the case where t_* belongs to the interval of integrability, because in this case the left-hand sides in (38) and (39) are either equal to $-\infty$ or undetermined. Hence, we assumed that in (37)–(40) we have $\delta > t_0$, respectively, $\Delta < 1/t_0$, where $t_0 := \sup_{\Phi_{n-1}(t)=0} t$, $t_0 = 0$ if $\Phi_{n-1}(0) > 0$.

The most interesting condition among the above is condition (39), which can be written in the form

$$\int_{\delta}^{\infty} \log \Phi(t)\, \frac{dt}{t^{n'}} = \infty, \tag{44}$$

where $\frac{1}{n'} + \frac{1}{n} = 1$, i.e., $n' = 2$ for $n = 2$, n' is strictly monotone decreasing in n, and $n' = n/(n-1) \to 1$ as $n \to \infty$.

The theory of the boundary behavior of lower Q-homeomorphisms applies, in particular, to mappings in the Sobolev and Orlicz–Sobolev classes and also to finitely bi–Lipschitz mappings, which are a far-reaching generalization of the well-known classes of isometries and quasiisometries, see, e.g., [21, 23, 28, 29, 30, 36] and [53].

§8. Lower Q-homeomorphisms and the Orlicz–Sobolev classes

Following Orlicz (see [44] and also the monographs [31] and [75]), given a monotone increasing function $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$, $\varphi(0) = 0$, we denote by L^{φ} the space of all functions $f\colon D \to \mathbb{R}$ such that

$$\int_D \varphi\left(\frac{|f(x)|}{\lambda}\right) dm(x) < \infty \tag{45}$$

for some $\lambda > 0$. As before, here m denotes Lebesgue measure in $\mathbb{R}^n$. The space L^{φ} is called the *Orlicz space*. In other words, L^{φ} is a cone over the class of all functions $g\colon D \to \mathbb{R}$ such that

$$\int_D \varphi\bigl(|g(x)|\bigr)\, dm(x) < \infty, \tag{46}$$

which is also called the *Orlicz class*, see [3].

The *Orlicz–Sobolev class* $W^{1,\varphi}(D)$ is the class of all locally integrable functions f defined on D with the first generalized derivatives whose gradient ∇f belongs to the Orlicz class in the domain D. Next, $f \in W^{1,\varphi}_{\mathrm{loc}}(D)$ if $f \in W^{1,\varphi}(D_*)$ for each domain D_* with a compact closure in D. Note that, by the definition, $W^{1,\varphi}_{\mathrm{loc}} \subseteq W^{1,1}_{\mathrm{loc}}$. As usual, we write $f \in W^{1,p}_{\mathrm{loc}}$ if $\varphi(t) = t^p$, $p \ge 1$. In what follows, we also write $f \in W^{1,\varphi}_{\mathrm{loc}}$ for all locally integrable vector-valued functions of n real variables $x_1, \dots, x_n$, $f = (f_1, \dots, f_m)$, if $f_i \in W^{1,1}_{\mathrm{loc}}$, $i = 1, \dots, m$, and

$$\int_D \varphi\left(|\nabla f(x)|\right)\, dm(x) < \infty \tag{47}$$

where $|\nabla f(x)| = \sqrt{\sum_{i,j} \left(\frac{\partial f_i}{\partial x_j}\right)^2}$. It should be noted that, in the present paper, we use the notation $W^{1,\varphi}_{\mathrm{loc}}$ for a more general class of functions φ than in the classical definition of the Orlicz class, often waiving the condition of convexity and the normalization of φ. Note also that the Orlicz–Sobolev classes have been studied intensively in various aspects, see, e.g., [29] and further references therein.

In this connection, we recall a few definitions related to Sobolev classes. Let U be an open set in $\mathbb{R}^n$, $n \ge 2$. The symbol $C_0^{\infty}(U)$ denotes the collection of all functions $\psi\colon U \to \mathbb{R}$ with compact support and with continuous first order partial derivatives. Let u and $v\colon U \to \mathbb{R}$ be locally integrable functions. Set $x = (x_1, x_2, \dots, x_n)$. The function v is called a *generalized derivative* of u in the variable x_i, $i = 1, 2, \dots, n$, written u_{x_i}, if

$$\int_U u\, \psi_{x_i}\, dm(x) = -\int_U v\, \psi\, dm(x) \quad \text{for all} \quad \psi \in C_0^{\infty}(U). \tag{48}$$

The *Sobolev class* $W^{1,p}(U)$ is defined as the collection of all functions $u\colon U \to \mathbb{R}$ in $L^p(U)$ whose first order generalized derivatives belong to $L^p(U)$. A function $u\colon U \to \mathbb{R}$ is said to belong to the space $W^{1,p}_{\rm loc}(U)$ if $u \in W^{1,p}(U_*)$ for every open set U_* with compact closure in U. In what follows, we use the notation $W^{1,p}_{\rm loc}$ in place of $W^{1,p}_{\rm loc}(U)$ if it is clear what U is. A similar definition can be given for a vector-function $f\colon U \to \mathbb{R}^m$ through its coordinate functions. As is well known, a continuous function f belongs to $W^{1,p}_{\rm loc}$ if and only if $f \in ACL^p$, i.e., if f is locally absolutely continuous along almost all straight lines parallel to coordinate axes and if all first order partial derivatives of f are locally integrable in the power p, see, e.g., [41, 1.1.3]. Recall that the notion of a generalized derivative was introduced by Sobolev in $\mathbb{R}^n$, $n \geq 2$, see [64], and at present it is developed in more general spaces by many authors, see, e.g., the corresponding references in [29].

In this section we show that every homeomorphism f with finite distortion in $\mathbb{R}^n$, $n \geq 3$, belonging the Orlicz–Sobolev class $W^{1,\varphi}_{\rm loc}$ and satisfying the Calderon type condition

$$\int_{t_*}^{\infty} \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{n-2}} dt < \infty \tag{49}$$

for some $t_* \in \mathbb{R}^+$, cf. [4], is a lower Q-homeomorphism, where Q is equal to one of the dilatations of f.

Recall that, for a mapping $f\colon D \to \mathbb{R}^n$ that has the first order partial derivatives a.e., $f'(x)$ denotes the Jacobi matrix of f at a point $x \in D$ where it exists, $J(x) = J(x, f) = \det f'(x)$ is the Jacobian of f at x, and $\|f'(x)\|$ is the operator norm of $f'(x)$, i.e.,

$$\|f'(x)\| = \max\{|f'(x)h| \,:\, h \in \mathbb{R}^n, |h| = 1\}. \tag{50}$$

We also denote

$$l(f'(x)) = \min\{|f'(x)h| \,:\, h \in \mathbb{R}^n,\, |h| = 1\}. \tag{51}$$

The *outer dilatation* of the mapping f at a point x is defined by the formula

$$K_O(x) = K_O(x, f) = \begin{cases} \frac{\|f'(x)\|^n}{|J(x,f)|} & \text{if } J(x, f) \neq 0, \\ 1 & \text{if } \ f'(x) = 0, \\ \infty & \text{at the other points,} \end{cases} \tag{52}$$

and the *inner dilatation* is defined by

$$K_I(x) = K_I(x, f) = \begin{cases} \frac{|J(x,f)|}{l(f'(x))^n} & \text{if } J(x, f) \neq 0, \\ 1 & \text{if } \ f'(x) = 0, \\ \infty & \text{at the other points.} \end{cases} \tag{53}$$

Moreover, we shall also use the dilatations P_O and P_I defined as follows:

$$P_O(x, f) = K_O^{\frac{1}{n-1}}(x, f) \quad \text{and} \quad P_I(x, f) = K_I^{\frac{1}{n-1}}(x, f). \tag{54}$$

Note that

$$P_O(x, f) \leq K_I(x, f) \quad \text{and} \quad P_I(x, f) \leq K_O(x, f), \tag{55}$$

see, e.g., [50, 1.2.1], and, in particular, $K_O(x, f)$ and $K_I(x, f)$, $P_O(x, f)$ and $P_I(x, f)$ are simultaneously finite or infinite. Moreover, the condition $K_O(x, f) < \infty$ a.e. is equivalent to the fact that a.e. we have either $\det f'(x) \neq 0$ or $f'(x) = 0$.

Recall that a homeomorphism f between domains D and D' in $\mathbb{R}^n$, $n \geq 2$, is called a *mapping with finite distortion* if $f \in W^{1,1}_{\rm loc}$ and

$$\|f'(x)\|^n \leq K(x) \cdot J_f(x) \quad \text{a.e.} \tag{56}$$

for some finite function $K(x) \geq 1$. In other words, inequality (56) means that the dilatations $K_O(x, f)$, $K_I(x, f)$, $P_O(x, f)$, and $P_I(x, f)$ are finite a.e.

For the first time, the notion of a mapping with finite distortion was introduced in the case of the plane for $f \in W^{1,2}_{\rm loc}$ in the paper [19]. Later, this condition was replaced in the monograph [17] by the requirement $f \in W^{1,1}_{\rm loc}$, but with the additional condition $J_f \in L^1_{\rm loc}$. The theory of mappings with finite distortion has many successors, see, e.g., the corresponding references in [13, 15] and [36]. As predecessors, the mappings with bounded distortion should be mentioned, see [50], and also [70], in other words, the quasiregular mappings, see, e.g., [14, 35] and [51]. The mappings with finite distortion are also closely related to the earlier mappings with bounded Dirichlet integral, and to the mappings quasiconformal in the mean, which had a reach history, see, e.g., references in [36].

Note that the additional condition $J_f \in L^1_{\rm loc}$ mentioned above is not necessary. Indeed, if a homeomorphism f between domains D and D' in $\mathbb{R}^n$ has first order partial derivatives a.e. in D, then there is a set E of zero Lebesgue measure such that f has Lusin's (N)-property in $D \setminus E$ and

$$\int_A J_f(x)\, dm(x) = |f(A)| \tag{57}$$

for every Borel set $A \subset D \setminus E$, see, e.g., Subsections 3.1.4, 3.1.8, and 3.2.5 in [8]. On that basis, it is easy to verify with the help of the Hölder inequality that, in particular, if $f \in W^{1,1}_{\rm loc}$ is a homeomorphism and $K_f \in L^q_{\rm loc}$ for some $q > n-1$, where $K_f = K_O(x,f)$, then also $f \in W^{1,p}_{\rm loc}$ for some $p > n-1$.

Using (57), it is easy to establish the following useful relations.

Proposition 4. *Let f be an ACL homeomorphic mapping in the domain D in $\mathbb{R}^n$, $n \geq 2$. Then*:

(i) $f \in W^{1,1}_{\rm loc}$ *if* $P_O \in L^1_{\rm loc}$;
(ii) $f \in W^{1,\frac{n}{2}}_{\rm loc}$ *if* $K_O \in L^1_{\rm loc}$;
(iii) $f \in W^{1,n-1}_{\rm loc}$ *if* $K_O \in L^{n-1}_{\rm loc}$;
(iv) $f \in W^{1,p}_{\rm loc}$ *with* $p > n-1$ *whenever* $K_O \in L^{\gamma}_{\rm loc}$, $\gamma > n-1$;
(v) $f \in W^{1,p}_{\rm loc}$ *with* $p = n\gamma/(1+\gamma) \geq 1$ *whenever* $K_O \in L^{\gamma}_{\rm loc}$, $\gamma \geq 1/(n-1)$.

These conclusions and estimates (58) *below are also valid for all ACL mappings* $f\colon D \to \mathbb{R}^n$ *with* $J_f \in L^1_{\rm loc}$.

Indeed, by the Hölder inequality on a compact set C in D, on the basis of (57) we obtain the following estimates for norms of the first order partial derivatives:

$$\|\partial_i f\|_p \leq \|f'\|_p \leq \|K_O^{1/n}\|_s \cdot \|J_f^{1/n}\|_n \leq \|K_O\|_\gamma^{1/n} \cdot |f(C)|^{1/n} < \infty \tag{58}$$

if $K_O \in L^{\gamma}_{\rm loc}$ for some $\gamma \in (0,\infty)$, because $\|f'(x)\| = K_O^{1/n}(x) \cdot J_f^{1/n}(x)$ a.e., where $\frac{1}{p} = \frac{1}{s} + \frac{1}{n}$ and $s = \gamma n$, i.e., $\frac{1}{p} = \frac{1}{n}\big(\frac{1}{\gamma} + 1\big)$.

The following statement plays a key role for deriving a number of important consequences of our theory as developed in §§5, 6, and 7, cf. [29, Theorem 5] and [28, Theorem 4.1].

Lemma 6. *Let D and D' be domains in $\mathbb{R}^n$, $n \geq 3$, and let $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a monotone nondecreasing function such that*

$$\int_{t_*}^{\infty} \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{n-2}} dt < \infty \tag{59}$$

for some $t_* \in \mathbb{R}^+$. Then every homeomorphism $f\colon D \to D'$ of finite distortion in the class $W^{1,\varphi}_{\rm loc}$ is a lower Q-homeomorphism at each point $x_0 \in \bar{D}$ with $Q(x) = P_I(x,f)$.

Proof. Let B be the (Borel) set of all points $x \in D$ where the mapping f has total differential $f'(x)$ with $J_f(x) \neq 0$. Applying the Kirszbraun theorem and the uniqueness of the approximative differential, see, e.g., Subsections 2.10.43 and 3.1.2 in [8], we conclude that the set B is the union of a countable collection of Borel sets B_l, $l = 1, 2, \dots$, such that $f_l = f|_{B_l}$ are bi-Lipschitz homeomorphisms, see, e.g., Lemmas 3.1.4 and 3.1.8 in [8]. With no loss of generality, we may assume that the sets B_l are mutually disjoint. Let B_* be the set of all points $x \in D$ where f has total differential with $f'(x) = 0$.

By construction, $B_0 := D \setminus (B \cup B_*)$ has Lebesgue measure zero, see [29, Theorem 1]. Consequently, $\mathcal{A}_S(B_0) = 0$ for a.e. hypersurface S in $\mathbb{R}^n$ and, in particular, for a.e. sphere $S_r := S(x_0, r)$ centered at the point $x_0 \in \bar{D}$, see [26, Theorem 2.11] or [36, Theorem 9.1]. Thus, by Corollary 4 in [29], we obtain $\mathcal{A}_{S_r^*}(f(B_0)) = 0$, and also $\mathcal{A}_{S_r^*}(f(B_*)) = 0$ for a.e. S_r, where $S_r^* = f(S_r)$.

Let Γ be the family of all intersections of the spheres S_r, $r \in (\varepsilon, \varepsilon_0)$, $\varepsilon_0 < d_0 = \sup_{x \in D} |x - x_0|$, with the domain D. For arbitrary function $\varrho_* \in \operatorname{adm} f(\Gamma)$ such that $\varrho_* \equiv 0$ outside of $f(D)$, set $\varrho \equiv 0$ outside of D and in $D \setminus B$, and

$$\varrho(x) := \Lambda(x) \cdot \varrho_*(f(x)) \quad \text{for} \quad x \in B,$$

where

$$\Lambda(x) = [J_f(x) \cdot P_I(x, f)]^{\frac{1}{n}} = \left[\frac{\det f'(x)}{l(f'(x))}\right]^{\frac{1}{n-1}} = [\lambda_2 \cdot \dots \cdot \lambda_n]^{\frac{1}{n-1}} \geq \left[J_{n-1}(x)\right]^{\frac{1}{n-1}}.$$

Here, as usual, $\lambda_n \geq \dots \geq \lambda_1$ are the principal dilatation coefficients of $f'(x)$, see, e.g., [50, Subsection I.4.1], $J_{n-1}(x)$ is the $(n-1)$-dimensional Jacobian of the restriction $f|_{S_r}$ at x, see [8, 3.2.1].

Arguing piecewise on every B_l, $l = 1, 2, \dots$, and taking the Kirszbraun theorem into account, we can use Theorem 3.2.5 on the change of variables in [8] to show that

$$\int_{S_r} \varrho^{n-1} \, d\mathcal{A} \geq \int_{S_*^r} \varrho_*^{n-1} \, d\mathcal{A} \geq 1$$

for a.e. S_r and, consequently, $\varrho \in \operatorname{ext} \operatorname{adm} \Gamma$.

Again applying [8, Theorem 3.2.5] on every B_l, $l = 1, 2, \dots$, and using the countable subadditivity of the integral, we get the estimate

$$\int_D \frac{\varrho^n(x)}{P_I(x)} \, dm(x) \leq \int_{f(D)} \varrho_*^n(x) \, dm(x),$$

which completes the proof. □

Corollary 5. *Every homeomorphism f with finite distortion in $\mathbb{R}^n$, $n \geq 3$, of class $W^{1,p}_{\mathrm{loc}}$ with $p > n - 1$ is a lower Q-homeomorphism with $Q = P_I$ at each point $x_0 \in \bar{D}$.*

Combining the last consequence and Proposition 4, we obtain the following conclusion.

Corollary 6. *Every homeomorphism f of the class $W^{1,1}_{\mathrm{loc}}$ in $\mathbb{R}^n$, $n \geq 3$, with $K_O \in L^q_{\mathrm{loc}}$ for some $q > n - 1$ is a lower Q-homeomorphism with $Q = P_I$ at each point $x_0 \in \bar{D}$.*

Proposition 3 shows that Lemma 6 has the following consequence.

Proposition 5. *Let $f \colon D \to \mathbb{R}^n$, $n \geq 3$, be a homeomorphism of the class $W^{1,\varphi}_{\mathrm{loc}}$ with $K_I \in L^1_{\mathrm{loc}}$, where $\varphi \colon \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone nondecreasing function such that*

$$\int_{t_*}^{\infty} \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{n-2}} dt < \infty. \tag{60}$$

Then f is a ring Q-homeomorphism with $Q = K_I$ at each point $x_0 \in \bar{D}$.

Corollary 7. *Every homeomorphism f of the class $W^{1,1}_{\rm loc}$ in $\mathbb{R}^n$, $n \geq 3$, with $K_O \in L^q_{\rm loc}$ for some $q > n-1$ is a ring Q-homeomorphism with $Q = K_I$ at each point $x_0 \in \bar{D}$.*

Remark 7. By Remark 1, the conclusion of Proposition 5 remains valid if K_I is integrable only over almost all spheres of sufficiently small radius centered at the point x_0, under the agreement that the function K_I is extended by zero outside of D.

§9. The boundary behavior in the Orlicz–Sobolev classes

In this section we assume that $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone nondecreasing function such that

$$\int_{t_*}^{\infty} \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{n-2}} dt < \infty \tag{61}$$

for some $t_* \in \mathbb{R}^+$. The continuous extendibility to the boundary of the inverse mappings has a criterion that is simpler than the corresponding criteria for the direct mappings. Hence we start precisely with the former. Namely, in view of Lemma 6 we have the following consequence of Theorem 1.

Theorem 7. *Let D and D' be regular domains in $\mathbb{R}^n$, $n \geq 3$, and let f be a homeomorphism of D onto D' of class $W^{1,\varphi}_{\rm loc}$ satisfying* (61) *and with $K_I \in L^1(D)$. Then f^{-1} extends to a continuous mapping of $\overline{D'}_P$ onto $\bar{D}_P$.*

However, the example in the proof of Proposition 6.3 in [36] shows that no power of integrability of $K_I \in L^q(D)$, $q \in [1, \infty)$, can guarantee the continuous extendibility of the direct mapping.

Using Lemma 6, we obtain the following consequence of Theorem 3.

Theorem 8. *Let D and D' be regular domains in $\mathbb{R}^n$, $n \geq 3$, and let f be a homeomorphism of finite distortion of D onto D' belonging to $W^{1,\varphi}_{\rm loc}$, satisfying* (61), *and such that*

$$\int_0^{\delta(x_0)} \frac{dr}{\|K_I\|^{\frac{1}{n-1}}(x_0, r)} = \infty \quad \text{for all} \quad x_0 \in \partial D \tag{62}$$

with some $\delta(x_0) \in (0, d(x_0))$, where $d(x_0) = \sup_{x \in D} |x - x_0|$ and

$$\|K_I\|(x_0, r) = \int_{D \cap S(x_0, r)} K_I(x, f)\, d\mathcal{A}.$$

Then f extends up to a homeomorphism of $\bar{D}_P$ onto $\overline{D'}_P$.

In particular, as a consequence of Theorem 8, we obtain the following generalization of the well-known theorems of Gehring–Martio and Martio–Vuorinen on a homeomorphic extension to the boundary of quasiconformal mappings between QED-domains, see [11] and [38].

Corollary 8. *Let D and D' be regular domains in $\mathbb{R}^n$, $n \geq 3$, and let f be a homeomorphism of finite distortion of D onto D' belonging to $W^{1,p}_{\rm loc}$, $p > n-1$, and, in particular, a homeomorphism of class $W^{1,1}_{\rm loc}$ with $K_O \in L^q_{\rm loc}$, $q > n-1$. If condition* (62) *is fulfilled, then f extends up to a homeomorphism of $\bar{D}_P$ onto $\overline{D'}_P$.*

By Lemma 6, as a consequence of Lemma 5, we get the next general lemma.

Lemma 7. *Let D and D' be regular domains in $\mathbb{R}^n$, $n \geq 3$, and let f be a homeomorphism of finite distortion of D onto D' belonging to $W^{1,\varphi}_{\rm loc}$, satisfying* (61), *and such that*

$$\int_{D(x_0,\varepsilon,\varepsilon_0)} K_I(x,f)\cdot \psi^n_{x_0,\varepsilon}(|x-x_0|)\, dm(x) = o(I^n_{x_0}(\varepsilon)) \quad \text{for all} \quad x_0\in\partial D \tag{63}$$

as $\varepsilon\to 0$, where $D(x_0,\varepsilon,\varepsilon_0)=\{x\in D : \varepsilon<|x-x_0|<\varepsilon_0\}$ for some $\varepsilon_0\in(0,\delta_0)$, $\delta_0=\delta(x_0)=\sup_{x\in D}|x-x_0|$, and $\psi_{x_0,\varepsilon}(t)$ is a family of nonnegative Lebesgue measurable functions on $(0,\infty)$ such that

$$0<I_{x_0}(\varepsilon):=\int_{\varepsilon}^{\varepsilon_0}\psi_{x_0,\varepsilon}(t)\,dt<\infty \quad \text{for all} \quad \varepsilon\in(0,\varepsilon_0)\ . \tag{64}$$

Then f extends up to a homeomorphism of $\bar{D}_P$ onto $\overline{D'}_P$.

Choosing $\psi(t)=1/(t\log 1/t)$ in Lemma 7 and applying [16, Corollary 2.3] on FMO, see also [36, Corollary 6.3], we obtain the following result.

Theorem 9. *Let D and D' be regular domains in $\mathbb{R}^n$, $n\geq 3$, and let f be a homeomorphism of D onto D' of class $W^{1,\varphi}_{\rm loc}$ with condition* (61) *such that*

$$K_I(x,f)\leq Q(x) \qquad a\,e. \text{ in } D \tag{65}$$

with a function $Q\in \mathrm{FMO}(x_0)$ for all $x_0\in\bar{D}$. Then f extends up to a homeomorphism of $\bar{D}_P$ onto $\overline{D'}_P$.

In the following statements we assume that $K_I(x,f)$ is extended by zero outside of D.

Corollary 9. *In particular, the conclusion of Theorem* 9 *holds true if*

$$\limsup_{\varepsilon\to 0} ⨍_{B(x_0,\varepsilon)} K_I(x,f)\,dm(x)<\infty \quad \text{for all} \quad x_0\in\bar{D}. \tag{66}$$

Similarly, choosing $\psi(t)=1/t$ in Lemma 7, we arrive at the following claim.

Theorem 10. *Let D and D' be regular domains in $\mathbb{R}^n$, $n\geq 3$, and let f be a homeomorphism of D onto D' of class $W^{1,\varphi}_{\rm loc}$ with condition* (61) *such that*

$$\int_{\varepsilon<|x-x_0|<\varepsilon_0} K_I(x,f)\,\frac{dm(x)}{|x-x_0|^n} = o\left(\left[\log\frac{\varepsilon_0}{\varepsilon}\right]^n\right) \quad \text{for all} \quad x_0\in\partial D \tag{67}$$

as $\varepsilon\to 0$ for some $\varepsilon_0\in(0,d_0)$, where $d_0=d(x_0)=\sup_{x\in D}|x-x_0|$. Then f extends up to a homeomorphism of $\bar{D}_P$ onto $\overline{D'}_P$.

Corollary 10. *Condition* (67) *and the conclusion of Theorem* 10 *are valid if*

$$K_I(x,f)=o\left(\left[\log\frac{1}{|x-x_0|}\right]^{n-1}\right) \tag{68}$$

as $x\to x_0$. The same is true if

$$k_f(r)=o\left(\left[\log\frac{1}{r}\right]^{n-1}\right) \tag{69}$$

as $r\to 0$, where $k_f(r)$ is the average of the function $K_I(x,f)$ over the sphere $|x-x_0|=r$.

Remark 8. Choosing the function $\psi(t)=1/(t\log 1/t)$ instead of $\psi(t)=1/t$ in Lemma 7, we are able to replace condition (67) by the condition

$$\int_{\varepsilon<|x-x_0|<1}\frac{K_I(x,f)\,dm(x)}{\left(|x-x_0|\log\frac{1}{|x-x_0|}\right)^n} = o\left(\left[\log\log\frac{1}{\varepsilon}\right]^n\right), \tag{70}$$

and (69) by the condition

$$k_f(r) = o\left(\left[\log\frac{1}{r}\log\log\frac{1}{r}\right]^{n-1}\right). \tag{71}$$

Thus, it suffices to require that

$$k_f(r) = O\left(\left[\log\frac{1}{r}\right]^{n-1}\right). \tag{72}$$

Of course, here we could give a series of other conditions in terms of log, choosing functions $\psi(t)$ of the form $1/(t\log\ldots\log 1/t)$.

Theorem 11. *Let D and D' be regular domains in $\mathbb{R}^n$, $n \geq 3$, and let f be a homeomorphism of D onto D' of class $W^{1,\varphi}_{\mathrm{loc}}$ with condition* (61) *such that*

$$\int_D \Phi(K_I(x,f))\,dm(x) < \infty \tag{73}$$

for a monotone nondecreasing convex function $\Phi\colon \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$. If for some $\delta > \Phi(0)$ we have

$$\int_\delta^\infty \frac{d\tau}{\tau\,[\Phi^{-1}(\tau)]^{\frac{1}{n-1}}} = \infty, \tag{74}$$

then f extends up to a homeomorphism of $\bar{D}_P$ onto $\overline{D'}_P$.

Indeed, by Theorem 3.1 and [60, Corollary 3.2], (73) and (74) imply (62) and, consequently, Theorem 11 is a direct consequence of Theorem 8.

Corollary 11. *In particular, the conclusion of Theorem* 11 *holds true if for some $\alpha > 0$ we have*

$$\int_D e^{\alpha K_I(x,f)}\,dm(x) < \infty. \tag{75}$$

Remark 9. Note that by Theorem 5.1 and Remark 5.1 in [27], condition (74) is not only sufficient but also necessary for a continuous extendibility to the boundary of mappings f with the integral restriction (73).

Note also that by Remark 6 condition (74) is equivalent to every one among conditions (37)–(42) and, in particular, to the condition

$$\int_\delta^\infty \log\Phi(t)\,\frac{dt}{t^{n'}} = +\infty \tag{76}$$

for some $\delta > 0$, where $\frac{1}{n'} + \frac{1}{n} = 1$, i.e., $n' = 2$ for $n = 2$, n' is strictly monotone decreasing in n, and $n' = n/(n-1) \to 1$ as $n \to \infty$.

Finally, note that all these results are valid, for example, if $f \in W^{1,p}_{\mathrm{loc}}$, $p > n-1$, and, in particular, if $f \in W^{1,1}_{\mathrm{loc}}$ and $K_O \in L^q_{\mathrm{loc}}$, $q > n-1$. Moreover, these results can be carried over to the case of Riemannian manifolds, see, e.g., [2] and [30].

§10. On finitely bi-Lipschitz mappings

As in [26], see also [36], given an open set $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, we say that a mapping $f\colon \Omega \to \mathbb{R}^n$ is *finitely bi-Lipschitz* if

$$0 < l(x,f) \leq L(x,f) < \infty \quad \text{for all} \quad x \in \Omega, \tag{77}$$

where

$$L(x,f) = \limsup_{y\to x} \frac{|f(y)-f(x)|}{|y-x|} \tag{78}$$

and

$$l(x,f)=\liminf_{y\to x}\frac{|f(y)-f(x)|}{|y-x|}, \tag{79}$$

cf. the definition of bi–Lipschitz mappings in §4.

By the classical Stepanoff theorem, see [65] and also [34], from the right-hand side of (77) we see that the finitely bi-Lipshitz mappings are differentiable a.e., and the left-hand side of (77) shows that $J_f(x)\neq 0$ a.e. Moreover, such mappings possess the (N)-property with respect to Hausdorff measures, see, e.g., either [26, Lemma 5.3] or [36, Lemma 10.6]. Thus, the proof of the following lemma repeats the proof of Lemma 6, so that we do not give it, cf. also the weaker but similar statements: [26, Corollary 5.15] and [36, Corollary 10.10].

Lemma 8. *Every finitely bi-Lipschitz homeomorphism $f\colon\Omega\to\mathbb{R}^n$, $n\geq 2$, is a lower Q-homeomorphism with $Q=P_I$.*

Using Proposition 3, we deduce the following statement from Lemma 8.

Proposition 6. *Every finitely bi-Lipschitz homeomorphism $f\colon\Omega\to\mathbb{R}^n$, $n\geq 2$, with $K_I\in L^1_{\rm loc}$ is a ring Q-homeomorphism with $Q=K_I$ at every point $x_0\in\bar{D}$.*

Remark 10. By Remark 1, the conclusion of Proposition 6 remains valid if K_I is only integrable over almost all spheres of sufficiently small radius centered at the point x_0, under the agreement that the function K_I is extended by zero outside of D.

Corollary 12. *All results on the boundary behavior of lower Q-homeomorphisms stated in §§5–7, are also true for the finitely bi-Lipschitz homeomorphisms $f\colon\Omega\to\mathbb{R}^n$, $n\geq 2$, with $Q=P_I(x,f)$.*

All these results for finitely bi-Lipschitz homeomorphisms are entirely similar to the corresponding results for homeomorphisms with finite distortion in the Orlicz–Sobolev classes. Therefore, we do not formulate all of them in an explicit form in terms of the inner dilatation K_I, restricting ourselves to only one such result.

Theorem 12. *Let D and D' be regular domains in $\mathbb{R}^n$, $n\geq 2$, and let $f\colon D\to D'$ be a finitely bi-Lipschitz homeomorphism such that*

$$\int_D \Phi(K_I(x,f))\,dm(x)<\infty \tag{80}$$

for a monotone nondecreasing convex function $\Phi\colon\overline{\mathbb{R}^+}\to\overline{\mathbb{R}^+}$. If for some $\delta>\Phi(0)$ we have

$$\int_\delta^\infty \frac{d\tau}{\tau\,[\Phi^{-1}(\tau)]^{\frac{1}{n-1}}}=\infty, \tag{81}$$

then f extends up to a homeomorphism of $\bar{D}_P$ onto $\overline{D'}_P$.

Corollary 13. *In particular, the conclusion of Theorem* 12 *holds true if for some $\alpha>0$ we have*

$$\int_D e^{\alpha K_I(x,f)}\,dm(x)<\infty. \tag{82}$$

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Institute of Applied Mathematics and Mechanics, National Academy of Sciences of Ukraine, Roze Luxemburg Str. 74, Donetsk 83114, Ukraine
E-mail address: denis_kovtonyuk@bk.ru

Institute of Applied Mathematics and Mechanics, National Academy of Sciences of Ukraine, Roze Luxemburg Str. 74, Donetsk 83114, Ukraine
E-mail address: vl_ryazanov1@mail.ru

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