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SPECTRAL ANALYSIS OF A FOURTH ORDER DIFFERENTIAL OPERATOR WITH PERIODIC AND ANTIPERIODIC BOUNDARY CONDITIONS

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Abstract. By the method of similar operators, the spectral properties of a fourth order differential operator are studied under periodic or semiperiodic boundary conditions. The spectrum asymptotics is obtained, together with some estimates for the spectral resolution for the operator in question. Also, the operator semigroup is constructed whose generator is equal to minus the operator under study.

§1. Introduction

Let $L_2[0,1]$ stand for the Hilbert space of complex functions square integrable on $[0,1]$ with the scalar product $(x,y)=\int_0^1 x(\tau)\overline{y(\tau)}\,d\tau$, $x,y\in L_2[0,1]$. We denote by $W_2^4[0,1]$ the Sobolev space $\{y\in L_2[0,1]\to\mathbb{C} : y$ has three continuous derivatives, y''' is absolutely continuous, and $y^{IV}\in L_2[0,1]\}$.

We shall consider an operator $L_{bc}\colon D(L_{bc})\subset L_2[0,1]\to L_2[0,1]$ determined by the differential expression

$$l(y)=y^{IV}-a(t)y''-b(t)y,\ \text{where } a,b\in L_2[0,1].$$

The domain $D(L_{bc})$ is determined by one of the following boundary conditions bc:

(a) periodic, $bc=\mathrm{per} : y^{(j)}(0)=y^{(j)}(1)$, $j=0,1,2,3$;

(b) semiperiodic, $bc=ap : y^{(j)}(0)=-y^{(j)}(1)$, $j=0,1,2,3$.

Specifically, we put $D(L_{bc})=\{y\in W_2^4[0,1] : y \text{ satisfies } bc\}$. The corresponding operators will be denoted by L_{per}, L_{ap}.

If $a=b=0$, we use the notation $\mathcal{L}_{bc}^0$ or $\mathcal{L}_{\mathrm{per}}^0$, $\mathcal{L}_{ap}^0$. The operator $\mathcal{L}_{bc}^0$ is said to be *free*. It will play the role of a nonperturbed operator when we study L_{bc}, whereas the operator $B : D(B)=D(L_{bc})\subset L_2[0,1]\to L_2[0,1]$, $By=a(t)y''+b(t)y$ will play the role of a perturbation. $\mathcal{L}_{bc}^0$ is a selfadjoint operator with compact resolvent. Since a, $b\in L_2[0,1]$, they expand into the series $a(t)=\sum_{l\in\mathbb{Z}}a_l e^{i2\pi lt}$, $b(t)=\sum_{l\in\mathbb{Z}}b_l e^{i2\pi lt}$, where a_l and b_l are the Fourier coefficients of a and b, respectively. We emphasize that no additional restrictions on a and b (like smoothness) beyond a, $b\in L_2[0,1]$ are imposed.

We describe the spectra $\sigma(\mathcal{L}_{bc}^0)$ and the eigenfunctions of $\mathcal{L}_{bc}^0$, $bc\in\{\mathrm{per},ap\}$:

(a) $\sigma(\mathcal{L}_{\mathrm{per}}^0)=\{(2\pi n)^4, n\in\mathbb{Z}_+=\mathbb{N}\cup\{0\}\}$; for $n\neq 0$, the corresponding eigenspace is $E_n^0=\mathrm{Span}\{e_n^1,e_n^2\}$, where $e_n^1(t)=e^{-i2\pi nt}$, $e_n^2(t)=e^{i2\pi nt}$, $t\in[0,1]$. For $n=0$, we have $E_0^0=\{\alpha e_0,\alpha\in\mathbb{C}\}$, where $e_0(t)=1$, $t\in[0,1]$;

(b) $\sigma(\mathcal{L}_{ap}^0)=\{\pi^4(2n+1)^4, n\in\mathbb{Z}_+=\mathbb{N}\cup\{0\}\}$; the corresponding eigenspaces are given by $E_n^0=\mathrm{Span}\{e_n^1,e_n^2\}$, where $e_n^1(t)=e^{-i\pi(2n+1)t}$, $e_n^2(t)=e^{i\pi(2n+1)t}$, $t\in[0,1]$.

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We denote by P_n, $n \in \mathbb{Z}_+$, the Riesz projection corresponding to the singleton $\{(2\pi n)^4\}$ or $\{\pi^4(2n+1)^4\}$. For every $x \in \mathrm{L}_2[0,1]$, we have

(a) $P_n x = (x, e_n^1)e_n^1 + (x, e_n^2)e_n^2$, $n \in \mathbb{N}$, $\quad P_0 x = (x, e_0)e_0$;

(b) $P_n x = (x, e_n^1)e_n^1 + (x, e_n^2)e_n^2$, $n \in \mathbb{Z}_+$.

Throughout, we assume that $b_0 = \int_0^1 b(t)\,\mathrm{d}t = 0$. This is not a restriction because the shift of the potential by a constant shifts the spectrum by the same constant. However, we take the value of $b_0 = 0$ into account when we calculate the spectral asymptotics for the operator in question.

The operators L_{bc} are interesting because they describe the vibration of beams and shells as well as of a compressed rod on an elastic base (see, e.g., [1, 2]). Presently, a considerable interest to this topic emerges from numerous applications to optics, acoustics (see [3]) and also to the study of nanotubes conductivity (see [4]).

A selfadjoint 4th order differential operator with periodic coefficients was studied in a series of papers by Badanin and Korotyaev. In [5], the operator $\frac{d^4}{dt^4} + V$ with real periodic potential V in L_1 was studied on the real line. In [6], the 4th order operator $H = \frac{d^4}{dt^4} + \frac{d}{dt} p \frac{d}{dt} + q$ with real periodic potentials p and q was treated under the assumption $p,\ p',\ q \in L_1(0,1)$. In [7], the same operator was considered in $L_2(0,1)$ with classical boundary conditions and with $p,\ p'',\ q \in L_1(0,1)$. In [8], a general periodic operator of even order in $L_2(\mathbb{R})$ and with coefficients in L_1 was studied. In all these papers, the asymptotic of the spectrum was determined and the spectral bands were explored, together with the high energy spectral characteristics. The method of the study was based on the construction of Lyapunov functions.

We also mention the paper [9], in which a differential operator of arbitrary order m with complex potential was treated and asymptotics formulas for its eigenvalues were derived.

Mikhaĭlets and Molyboga (see [10, 11, 12]) obtained asymptotic estimates for the operator $(-1)^N \frac{d^{2N}}{dt^{2N}} + q$ with periodic and semiperiodic boundary conditions, where q is a periodic distribution belonging to a Sobolev space.

Note also that in Naĭmark's monograph [13] the eigenvalue asymptotics was described for an nth order differential operator with regular boundary conditions in the space of continuous functions and also in the space of vector-valued functions.

The theory of perturbed differential operators determined by boundary conditions on a finite interval involves various methods. Thus, in the paper [14] by Dzhakov and Mityagin, the Schrödinger and Dirak operators were studied by resolvent methods, see [14, 15, 16]. These methods make it possible to calculate the first approximation for eigenvalues of the perturbed operator and its projections.

In [17], Agranovich obtained certain results about the equiconvergence of spectral resolutions for a perturbation subordinate to a fractional power of the nonperturbed operator; also, he gave asymptotic estimates for the equiconvergence of spectral resolutions. However, the results of that paper are applicable only in the case where a is a bounded function.

In the present paper, we obtain second approximations for the eigenvalues, and also estimate spectral projections for L_{bc}. Unlike Theorem V.4.15 in [15], the function a is not assumed to be bounded. We employ the *similar operators method* (see [18, 19, 20, 21]) to study L_{bc}. This method emerged in the construction of an analog of the Bogolyubov–Krylov substitution for nonlinear equations in a Banach space, see [18, 19, 20, 21]. It is intimately related to the Friedrichs method (see [16]), which is used for perturbed operators with continuous spectrum. To develop the method of similar operators, some patterns of harmonic analysis were invoked.

The version of the similar operators method employed in the present paper was developed in [22, 23]. In [23, 24] it was used by the author to study L_{bc} under other classical boundary conditions. The main idea of the method is in a similarity transformation of L_{bc} to an operator whose spectral properties are close to those of the unperturbed operator $\mathcal{L}_{bc}^0$. Specifically, it is shown that L_{bc} is similar to an operator of block-diagonal form in the basis of eigenvectors for $\mathcal{L}_{bc}^0$ (this is an analog of the Jordan theorem for a linear operator in a finite-dimensional space). This simplifies the study of L_{bc} a lot.

Among the main results of the paper, we mention Theorem 7, which says that L_{bc} is similar to an operator of the form indicated above and having the same eigenvalues as $\mathcal{L}_{bc}^0$, except a finite set. This statement provides a basis for the deduction of the eigenvalue asymptotics and the proof of the equiconvergence of spectral resolutions.

The main results of the paper are stated below.

Theorem 1. *The differential operator L_{bc} has compact resolvent and its spectrum has the form*

$$\sigma(L_{bc}) = \widetilde{\sigma}_m \cup \bigl\{\widetilde{\lambda}_n^{\mp}, n \geq m+1\bigr\}$$

for some $m \in \mathbb{N}$, where $\widetilde{\sigma}_m$ is a finite set of at most m points. The eigenvalues $\widetilde{\lambda}_n^{\mp}$, $n \geq m+1$, of L_{per} have the following asymptotic representation:

$$\widetilde{\lambda}_n^{\mp} = \bigl(2\pi n\bigr)^4 + (2\pi n)^2 a_0 - n^2 \sum_{\substack{l=1\\ l\neq n}}^{\infty} \frac{(a_{n+l}a_{-n-l} + a_{n-l}a_{l-n})l^2}{l^4 - n^4}$$
$$\mp (2\pi n)^2 \left(a_{-2n}a_{2n} + \frac{1}{4\pi^4} \left(\sum_{\substack{l=1\\ l\neq n}}^{\infty} \frac{a_{l-n}a_{-n-l}l^2}{l^4 - n^4} \right) \left(\sum_{\substack{l=1\\ l\neq n}}^{\infty} \frac{a_{l+n}a_{n-l}l^2}{l^4 - n^4} \right) \right.$$
$$\left. - \frac{a_{-2n}}{2\pi^2} \sum_{\substack{l=1\\ l\neq n}}^{\infty} \frac{a_{l+n}a_{n-l}l^2}{l^4 - n^4} - \frac{a_{2n}}{2\pi^2} \sum_{\substack{l=1\\ l\neq n}}^{\infty} \frac{a_{l-n}a_{-n-l}l^2}{l^4 - n^4} \right)^{\frac{1}{2}} + \gamma_n n^2, \quad n \geq m+1.$$

Or, in an abridged form:

$$\widetilde{\lambda}_n^{\mp} = \bigl(2\pi n\bigr)^4 + (2\pi n)^2 a_0 + \gamma_n^{\mp} n^2, \quad n \geq m+1.$$

Here (γ_n) and $(\gamma_n^{\mp})$ are sequences summable with the powers $\frac{4}{3}$ and 2, respectively, and the a_k, $k \in \mathbb{Z}$, are the Fourier coefficients of a.

For L_{ap}, we have the following asymptotic representation:

$$\widetilde{\lambda}_n^{\mp} = \bigl(\pi(2n+1)\bigr)^4 + \bigl(\pi(2n+1)\bigr)^2 a_0 - (2n+1)^2 \sum_{\substack{l=0\\ l\neq n}}^{\infty} \frac{(a_{n+l+1}a_{-n-l-1} + a_{n-l}a_{l-n})(2l+1)^2}{(2l+1)^4 - (2n+1)^4}$$
$$\mp (\pi(2n+1))^2 \left(a_{-2n-1}a_{2n+1} + \frac{4}{\pi^4} \sum_{\substack{l=0\\ l\neq n}}^{\infty} \frac{a_{l-n}a_{-n-l-1}(2l+1)^2}{(2l+1)^4 - (2n+1)^4} \sum_{\substack{l=0\\ l\neq n}}^{\infty} \frac{a_{n+l+1}a_{n-l}(2l+1)^2}{(2l+1)^4 - (2n+1)^4} \right.$$
$$\left. - \frac{2a_{-2n-1}}{\pi^2} \sum_{\substack{l=0\\ l\neq n}}^{\infty} \frac{a_{n+l+1}a_{n-l}(2l+1)^2}{(2l+1)^4 - (2n+1)^4} - \frac{2a_{2n+1}}{\pi^2} \sum_{\substack{l=0\\ l\neq n}}^{\infty} \frac{a_{l-n}a_{-n-l-1}(2l+1)^2}{(2l+1)^4 - (2n+1)^4} \right)^{\frac{1}{2}} + \widetilde{\gamma}_n n^2,$$

$n \geq m+1$.

Or, in an abridged form:

$$\widetilde{\lambda}_n^{\mp} = \bigl(\pi(2n+1)\bigr)^4 + (\pi(2n+1))^2 a_0 + \widetilde{\gamma}_n^{\mp} n^2, \quad n \geq m+1.$$

Here $(\widetilde{\gamma}_n)$ and $(\widetilde{\gamma}_n^{\mp})$ are sequences summable with the powers $\frac{4}{3}$ and 2, respectively.

Theorem 2. *If a is a function of bounded variation, then*

$$\widetilde{\lambda}_n^{\pm} = (2\pi n)^4 + (2\pi n)^2 a_0 + O(1)$$

and

$$\widetilde{\lambda}_n^{\pm} = \bigl(\pi(2n+1)\bigr)^4 + \bigl(\pi(2n+1)\bigr)^2 a_0 + O(1)$$

for the cases of $bc = \mathrm{per}$ and $bc = ap$, respectively.

In the next theorem, the symbol $\widetilde{P}_n$, $n \geq m+1$ ($m \in \mathbb{N}$ is taken from Theorem 1), stands for the Riesz projection constructed for the subsets $\{\widetilde{\lambda}_n^{\mp}\}$ of the spectrum $\sigma(L_{bc})$. If Ω is an arbitrary subset of $\mathbb{N}\setminus\{0,\dots,m\}$, then $\widetilde{P}(\Omega) = \sum_{k\in\Omega} \widetilde{P}_k$ is the Riesz projection constructed for the set $\{\widetilde{\lambda}_k^{\mp}, k \in \Omega\}$. Similarly, $P(\Omega) = \sum_{k\in\Omega} P_k$. Next, we denote by $\widetilde{P}_{(m)}$ the Riesz projection for the operator L_{bc} that corresponds to the spectral set $\widetilde{\sigma}_m$, and we put $P_{(m)} = P_1 + \cdots + P_m$.

Theorem 3. *The system of projections $\widetilde{P}_n$, $n \in \mathbb{N}$, has the following property*:

$$\|\widetilde{P}(\Omega) - P(\Omega)\|_2 \leq \frac{\widetilde{M}\,(\ln k(\Omega))^{\frac{1}{2}}}{k(\Omega)},$$

where $k(\Omega) = \min_{k\in\Omega} k$ and $\widetilde{M} > 0$ is a constant independent of $k(\Omega)$.

This theorem shows that the eigenfunctions and generalized eigenfunctions of the operators in question form an unconditional basis. It should be noted that such estimates cannot be obtained with the help of the resolvent method used in [14, 15, 16], because the choice of an integration contour presents a problem.

Theorem 3 readily implies the following statement.

Theorem 4. *The following equiconvergence estimates are true for the spectral resolutions for L_{bc} and $\mathcal{L}_{bc}^0$:*

$$\left\|\widetilde{P}_{(m)} + \sum_{k=m+1}^{n} \widetilde{P}_k - P_{(m)} - \sum_{k=m+1}^{n} P_k\right\|_2 \leq \frac{\widetilde{M}(\ln n)^{\frac{1}{2}}}{n}, \quad n \geq m+1,$$

where $\widetilde{M} > 0$ is the constant from Theorem 3.

Remark 1. If a is bounded, the estimates of [17] are applicable, yielding the same as Theorem 4 but without the factor of $(\ln n)^{\frac{1}{2}}$. This sharper estimate for bounded a also can be obtained by the method of similar operators.

Theorem 8 claims that $-L_{bc}$ is a sectorial operator and generates an analytic semigroup of operators. Moreover, this semigroup is similar to a semigroup of the form $T_{(m)}(t) \oplus T^{(m)}(t)$ acting in $\mathrm{L}_2[0,1] = \mathcal{H}_{(m)} \oplus \mathcal{H}^{(m)}$, where $\mathcal{H}_{(m)} = \operatorname{Im} P_{(m)}$, $\mathcal{H}^{(m)} = \operatorname{Im}(I - P_{(m)})$, and $T^{(m)}(t)$ admits a representation of the form

$$T^{(m)}(t)x = \sum_{k=m+1}^{\infty} \mathrm{e}^{C_k t} P_k x, \quad x \in \mathrm{L}_2[0,1],$$

where $C_k \in \operatorname{End} \mathcal{H}_k$, $\mathcal{H}_k = \operatorname{Im} P_k$ (the matrix of this operator will be described in Theorem 8).

These results were announced in the short note [25].

§2. Construction of an admissible triple

The general outline of the method of similar operators (see [22]) requires to start with the construction of an admissible triple. In this section, we construct an admissible triple for an abstract operator with the properties most close to those of the operators $L_{\rm per}$ and L_{ap} in question. In the next section, this construction will be specified for the operators L_{bc}, $bc \in \{{\rm per}, ap\}$.

For a complex Banach space $\mathcal{X}$, let $\operatorname{End}\mathcal{X}$ denote the Banach algebra of all bounded linear operators on $\mathcal{X}$. Let $A : \mathrm{D}(A) \subset \mathcal{X} \to \mathcal{X}$ be a closed linear operator. We denote by $\mathfrak{L}_A(\mathcal{X})$ the Banach space of operators acting in $\mathcal{X}$ and subordinate to A. Thus, a linear operator $X\colon \mathrm{D}(X) \subset \mathcal{X} \to \mathcal{X}$ belongs to $\mathfrak{L}_A(\mathcal{X})$ if $\mathrm{D}(X) \supseteq \mathrm{D}(A)$ and the quantity $\|X\|_A = \inf\{C > 0 : \|Xx\| \le C(\|x\| + \|Ax\|), x \in \mathrm{D}(A)\}$ is finite. This quantity is taken for the norm on $\mathfrak{L}_A(\mathcal{X})$.

We recall the general notions of the method of similar operators (see [22, 23]).

Definition 1. Two linear operators $A_i\colon \mathrm{D}(A_i) \subset \mathcal{X} \to \mathcal{X}$ are said to be *similar* if there exists a continuously invertible operator $U \in \operatorname{End}\mathcal{X}$ such that $U\mathrm{D}(A_2) = \mathrm{D}(A_1)$ and $A_1Ux = UA_2x$, $x \in \mathrm{D}(A_2)$. In this case we say that U *transforms* A_1 to A_2.

It is well known (see [22, Lemma 1]) that many spectral properties of similar operators coincide.

Definition 2. Let $\mathfrak{U}$ be a linear subspace of $\mathfrak{L}_A(\mathcal{X})$, and let $J\colon \mathfrak{U} \to \mathfrak{U}$ and $\Gamma\colon \mathfrak{U} \to \operatorname{End}\mathcal{X}$ be transformes, i.e., linear operators acting on linear operators. The *triple* $(\mathfrak{U}, J, \Gamma)$ is said to be *admissible* for a (nonperturbed) operator $A : \mathrm{D}(A) \subset \mathcal{X} \to \mathcal{X}$ (then $\mathfrak{U}$ is called the *space of admissible perturbations*) if the following conditions are fulfilled:

1) $\mathfrak{U}$ is a Banach space (with its own norm $\|\cdot\|_*$) embedded in $\mathfrak{L}_A(\mathcal{X})$ continuously;
2) J and Γ are continuous transformes, moreover, J is a projection;
3) $(\Gamma X)\mathrm{D}(A) \subset \mathrm{D}(A)$, moreover, $A(\Gamma X) - (\Gamma X)A = X - JX$ for all $X \in \mathfrak{U}$;
4) $X\Gamma Y, (\Gamma X)Y \in \mathfrak{U}$ for all $X, Y \in \mathfrak{U}$, and there exists a constant $\gamma > 0$ such that

$$\|\Gamma\| \le \gamma,\ \max\{\|X\Gamma Y\|_*, \|(\Gamma X)Y\|_*\} \le \gamma\|X\|_*\|Y\|_*;$$

5) for every $X \in \mathfrak{U}$ and $\varepsilon > 0$, there exists $\lambda_\varepsilon \in \rho(A)$ with $\|X(A - \lambda_\varepsilon I)^{-1}\| < \varepsilon$.

Theorem 5 (see [22]). *Let $(\mathfrak{U}, J, \Gamma)$ be a triple admissible for an operator $A : \mathrm{D}(A) \subset \mathcal{X} \to \mathcal{X}$, and let B be an operator that belongs to the space $\mathfrak{U}$ of admissible perturbations for A. If $\|J\|\|B\|_*\|\Gamma\| < \frac{1}{4}$, then $A - B$ is similar to $A - JX_*$, where $X_* \in \mathfrak{U}$ is a solution of the* (*nonlinear*) *equation*

$$X = B\Gamma X - (\Gamma X)(JB) - (\Gamma X)J(B\Gamma X) + B = \Phi(X). \tag{2.1}$$

The solutions of (2.1) *can be found by the method of simple iterations, taking $X_0 = 0$, $X_1 = B$, and so on* (*the mapping $\Phi\colon \mathfrak{U} \to \mathfrak{U}$ is a contraction on the ball $\{X \in \mathfrak{U} : \|X - B\| \le 3\|B\|\}$*)*. A similarity transformation taking $A - B$ to $A - JX_*$ is done by the operator $I + \Gamma X_* \in \operatorname{End}\mathcal{X}$.*

In our case, we take a complex Hilbert space $\mathcal{H}$ for $\mathcal{X}$. Let $\mathfrak{S}_2(\mathcal{H})$ be the ideal of Hilbert–Schmidt operators belonging to $\operatorname{End}\mathcal{H}$ (see [26]). We recall the definition.

Definition 3. By a *Hilbert–Schmidt operator* $X \in \operatorname{End}\mathcal{H}$ we mean an operator satisfying $\sum_{j=0}^{\infty} \|Xf_j\|^2 < \infty$ for every orthonormal basis $f_0, f_1, \dots$ in $\mathcal{H}$.

Introducing the matrix (x_{kj}) of $X \in \operatorname{End}\mathcal{H}$ in the orthonormal basis $f_0, f_1, \dots$, i.e., $x_{kj} = (Xf_j, f_k)$, $k, j \ge 0$, we can rewrite the above inequality as $\sum_{k,j=0}^{\infty} |x_{kj}|^2 < \infty$. This quantity is a norm on the Hilbert–Schmidt ideal $\mathfrak{S}_2(\mathcal{H})$; it will be denoted by $\|\cdot\|_2$.

Let $A\colon \mathrm{D}(A) \subset \mathcal{H} \to \mathcal{H}$ be a selfadjoint operator with compact resolvent $R(\,\cdot\,, A) : \rho(A) \to \operatorname{End}\mathcal{H}$ and with spectrum $\sigma(A)$ formed by a sequence of eigenvalues $\lambda_{n,\theta}$, $n \in \mathbb{Z}_+$, of the form

$$\lambda_{n,\theta} = \pi^4(2n+\theta)^4, \quad n \in \mathbb{Z}_+,$$

where $\theta = 0$ if $A = \mathcal{L}^0_{\mathrm{per}}$ and $\theta = 1$ if $A = \mathcal{L}^0_{ap}$.

The eigenvalues of A possess the following property:

$$|\lambda_{k,\theta} - \lambda_{j,\theta}| \geq \frac{1}{c}|k^4 - j^4|, \quad |\lambda_{k,\theta}| \leq ck^4, \;\; k, j \geq 0, \;\; k \neq j, \tag{2.2}$$

where $c = (2\pi)^4$ for $\theta = 0$ and $c = (3\pi)^4$ for $\theta = 1$.

Let e_0, e_n^1, e_n^2, $n \in \mathbb{N}$, be an orthonormal basis. Denote by P_n, $n \in \mathbb{Z}_+$, the orthogonal projection corresponding to the set $\{\lambda_{n,\theta}\} \subset \sigma(A)$ and defined by the formula

$$P_n x = (x, e_n^1)e_n^1 + (x, e_n^2)e_n^2, \;\; n \in \mathbb{N}, \quad P_0 x = (x, e_0)e_0, \;\; n = 0, \;\text{ for } \theta = 0,$$

$$P_n x = (x, e_n^1)e_n^1 + (x, e_n^2)e_n^2, \;\; n \in \mathbb{Z}_+, \;\text{ for } \theta = 1.$$

We shall consider an operator $\mathcal{A}$ such that $\mathcal{A}P_n = AP_n = \lambda_{n,0}P_n$, $n \in \mathbb{N}$, $\mathcal{A}P_0 = P_0$, $\theta = 0$ and $\mathcal{A}P_n = AP_n = \lambda_{n,1}P_n$, $n \in \mathbb{Z}_+$, for $\theta = 1$.

Consider the operator matrix $(\mathcal{X}_{kj})$ whose entries are the operator blocks $\mathcal{X}_{kj} = P_k X P_j$, $k, j \in \mathbb{Z}_+$. Given an operator $X \in \mathfrak{L}_{\mathcal{A}}(\mathcal{H})$, we consider the blocks of this matrix separately in the case where $\theta = 0$.

If $k = j = 0$, then $(\mathcal{X}_{00})$ is a matrix of size 1×1 with only one entry $\mathcal{X}_{00} = (Xe_0, e_0)$ (see [23]). The entries of the matrix $(\mathcal{X}_{k0})$, $k \geq 1$, of size 2×1 are of the form

$$\begin{pmatrix} (Xe_0, e_k^1) \\ (Xe_0, e_k^2) \end{pmatrix}.$$

Accordingly, the entries of the matrix $(\mathcal{X}_{0j})$, $j \geq 1$, of size 1×2 are of the form $((Xe_j^1, e_0), (Xe_j^2, e_0))$.

Finally, since $\dim \operatorname{Im} P_n = 2$ for $n \in \mathbb{N}$, the matrix $(\mathcal{X}_{kj})$, k, $j \geq 1$, has the form

$$\mathcal{X}_{kj} = \begin{pmatrix} (Xe_j^1, e_k^1) & (Xe_j^2, e_k^1) \\ (Xe_j^1, e_k^2) & (Xe_j^2, e_k^2) \end{pmatrix}. \tag{2.3}$$

Note that for $\theta = 1$, the matrix $(\mathcal{X}_{kj})$, k, $j \geq 0$, is also of the form (2.3). In the subsequent estimates, we use the identities $\|X\|_2^2 = \sum_{k,j=0}^{\infty} \|P_k X P_j\|_2^2$.

We introduce the selfadjoint operator $\mathcal{A}^{\frac12} : \mathrm{D}(\mathcal{A}^{\frac12}) \subset \mathcal{H} \to \mathcal{H}$ by putting

$$\mathcal{A}^{\frac12} x = \sum_{n=0}^{\infty} \lambda_{n,\theta}^{\frac12} P_n x;$$

the domain of this operator is $\mathrm{D}(\mathcal{A}^{\frac12}) = \{x \in \mathcal{H} : \sum_{n=0}^{\infty} |\lambda_{n,\theta}| \|P_n x\|^2 < \infty\}$.

The Banach space $\mathfrak{U}$ of admissible perturbations will consist of all $X \in \mathfrak{L}_{\mathcal{A}}(\mathcal{H})$ representable in the form

$$X = X_0 \mathcal{A}^{\frac12}, \quad X_0 \in \mathfrak{S}_2(\mathcal{H}).$$

The norm of X in $\mathfrak{U}$ is defined to be the quantity $\|X\|_* = \|X_0\|_2$.

Next, in accordance with the general outline described in [23], we construct transformers $J, \Gamma\colon \mathfrak{L}_{\mathcal{A}}(\mathcal{H}) \to \mathfrak{L}_{\mathcal{A}}(\mathcal{H})$. First, we define them on $\mathfrak{S}_2(\mathcal{H})$.

Specifically, for every $X \in \mathfrak{S}_2(\mathcal{H})$ we put

$$JX = \sum_{n=0}^{\infty} P_n X P_n, \quad \Gamma X = \sum_{\substack{k,j=0 \\ k \neq j}}^{\infty} \frac{P_k X P_j}{\lambda_{k,\theta} - \lambda_{j,\theta}}. \tag{2.4}$$

The consistency of this definition and the boundedness of the operators JX and ΓX will be established in the following lemma.

Lemma 1. *The transformers J, Γ: $\mathfrak{S}_2(\mathcal{H}) \to \mathfrak{S}_2(\mathcal{H})$ are well defined, bounded, and possess the following properties*:

1) *J is a projection, $\|J\| = 1$;*
2) *we have $\|\Gamma\| \le \frac{1}{\inf_{k\neq j} |\lambda_{k,\theta}-\lambda_{j,\theta}|} \le \frac{c}{15}$, where c is defined in* (2.2).

Proof. We verify 1). By orthogonality, we obtain

$$\|(JX)x\|^2 = \Big\|\sum_{n=0}^{\infty}(P_nXP_n)x\Big\|^2 = \sum_{n=0}^{\infty}\|P_n(XP_nx)\|^2 = \sum_{n=0}^{\infty}\|P_nXP_nx\|^2 \le \|X\|^2\|x\|^2,$$

where $x \in \mathrm{D}(\mathcal{A})$ and $X \in \mathfrak{S}_2(\mathcal{H})$. Therefore, $\|JX\| \le \|X\|$, $X \in \mathfrak{S}_2(\mathcal{H})$. Thus, J is well defined, bounded, and $\|J\| \le 1$. Note that $\|J\| = 1$ if and only if X coincides with some P_n, $n \in \mathbb{Z}_+$.

We verify 2), i.e., the boundedness of Γ together with the consistency if its definition. We have

$$\|\Gamma X\|_2^2 = \Big\|\sum_{\substack{k,j=0\\k\neq j}}^{\infty}\frac{P_kXP_j}{\lambda_{k,\theta}-\lambda_{j,\theta}}\Big\|_2^2 \le \frac{1}{\inf_{k\neq j}|\lambda_{k,\theta}-\lambda_{j,\theta}|^2}\Big\|\sum_{\substack{k,j=0\\k\neq j}}^{\infty}P_kXP_j\Big\|_2^2$$
$$= \frac{1}{\inf_{k\neq j}|\lambda_{k,\theta}-\lambda_{j,\theta}|^2}\sum_{\substack{k,j=0\\k\neq j}}^{\infty}\|P_kXP_j\|_2^2 = \frac{1}{\inf_{k\neq j}|\lambda_{k,\theta}-\lambda_{j,\theta}|^2}\|X\|_2^2.$$

Thus,

$$\|\Gamma X\|_2 \le \frac{1}{\inf_{k\neq j}|\lambda_{k,\theta}-\lambda_{j,\theta}|}\|X\|_2 \le \frac{c}{15}\|X\|_2, \quad X \in \mathfrak{S}_2(\mathcal{H}).$$

Consequently, Γ is well defined, bounded, and satisfies

$$\|\Gamma\| \le \frac{1}{\inf_{k\neq j}|\lambda_{k,\theta}-\lambda_{j,\theta}|} \le \frac{c}{15}. \qquad \square$$

The extensions of the transformers J and Γ to the spaces $\mathfrak{L}_{\mathcal{A}}(\mathcal{H})$ and $\mathfrak{U}$ (denoted below by the same symbols) will be given as follows:

$$\begin{aligned} &JX = J(X\mathcal{A}^{-1})\mathcal{A}, \quad \Gamma X = (\Gamma X\mathcal{A}^{-1})\mathcal{A}, \quad X \in \mathfrak{L}_{\mathcal{A}}(\mathcal{H}),\\ &JX = J(X\mathcal{A}^{-\frac{1}{2}})\mathcal{A}^{\frac{1}{2}}, \quad \Gamma X = (\Gamma X\mathcal{A}^{-\frac{1}{2}})\mathcal{A}^{\frac{1}{2}}, \quad X \in \mathfrak{U}. \end{aligned} \tag{2.5}$$

Lemma 2. *Every operator ΓX, $X \in \mathfrak{U}$, admits an extension to the entire space $\mathcal{H}$ up to an operator belonging to $\mathfrak{S}_2(\mathcal{H})$ (and denoted by the same symbol ΓX) is such a way that*

$$\|\Gamma X\|_2 \le \frac{c^{\frac{3}{2}}}{3}\|X\|_*, \quad X \in \mathfrak{U}, \tag{2.6}$$

where c is taken from (2.2).

Proof. Since $X \in \mathfrak{U}$, we have $X = X_0\mathcal{A}^{\frac{1}{2}}$ with $X_0 \in \mathfrak{S}_2(\mathcal{H})$. Using (2.2), we obtain

$$\|\Gamma X\|_2^2 = \Big\|\sum_{\substack{k,j=0\\k\neq j}}^{\infty}\frac{P_kXP_j}{\lambda_{k,\theta}-\lambda_{j,\theta}}\Big\|_2^2 = \Big\|\sum_{\substack{k,j=0\\k\neq j}}^{\infty}\frac{(P_kX_0P_j)\lambda_{j,\theta}^{\frac{1}{2}}}{\lambda_{k,\theta}-\lambda_{j,\theta}}\Big\|_2^2$$
$$\le c^3\sup_{\substack{k,j\ge 0\\k\neq j}}\frac{j^4}{(k^4-j^4)^2}\sum_{\substack{k,j=0\\k\neq j}}^{\infty}\|P_kX_0P_j\|_2^2 \le c^3\sup_{\substack{k,j\ge 0\\k\neq j}}\frac{1}{(k^2-j^2)^2}\|X_0\|_2^2 \le \frac{c^3}{9}\|X\|_*^2.$$

Thus, ΓX is a Hilbert–Schmidt operator. Consequently, ΓX admits a bounded extension to $\mathcal{H}$ that satisfies (2.6). $\square$

Remark 2. In accordance with Lemma 2, the transformer Γ defined by (2.6) will be regarded as a linear operator from $\mathfrak{U}$ to $\mathfrak{S}_2(\mathcal{H})$ and will be denoted by the same symbol. Furthermore, Lemma 2 shows that $\|\Gamma\| \le \frac{c^{\frac{3}{2}}}{3}$.

For every $m \in \mathbb{N}$, we introduce two transformers $J_m : \mathfrak{U} \to \mathfrak{U}$ and $\Gamma_m : \mathfrak{U} \to \mathfrak{S}_2(\mathcal{H})$ in the following way:

$$J_m X = JX - J(P_{(m)} X P_{(m)}) + P_{(m)} X P_{(m)}, \quad X \in \mathfrak{U}, \tag{2.7}$$

$$\Gamma_m X = \Gamma X - P_{(m)}(\Gamma X) P_{(m)}, \quad X \in \mathfrak{U}, \tag{2.8}$$

where $P_{(m)} = \sum_{k \le m} P_k$. Note that $J_1 X = JX$ and $\Gamma_1 X = \Gamma X$, $X \in \mathfrak{U}$.

The next statement is an immediate consequence of the definition of J_m and Γ_m, Lemma 2, and Remark 2.

Lemma 3. *All transformers J_m, Γ_m, $m \in \mathbb{N}$, admit a bounded extension to $\mathfrak{L}_{\mathcal{A}}(\mathcal{H})$* (*consequently, to* $\mathfrak{U}$). *Also, we have*

$$\|J_m\| = 1, \quad \|\Gamma_m\| \le \frac{c^{\frac{3}{2}}}{m},$$

where c was defined in (2.2).

Remark 3. Formulas (2.7) and (2.8) show directly that ΓX (respectively, JX), $X \in \mathfrak{U}$, differs from $\Gamma_m X$ (respectively, from $J_m X$) by the finite rank operator $P_{(m)}(\Gamma X)P_{(m)}$ (respectively, $P_{(m)}(JX)P_{(m)}$). Therefore, in what follows we shall verify all required properties for ΓX and JX.

Now we show that $(\mathfrak{U}, J_m, \Gamma_m)$ is an admissible triple.

Lemma 4. *The triple $(\mathfrak{U}, J_m, \Gamma_m)$ is admissible for $\mathcal{A}$; moreover, the quantity $\gamma = \gamma_m$* (*see the definition of an admissible triple*) *obeys the inequality $\gamma_m \le \frac{c^{\frac{3}{2}}}{m}$, where c was defined in* (2.2).

Proof. We verify all properties of an admissible triple. The first two properties follow from the representation of the space of admissible perturbations, Lemma 3, and formulas (2.7) and (2.8).

We prove property 3), i.e., the relation $(\Gamma_m X)\mathrm{D}(\mathcal{A}) \subset \mathrm{D}(\mathcal{A})$ for every $X \in \mathfrak{U}$. By Remark 3, we can consider $\Gamma_m X$ in place of ΓX. We represent X in the form $X = X_0 \mathcal{A}^{\frac{1}{2}}$, where $X_0 \in \mathfrak{S}_2(\mathcal{H})$. Take an arbitrary vector $x \in \mathrm{D}(\mathcal{A})$, then $x = \mathcal{A}^{-1} y$, where $y \in \mathcal{H}$. We have

$$(\Gamma X)\mathcal{A}^{-1} y = \sum_{\substack{k,j=0 \\ k \ne j}}^{\infty} \frac{(P_k X P_j)\mathcal{A}^{-1} y}{\lambda_{k,\theta} - \lambda_{j,\theta}} = \sum_{\substack{k,j=0 \\ k \ne j}}^{\infty} \frac{(P_k X_0 \mathcal{A}^{\frac{1}{2}} P_j) y}{(\lambda_{k,\theta} - \lambda_{j,\theta})\lambda_{j,\theta}} = \sum_{\substack{k,j=0 \\ k \ne j}}^{\infty} \frac{(P_k X_0 P_j) y}{(\lambda_{k,\theta} - \lambda_{j,\theta})\lambda_{j,\theta}^{\frac{1}{2}}}.$$

By (2.2) and the inequality $\sup_{k,j\ge 1 k\ne j} \frac{k^2}{(k-j)^2 j^4} \le 4$, it follows that

$$\left\|\mathcal{A}(\Gamma X)\mathcal{A}^{-1} y\right\|^2 = \left\|\mathcal{A} \sum_{\substack{k,j=0 \\ j \ne k}}^{\infty} \frac{(P_k X_0 P_j) y}{(\lambda_{k,\theta} - \lambda_{j,\theta})\lambda_{j,\theta}^{\frac{1}{2}}}\right\|^2 = \left\| \sum_{\substack{k,j=0 \\ j \ne k}}^{\infty} \frac{\lambda_{k,\theta}(P_k X_0 P_j) y}{(\lambda_{k,\theta} - \lambda_{j,\theta})\lambda_{j,\theta}^{\frac{1}{2}}}\right\|^2$$

$$\le c^3 \left\| \sum_{\substack{k,j=0 \\ j \ne k}}^{\infty} \frac{k^4 (P_k X_0 P_j) y}{(k^4 - j^4) j^2} \right\|^2 \le c^3 \sup_{\substack{k,j \ge 1 \\ k \ne j}} \frac{k^8}{(k^4 - j^4)^2 j^4} \sum_{k,j=1}^{\infty} \|(P_k X_0 P_j) y\|^2$$

$$\le c^3 \sup_{\substack{k,j \ge 1 \\ k \ne j}} \frac{k^2}{(k-j)^2 j^4} \|X_0\|_2^2 \|y\|^2 \le 4c^3 \|X_0\|_2^2 \|y\|^2.$$

Thus, $(\Gamma X)\mathcal{A}^{-1}x \in \mathrm{D}(\mathcal{A})$, and the above estimates show that the operator $\mathcal{A}(\Gamma X)\mathcal{A}^{-1}$ is bounded. Consequently, $(\Gamma X)\mathrm{D}(\mathcal{A}) \subset \mathrm{D}(\mathcal{A})$, whence $(\Gamma_m X)\mathrm{D}(\mathcal{A}) \subset \mathrm{D}(\mathcal{A})$. It remains to show that the operators $\mathcal{A}(\Gamma_m X) - (\Gamma_m X)\mathcal{A}$ and $X - J_m X$ possess equal matrices. For $k \neq j$, we have

$$\left(\frac{\lambda_{k,\theta}\widetilde{x}_{kj}(1-\delta_{kj})}{\lambda_{k,\theta}-\lambda_{j,\theta}}\right)-\left(\frac{\widetilde{x}_{kj}(1-\delta_{kj})\lambda_{j,\theta}}{\lambda_{k,\theta}-\lambda_{j,\theta}}\right)=\left(\frac{\widetilde{x}_{kj}(\lambda_{k,\theta}-\lambda_{j,\theta})(1-\delta_{kj})}{\lambda_{k,\theta}-\lambda_{j,\theta}}\right)=\widetilde{x}_{kj}-\delta_{kj}\widetilde{x}_{kj},$$

where $(\widetilde{x}_{kj})$ is the matrix of X, and property 3) follows.

We verify property 4). Taking X, $Y \in \mathfrak{U}$, we write them in the form $X = X_0\mathcal{A}^{\frac{1}{2}}$ and $Y = Y_0\mathcal{A}^{\frac{1}{2}}$. Then $X\Gamma_m Y$ can be written as $X\Gamma_m Y = X_0\mathcal{A}^{\frac{1}{2}}\Gamma_m Y_0\mathcal{A}^{\frac{1}{2}} = Z_0\mathcal{A}^{\frac{1}{2}}$. We show that Z_0 is a Hilbert–Schmidt operator. By Lemma 3, it suffices to prove this for some $m \in \mathbb{N}$. Using (2.2), we obtain

$$\begin{aligned}\|Z_0\|_2^2 = \|X_0\mathcal{A}^{\frac{1}{2}}\Gamma Y_0\|_2^2 &= \left\| X_0\left(\sum_{\substack{k,j=m\\k\neq j}}^{\infty}\frac{(P_kY_0P_j)\lambda_{j,\theta}^{\frac{1}{2}}}{\lambda_{k,\theta}-\lambda_{j,\theta}}\right)\right\|_2^2\\
&\le c^3\sup_{\substack{k,j\ge m\\k\neq j}}\frac{j^4}{(k^4-j^4)^2}\left\|X_0\sum_{k,j=m}^{\infty}P_kY_0P_j\right\|_2^2\\
&\le c^3\sup_{\substack{k,j\ge m\\k\neq j}}\frac{1}{(k^2-j^2)^2}\|X_0\|_2^2\sum_{k,j=m}^{\infty}\|P_kY_0P_j\|_2^2 \le \frac{c^3\|X\|_*^2\|Y\|_*^2}{m^2},\end{aligned}$$

where c is taken from (2.2). Therefore, $Z_0 \in \mathfrak{S}_2(\mathcal{H})$, and the operator $X\Gamma_m Y$ belongs to the space $\mathfrak{U}$ of admissible perturbations, moreover, $\|X\Gamma_m Y\|_* \le \frac{c^{\frac{3}{2}}}{m}\|X\|_*\|Y\|_*$. A similar argument yields a similar estimate for $(\Gamma_m X)Y$. Now, a direct calculation shows that $\|\Gamma_m\| \le \frac{c^{\frac{3}{2}}}{m}$.

We verify the last property of admissible triples. Let $X = X_0\mathcal{A}^{\frac{1}{2}}$ be an arbitrary operator in $\mathfrak{U}$, and let $\varepsilon > 0$. For the role of λ_ε, we take $-cn$, $n \in \mathbb{N}$, where $c > 0$ is the quantity from (2.2) and $n \in \mathbb{N}$ satisfies $\frac{1}{2}c^{\frac{3}{2}}n^{-\frac{1}{2}}\|X_0\|_2 < \varepsilon$. Then

$$\begin{aligned}\|X(\mathcal{A}-\lambda_\varepsilon I)^{-1}\| \le \|X_0\|_2\|\mathcal{A}^{\frac{1}{2}}(\mathcal{A}-\lambda_\varepsilon I)^{-1}\| &= \|X_0\|_2\max_{k\ge 1}\frac{|\lambda_{k,\theta}^{\frac{1}{2}}|}{|\lambda_{k,\theta}-\lambda_\varepsilon|}\\
&\le c^{\frac{3}{2}}\|X_0\|_2\max_{k\ge1}\frac{k^2}{k^4+n} \le \frac{\|X_0\|_2c^{\frac{3}{2}}}{2n^{\frac{1}{2}}} < \varepsilon.\end{aligned}$$

Thus, $(\mathfrak{U}, J_m, \Gamma_m)$ is an admissible triple, and the lemma follows. □

§3. A preliminary similarity transformation

We return to the study of the operator L_{bc} defined in the Introduction. We apply the abstract method described in the preceding section to the study of the spectral properties of the operators L_{bc}, $bc \in \{\mathrm{per}, ap\}$. The role of $\mathcal{A}$ will be played by the operators L^0_{per} and L^0_{ap} defined as follows:

$$\begin{aligned}&L^0_{\mathrm{per}}P_n = \mathcal{L}^0_{\mathrm{per}}P_n = \lambda_nP_n,\ \ n\in\mathbb{N}, \quad L^0_{\mathrm{per}}P_0 = \mathcal{L}^0_{\mathrm{per}}P_0 = P_0,\\
&L^0_{ap}P_n = \mathcal{L}^0_{ap}P_n = \lambda_nP_n,\ \ n\in\mathbb{Z}_+,\end{aligned}$$

where the orthogonal projections P_n, $n \in \mathbb{Z}_+$, are those described in the Introduction. Now, L^0_{per} and L^0_{ap} are selfadjoint operators with compact resolvent and with eigenvalues satisfying (2.2), where $c = (2\pi)^4$ in the case of L^0_{per} and $c = (3\pi)^4$ in the case of L^0_{ap}. In what follows, we put $\mathcal{H} = \mathrm{L}_2[0,1]$ throughout, and we identify this space with the space $\mathrm{L}_2(\mathbb{R},\mathbb{C})$ of 1-periodic functions on $\mathbb{R}$ that are square integrable on $[0,1]$.

The perturbation B described in the Introduction belongs to the space $\mathfrak{L}_{L^0_{bc}}(\mathcal{H})$, $bc \in \{\mathrm{per}, ap\}$. Consequently, the operators JB, ΓB, $J_m B$, and $\Gamma_m B$ defined by (2.4), (2.7), (2.8) are well defined.

Since B does not belong to the space of admissible perturbations constructed in the preceding section, we must do a preliminary similarity transformation (see [22]) of L_{bc} to the operator $\widetilde{L}_{bc} = L^0_{bc} - \widetilde{B}$, $bc \in \{\mathrm{per}, ap\}$, where $\widetilde{B}$ belongs to $\mathfrak{U}$. This is our objective in this section.

First, we consider the operator $B = B_1 + B_2$, where $B_1 y = ay''$ and $B_2 y = by$, $y \in \mathrm{D}(L^0_{bc})$, $bc \in \{\mathrm{per}, ap\}$ and a, $b \in \mathcal{H}$. We represent the perturbation B in the form

$$B = \big(B(L^0_{bc})^{-\frac{1}{2}}\big)(L^0_{bc})^{\frac{1}{2}} = \big(B_1(L^0_{bc})^{-\frac{1}{2}}\big)(L^0_{bc})^{\frac{1}{2}} + \big(B_2(L^0_{bc})^{-\frac{1}{2}}\big)(L^0_{bc})^{\frac{1}{2}}.$$

Since a and b belong to $\mathcal{H}$, we have

$$a(t) = \sum_{l\in\mathbb{Z}} a_l \mathrm{e}^{2i\pi l t}, \quad b(t) = \sum_{l\in\mathbb{Z}} b_l \mathrm{e}^{2i\pi l t}.$$

Consider the case of $bc = \mathrm{per}$. The numerical block matrices $(\mathfrak{A}^{\mathrm{per}}_{kj})$, $(\mathfrak{B}^{\mathrm{per}}_{kj})$, $k, j \geq 0$, are defined in the same way as in the preceding section (in particular, formula (2.3) is true). We calculate the entries of these matrices. For the operator of multiplication by a, the entries are found in the following way:

$$\big(ae^1_j, e^1_k\big) = \int_0^1 a(t)e^1_j(t)\overline{e^1_k(t)}\,\mathrm{d}t = \int_0^1 \sum_{l\in\mathbb{Z}} a_l \mathrm{e}^{2i\pi l t}\cdot \mathrm{e}^{i2\pi(k-j)t}\,\mathrm{d}t = a_{j-k}.$$

Arguing similarly, we obtain

$$\big(ae^2_j, e^1_k\big) = a_{-j-k}, \quad \big(ae^1_j, e^2_k\big) = a_{j+k}, \quad \big(ae^2_j, e^2_k\big) = a_{-j+k}.$$

If $k = 0$, $j \geq 1$, or $k \geq 1$, $j = 0$, we have

$$(ae_0, e_0) = 0, \quad (ae_0, e^1_k) = 0, \quad (ae_0, e^2_k) = 0,$$
$$(ae^1_j, e_0) = a_j, \quad (ae^2_j, e_0) = a_{-j}.$$

Thus, the matrix of B_1 has the following form:

$$\mathfrak{A}^{\mathrm{per}}_{kj} = -(2\pi j)^2 \begin{pmatrix} 0 & a_j & a_{-j} \\ 0 & a_{j-k} & a_{-j-k} \\ 0 & a_{j+k} & a_{-j+k} \end{pmatrix}, \quad k, j \geq 1. \tag{3.1}$$

Since B_2 is multiplication by a function b in $\mathcal{H}$, the matrix $(\mathfrak{B}^{\mathrm{per}}_{kj})$, k, $j \geq 1$ looks like this:

$$\mathfrak{B}^{\mathrm{per}}_{kj} = \begin{pmatrix} 0 & b_j & b_{-j} \\ b_{-k} & b_{j-k} & b_{-j-k} \\ b_k & b_{j+k} & b_{-j+k} \end{pmatrix}. \tag{3.2}$$

The matrix entries of $\mathfrak{U}^{ap}_{kj}$ for $bc = ap$ are calculated similarly. The matrices $(\mathfrak{A}^{ap}_{kj})$, $(\mathfrak{B}^{ap}_{kj})$, k, $j \geq 0$, have the following form:

$$\begin{aligned} \mathfrak{A}^{ap}_{kj} &= -(\pi(2j+1))^2 \begin{pmatrix} a_{j-k} & a_{-j-k-1} \\ a_{j+k+1} & a_{-j+k} \end{pmatrix}, \\ \mathfrak{B}^{ap}_{kj} &= \begin{pmatrix} b_{j-k} & b_{-j-k-1} \\ b_{j+k+1} & b_{-j+k} \end{pmatrix}. \end{aligned} \tag{3.3}$$

Next, we prove several technical lemmas.

Lemma 5. *The operators* ΓB *and* $\Gamma_m B$, $m \in \mathbb{N}$, *are Hilbert–Schmidt, and moreover, we have* $\lim_{m\to\infty} \|\Gamma_m B\|_2^2 = 0$.

Proof. We show that ΓB is a Hilbert–Schmidt operator. Since B_2 is multiplication by a function b in $\mathcal{H}$, it suffices to consider ΓB_1 in place of ΓB. First, we consider the case of $bc = \text{per}$. By (3.1), we have

$$\begin{aligned}
\|\Gamma B_1\|_2^2 &= \sum_{j=1}^{\infty} |(\Gamma B_1 e_j^1, e_0)|^2 + \sum_{j=1}^{\infty} |(\Gamma B_1 e_j^2, e_0)|^2 + \sum_{k,j=1}^{\infty} \left|(\Gamma B_1 e_j^1, e_k^1)\right|^2 \\
&\quad + \sum_{k,j=1}^{\infty} \left|(\Gamma B_1 e_j^1, e_k^2)\right|^2 + \sum_{k,j=1}^{\infty} \left|(\Gamma B_1 e_j^2, e_k^1)\right|^2 + \sum_{k,j=1}^{\infty} \left|(\Gamma B_1 e_j^2, e_k^2)\right|^2 \\
&\le \sum_{j=1}^{\infty} \frac{|a_j|^2 (2\pi j)^4}{(2\pi j)^8} + \sum_{j=1}^{\infty} \frac{|a_{-j}|^2 (2\pi j)^4}{(2\pi j)^8} + \sum_{\substack{k,j=1\\k\neq j}}^{\infty} \frac{|a_{j-k}|^2 (2\pi j)^4}{((2\pi k)^4 - (2\pi j)^4)^2} \\
&\quad + \sum_{\substack{k,j=1\\k\neq j}}^{\infty} \frac{|a_{-j-k}|^2 (2\pi j)^4}{((2\pi k)^4 - (2\pi j)^4)^2} + \sum_{\substack{k,j=1\\k\neq j}}^{\infty} \frac{|a_{j+k}|^2 (2\pi j)^4}{((2\pi k)^4 - (2\pi j)^4)^2} + \sum_{\substack{k,j=1\\k\neq j}}^{\infty} \frac{|a_{-j+k}|^2 (2\pi j)^4}{((2\pi k)^4 - (2\pi j)^4)^2} \\
&\le \frac{1}{(2\pi)^4} \left(\sum_{j=1}^{\infty} \frac{|a_j|^2 + |a_{-j}|^2}{j^4} + \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{\substack{j=1\\j\neq k}}^{\infty} \frac{|a_{j-k}|^2}{(k-j)^2} \right. \\
&\quad \left. + \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{\substack{j=1\\j\neq k}}^{\infty} \frac{|a_{-j-k}|^2}{(k-j)^2} + \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{\substack{j=1\\j\neq k}}^{\infty} \frac{|a_{j+k}|^2}{(k-j)^2} + \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{\substack{j=1\\j\neq k}}^{\infty} \frac{|a_{-j+k}|^2}{(k-j)^2} \right) < \infty.
\end{aligned}$$

Consequently, $\Gamma B \in \mathfrak{S}_2(\mathcal{H})$. By Remark 3, the $\Gamma_m B$, $m \in \mathbb{N}$, are also Hilbert–Schmidt operators.

Similar arguments based on (3.3) show that $\Gamma B \in \mathfrak{S}_2(\mathcal{H})$ for $bc = ap$. Thus, $\Gamma_m B$ belongs to $\mathfrak{S}_2(\mathcal{H})$ also in this case.

Next, from (2.8) we deduce that

$$\begin{aligned}
\lim_{m\to\infty} \|\Gamma_m B\|_2^2 &= \lim_{m\to\infty} \|\Gamma B - P_{(m)}(\Gamma B) P_{(m)}\|_2^2 \\
&= \lim_{m\to\infty} \sum_{\max\{k,j\}\ge m+1}^{\infty} \|P_k(\Gamma_m B)P_j\|_2^2 = 0.
\end{aligned}$$

□

Lemma 6. *The operator* $J_m B$ *belongs to the space* $\mathfrak{U}$ *of admissible perturbations.*

Proof. By Remark 3, we may consider JB instead. We represent it in the form

$$JB = (JB(L_{bc}^0)^{-\frac{1}{2}})(L_{bc}^0)^{\frac{1}{2}} = B_{JB}(L_{bc}^0)^{\frac{1}{2}}$$

and show that B_{JB} belongs to $\mathfrak{S}_2(\mathcal{H})$. Using (3.1) and (3.2), for $bc = \text{per}$ we obtain

$$\begin{aligned}
\|JB(L_{\text{per}}^0)^{-\frac{1}{2}}\|_2^2 &= |(JB(L_{\text{per}}^0)^{-\frac{1}{2}} e_0, e_0)|^2 + \sum_{n=1}^{\infty} |(JB(L_{\text{per}}^0)^{-\frac{1}{2}} e_n^1, e_n^1)|^2 \\
&\quad + \sum_{n=1}^{\infty} |(JB(L_{\text{per}}^0)^{-\frac{1}{2}} e_n^1, e_n^2)|^2 + \sum_{n=1}^{\infty} |(JB(L_{\text{per}}^0)^{-\frac{1}{2}} e_n^2, e_n^1)|^2 + \sum_{n=1}^{\infty} |(JB(L_{\text{per}}^0)^{-\frac{1}{2}} e_n^2, e_n^2)|^2 \\
&= \sum_{n=1}^{\infty} \frac{|a_{2n}|^2 (2\pi n)^2}{(2\pi n)^2} + \sum_{n=1}^{\infty} \frac{|a_{-2n}|^2 (2\pi n)^2}{(2\pi n)^2} + 2|a_0|^2 + \sum_{n=1}^{\infty} \frac{|b_{2n}|^2}{(2\pi n)^2} + \sum_{n=1}^{\infty} \frac{|b_{-2n}|^2}{(2\pi n)^2} < \infty.
\end{aligned}$$

Arguing similarly on the basis of (3.3), we analyze also the case of $bc = ap$. Thus, $B_{JB} \in \mathfrak{S}_2(\mathcal{H})$. Therefore, the $J_m B$, $m \in \mathbb{N}$, belong to $\mathfrak{U}$. □

Lemma 7. *The operators $B\Gamma_m B$ and $(\Gamma_m B)J_m B$ belong to the space $\mathfrak{U}$ of admissible perturbations.*

Proof. In accordance with Remark 3, first we prove that $B\Gamma B \in \mathfrak{U}$. For this, we write

$$B\Gamma B = (B\Gamma B(L^0_{bc})^{-\frac{1}{2}})(L^0_{bc})^{\frac{1}{2}} = B_0(L^0_{bc})^{\frac{1}{2}}$$

and prove that $B_0 \in \mathfrak{S}_2(\mathcal{H})$. In its turn, B_0 can be represented in the form

$$B_0 = B_1\Gamma B_1(L^0_{bc})^{-\frac{1}{2}} + B_1\Gamma B_2(L^0_{bc})^{-\frac{1}{2}} + B_2\Gamma B_1(L^0_{bc})^{-\frac{1}{2}} + B_2\Gamma B_2(L^0_{bc})^{-\frac{1}{2}}.$$

We prove that $\|B_1\Gamma B_1(L^0_{bc})^{-\frac{1}{2}}\|_2^2 < \infty$. First, we consider the case of $bc = \mathrm{per}$. By (3.1) and (3.2), we have

$$\begin{aligned}\|B_1\Gamma B_1(L^0_{\mathrm{per}})^{-\frac{1}{2}}\|_2^2 &= \sum_{k,j=1}^{\infty} |(B_1\Gamma B_1(L^0_{\mathrm{per}})^{-\frac{1}{2}}e^1_j, e^1_k)|^2 \\ &+ \sum_{k,j=1}^{\infty} |(B_1\Gamma B_1(L^0_{\mathrm{per}})^{-\frac{1}{2}}e^2_j, e^1_k)|^2 + \sum_{k,j=1}^{\infty} |(B_1\Gamma B_1(L^0_{\mathrm{per}})^{-\frac{1}{2}}e^1_j, e^2_k)|^2 \\ &+ \sum_{k,j=1}^{\infty} |(B_1\Gamma B_1(L^0_{\mathrm{per}})^{-\frac{1}{2}}e^2_j, e^2_k)|^2 + \sum_{j=1}^{\infty} |(B_1\Gamma B_1(L^0_{\mathrm{per}})^{-\frac{1}{2}}e^1_j, e_0)|^2 \\ &+ \sum_{j=1}^{\infty} |(B_1\Gamma B_1(L^0_{\mathrm{per}})^{-\frac{1}{2}}e^2_j, e_0)|^2.\end{aligned}$$

We estimate the first summand (the remaining ones are treated similarly):

$$\begin{aligned}&\sum_{k,j=1}^{\infty} |(B_1\Gamma B_1(L^0_{\mathrm{per}})^{-\frac{1}{2}}e^1_j, e^1_k)|^2 \\ &= \sum_{k,j=1}^{\infty} \left| \sum_{\substack{l=1\\ l\neq j}}^{\infty} \frac{(a_{l-k}a_{j-l} + a_{-l-k}a_{j+l})(2\pi j)^2(2\pi l)^2}{((2\pi l)^4 - (2\pi j)^4)(2\pi j)^2} \right|^2 \\ &\le \frac{1}{8\pi^4}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\Bigg(\sum_{\substack{l=1\\ l\neq j}}^{\infty} \frac{|a_{l-k}|\,|a_{j-l}|}{|l^2-j^2|}\Bigg)^2 + \frac{1}{8\pi^4}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\Bigg(\sum_{\substack{l=1\\ l\neq j}}^{\infty} \frac{|a_{-l-k}|\,|a_{j+l}|}{|l^2-j^2|}\Bigg)^2.\end{aligned}$$

We estimate only one of the two terms on the right (the other one is estimated in the same way). For definiteness, we choose the term $\frac{1}{8\pi^4}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\big(\sum_{l=1, l\neq j}^{\infty}\frac{|a_{l-k}|\,|a_{j-l}|}{|l^2-j^2|}\big)^2$ and denote it by γ_1. Consider the sequences $f_j\colon \mathbb{N} \to \mathbb{R}_+ = [0,\infty)$, $j \ge 1$, of the form $f_j(l) = \frac{|a_{j-l}|}{|l^2-j^2|}$ for $l \neq j$ and 0 for $l = j$ and estimate their norms in ℓ^1:

$$\|f_j\|_{\ell^1} = \sum_{\substack{l=1\\ l\neq j}}^{\infty}|f_j(l)| = \sum_{\substack{l=1\\ l\neq j}}^{\infty}\frac{|a_{j-l}|}{|l^2-j^2|} \le \frac{\|a\|_{\ell^2}}{j}\left(2\sum_{l=1}^{\infty}\frac{1}{l^2}\right)^{\frac{1}{2}} = \frac{\|a\|_{\ell^2}\pi}{j\sqrt{3}}, \quad j \ge 1.$$

Since the sequences $k \mapsto |a_{|k-l|}|\colon \mathbb{N} \to \mathbb{R}_+$, $l \ge 1$, denoted by $\widetilde{a}_l$ in the sequel, belong to ℓ^2, we have

$$\Big\|\sum_{l=1}^{\infty}\widetilde{a}_l f_j(l)\Big\|_{\ell^2} \le \sum_{l=1}^{\infty}\|\widetilde{a}_l\|_{\ell^2}|f_j(l)| \le \|a\|_{\ell^2}\sum_{l=1}^{\infty}|f_j(l)| \le \frac{\|a\|^2_{\ell^2}\pi}{j\sqrt{3}}.$$

Consequently,

$$\gamma_1 = \frac{1}{8\pi^4}\sum_{j=1}^{\infty}\Big\|\sum_{l=1}^{\infty}\widetilde{a}_l f_j(l)\Big\|^2_{\ell^2} \le \frac{\|a\|^4_{\ell^2}}{24\pi^2}\sum_{j=1}^{\infty}\frac{1}{j^2} = \frac{\|a\|^4_{\ell^2}}{144}.$$

Similar arguments and calculations are applicable in the case where $bc = ap$. Therefore, $B_1\Gamma B_1(L^0_{bc})^{-\frac{1}{2}}$ belongs to $\mathfrak{S}_2(\mathcal{H})$. Since B_2 is multiplication by a function b in $\mathcal{H}$, the operators $B_1\Gamma B_2(L^0_{bc})^{-\frac{1}{2}}$, $B_2\Gamma B_1(L^0_{bc})^{-\frac{1}{2}}$, $B_2\Gamma B_2(L^0_{bc})^{-\frac{1}{2}}$ also belong to $\mathfrak{S}_2(\mathcal{H})$. Consequently, $B\Gamma B$ belongs to $\mathfrak{U}$, and so $B\Gamma_m B \in \mathfrak{U}$.

It remains to prove that $(\Gamma_m B)J_m B \in \mathfrak{U}$. By Lemma 5, the operator $\Gamma_m B$ belongs to $\mathfrak{S}_2(\mathcal{H})$, and Lemma 6 shows that $J_m B$ belongs to $\mathfrak{U}$. Since the product of two Hilbert–Schmidt operators is of trace class (see [26]), we see that $(\Gamma_m B)J_m B$ belongs to $\mathfrak{U}$. □

Lemma 8. *There exists $m \in \mathbb{N}$ such that the operators B, $J_m B$, $\Gamma_m B$ satisfy the following conditions*:

(a) $\Gamma_m B \in \operatorname{End}\mathcal{H}$ *and* $\|\Gamma_m B\|_2 < 1$;
(b) $(\Gamma_m B)\mathrm{D}(L^0_{bc}) \subset \mathrm{D}(L^0_{bc})$;
(c) $B\Gamma_m B, (\Gamma_m B)J_m B \in \mathfrak{U}$, *where* $\mathfrak{U}$ *is the space of admissible perturbations*;
(d) $L^0_{bc}(\Gamma_m B)x - (\Gamma_m B)L^0_{bc}x = Bx - (J_m B)x$, $x \in \mathrm{D}(L^0_{bc})$;
(e) *for every* $\varepsilon > 0$ *there exists* $\lambda_\varepsilon \in \rho(L^0_{bc})$ *with* $\|B(L^0_{bc} - \lambda_\varepsilon I)^{-1}\| < \varepsilon$.

Proof. Lemma 5 shows that $\Gamma_m B \in \mathfrak{S}_2(\mathcal{H}) \subset \operatorname{End}\mathcal{H}$; moreover, by (2.8), $\|\Gamma_m B\|_2 < 1$ for sufficiently large $m \in \mathbb{N}$. Thus, (a) is fulfilled.

To verify (b) for ΓB (see Remark 3), we argue as in the proof of property 3 in Lemma 4. Thus, $(\Gamma_m B)\mathrm{D}(L^0_{bc}) \subset \mathrm{D}(L^0_{bc})$.

Property (c) follows from Lemma 7.

To verify (d), we argue as in the proof of property 4 in Lemma 4. Furthermore, by (2.7) and (2.8), for $x \in \mathrm{D}(L^0_{bc})$ we have

$$\begin{aligned} L^0_{bc}(\Gamma_m B)x &= L^0_{bc}\Gamma Bx - L^0_{bc}P_{(m)}(\Gamma B)P_{(m)} = L^0_{bc}\Gamma Bx - P_{(m)}(L^0_{bc}\Gamma B)P_{(m)} \\ &= (B - JB)x + (\Gamma B)L^0_{bc}x - P_{(m)}(B - JB)P_{(m)}x - P_{(m)}(\Gamma B)L^0_{bc}P_{(m)}x \\ &= (B - J_m B)x + (\Gamma_m B)L^0_{bc}x. \end{aligned}$$

Thus, (d) follows.

It remains to establish (e). Consider the case of $bc = \mathrm{per}$. For arbitrary $\varepsilon > 0$, we choose $n \in \mathbb{N}$ in such a way that

$$\left(\frac{4\|a\|^2_{\ell^2}}{3} + \frac{\|b\|^2_{\ell^2}}{1890}\right)^{\frac{1}{2}} \frac{3^{\frac{3}{4}}}{4\pi n^{\frac{1}{4}}} < \varepsilon. \tag{3.4}$$

Also, we take $-\pi^4 n$ for the role of λ_ε. Direct inspection shows that $B(L^0_{\mathrm{per}})^{-\frac{3}{4}}$ is bounded and $\|B(L^0_{\mathrm{per}})^{-\frac{3}{4}}\|^2_2 \le \frac{4\|a\|^2_{\ell^2}}{3} + \frac{\|b\|^2_{\ell^2}}{1890}$. Then for every $n \in \mathbb{N}$ satisfying (3.4) we have

$$\begin{aligned} \|B(L^0_{\mathrm{per}} - \lambda_\varepsilon I)^{-1}\| &\le \|B(L^0_{\mathrm{per}})^{-\frac{3}{4}}\|_2 \max_{k\ge1} \frac{\lambda_k^{\frac{3}{4}}}{|\lambda_k - \lambda_\varepsilon|} \\ &\le \left(\frac{4\|a\|^2_{\ell^2}}{3} + \frac{\|b\|^2_{\ell^2}}{1890}\right)^{\frac{1}{2}} \frac{1}{\pi} \max_{k\ge1} \frac{k^3}{k^4 + n} \le \left(\frac{4\|a\|^2_{\ell^2}}{3} + \frac{\|b\|^2_{\ell^2}}{1890}\right)^{\frac{1}{2}} \frac{3^{\frac{3}{4}}}{4\pi n^{\frac{1}{4}}} < \varepsilon. \end{aligned}$$

This proves (e) for L^0_{per}, and the case of $bc = ap$ is treated similarly. □

Theorem 6. *If $m \in \mathbb{N}$ satisfies*

$$\|\Gamma_m B\|_2 < 1, \tag{3.5}$$

then the operator $L_{bc} = L^0_{bc} - B$ is similar to the operator $\widetilde{L}_{bc} = L^0_{bc} - \widetilde{B}$, where

$$\widetilde{B} = J_m B_1 + (I + \Gamma_m B)^{-1}(B_1\Gamma_m B_1 - (\Gamma_m B_1)J_m B_1) + \widetilde{C}. \tag{3.6}$$

The operator $\widetilde{C}$ *is defined by* $\widetilde{C} = J_m B_2 + (I+\Gamma_m B)^{-1}(B_1\Gamma_m B_2 + B_2\Gamma_m B_1 + B_2\Gamma_m B_2 - (\Gamma_m B_1)J_m B_2 - (\Gamma_m B_2)J_m B_1 - (\Gamma_m B_2)J_m B_2)$*, and moreover,*

$$(L^0_{bc} - B)(I + \Gamma_m B) = (I + \Gamma_m B)(L^0_{bc} - \widetilde{B}). \tag{3.7}$$

The operator $\widetilde{B}$ *in* (3.7) *is representable in the form*

$$\widetilde{B} = JB_1 + B_1\Gamma B_1 - (\Gamma B_1)JB_1 + C \in \mathfrak{U}, \tag{3.8}$$

where $C = C_0(L^0_{bc})^{\frac{1}{2}}$*,* C_0 *belongs to the trace class* $\mathfrak{S}_1(\mathcal{H})$ *on* $\mathrm{L}_2[0,1]$ (see [26]).

Proof. The existence of $m \in \mathbb{N}$ satisfying (3.5) was proved in Lemma 5. By Theorem 2 in [22] and Lemma 8, the operator $L_{bc} = L^0_{bc} - B$ is similar to $\widetilde{L}_{bc} = L^0_{bc} - \widetilde{B}$ and (3.6), (3.7) hold true. The operator C in (3.8) has the form

$$C = -(I + \Gamma_m B)^{-1}(\Gamma_m B)(B_1\Gamma_m B_1 - (\Gamma_m B_1)J_m B_1) + C_1 + \widetilde{C},$$

where

$$C_1 = B_1\Gamma_m B_1 - B_1\Gamma B_1 - (\Gamma_m B_1)J_m B_1 + (\Gamma B_1)JB_1 + J_m B_1 - JB_1$$

is of finite rank and, consequently, belongs to $\mathfrak{S}_1(\mathcal{H})$.

From Lemma 7, it follows that the operators $(\Gamma_m B)J_m B$ and $B\Gamma_m B$ belong to $\mathfrak{U}$. Consequently, $\widetilde{C} \in \mathfrak{U}$ and $B_1\Gamma_m B_1$, $(\Gamma_m B_1)J_m B_1 \in \mathfrak{U}$. Thus, C is representable in the form $C = C_0(L^0_{bc})^{\frac{1}{2}}$, where C_0 belongs to $\mathfrak{S}_1(\mathcal{H})$ (as the sum of a finite rank operator and the product of two Hilbert–Schmidt operators, see [26])). Thus, $\widetilde{B} \in \mathfrak{U}$. □

§4. Proofs of the main results

Theorem 6 allows us to reduce the study of L_{bc} to the study of $\widetilde{L}_{bc}$, and the latter will be done on the basis of Theorem 5 by the method of similar operators.

In the next statement, the number m is chosen in such a way that

$$\|\Gamma_m B\|_2 < 1, \quad \frac{c^{\frac{3}{2}}\|B\|_*}{m} < \frac{1}{4}, \tag{4.1}$$

where $c = (2\pi)^4$ if $bc = \text{per}$ and $c = (3\pi)^4$ if $bc = ap$. This statement is a principal result of the paper.

Theorem 7. *Let* $m \in \mathbb{N}$ *satisfy* (4.1)*. Then the operator* $L_{bc} = L^0_{bc} - B$ (*consequently, also* $\widetilde{L}_{bc}$) *is similar to an operator of the form*

$$L^0_{bc} - J_m X_* = L^0_{bc} - P_{(m)}X_*P_{(m)} - \sum_{j\geq m+1} P_j X_* P_j. \tag{4.2}$$

The operator $X_* \in \mathfrak{U}$ *is a solution of the equation*

$$X = \widetilde{B}\Gamma_m X - (\Gamma_m X)(J_m\widetilde{B}) - (\Gamma_m X)J_m(\widetilde{B}\Gamma_m X) + \widetilde{B}, \tag{4.3}$$

in $\mathfrak{U}$*. The operator* $I + \Gamma_m X_*$ *is invertible, and the similarity transformation taking* $L_{bc} = L^0_{bc} - B$ *to* $L^0_{bc} - J_m X_*$ *is done with the help of the operator*

$$U_m = (I + \Gamma_m B)(I + \Gamma_m X_*) = I + V_m, \tag{4.4}$$

where $V_m \in \mathfrak{S}_2(\mathcal{H})$*. Moreover,* $J_m X_*$ *is representable in the form*

$$J_m X_* = J\widetilde{B} + J(\widetilde{B}\Gamma\widetilde{B}) + T_0, \tag{4.5}$$

where $T_0 = T'_0(L^0_{bc})^{\frac{1}{2}}$*,* $T'_0 \in \mathfrak{S}_1(\mathcal{H})$*.*

Proof. By (3.5) (see the first condition in (4.1)), the operator $I+\Gamma_m B$ is invertible. Theorem 6 shows that $L_{bc}=L_{bc}^0-B$ is similar to $\widetilde{L}_{bc}=L_{bc}^0-\widetilde{B}$, where $\widetilde{B}$ is given by (3.8). Since $\widetilde{B}$ belongs to $\mathfrak{U}$ (by Theorem 6), we see that $\widetilde{L}_{bc}=L_{bc}^0-\widetilde{B}$ (consequently, also $L_{bc}=L_{bc}^0-B$) is similar to an operator $L_{bc}^0-J_mX_*$ of the form (4.2), where $X_*\in\mathfrak{U}$ is a solution of equation (4.3). Applying J_m to the two sides of this equation, we obtain

$$\begin{aligned} J_mX_* &= J_m(\widetilde{B}\Gamma_m X_*)+J_m\widetilde{B}\\ &= J_m\widetilde{B}+J_m(\widetilde{B}\Gamma_m\widetilde{B})+J_m(\widetilde{B}\Gamma_m(X_*-\widetilde{B}))=J\widetilde{B}+J(\widetilde{B}\Gamma\widetilde{B})+T_0, \end{aligned}$$

where $T_0=T_0'(L_{bc}^0)^{\frac{1}{2}}$, $T_0'\in\mathfrak{S}_1(\mathcal{H})$. We have used the fact that the product of two Hilbert–Schmidt operators is of trace class and that the operators J_mX-JX, $\Gamma_mX-\Gamma X$, $X\in\mathfrak{U}$, $m\in\mathbb{N}$, are of finite rank.

Clearly, the operator establishing similarity between L_{bc} and $L_{bc}^0-J_mX_*$ coincides with the operator U_m in (4.4). Since $\Gamma_m B,\Gamma_m X_*\in\mathfrak{S}_2(\mathcal{H})$, we see that the operator V_m in (4.4) belongs to $\mathfrak{S}_2(\mathcal{H})$. □

Below ℓ^p, $p\geq 1$, stands for the space of sequences summable with power p. Before proving the main results, we state a lemma.

Lemma 9. *The eigenvalues $\widetilde{\mu}_n^{\pm}$, $n\in\mathbb{N}$, of the matrix*

$$\begin{pmatrix} c_1(n) & c_2(n)\\ c_3(n) & c_4(n)\end{pmatrix}+\begin{pmatrix} d_1(n) & d_2(n)\\ d_3(n) & d_4(n)\end{pmatrix},$$

where $c_j\in\ell^2$, $d_j\in\ell^1$, $1\leq j\leq 4$, admit a representation of the form

$$\widetilde{\mu}_n^{\pm}=\frac{c_1(n)+c_4(n)}{2}\pm\frac{1}{2}\sqrt{(c_1(n)-c_4(n))^2+4c_2(n)c_3(n)}+\varepsilon_n^{\pm},$$

where the sequences $(\varepsilon_n^{\pm})$ belong to $\ell^{\frac{4}{3}}$, i.e., $\sum_{n=1}^{\infty}|\varepsilon_n^{\pm}|^{\frac{4}{3}}<\infty$.

We proceed to the proof of the main results.

Proof of Theorem 1. Theorem 7 (already proved) makes it possible to establish the asymptotics for the eigenvalues of L_{bc}. Lemma 1 in [22] and Theorem 7 show that the operator $L_{bc}^0-J_mX_*$ of the form (4.2) commutes with all projections $P_{(m)}$, P_k, $k\geq m+1$ (see the Introduction). Consequently, the spaces $\mathcal{H}_{(m)}=\operatorname{Im}P_{(m)}$ (where $P_{(m)}=\sum_{j\leq m}P_j$) and $\mathcal{H}_j=\operatorname{Im}P_j$, $j\geq m+1$, are invariant for $L_{bc}^0-J_mX_*$. Since the operators L_{bc} and $L_{bc}^0-J_mX_*$ are similar, we have $\sigma(L_{bc})=\sigma(\widetilde{L}_{bc})=\sigma(L_{bc}^0-J_mX_*)$. It can easily be shown that $L_{bc}=L_{bc}^0-B$ (like $L_{bc}^0-J_mX_*$) has compact resolvent. Thus, if $\lambda_0\in\sigma(L_{bc}^0-J_mX_*)$, then there exists a vector $x_0\in\mathrm{D}(L_{bc}^0)$ with $(L_{bc}^0-J_mX_*)x_0=\lambda_0x_0$. Now, the form of J_mX_* implies

$$A_{(m)}P_{(m)}x_0=\lambda_0P_{(m)}x_0,\quad A_jP_jx_0=\lambda_0P_jx_0,\ \ j\geq m+1, \tag{4.6}$$

where

$$A_{(m)}=\left(L_{bc}^0-J_mX_*\big|\,\mathcal{H}_{(m)}\right)$$

is the restriction of $L_{bc}^0-J_mX_*$ to $\mathcal{H}_{(m)}$;

$$A_j=\left(L_{bc}^0-J_mX_*\big|\,\mathcal{H}_j\right)$$

is the restriction of $L_{bc}^0-J_mX_*$ to $\mathcal{H}_j$, $j\geq m+1$. Since $I=P_{(m)}+\sum_{j=m+1}^{\infty}P_j$ (the projections P_j, $j\geq m+1$, constitute a partition of unity), by (4.6) we see that at least one of the vectors P_jx_0, $j\geq m+1$, $P_{(m)}x_0$, is nonzero. Thus, λ_0 is an eigenvalue of the corresponding operator among A_j, $j \geq m+1$, and $A_{(m)}$. Thus, we have proved the inclusion

$$\sigma(L_{bc}) = \sigma(\widetilde{L}_{bc}) = \sigma(L^0_{bc} - J_m X_*) \subset \sigma(A_{(m)}) \cup \Bigl(\bigcup_{j\geq m+1} \sigma(A_j)\Bigr).$$

The reverse inclusion is obvious. Consequently,

$$\sigma(L_{bc}) = \sigma(\widetilde{L}_{bc}) = \sigma(L^0_{bc} - J_m X_*) = \sigma(A_{(m)}) \cup \Bigl(\bigcup_{j\geq m+1} \sigma(A_j)\Bigr). \tag{4.7}$$

Since $\mathcal{H}_{(m)}$ is finite-dimensional, $\dim \mathcal{H}_{(m)} = m$, (4.7) implies that the set $\sigma(A_{(m)}) = \sigma_{(m)}$ is finite. Also, the spaces $\mathcal{H}_j$, $j \geq m+1$, are two-dimensional. Thus, the operators $A_{(m)}$ and A_j, $j \geq m+1$, are well defined.

Since each L_{bc} is similar to the corresponding operator $\widetilde{L}_{bc}$, all subsequent calculations will be done for $\widetilde{L}_{bc}$.

We calculate the eigenvalues of L_{bc}. For this, we use the representation (3.1), (3.2) of matrices, and also the Fourier series expansions for a and b:

$$a(t) = \sum_{l\in\mathbb{Z}} a_l \mathrm{e}^{2i\pi lt}, \quad b(t) = \sum_{l\in\mathbb{Z}} b_l \mathrm{e}^{2i\pi lt}.$$

Suppose that $bc = \mathrm{per}$. Formula (3.1) shows that the block matrix $(\mathfrak{A}^{\mathrm{per}}_{nn})$, $n \in \mathbb{N}$, of B_1 has the form

$$\mathfrak{A}^{\mathrm{per}}_{nn} = -(2\pi n)^2 \begin{pmatrix} 0 & a_n & a_{-n} \\ 0 & a_0 & a_{-2n} \\ 0 & a_{2n} & a_0 \end{pmatrix}.$$

Accordingly, for $bc = ap$ the block matrix $(\mathfrak{A}^{ap}_{nn})$, $n \in \mathbb{N}$, has the form

$$\mathfrak{A}^{ap}_{nn} = -(\pi(2n+1))^2 \begin{pmatrix} a_0 & a_{-2n-1} \\ a_{2n+1} & a_0 \end{pmatrix}.$$

The block-diagonal entries $\mathcal{C}^{\mathrm{per}}_{nn}$, $n \in \mathbb{N}$, of the matrix of the operator $B\Gamma B$ in the case of $bc = \mathrm{per}$ have the form

$$\mathcal{C}^{\mathrm{per}}_{nn} = n^2 \sum_{\substack{l=1\\ l\neq n}}^{\infty} \frac{l^2}{l^4 - n^4} \begin{pmatrix} 0 & a_l a_{n-l} + a_{-l} a_{n+l} & a_l a_{-n-l} + a_{-l} a_{-n+l} \\ 0 & a_{n-l} a_{l-n} + a_{n+l} a_{-n-l} & 2a_{l-n} a_{-n-l} \\ 0 & 2a_{n+l} a_{n-l} & a_{n+l} a_{-n-l} + a_{n-l} a_{l-n} \end{pmatrix}.$$

Accordingly, for $bc = ap$, the matrix has the form

$$\mathcal{C}^{ap}_{nn} = (2n+1)^2 \sum_{\substack{l=0\\ l\neq n}}^{\infty} \frac{(2l+1)^2}{(2l+1)^4 - (2n+1)^4} \times \begin{pmatrix} a_{n-l} a_{l-n} + a_{n+l+1} a_{-n-l-1} & 2a_{l-n} a_{-n-l-1} \\ 2a_{n+l+1} a_{n-l} & a_{n+l+1} a_{-n-l-1} + a_{n-l} a_{l-n} \end{pmatrix}.$$

Using Theorem 7, formula (4.5), and Lemma 9, we deduce that the remainder has the form $\gamma_n n^2$, where

$$\sum_{n=m+1}^{\infty} |\gamma_n|^{\frac{4}{3}} < \infty.$$

Thus, for $n \geq m+1$ the eigenvalues of L_{per} have the following asymptotics:

$$\widetilde{\lambda}_n^{\mp} = \big(2\pi n\big)^4 + (2\pi n)^2 a_0 - n^2 \sum_{\substack{l=1\\ l\neq n}}^{\infty} \frac{(a_{n+l}a_{-n-l} + a_{n-l}a_{l-n})l^2}{l^4 - n^4}$$
$$\mp (2\pi n)^2 \Bigg(a_{-2n}a_{2n} + \frac{1}{4\pi^4}\Bigg(\sum_{\substack{l=1\\ l\neq n}}^{\infty} \frac{a_{l-n}a_{-n-l}l^2}{l^4-n^4}\Bigg)\Bigg(\sum_{\substack{l=1\\ l\neq n}}^{\infty} \frac{a_{l+n}a_{n-l}l^2}{l^4-n^4}\Bigg)$$
$$- \frac{a_{-2n}}{2\pi^2}\sum_{\substack{l=1\\ l\neq n}}^{\infty} \frac{a_{l+n}a_{n-l}l^2}{l^4-n^4} - \frac{a_{2n}}{2\pi^2}\sum_{\substack{l=1\\ l\neq n}}^{\infty} \frac{a_{l-n}a_{-n-l}l^2}{l^4-n^4}\Bigg)^{\frac{1}{2}} + \gamma_n n^2,$$
$$n \geq m+1.$$

This can also be written in the form

$$\widetilde{\lambda}_n^{\mp} = \big(2\pi n\big)^4 + (2\pi n)^2 a_0 + \gamma_n^{\mp} n^2, \tag{4.8}$$

where $(\gamma_n^{\mp})$ belongs to ℓ^2. Indeed, we have

$$\Bigg|\sum_{\substack{l=1\\ l\neq n}}^{\infty} \frac{a_{l+n}a_{n-l}l^2}{l^4-n^4}\Bigg| \leq \sum_{k=1}^{\infty} \frac{|a_k|\,|a_{2n+k}|}{k(k+2n)} = \sum_{k=1}^{\infty} \frac{\widetilde{\xi}_k}{k+2n} = \sum_{k=1}^{\infty} \widetilde{\xi}_k \xi_{k+2n},$$

where the sequence $(\widetilde{\xi}_k)$ belongs to ℓ^1 and the sequence ξ_{k+2n} is summable with a power greater than 1. Then

$$\Bigg\|\sum_{k=1}^{\infty} \widetilde{\xi}_k \xi_{k+2n}\Bigg\|_{\ell^2} \leq \sum_{k=1}^{\infty} |\widetilde{\xi}_k| \|\xi\|_{\ell^2} < \infty.$$

Estimating all summands in the asymptotic formula for eigenvalues in a similar way, we arrive at (4.8).

In the case of $bc = ap$, we argue much as above to obtain the following asymptotic formula for the eigenvalues of L_{ap}:

$$\widetilde{\lambda}_n^{\mp} = \big(\pi(2n+1)\big)^4 + (\pi(2n+1))^2 a_0 - (2n+1)^2 \sum_{\substack{l=0\\ l\neq n}}^{\infty} \frac{(a_{n+l+1}a_{-n-l-1} + a_{n-l}a_{l-n})(2l+1)^2}{(2l+1)^4 - (2n+1)^4}$$
$$\mp (\pi(2n+1))^2 \Bigg(a_{-2n-1}a_{2n+1} + \frac{4}{\pi^4}\sum_{\substack{l=0\\ l\neq n}}^{\infty} \frac{a_{l-n}a_{-n-l-1}(2l+1)^2}{(2l+1)^4-(2n+1)^4}\sum_{\substack{l=0\\ l\neq n}}^{\infty} \frac{a_{n+l+1}a_{n-l}(2l+1)^2}{(2l+1)^4-(2n+1)^4}$$
$$- \frac{2a_{-2n-1}}{\pi^2}\sum_{\substack{l=0\\ l\neq n}}^{\infty} \frac{a_{n+l+1}a_{n-l}(2l+1)^2}{(2l+1)^4-(2n+1)^4} - \frac{2a_{2n+1}}{\pi^2}\sum_{\substack{l=0\\ l\neq n}}^{\infty} \frac{a_{l-n}a_{-n-l-1}(2l+1)^2}{(2l+1)^4-(2n+1)^4}\Bigg)^{\frac{1}{2}} + \widetilde{\gamma}_n n^2,$$
$$n \geq m+1,$$

where $\sum_{n=m+1}^{\infty} |\widetilde{\gamma}_n|^{\frac{4}{3}} < \infty$. We also have the following abridged asymptotic formula:

$$\widetilde{\lambda}_n^{\mp} = \big(\pi(2n+1)\big)^4 + \big(\pi(2n+1)\big)^2 a_0 + \widetilde{\gamma}_n^{\mp} n^2,$$

where $(\widetilde{\gamma}_n^{\mp})$ belongs to ℓ^2. This completes the proof of the theorem. □

Proof of Theorem 2. If a is a function of bounded variation, then its Fourier coefficients admit the estimate $|a_n| \leq \frac{c_0}{|n|}$, where c_0 is some constant (see [27]). Now, arguing as in the proof of Theorem 1, we establish the claim. □

Corollary 1. *The operator* $L_{bc} = L^0_{bc} - B$, $bc \in \{\mathrm{per}, ap\}$ *is spectral in the sense of Dunford* (*see* [16]).

In what follows, we consider the spectral projections described in the Introduction. Note that we have the following partition of unity:

$$I = \sum_{k \geq m+1} P_k + P_{(m)}, \quad I = \sum_{k \geq m+1} \widetilde{P}_k + \widetilde{P}_{(m)},$$

where $\widetilde{P}_{(m)} = (I + V_m)P_{(m)}(I + V_m)^{-1}$.

Proof of Theorem 3. Theorem 7 implies that L_{bc} is similar to $L^0_{bc} - J_m X_*$ (see formula (4.2)) and the corresponding transformation is done by the operator U_m described in (4.4). The operator V_m in (4.4) has the form $V_m = \Gamma_m B + \Gamma_m X_* + (\Gamma_m B)(\Gamma_m X_*)$. Since $L^0_{bc} - B = (I + V_m)(L^0_{bc} - J_m X_*)(I + V_m)^{-1}$, by Lemma 1 in [22] the spectral projections $\widetilde{P}(\Omega)$ and $P(\Omega)$ are similar, moreover, $\widetilde{P}(\Omega)$ admits the representation $\widetilde{P}(\Omega) = (I + V_m)P(\Omega)(I + V_m)^{-1}$. Consequently, $\widetilde{P}(\Omega) - P(\Omega)$ is representable in the form

$$\widetilde{P}(\Omega) - P(\Omega) = (I + V_m)P(\Omega)(I + V_m)^{-1} - P(\Omega) = \big(V_m P(\Omega) - P(\Omega)V_m\big)(I + V_m)^{-1}.$$

First, we consider the case of $bc = \mathrm{per}$. For further use, we need to estimate the quantities $\|\Gamma_m X_* P(\Omega)\|_2$, $\|P(\Omega)\Gamma_m X_*\|_2$, $\|\Gamma_m B P(\Omega)\|_2$, and $\|P(\Omega)\Gamma_m B\|_2$. By (3.1) and Remark 3, we have

$$\begin{aligned}\|\Gamma_m X_* P(\Omega)\|_2^2 = \|\Gamma_m X_0 (L^0_{bc})^{\frac{1}{2}} P(\Omega)\|_2^2 &= \Bigg\| \sum_{\substack{p \geq 0,\, j \geq k(\Omega) \\ p \neq j}}^{\infty} \frac{(P_p X_0 P_j)\lambda_j^{\frac{1}{2}}}{\lambda_p - \lambda_j} \Bigg\|_2^2 \\ &\leq c^3 \max_{\substack{p \geq 0,\, j \geq k(\Omega) \\ p \neq j}} \frac{j^4}{(p^4 - j^4)^2} \sum_{\substack{p \geq 0,\, j \geq k(\Omega) \\ p \neq j}}^{\infty} \|P_p X_0 P_j\|_2^2 \\ &\leq c^3 \max_{\substack{p \geq 0,\, j \geq k(\Omega) \\ p \neq j}} \frac{1}{(p^2 - j^2)^2} \|X_0\|_2^2 \leq \frac{c^3}{(2k(\Omega) - 1)^2} \|X_0\|_2^2.\end{aligned}$$

Thus,

$$\|\Gamma_m X_* P(\Omega)\|_2 \leq \frac{c^{\frac{3}{2}} \|X_0\|_2}{2k(\Omega) - 1},$$

where $X_0 \in \mathfrak{S}_2(\mathcal{H})$ is taken from the representation $X_* = X_0 (L^0_{bc})^{\frac{1}{2}}$, and $c = (2\pi)^4$ for $bc = \mathrm{per}$, $c = (3\pi)^4$ for $bc = ap$.

Similarly, we prove the estimate

$$\|P(\Omega)\Gamma_m X_*\|_2 \leq \frac{c^{\frac{3}{2}} \|X_0\|_2}{2k(\Omega) - 1}.$$

Next, we estimate the quantity $\|\Gamma_m B P(\Omega)\|_2$. Since B_2 is multiplication by the function b in $\mathcal{H}$, it suffices to estimate the quantities $\|\Gamma_m B_1 P(\Omega)\|_2$ and $\|P(\Omega)\Gamma_m B_1\|_2$. Using the matrix representations (3.1), (3.2), and Remark 3, in the case of $bc = \mathrm{per}$ we obtain

$$\begin{aligned}\|\Gamma B_1 P(\Omega)\|_2^2 = &\sum_{j \geq k(\Omega)}^{\infty} |(\Gamma B_1 e_j^1, e_0)|^2 + \sum_{j \geq k(\Omega)}^{\infty} |(\Gamma B_1 e_j^2, e_0)|^2 + \sum_{\substack{p=1 \\ p \neq j, j \geq k(\Omega)}}^{\infty} |(\Gamma B_1 e_j^1, e_p^1)|^2 \\ &+ \sum_{\substack{p=1 \\ p \neq j, j \geq k(\Omega)}}^{\infty} |(\Gamma B_1 e_j^2, e_p^1)|^2 + \sum_{\substack{p=1 \\ p \neq j, j \geq k(\Omega)}}^{\infty} |(\Gamma B_1 e_j^1, e_p^2)|^2 + \sum_{\substack{p=1 \\ p \neq j, j \geq k(\Omega)}}^{\infty} |(\Gamma B_1 e_j^2, e_p^2)|^2\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j\geq k(\Omega)}^{\infty} \frac{|a_j|^2(2\pi j)^4}{(2\pi j)^8} + \sum_{j\geq k(\Omega)}^{\infty} \frac{|a_{-j}|^2(2\pi j)^4}{(2\pi j)^8} + \sum_{\substack{p=1\\ p\neq j, j\geq k(\Omega)}}^{\infty} \frac{|a_{j-p}|^2(2\pi j)^4}{((2\pi p)^4-(2\pi j)^4)^2} \\
&\quad + \sum_{\substack{p=1\\ p\neq j, j\geq k(\Omega)}}^{\infty} \frac{|a_{-j-p}|^2(2\pi j)^4}{((2\pi p)^4-(2\pi j)^4)^2} + \sum_{\substack{p=1\\ p\neq j, j\geq k(\Omega)}}^{\infty} \frac{|a_{j+p}|^2(2\pi j)^4}{((2\pi p)^4-(2\pi j)^4)^2} + \sum_{\substack{p=1\\ p\neq j, j\geq k(\Omega)}}^{\infty} \frac{|a_{-j+p}|^2(2\pi j)^4}{((2\pi p)^4-(2\pi j)^4)^2} \\
&\leq \frac{1}{(2\pi)^4}\sum_{j\geq k(\Omega)}^{\infty} \frac{|a_j|^2}{j^4} + \frac{1}{(2\pi)^4}\sum_{j\geq k(\Omega)}^{\infty} \frac{|a_{-j}|^2}{j^4} + \frac{1}{(2\pi)^4}\sum_{\substack{p=1\\ p\neq j, j\geq k(\Omega)}}^{\infty} \frac{|a_{j-p}|^2}{(p^2-j^2)^2} \\
&\quad + \frac{1}{(2\pi)^4}\sum_{\substack{p=1\\ p\neq j, j\geq k(\Omega)}}^{\infty} \frac{|a_{-j-p}|^2}{(p^2-j^2)^2} + \frac{1}{(2\pi)^4}\sum_{\substack{p=1\\ p\neq j, j\geq k(\Omega)}}^{\infty} \frac{|a_{j+p}|^2}{(p^2-j^2)^2} + \frac{1}{(2\pi)^4}\sum_{\substack{p=1\\ p\neq j, j\geq k(\Omega)}}^{\infty} \frac{|a_{-j+p}|^2}{(p^2-j^2)^2} \\
&\leq \frac{1}{2\pi^4}\|a\|_{\ell^2}^2 \left(\sum_{p=1}^{k(\Omega)-1} \frac{1}{(p+k(\Omega))^2(k(\Omega)-p)} + \sum_{p=k(\Omega)+1}^{\infty} \frac{1}{(p+k(\Omega))^2(p-k(\Omega))} \right) \\
&\leq \frac{\|a\|_{\ell^2}^2 c_1^2}{k^2(\Omega)} \ln\left(\frac{(k(\Omega)-1)(2k(\Omega)+1)}{k(\Omega)+1} \right),
\end{aligned}
$$

where $c_1 > 0$ is a constant. Consequently,

$$
\|\Gamma_m B P(\Omega)\|_2 \leq \frac{c_1\|a\|_{\ell^2}}{k(\Omega)} \left(\ln\left(\frac{(k(\Omega)-1)(2k(\Omega)+1)}{k(\Omega)+1} \right) \right)^{\frac{1}{2}}.
$$

Similar arguments yield the same estimate (with a constant $c_2 > 0$) in the case of $bc = ap$. Similar inequalities hold for $\|P(\Omega)\Gamma_m B\|_2$ in both cases.

Using the above estimates, inequality (3.5), and also the representation of the operator V_m, we arrive at

$$
\begin{aligned}
\|\widetilde{P}(\Omega) - P(\Omega)\|_2 &\leq \|V_m P(\Omega)\|_2 + \|P(\Omega)V_m\|_2 \\
&\leq \|\Gamma_m B_1 P(\Omega)\|_2 + \|\Gamma_m X_* P(\Omega)\|_2 + \|P(\Omega)\Gamma_m B_1\|_2 + \|P(\Omega)\Gamma_m X_*\|_2 \\
&\qquad + \|\Gamma_m X_* P(\Omega)\|_2 + \|P(\Omega)\Gamma_m B_1\|_2 \\
&\leq \frac{3c_1\|a\|_{\ell^2}}{k(\Omega)} \left(\ln\left(\frac{(k(\Omega)-1)(2k(\Omega)+1)}{k(\Omega)+1} \right) \right)^{\frac{1}{2}} + \frac{3c^{\frac{3}{2}}\|X_0\|_2}{2k(\Omega)-1} \leq \frac{\widetilde{M}\big(\ln k(\Omega)\big)^{\frac{1}{2}}}{k(\Omega)},
\end{aligned}
$$

where $\widetilde{M} > 0$ is a constant independent of $k(\Omega)$, with $c = (2\pi)^4$ for $bc =$ per, and $c = (3\pi)^4$ for $bc = ap$. □

Corollary 2. *Under the assumptions of Theorem* 3, *we have*

$$
\|\widetilde{P}_n - P_n\|_2 \leq \frac{M_1}{n}, \quad n \in \mathbb{N},
$$

where $M_1 > 0$ *is a constant. In this case* Ω *is a singleton* $\{n\}$ *and there is no summation on* j.

Corollary 3. *Under the assumptions of Theorem* 3, *we have*

$$
\sum_{n=m+1}^{\infty} \|\widetilde{P}_n - P_n\|_2^2 < \frac{M_1^2}{m^2}.
$$

Proof of Theorem 4. The partition of unity formulas and Theorem 3 imply

$$\begin{aligned}\Big\|\widetilde{P}_{(m)} &+ \sum_{k=m+1}^{n} \widetilde{P}_k - P_{(m)} - \sum_{k=m+1}^{n} P_k\Big\|_2 \\ &= \Big\|\widetilde{P}_{(m)} + \sum_{k=m+1}^{\infty} \widetilde{P}_k - \sum_{k=n+1}^{\infty} \widetilde{P}_k - P_{(m)} - \sum_{k=m+1}^{\infty} P_k + \sum_{k=n+1}^{\infty} P_k\Big\|_2 \\ &= \Big\|\sum_{k=n+1}^{\infty} \widetilde{P}_k - \sum_{k=n+1}^{\infty} P_k\Big\|_2 \le \frac{\widetilde{M}\big(\ln n\big)^{\frac{1}{2}}}{n},\end{aligned}$$

and the proof is complete. □

Corollary 4. *The spectral resolutions for $L_{bc} = L_{bc}^0 - B$ and L_{bc}^0 converge uniformly*:

$$\lim_{n\to\infty}\Big\|\widetilde{P}_{(m)} + \sum_{k=m+1}^{n} \widetilde{P}_k - P_{(m)} - \sum_{k=m+1}^{n} P_k\Big\|_2 = 0.$$

§5. Construction of an analytic semigroup of operators

In this section, the results about the spectral properties of $L_{bc} = L_{bc}^0 - B$ obtained above (especially, Theorem 7), will be used to show that the operator $-L_{bc} = -L_{bc}^0 + B$ is sectorial and to construct the analytic semigroup whose generator is this operator.

Definition 4 (see [28])**.** A linear operator

$$C\colon \mathrm{D}(C) \subset \mathcal{X} \to \mathcal{X}$$

in a Banach space $\mathcal{X}$ is said to be *sectorial* if it is closed and densely defined and, moreover, for some $\varphi \in (\frac{\pi}{2}, \pi)$, $M \ge 1$, and real a, the sector $S_{a,\varphi} = \{\lambda \in \mathbb{C} : |\arg(\lambda - a)| < \varphi, \lambda \ne a\}$ is included in the resolvent set for C and $\|(\lambda - C)^{-1}\| \le \frac{M}{|\lambda - a|}$ for all $\lambda \in S_{a,\varphi}$.

In the next theorem and its proof, we use the notation of Theorem 7.

Theorem 8. *The differential operator $-L_{bc} = -L_{bc}^0 + B$ is sectorial and generates an analytic semigroup of operators $T\colon \mathbb{R}_+ \to \operatorname{End}\mathcal{H}$. Furthermore,*

$$T(t) = U_m \widetilde{T}(t) U_m^{-1},$$

where $U_m = (I + \Gamma_m B)(I + \Gamma_m X_*)$ and $\widetilde{T}\colon \mathbb{R}_+ \to \operatorname{End}\mathcal{H}$ is the semigroup generated by $-L_{bc}^0 + J_m X_*$. Moreover, this semigroup is similar to a semigroup of the form $T_{(m)}(t) \oplus T^{(m)}(t)$ acting on $\mathrm{L}_2[0,1] = \mathcal{H}_{(m)} \oplus \mathcal{H}^{(m)}$, where $\mathcal{H}_{(m)} = \operatorname{Im} P_{(m)}$, $\mathcal{H}^{(m)} = \operatorname{Im}(I - P_{(m)})$, and $T^{(m)}(t)$ is representable in the form

$$T^{(m)}(t)x = \sum_{k=m+1}^{\infty} \mathrm{e}^{C_k t} P_k x, \quad x \in \mathrm{L}_2[0,1],$$

where $C_k \in \operatorname{End}\mathcal{H}_k$, $\mathcal{H}_k = \operatorname{Im} P_k$. The natural number m is chosen so that the claim of Theorem 7 be true.

Proof. By Theorem 7 and formula (4.7), the operator L_{bc} (respectively, $-L_{bc}$) is similar to $L_{bc}^0 - J_m X_*$ (respectively, to $-L_{bc}^0 + J_m X_*$), where $X_* = X_0 (L_{bc}^0)^{\frac{1}{2}}, X_0 \in \mathfrak{S}_2(\mathcal{H})$. Consequently,

1) $$\sigma(-L_{bc}) = \sigma(-L_{bc}^0 + J_m X_*) = \sigma_m \cup \Big(\bigcup_{j\ge m+1} \sigma_j\Big),$$

where σ_m is a finite set;

2) $$R(\lambda, -L_{bc}) = U_m R(\lambda, -L^0_{bc} + J_m X_*) U_m^{-1},$$

where U_m is the transformation operator, $\lambda \in \rho(-L_{bc}) = \rho(-L^0_{bc} + J_m X_*)$, $\lambda \notin \sigma(-L_{bc})$.

To estimate the resolvent of $-L^0_{bc} + J_m X_*$ we consider the identities

$$\begin{aligned}-L^0_{bc} + J_m X_* - \lambda I &= (I + J_m X_*(-L^0_{bc} - \lambda I)^{-1})(-L^0_{bc} - \lambda I)\\ &= (I + J_m X_0 (L^0_{bc})^{\frac{1}{2}} (-L^0_{bc} - \lambda I)^{-1})(-L^0_{bc} - \lambda I).\end{aligned}$$

We exhibit a sector containing the spectrum of $-L^0_{bc} + J_m X_*$ and such that the operator $I + J_m X_0 (L^0_{bc})^{\frac{1}{2}} (-L^0_{bc} - \lambda I)^{-1}$ is invertible. By 1), the spectrum of $-L^0_{bc} + J_m X_*$ is the union of a finite set and the sets σ_j, $j \geq m+1$, where $\sigma_j = \{-\widetilde{\lambda}_j\}$. All eigenvalues of $-L^0_{bc} + J_m X_*$ belong to the sector $\gamma = \gamma_0 + 2\|X_0\|_2^2$, where γ_0 is the sector with vertex at zero such that the argument obeys the condition $\frac{3\pi}{4} \leq \arg z \leq \frac{5\pi}{4}$. For every λ in γ, the operator $I + J_m X_0 (L^0_{bc})^{\frac{1}{2}} (-L^0_{bc} - \lambda I)^{-1}$ is invertible and we have $\|J_m X_0 (L^0_{bc})^{\frac{1}{2}} (-L^0_{bc} - \lambda I)^{-1}\| \leq \frac{1}{2}$. A direct calculation shows that the resolvent satisfies the inequality

$$\|R(\lambda, -L^0_{bc} + J_m X_*)\| \leq \frac{2}{|\pi^4 + \lambda|} \leq \frac{2}{|\lambda - 2\|X_0\|_2^2|}.$$

Consequently, the operator $-L^0_{bc} + J_m X_*$ (thus, also $-L_{bc}$) is sectorial. By Theorem II.4.6 in [29], the operator $-L_{bc}$ is the generator of an analytic semigroup $T(t) = U_m \widetilde{T}(t) U_m^{-1}$ (because $-L_{bc}$ and $-L^0_{bc} + J_m X_*$ are similar), where

$$\widetilde{T}(t) = \frac{1}{2\pi i} \int_\Gamma \mathrm{e}^{\lambda t} R(\lambda, -L^0_{bc} + J_m X_*)\, \mathrm{d}\lambda.$$

Consider the orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{(m)} \oplus \mathcal{H}^{(m)}, \text{ where } \mathcal{H}_{(m)} = \operatorname{Im} P_{(m)}, \quad \mathcal{H}^{(m)} = \operatorname{Im}\Bigg(\sum_{k \geq m+1} P_k\Bigg).$$

Accordingly, the operator $-L^0_{bc} + J_m X_*$ decomposes as follows:

$$-L^0_{bc} + J_m X_* = \big(-\widetilde{A}_{(m)} + P_{(m)} \mid \mathcal{H}_{(m)}\big) \oplus \widetilde{A}^{(m)},$$

where $\widetilde{A}_{(m)}$ is the restriction of $L^0_{bc} - J_m X_*$ to $\mathcal{H}_{(m)}$ and $\widetilde{A}^{(m)}$ is the restriction of $-L^0_{bc} + J_m X_*$ to $\mathcal{H}^{(m)}$. By [29], the semigroup $T(t)$ is similar to $T_{(m)}(t) \oplus T^{(m)}(t)$, where $T^{(m)}(t)$ is representable in the form

$$T^{(m)}(t)x = \sum_{k=m+1}^{\infty} \mathrm{e}^{C_k t} P_k x, \quad x \in \mathcal{H},$$

with $C_k \in \operatorname{End} \mathcal{H}_k$, $\mathcal{H}_k = \operatorname{Im} P_k$. Using the calculations made in the proof of Theorem 1, we obtain the following formula for the matrix of C_k in the case of L_{per}:

$$-(2\pi k)^4 I - \begin{pmatrix} 0 & a_k - a_l a_{k-l} - a_{-l} a_{k+l} & a_{-k} - a_l a_{-k-l} - a_{-l} a_{-k+l} \\ 0 & v_1^{\mathrm{per}}(k) + k^2 \xi_n^1 & v_2^{\mathrm{per}}(k) + k^2 \xi_n^2 \\ 0 & v_3^{\mathrm{per}}(k) + k^2 \xi_n^3 & v_4^{\mathrm{per}}(k) + k^2 \xi_n^4 \end{pmatrix},$$

where I is the unit matrix and

$$v_1^{\rm per}(k) = (2\pi k)^2 a_0 - k^2 \sum_{\substack{l=1\\ l\neq k}}^{\infty} \frac{(a_{k-l}a_{l-k} + a_{k+l}a_{-k-l})l^2}{l^4 - k^4},$$

$$v_2^{\rm per}(k) = (2\pi k)^2 a_{-2k} - 2k^2 \sum_{\substack{l=1\\ l\neq k}}^{\infty} \frac{a_{l-k}a_{-l-k}l^2}{l^4 - k^4},$$

$$v_3^{\rm per}(k) = (2\pi k)^2 a_{2k} - 2k^2 \sum_{\substack{l=1\\ l\neq k}}^{\infty} \frac{a_{k-l}a_{l+k}l^2}{l^4 - k^4},$$

$$v_4^{\rm per}(k) = (2\pi k)^2 a_0 - k^2 \sum_{\substack{l=1\\ l\neq k}}^{\infty} \frac{(a_{k+l}a_{-l-k} + a_{k-l}a_{l-k})l^2}{l^4 - k^4},$$

with (ξ_n^1), (ξ_n^2), (ξ_n^3), $(\xi_n^4) \in \ell^1$.

Accordingly, for L_{ap} this matrix has the following form:

$$-(\pi(2k+1))^4 I - \begin{pmatrix} v_1^{ap}(k) + (2k+1)^2\xi_n^5 & v_2^{ap}(k) + (2k+1)^2\xi_n^6 \\ v_3^{ap}(k) + (2k+1)^2\xi_n^7 & v_4^{ap}(k) + (2k+1)^2\xi_n^8 \end{pmatrix},$$

where I is the unit matrix and

$$v_1^{ap}(k) = (\pi(2k+1))^2 a_0 - (2k+1)^2 \sum_{\substack{l=0\\ l\neq k}}^{\infty} \frac{(a_{l-k}a_{k-l} + a_{k+l+1}a_{-k-l-1})(2l+1)^2}{(2l+1)^4 - (2k+1)^4},$$

$$v_2^{ap}(k) = (\pi(2k+1))^2 a_{-2k-1} - 2(2k+1)^2 \sum_{\substack{l=0\\ l\neq k}}^{\infty} \frac{a_{l-k}a_{-l-k-1}(2l+1)^2}{(2l+1)^4 - (2k+1)^4},$$

$$v_3^{ap}(k) = (\pi(2k+1))^2 a_{2k+1} - 2(2k+1)^2 \sum_{\substack{l=0\\ l\neq k}}^{\infty} \frac{a_{k-l}a_{l+k+1}(2l+1)^2}{(2l+1)^4 - (2k+1)^4},$$

$$v_4^{ap}(k) = (\pi(2k+1))^2 a_0 - (2k+1)^2 \sum_{\substack{l=0\\ l\neq k}}^{\infty} \frac{(a_{k+l+1}a_{-l-k-1} + a_{k-l}a_{l-k})(2l+1)^2}{(2l+1)^4 - (2k+1)^4},$$

with (ξ_n^5), (ξ_n^6), (ξ_n^7), $(\xi_n^8) \in \ell^1$. This proves the theorem. □

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