REMARKS ON $A_p$-REGULAR LATTICES OF MEASURABLE FUNCTIONS

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Abstract. A Banach lattice $X$ of measurable functions on a space of homogeneous type is said to be $A_p$-regular if every $f \in X$ admits a majorant $g \geq |f|$ belonging to the Muckenhoupt class $A_p$ with suitable control on the norm and the constant. It is well known that the $A_p$-regularity of the order dual $X'$ of $X$ implies the boundedness of the Hardy–Littlewood maximal operator on $X^{1/p}$ for $p > 1$ (equivalently, the $A_1$-regularity of this lattice), provided that $X'$ is norming for $X$. This result admits a partial converse and an interesting characterization: the $A_1$-regularity of $X^{1/p}$ implies the $A_p$-regularity of $X'$, and for lattices $X$ with the Fatou property these conditions are equivalent to the $A_1$-regularity of both $X^{1/p}$ and $(X^{1/p})'$. As an application, an exact form of the self-duality of BMO-regularity is obtained, the $A_q$-regularity of the lattices $L_\infty(\ell_p)$ for all $1 < p, q < \infty$ is established, and in many cases it is shown that the $A_1$-regularity of both $Y$ and $Y'$ yields the $A_1$-regularity of $Y(\ell^s)$ for all $1 < s < \infty$, which implies the boundedness of the Calderón–Zygmund operators in $Y(\ell^s)$.

Introduction

Let a quasimetric space $S$ endowed with a measure $\nu$ be a space of homogeneous type, e.g., $S = \mathbb{R}^n$ or $S = T^n$ with the Lebesgue measure, and let $\Omega$ be a $\sigma$-finite measurable space with measure $\mu$. The generic point $\omega \in \Omega$ will be regarded as an additional variable. We consider quasinormed lattices $X$ of measurable functions on $S \times \Omega$. For more details on lattices of measurable functions see, e.g., [11]; the definitions of most of the (standard) notions and properties can be found, e.g., in [14].

Let $p \geq 1$. A lattice $X$ is said to be $A_p$-regular with constants $(C, m)$ if for any $f \in X$ there exists a majorant $g \geq |f|$ such that $\|g\|_X \leq m\|f\|_X$ and $g(\cdot, \omega) \in A_p$ with constant $C$ for almost all $\omega \in \Omega$, where $A_p$ is the Muckenhoupt class (see, e.g., [9, Chapter 5]).

As was demonstrated in [4], the mere existence of majorants of class $A_1$ already characterizes the natural ambient space $\bigcup_{p > 1} L_p(\mathbb{T}^n) = \bigcup_{w \in \mathbb{A}_2} L_2(\mathbb{T}^n, w)$; there are also some generalizations of this result to spaces on $\mathbb{R}^n$ and also to the Hardy classes. The $A_1$-regularity property, which is equivalent to the boundedness of the Hardy–Littlewood maximal operator $M$ (see, e.g., [14] Proposition 1)), was found to be useful in the study of some properties related to the Calderón–Zygmund operators (see [14] 7 8 15 6).

The $A_p$-regularity property was introduced as a refinement of the following notion, which is related to the interpolation of Hardy-type spaces (see, e.g., [2]): a lattice $X$ is said to be BMO-regular with constants $(C, m)$ if for any $f \in X$ there exists a majorant $g \geq |f|$ such that $\|g\|_X \leq m\|f\|_X$ and $\log g(\cdot, \omega) \in \text{BMO}$ with norm of at most $C$ for almost all $\omega \in \Omega$.

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An important feature of BMO-regularity is its self-duality: under suitable assumptions a lattice $X$ is BMO-regular if and only if its order dual lattice $X'$ is also BMO-regular. For the first time this property was proved, apparently, in [1] for the case of super-reflexive spaces on the circle (see the remarks in the proof of [1] Theorem 5.12]). Later, it was extended in [10] to the general case of Banach lattices on the circle satisfying the Fatou property, by using real interpolation of Hardy-type spaces with an additional variable, and this generalization yielded the BMO-regularity of the lattices $L_1(\ell^q)$ (see also Corollary 6 below). Finally, in [14] an equivalent result on the divisibility of the BMO-regularity property was established by using only the real-variable techniques for lattices with the Fatou property on a space of homogeneous type.

We note that the proofs in both [10] and [14] are rather involved and rely on a fixed-point theorem. In [1] we give a simple and short proof of the self-duality of the BMO-regularity property for lattices on $\mathbb{R}^n$ and $\mathbb{T}$. This argument is based on well-known results of Rubio de Francia [5]; thus, in this case everything follows from the Hahn–Banach separation theorem and the Grothendieck theorem, without using fixed-point theorems or the divisibility property.

Furthermore, the results presented below yield an exact version for the self-duality of the BMO-regularity property, which can be stated in terms of the $A_p$-regularity property as follows. Recall that for every $1 < p < \infty$ the BMO-regularity of a lattice $X$ is equivalent to the $A_p$-regularity of the lattice $X^\delta$ for a sufficiently small $\delta > 0$ with suitable estimates (see, e.g., the remarks after [14, Definition 1]).

**Theorem 1.** Suppose that $X$ is a Banach lattice of measurable functions on $S \times \Omega$ satisfying the Fatou property, and let $\alpha, \beta > 0$. The following conditions are equivalent:

1. $X^{\frac{1}{1+\alpha}}$ is $A_{\frac{1}{1+\alpha}+\beta+1}$-regular;
2. $X'_{\frac{1}{1+\alpha}}$ is $A_{\frac{1}{1+\alpha}+\beta+1}$-regular.

We note that, in general, Theorem 1 fails if either $\alpha$ or $\beta$ is zero; see the paragraph below. Theorem 1 is a natural reformulation of Theorem 1 in [14] given in [1] below, which expresses this result rather concisely in terms of an $F^\alpha_\beta$-regularity property introduced therein. The proof of Theorem 1 taken in complete detail is quite elementary, and it is based only on the Hahn-Banach separation theorem, without any need for either the Grothendieck theorem or a fixed point theorem.

The main property of the Muckenhoupt weights shows at once that if $X'$ is a norming space for a lattice $X$ (e.g., if $X$ has either the Fatou property or an order continuous norm), then the $A_p$-regularity of $X'$ implies the $A_1$-regularity of $X^{\frac{1}{p}}$ (see, e.g., [14, Proposition 4]). The converse is false generally: for example, if $X = L_{\infty}$, then $X^{\frac{1}{p}} = L_{\infty}$ is an $A_1$-regular lattice for all $1 \leq p < \infty$, but $X' = L_1$ is not $A_p$-regular for any $p$ if $S = \mathbb{T}$ or $S = \mathbb{R}^n$ (see, e.g., [14, Proposition 3]). Nevertheless, we establish the following characterization whose proof is given in [2] below.

**Theorem 2.** Let $X$ be a normed lattice of measurable functions on $S \times \Omega$ such that $X'$ is a norming space for $X$. The following conditions are equivalent for all $1 < p < \infty$:

1. $X^{\frac{1}{p}} (p') = [X (\ell^1)]^{\frac{1}{p}}$ is $A_1$-regular;
2. $X'$ is $A_p$-regular.

If $X$ has the Fatou property then these conditions are also equivalent to the following.

3. Both $X^{\frac{1}{p}}$ and $(X^{\frac{1}{p}})' = X'_{\frac{1}{p}} L_1^{1-\frac{1}{p}}$ are $A_1$-regular.

Thus $A_p$-regularity of lattices is closely related to the $A_1$-regularity of some derived lattices. The $A_1$-regularity of both $Y$ and $Y'$ implies (and often characterizes) the boundedness of the Calderón–Zygmund operators in $Y$ and some other interesting properties.
The following conditions are equivalent. In this regard, the following observations should be noted, which follow immediately from the equivalence of conditions 2 and 3 of Theorem \([2]\) we also make use of the fact that the \(A_{\infty}\)-regularity of a lattice is equivalent to its \(A_p\)-regularity for sufficiently large values of \(p\).

**Corollary 3.** Suppose that a normed lattice \(Y\) of measurable functions on \(S \times \Omega\) satisfies the Fatou property and is \(p\)-convex for some (finite) \(p > 1\). The following conditions are equivalent.

1. Both \(Y\) and \(Y'\) are \(A_1\)-regular.
2. \((Y^p)'\) is \(A_p\)-regular.

**Corollary 4.** Let \(X\) be a lattice of measurable functions on \(S \times \Omega\) with the Fatou property. The following conditions are equivalent.

1. \(X'\) is \(A_{\infty}\)-regular.
2. Both \(X^\delta\) and \((X^\delta)' = X^\delta L_1^{1-\delta}\) are \(A_1\)-regular for some \(0 < \delta < 1\) (equivalently, for all sufficiently small \(\delta\)).

As an interesting example, consider the following question: for which weights \(w\), is the lattice \(X = L_1(w)\) \(A_p\)-regular? We define (as in \([14]\)) a weighted lattice \(Z(w)\) to be the set \(\{w f \mid f \in Z\}\) endowed with the norm \(\|g\|_{Z(w)} = \|gw^{-1}\|_Z\). Thus, the weighted Lebesgue spaces with the “classical” weighted norm \(\|f\| = (\int |f|^p w)\frac{1}{p}\) look like \(L_p(w^{-\frac{1}{p}})\) in our notation. It is well known that in the case of \(p = 1\) the necessary and sufficient condition is \(w^{-q} \in A_q\), and such lattices \(X\) and \(X'\) are \(A_1\)-regular only simultaneously. We have already noted that with \(q = 1\) there are no \(A_p\)-regularity in the typical cases; see, e.g., \([14]\) Proposition 3]. The equivalence of conditions 1 and 3 in Theorem \([2]\) yields (after a simple computation) the following characterization (see also Proposition \([12]\) below).

**Corollary 5.** Suppose that \(1 \leq p \leq \infty\), \(1 < q \leq \infty\), and \(w\) is a weight. Then \(X = L_q(w)\) is \(A_p\)-regular if and only if \(w^q \in A_{q^p}\).

Theorem \([2]\) allows us to refine the BMO-regularity property of the lattices \(L_\infty(\ell^q)\), which was first established, apparently, in \([10]\) in the case of \(S = \mathbb{T}\) by using the self-duality of the BMO-regularity.

**Corollary 6.** The lattices \(L_\infty(\ell^q)\) on a measurable space \(S \times \Omega \times \mathbb{Z}\) are \(A_p\)-regular for all \(1 < p, q < \infty\).

It suffices to apply implication \(3 \Rightarrow 2\) of Theorem \([2]\) to \(X = L_1(\ell^q);\) the \(A_1\)-regularity of the lattices \(X^p = L_p(\ell^{q'})\) and \((X^p)' = L_{p'}(\ell^{q'})\) is well known (see, e.g., \([9]\) Chapter 2, \(\S 1.3.1\) or Corollary \([8]\) below).

Earlier, in \([14]\), §1, Proposition 10] the \(A_p\)-regularity of the lattices \(L_\infty(\ell^q)\) was only proved for \(q > 1 + \frac{1}{p}\). We mention that (at least for \(S = \mathbb{R}^n\) or \(S = \mathbb{T}\) the result of Proposition \([6]\) is sharp in the sense of the admissible values of \(p\) and \(q\): with \(q = 1\) the conclusion of Proposition \([6]\) is false for all \(p\) (see \([14]\), §1, Proposition 10]), and its falseness for \(p = 1\) and all \(q\) follows from the nonboundedness of the maximal operator on \(L_\infty(\ell^q)\) (see, e.g., \([9]\) Chapter 2, \(\S 5.2\)).

Theorem \([2]\) also has some interesting applications concerning the boundedness of operators on lattices with an additional variable. The proof of the following result is given in \([2]\) below.

**Theorem 7.** Let \(Y\) be a normed lattice of measurable functions on \(S \times \Omega\) satisfying the Fatou property. Suppose also that \(Y\) is \(p\)-convex with some \(p > 1\). If both \(Y\) and \(Y'\) are \(A_1\)-regular, then \(Y(\ell^s)\) is also \(A_1\)-regular for all \(1 < s \leq \infty\).
It is unclear whether the p-convexity assumption is indispensable in the statement of Theorem 7; it might already follow from the assumed $A_1$-regularity of $Y$.

Combined with [7 Proposition 5], duality yields the following result. For the generalities concerning the Calderón–Zygmund operators, see, e.g., [9].

**Corollary 8.** Let $Y$ be a normed lattice of measurable functions on $S \times \Omega$ satisfying the Fatou property. Suppose also that $Y$ is p-convex and q-concave with some $1 < p, q < \infty$. If both $Y$ and $Y'$ are $A_1$-regular, then $Y(\ell^p)$ and $Y'(\ell^q)$ are also $A_1$-regular for all $1 < s < \infty$, and, thus, any Calderón–Zygmund operator is bounded in $Y(\ell^p)$ for all $1 < s < \infty$.

The main result of [7] and [8] yields yet another corollary. The definition of a nondegenerate operator can be found in [8]; we only note that the Hilbert transform on the circle and all Riesz transforms on $\mathbb{R}$ are nondegenerate.

**Corollary 9.** Suppose that $Y$ is a normed lattice of measurable functions on $\mathbb{R}^n \times \Omega$ or $\mathbb{T} \times \Omega$ such that $Y$ is p-convex and q-concave with some $1 < p, q < \infty$ and $Y$ satisfies the Fatou property. Then the boundedness of any nondegenerate Calderón–Zygmund operator $T$ on $Y$ implies the $A_1$-regularity of both $Y$ and $Y'$, and the boundedness of all Calderón–Zygmund operators on $Y(\ell^p)$ for all $1 < s < \infty$.

It is not clear for what lattices $E$ other than $E = \ell^s$ the results of Theorem 7 and its corollaries hold true. The proof suggests that this class of lattices probably includes all symmetric lattices on $\mathbb{Z}$ that are p-convex and q-concave with some $1 < p, q < \infty$ (because it is well known that such lattices are interpolation spaces for the couple $(\ell^p, \ell^q)$). Is this also true for fairly arbitrary UMD spaces $E$?

§1. **Duality and factorisable weights**

It is well known that the $A_p$ weights are characterized in terms of the P. Jones factorization theorem: $w \in A_p$ if and only if $w = w_0w_1^{1-p}$ with some weights $w_0, w_1 \in A_1$ and with some estimates for the constants; see, e.g., [9 Chapter 5, §5.3]. It is also well known that $\log w \in \BMO$ is equivalent to $w^\delta \in A_2$ for some $\delta > 0$ with some estimates for the constants (see, e.g., [9 Chapter 5, §6.2]). These observations motivate the following notions, which appear to be quite convenient for studying BMO-regularity.

**Definition 10.** Let $\alpha, \beta \geq 0$. A weight $w$ on $S \times \Omega$ is said to belong to $F_\alpha^\beta$ with constant $C$ if there exist weights $\omega_0, \omega_1 \in A_1$ with constant $C$ such that $w = \frac{\omega_0}{\omega_1}$.

**Definition 11.** Let $\alpha, \beta \geq 0$, and suppose that $X$ is a quasinormed lattice of measurable functions on $S \times \Omega$. $X$ is said to be $F_\alpha^\beta$-regular with constants $(C, m)$ if for any $f \in X$ there exists a majorant $w \in X$, $w \geq |f|$, such that $\|w\|_X \leq m\|f\|_X$ and $w \in F_\alpha^\beta$ with constant $C$.

We note that, like the Muckenhoupt classes $A_p$, the weights belonging to $F_\alpha^\beta$ and the $F_\alpha^\beta$-regularity condition have quite natural algebraic and order properties, and $F_\alpha^\beta$-regularity admits an exact version of the divisibility theorem; see [8 §3].

In the present paper, however, we shall only need the following elementary properties. An application of the Hölder inequality shows that $w \in A_1$ with the constant $C$ implies $w^\delta \in A_1$ with a constant $C^\delta$ for all $0 < \delta < 1$, so the classes are monotone in the parameters: $F_{\alpha_1}^{\beta_1} \subset F_{\alpha_2}^{\beta_2}$ for all $0 \leq \alpha_1 \leq \alpha$ and $0 \leq \beta_1 \leq \beta$ with some estimates for the constants, and the $F_{\alpha_1}^{\beta_1}$-regularity of a lattice $X$ implies its $F_{\alpha_2}^{\beta_2}$-regularity. An example of the weights $w(t) = \frac{t^n}{(t-1)^\gamma}$ on the line (with suitable generalizations for the cases of...
$\mathbb{R}^n$ and $T$) shows that the classes $F^\alpha_{\beta}$ are distinct for distinct values of the parameters $\alpha$ and $\beta$.

For all $\alpha, \beta \geq 0$, $\delta > 0$, and weights $w$, the conditions $w \in F^\alpha_{\beta}$, $w^\delta \in F^{\delta \alpha}_{\delta \beta}$ and $w^{-\delta} \in F^{\delta \beta}_{\delta \alpha}$ are equivalent. For the latter equivalence we also need to suitably clarify its meaning for the case of weights taking zero values on sets of positive measure; however, for simplicity we shall assume that all weights are nonnegative almost everywhere (we may always assume this in the $F^\alpha_{\beta}$-property when majorizing nonzero functions at the expense of an arbitrarily small increase of the constant $m$). A lattice $X$ is $F^\alpha_{\beta}$-regular if and only if $X^\delta$ is $F^{\delta \beta}_{\delta \alpha}$-regular with appropriate estimates for the constants. By the factorization theorem already mentioned above, $w \in A_p$ with some $p > 1$ if and only if $w \in F^1_{p-1}$ (in the case where $p = 1$ this equivalence is trivial), and a lattice $X$ is $A_p$-regular if and only if $X$ is $F^1_{p-1}$-regular. Accordingly, $\log w \in \text{BMO}$ if and only if $w \in F^\alpha_{\beta}$ for some some $\alpha, \beta > 0$ with suitable estimates for the constants, and a lattice $X$ is BMO-regular if and only if it is $F^\alpha_{\beta}$-regular for some $\alpha, \beta > 0$.

As a typical example, we consider the $F^\alpha_{\beta}$-regularity property for weighted Lebesgue spaces.

**Proposition 12.** Let $\alpha, \beta > 0$ and $1 < q < \infty$ be such that $\alpha q > 1$, and let $w$ be a weight. The space $L_q(w)$ is $F^\alpha_{\beta}$-regular if and only if $w \in F^{\alpha^{-\frac{1}{q}}}_{\beta^{\frac{1}{q}}}$.

Indeed, the $F^\alpha_{\beta}$-regularity of $L_q(w)$ is equivalent to the $F^1_{\frac{\alpha}{q}}$-regularity of $[L_q(w)]^\frac{1}{\alpha} = L_{\alpha q}(w^\frac{1}{\alpha})$, that is, to its $A_{\frac{\alpha}{q} + 1}$-regularity, which by Corollary 5 is equivalent to

$$w^\frac{1}{\alpha}(\alpha q') \in A_{(\alpha q')(\frac{\alpha}{q} + 1)} = A_{1 + \frac{\alpha q}{q'-1}} = F^1_{\frac{q}{q'-1}}.$$  

A simple computation shows that the latter is equivalent to $w \in F^{\alpha^{-\frac{1}{q}}}_{\beta^{\frac{1}{q}}}$.

Following [14, §2, Definition 2], we say that a mapping $T$ is $A_p$-bounded with constants $(C, m)$ if it is defined on a set $\Omega_T$ of measurable functions on $S \times \Omega$ such that the $(\nu \times \mu)$-closure of $\Omega_T$ (i.e., its closure with respect to convergence in measure on all sets of finite measure) contains $L_\infty$, and for any weight $w \in A_p$ with constant $C$ we have

$$\|T(f)\|_{L_p(w^{-\frac{1}{p}})} \leq m\|f\|_{L_p(w^{-\frac{1}{p}})}$$

for all $f \in \Omega_T$. It is well known that the maximal operator and all Calderón–Zygmund operators are $A_p$-bounded for all $1 < p < \infty$. It is easy to show that (see, e.g., [14, §2, Proposition 13]) the $A_p$-regularity of $X'$ implies (under suitable conditions) the boundedness of the $A_p$-bounded operators in $X'$, and, in particular, it implies the $A_1$-regularity of $X'$. Together with the divisibility property, this was used in [14] in order to verify the self-duality of the BMO-regularity property.

However, a similar result can be established for the lattices $XL_p$ by using the lattice product instead of duality (see also Proposition 18 below).

**Proposition 13.** Suppose that $Z$ is a quasi-normed lattice of measurable functions on $S \times \Omega$, $1 < p < \infty$, $\beta = \frac{1}{p}$, and $Z$ is $F^{1-\beta}_{\beta}$-regular with constants $(C, m)$. Then all $A_p$-bounded operators $T$ are bounded on $ZL_p$.

Indeed, due to order continuity, $ZL_p \cap \Omega_T$ is dense in $ZL_p$. Suppose that $f \in ZL_p \cap \Omega_T$ with norm 1. Then there exist $g \in Z$, $h \in L_p$ such that $f = gh$ and $\|g\|_Z \leq 2$, $\|h\|_{L_p} \leq 1$. For simplicity we may assume (see, e.g., [14, §3, Proposition 14]) that $g > 0$ almost everywhere. The $F^{1-\beta}_{\beta}$-regularity of $Z$ implies that there exists a majorant $u \geq |g|$ such
that \( \|u\|_Z \leq 2m \) and \( u \in F_1^{1-\beta} \) with constant \( C \), whence

\[
u^{-p} \in F_{\rho(1-\beta)}^\rho = F_{p-1}^1 = A_p
\]

with some constants independent of \( f \). Thus,

\[
\|Tf\|_{ZL_p} = \|u \cdot u^{-1}(Tf)\|_{ZL_p} \leq \|u\|_Z \|u^{-1}(Tf)\|_{L_p} \\
\leq 2m\|Tf\|_{L_p}\left(\frac{1}{|u^{-1}|} \right) \leq c\|f\|_{L_p}\left(\frac{1}{|u^{-1}|} \right) \\
= c\|h \cdot gu^{-1}\|_{L_p} \leq c\|h\|_{L_p} \leq c
\]

with a constant \( c \) independent of \( f \). We see that \( T \) is indeed bounded on \( ZL_p \).

Considering the case where \( Z = L_\infty(w) \) and \( T = M \) shows that the conditions of Proposition 15 are sharp in the sense that the parameters \( 1 - \beta \) and \( \beta \) cannot be replaced by larger numbers. With the help of Proposition 12 it is easy to check that in the case where \( Z = L_q(w) \) (with \( \frac{1}{q} + \frac{1}{p} < 1 \)) and \( T = M \) the converse to Proposition 13 is also true. In general, however, the \( A_1 \)-regularity of \( ZL_p \) is weaker than the \( F_1^{1-\beta} \)-regularity of \( Z \). For example, should the equivalence be true for \( Z = L_\infty(\ell^1) \), this lattice would be \( F_{\frac{p}{p}}^{\beta} \)-regular for all \( 1 < p < \infty \) and (by raising to the power \( q \)) we would have the \( F_{\frac{p}{p}}^{\beta} \)-regularity of \( L_\infty(\ell^1) \), which is false for \( \frac{q}{p'} \leq 1 \) (see [14], §1, Proposition 10).

Now we are ready to state the main result concerning the self-duality of \( F_{\alpha}^\beta \)-regularity.

**Theorem 14.** Suppose that \( X \) is a Banach lattice of measurable functions on \( S \times \Omega \) satisfying the Fatou property and \( \alpha > 1, \beta > 0 \). Then \( X \) is \( F_{\alpha}^\beta \)-regular if and only if the lattice \( X' \) is \( F_{\alpha-1}^{\beta+1} \)-regular.

As an illustration to Theorem 14, now we deduce Corollary 6 from this result. Indeed, the \( A_1 \)-regularity of \( L_t(\ell^p) \) for all \( 1 < t, s < \infty \) (see, e.g., [9], Chapter 2, §1.3.1), or Corollary 8 implies that under the assumptions of Corollary 6 the lattice \( X = L_1(\ell^p) \) is \( F_0^{1+\delta} \)-regular for any \( \delta > 0 \), which by Theorem 14 yields the \( F_1^{1-\delta} \)-regularity of \( X' = L_\infty(\ell^q) \), i.e., its \( A_p \)-regularity for all \( p = \delta + 1 > 1 \).

The proof of Theorem 14 is given in [12] below. For now we present a relatively simple argument (but with coarser estimates) that proves the self-duality of the BMO-regularity property for lattices \( X \) on spaces of homogeneous type \( S \) such that \( L_2(S) \) admits a linear operator \( T \) that is \( A_\infty \)-bounded for all \( 1 < s < \infty \) and \( A_2 \)-nondegenerate (concerning \( A_2 \)-nondegeneracy see, e.g., [14], Definition 3). For example, in the case of \( S = \mathbb{T} \) we can take the Hilbert transform \( T = H \), and in the case where \( S = \mathbb{R}^n \) any Riesz transform \( R_j \) will do for \( T \).

We shall need the following known result (for the proof in the given form and some discussion see, e.g., [14], §6).

**Theorem 15.** Suppose that a Banach lattice \( Y \) of measurable functions on a measurable space \( S \times \Omega \) has order continuous norm. If a linear operator \( T \) is bounded in \( Y_\frac{1}{2} \), then for any \( f \in Y' \) there exists a majorant \( w \geq |f| \), \( \|w\|_{Y'} \leq 2\|f\|_{Y'} \), such that \( \|T\|_{L_2(w^{-\frac{1}{2}}) \rightarrow L_2(w^{-\frac{1}{2}})} \leq C \) with a constant \( C \) independent of \( f \).

To verify the self-duality of BMO-regularity, suppose that a Banach lattice \( X \) on \( S \times \Omega \) satisfies the Fatou property and \( X \) is BMO-regular, so that it is \( F_1^{\beta} \)-regular with some \( \alpha, \beta > 0 \). We want to apply Theorem 15 to the lattice \( Y = X_\delta L_1^{1-\delta} \) and to the operator \( T \) with some sufficiently small \( 0 < \delta < 1 \). If the conditions of Theorem 15 are satisfied in this case, then by the assumed \( A_2 \)-nondegeneracy of \( T \), the lattice \( Y' = X_\delta \) is \( A_2 \)-regular, and so \( X' = Y'_{\frac{1}{2}} \) is BMO-regular.
Thus, it suffices to prove that $T$ is bounded on
\[ Y^{\frac{1}{2}} = (X^{\frac{1}{2}} L_{1}^{1-\delta})^{\frac{1}{2}} = X^{\frac{1}{2}} L_{1}^{1-\delta}. \]

For that, in its turn, it suffices to verify that $Z = X^{\frac{1}{2}}$ satisfies the conditions of Proposition 13 with $p = 2^{\frac{2}{1-\delta}}$, i.e., that $Z$ is $F^{1-\beta}_{\beta}$-regular with $\beta = \frac{1-\delta}{2}$. The latter is equivalent to the $F^{\frac{2}{1-\beta}}_{\beta}$-regularity of $X = Z^{\frac{1}{2}}$, which is the same as the $F^{\frac{1}{2}+1}_{\beta}$-regularity of $X$.

Choosing $\delta$ so small that $\frac{1}{\delta} + 1 \geq \alpha$ and $\frac{1}{\delta} - 1 \geq \beta$, we see that this assumption is satisfied.

The example of $X = L_{\infty}$ shows that the conclusion of the “only if” part of Theorem 14 is false for $\beta = 0$ and any $\alpha$, because $X' = L_{1}$ is not $A_{p}$-regular with any $p > 1$ (see, e.g., [14 §1, Proposition 3]). It is not clear, however, whether the $F^{\alpha}_{\beta}$-regularity of $X$ with $\alpha \leq 1$ provides any additional information about the BMO-regularity of $X'$.

§2. Proof of the main results

The implication $2 \Rightarrow 1$ of Theorem 2 is established in the same way as [14 §1, Proposition 4]. To verify the other implications we introduce the following construction.

We fix some sequence $\{x_{k}\}_{k \in \mathbb{Z}}$ dense in $S$. For convenience, we enumerate all balls $B_{j}$, $j \in \mathbb{Z}$, of $S$ with centers at these points and rational radii. Now we define a linear operator $\mathcal{M} = \{\mathcal{M}_{j}\}_{j \in \mathbb{Z}}$ on the functions $f = \{f_{j}\}_{j \in \mathbb{Z}}$ on $S \times \Omega \times \mathbb{Z}$ that are locally integrable in the first variable by
\[ \mathcal{M}_{j} f_{j}(\cdot, \omega) = \left[ \frac{1}{\nu(B_{j})} \int_{B_{j}} f_{j}(t, \omega) \, dt \right] \chi_{B_{j}}(\cdot) \]
for all $j \in \mathbb{Z}$ and almost all $\omega \in \Omega$. $\mathcal{M}$ is a positive linear operator closely related to the Hardy–Littlewood maximal operator $\mathcal{M}$: it is easily seen $\mathcal{M} f \leq \mathcal{M} f \leq c \mathcal{M} f$ with a constant $c$, where $\mathcal{M}$ is the noncentered Hardy–Littlewood maximal operator, and $\|\mathcal{M} f(x, \omega, \cdot)\|_{1, \infty} = \sqrt{\frac{1}{\nu(B_{j})}} \int_{B_{j}} (\mathcal{M}_{j} f(x, \omega))$ is pointwise equivalent to $\mathcal{M} f(x, \omega)$ for almost all $x \in S$ and $\omega \in \Omega$ provided $f$ is nonnegative.

We shall show that the conditions of Theorem 2 are equivalent to the following auxiliary condition.

4. $\mathcal{M}$ is bounded on $X^{\frac{1}{2}}(\ell^{p}) = [X(\ell^{1})]^{\frac{1}{2}}$.

The implication $1 \Rightarrow 4$ follows at once from the estimate $\mathcal{M} f \leq c \mathcal{M} f$. To establish $4 \Rightarrow 2$, we need the following known generalization [5 §3] of Theorem 15.

**Theorem 16.** Suppose that a Banach lattice $Y$ of measurable functions on $(S \times \Omega, \nu \times \mu)$ has order continuous norm, and let $1 < p < \infty$. If a linear operator $T : Y^{\frac{1}{2}} \rightarrow Y^{\frac{1}{2}}$ is bounded (as an operator acting in the first variable) on $Y^{\frac{1}{2}}(\ell^{p}) = [Y(\ell^{1})]^{\frac{1}{2}}$, then for any $f \in Y'$ there exists a majorant $w \geq |f|$, $\|w\|_{Y'} \leq 2\|f\|_{Y'}$, such that
\[ \|T\|_{L_{p}(w^{-\frac{1}{2}}) \rightarrow L_{p}(w^{-\frac{1}{2}})} \leq C \]
with a constant $C$ independent of $f$.

The proof is essentially contained in the proof for the case of $p = 2$ ([14 §2, Theorem 6]), we only need to replace 2 with $p$ in the arguments and make direct use of the assumption that $T$ is bounded on $Y^{\frac{1}{2}}(\ell^{p})$ rather than applying the Grothendieck theorem. We omit the details.

Now suppose that $\mathcal{M}$ is bounded on $X^{\frac{1}{2}}(\ell^{p})$ under the assumptions of Theorem 2 and let $f \in X'$, $\|f\|_{X'} = 1$; we need to construct a suitable $A_{p}$-majorant for $f$. First, we additionally assume that $X$ has order continuous norm. Let $\tilde{Y} = X(\ell^{1})$, which is a lattice of measurable functions on $S \times \Omega \times \mathbb{Z}$. Since $\mathcal{M}$ is a positive operator, $\mathcal{M}$ is bounded on
the lattice $Y\frac{1}{2}(\ell^p)$ of measurable functions on $S \times \Omega \times \mathbb{Z} \times \mathbb{Z}$ as well as on $Y\frac{1}{2}$ (see, e.g., [3, Volume 2, Proposition 1.d.9]). Then, by Theorem [16] applied to $\mathcal{M}$ and $Y$, for any function $g_k \in X'$ (to be exact, for the sequence $\{g_k\}_{j \in \mathbb{Z}}$; we construct the functions $g_k$ inductively starting with $g_0 = f$), there exists a majorant $G_{k+1} = \{g_{k+1,j}\}_{j \in \mathbb{Z}} \in Y' = X'(\ell^{\infty})$, $g_{k+1,j} \geq |g_k|$ for all $j$, such that $\left\| \int g_{k+1,j} \right\|_{Y'} = \|G_{k+1}\|_{Y'} \leq 2\|g_k\|_{X'}$ and
\[
(1) \quad \left\| \mathcal{M} \right\|_{L_p(G_{k+1}^{-\frac{1}{2}}) \to L_p(G_{k+1}^{-\frac{1}{2}})} \leq C.
\]
We choose $g_0 = f$ and set inductively
\[
g_{k+1} = \sqrt{\int g_{k+1,j}}.
\]
Now let $w = \sum_{k \geq 0} 4^{-k}g_k$. It is easily seen that $w \geq |f|$ and
\[
\left\| w \right\|_{X'} \leq \sum_{k \geq 0} 2^{-k} = 2.
\]
Estimate (1) implies
\[
(2) \quad \int |\mathcal{M}h|^p g_k \leq \int |\mathcal{M}h|^p G_{k+1} \leq C \int |h|^p G_{k+1} \leq C \int |h|^p g_{k+1}
\]
for any $h \in L_p(w^{-\frac{1}{p}})(\ell^p) \subset L_p(G_{k+1}^{-\frac{1}{2}})$. Multiplying inequalities (2) by $4^{-k}$ and summing yields
\[
(3) \quad \left\| \mathcal{M} \right\|_{L_p(w^{-\frac{1}{p}})(\ell^p) \to L_p(w^{-\frac{1}{p}})(\ell^p)} \leq 4C.
\]
Thus, by (3) we have
\[
\left\| \mathcal{M}_j \right\|_{L_p(w^{-\frac{1}{p}})(\ell^p) \to L_p(w^{-\frac{1}{p}})(\ell^p)} \leq 4C
\]
for all $j \in \mathbb{Z}$. This implies that (see the proof of [14, §3, Proposition 19])
\[
\left\| \mathcal{M}_j \right\|_{L_p(w^{-\frac{1}{p}}(\cdot, \omega) \to L_p(w^{-\frac{1}{p}}(\cdot, \omega))} \leq c
\]
for all $j \in \mathbb{Z}$ and almost all $\omega \in \Omega$ with a constant $c$ independent of $f$. Fixing such $\omega \in \Omega$ and applying this norm estimate to the functions $\chi_{B_j} h(\cdot, \omega)$ for arbitrary nonnegative $h \in L_p(w^{-\frac{1}{p}}(\cdot, \omega))$ shows that
\[
(4) \quad \left[ \frac{1}{\nu(B_j)} \int_{B_j} h(\cdot, \omega) \, d\nu(\cdot) \right]^p \int_{B_j} w(\cdot, \omega) \leq c^p \int_{B_j} [h(\cdot, \omega)]^p w(\cdot, \omega)
\]
for every $j \in \mathbb{Z}$. It is easy to check (using the local integrability of $w$ in the first variable, which follows from the estimates) that (4) implies the same estimate for arbitrary balls $B$ of $S$, which is equivalent to the fact that $w \in A_p$ with constant $c^p$ (see, e.g., [9, Chapter 5, §1.4]). Thus, $w$ is a suitable $A_p$-majorant for $f$, which proves $4 \Rightarrow 2$ under an additional assumption.

Now, we lift the assumption that the norm of $X$ is order continuous. Suppose that $\mathcal{M}$ is bounded on $Z = \left[ X(\ell^1) \right]^{\frac{1}{2}}$ under the assumptions of Theorem [2]. The boundedness of $M$ in $L_p$ implies that $\mathcal{M}$ is also bounded on $L_p(\ell^p) = \left[ L_1(\ell^1) \right]^{\frac{1}{2}}$. By complex interpolation (see, e.g., [14, Chapter 4, Theorem 1.14]), $\mathcal{M}$ is bounded on $Z^\theta \left[ L_p(\ell^p) \right]^{1-\theta} = \left[ X_\theta(\ell^1) \right]^{\frac{1}{2}}$ uniformly in $0 < \theta < 1$, where $X_\theta = X^\theta L_1^{1-\theta}$. The norm of $X_\theta$ is order continuous, and by the result already established we see that the lattices $X_\theta = X^\theta$ are $A_p$-regular uniformly in $0 < \theta < 1$. To deduce the $A_p$-regularity of $X'$ from this, we use the following proposition, which will conclude the proof of the implication $4 \Rightarrow 2$ in Theorem [2].
Proposition 17. Let $X$ be a quasi-normed lattice of measurable functions on $S \times \Omega$ such that the lattices $X^\theta$ are $A_p$-regular uniformly on $0 < \theta < 1$. Then $X$ is also $A_p$-regular.

Indeed, suppose that $f \in X$, $f \geq 0$, and $\|f\|_X = 1$; we need to show that $f$ admits a suitable $A_p$-majorant. By assumption, for every $0 < \theta < 1$ there exists a majorant $g \geq f^\theta$, $\|g\|_{X^\theta} \leq m$, such that $g \in A_p$ with a constant $C$ for some $C$ and $m$ independent of $f$. There exists $\rho > 1$ such that $g^\rho \in A_p$ with a constant $C_1$ independent of $f$ and $\theta$ (see, e.g., [9, Chapter 5, §6.1]). Setting $\theta = \frac{1}{p}$, we see that the function $h = g^{\frac{1}{p}} \in X$, $\|g\|_X \leq m^{\frac{1}{p}}$, is a suitable majorant for $f$.

Now we suppose that, under the assumptions of Theorem 2, the lattice $X$ has the Fatou property. If condition 2 is satisfied, then we have the $A_1$-regularity of $X^\frac{1}{2}$ by 2 $\Rightarrow$ 1, and the $A_1$-regularity of

$$(X^\frac{1}{2})' = X' \frac{1}{2} L_1^{\frac{1}{2}} = X' \frac{1}{2} L_p'$$

follows from Proposition 13 because the $A_p$-regularity of $X'$ is equivalent to its $F^1$-$\beta$-regularity and the $F^1_{\beta}$-$\beta$-regularity of $X' \frac{1}{2}$ with $\beta = \frac{1}{p'}$. Thus, the implication 2 $\Rightarrow$ 3 is verified.

Finally, we establish the implication 3 $\Rightarrow$ 4. The $A_1$-regularity of $X^\frac{1}{2}$ and $(X^\frac{1}{2})'$ implies at once the $A_1$-regularity of the lattices $X^\frac{1}{2}(\ell^\infty)$ and $(X^\frac{1}{2})'(\ell^\infty)$, and thus the boundedness of $M$ on these lattices. Since $M$ is a positive integral operator, its boundedness on an arbitrary lattice $Z$ is equivalent to its boundedness on $Z'$ if $Z'$ is a norming lattice for $Z$; this follows at once from the Fubini theorem and the fact that it suffices to verify the boundedness on positive functions. Therefore, $M$ is also bounded on $[(X^\frac{1}{2})'(\ell^\infty)]' = X \frac{1}{2}(\ell^1)$. The Calderón–Lozanovsky products are exact interpolation spaces for positive operators (see, e.g., [12]), so $M$ is also bounded on $X \frac{1}{2}(\ell^p) = [X \frac{1}{2}(\ell^1)]^\frac{1}{2} [X \frac{1}{2}(\ell^\infty)]^{1-\frac{1}{2}}$, which means that condition 4 is satisfied as claimed. The proof of Theorem 2 is complete.

Now we prove Theorem 7. Suppose that a Banach lattice $Y$ satisfies its assumptions: $Y$ is $p$-convex with some $p > 1$, $Y$ satisfies the Fatou property, and both $Y$ and $Y'$ are $A_1$-regular. Then $Y = X^\frac{1}{2}$ with a Banach lattice $X = Y^p$. Since $X$ satisfies condition 3 of Theorem 2, it also satisfies condition 1 of the same theorem, i.e., $X \frac{1}{2}(\ell^p) = Y(\ell^p)$ is $A_1$-regular for all values of $p > 1$ sufficiently close to 1. Since $Y(\ell^\infty)$ is also $A_1$-regular, the logarithmic convexity of the respective sets of $A_1$-majorants (or a direct application of the Hölder inequality; see, e.g., [14, §3, Proposition 16]) yields the $A_1$-regularity of $[Y(\ell^p)]^\delta [Y(\ell^\infty)]^{1-\delta} = Y(\ell^\frac{1}{2})$ for all values of $p$ sufficiently close to 1 and any $0 < \delta < 1$, which implies that the lattices $Y(\ell^s)$ are $A_1$-regular for all $1 < s < \infty$, as claimed.

Now it remains to prove Theorem 14. By symmetry, it suffices to verify the direct statement. First, we establish the following simple generalization of Proposition 13.

Proposition 18. Suppose that $Z$ is a quasinormed lattice of measurable functions on $S \times \Omega$, $1 < p < \infty$, $\beta = \frac{1}{p}$, and $Z$ is $F^1_{\beta}$-regular with constants $(C, m)$. Then $(Z L_p)(\ell^s)$ is $A_1$-regular for all $1 < s < \infty$.

Compared to the proof of Proposition 13, it suffices to observe that, by Corollary 8, the lattices $L_p(w^{-\frac{1}{p}})(\ell^s)$ are $A_1$-regular for all $w \in A_p$ and $1 < p < \infty$, $1 < s < \infty$. However, we give a complete proof for clarity.

Let $f \in (Z L_p)(\ell^s) = Z(\ell^\infty) L_p(\ell^s)$ with norm 1. Then there exist $g = \{g_j\}_{j \in Z} \in Z(\ell^\infty)$ and $h = \{h_j\}_{j \in Z} \in L_p(\ell^s)$ such that $f = gh$ and $\|\bigvee_j g_j\|_Z = \|g\|_{Z(\ell^\infty)} \leq 2$, $\|h\|_{L_p(\ell^s)} \leq 1$. For simplicity we may assume that $g > 0$ almost everywhere. By replacing $g$ with $\bigvee_j g_j$
and $h$ with $\frac{g}{\sqrt{g^2}}$ we may assume that $g$ does not depend on the last variable while retaining all estimates on its norm. By the $F^{1-\beta}_p$-regularity of $Z$, there exists a majorant $u \geq |g|$ such that $\|u\|_Z \leq 2m$ and $u \in F^{1-\beta}_p$ with constant $C$, and thus

$$u^{-p} \in F^{p\beta}_p(1-\beta) = F^{1}_p = A_p$$

with some constants independent of $f$. Therefore,

$$\|Mf\|_{(Z^{1+p})((p^*)')} = \|u \cdot u^{-1}(Mf)\|_{Z^{(1+p)}(p^*)} \leq \|\{u\}_{j \in Z} u^{-1}(Mf)\|_{L^p(p^*)} = \|u\|_Z \|Mf\|_{L^p([u^{-p}]^{-1})^p(p^*)} \leq c\|\|h\|_p^\beta \|L^p(p^*) \leq c$$

with a constant $c$ independent of $f$. Thus, the maximal operator $M$ is bounded on $(Z^{1+p})((p^*)'$, and, hence, this lattice is $A_1$-regular, as claimed.

Now suppose that $X$ is $F^{\delta}_{\gamma}$-regular with some $\alpha > 1$ and $\beta > 0$ under the assumptions of Theorem 14. We want to invoke Proposition 18 to establish the $A_1$-regularity of $Z = Y^{\frac{\delta}{\gamma}}((p^*)'$ with $Y = X^\delta L^{1-\delta}_p$ for some suitable $0 < \delta < 1$ and $1 < p < \infty$. Since $Y^{\frac{\delta}{\gamma}} = X^\delta L^{\frac{1-\delta}{p}}$, we need to check that $X^\delta$ is $F^{1-\beta}$-regular with $\beta = \frac{1-\delta}{p}$, which is equivalent to the $F^{\frac{\delta}{\gamma}}_\frac{1-\delta}{p}$-regularity of $X$. Comparing this with the assumptions of the theorem yields the conditions $\alpha = \frac{p}{\gamma} - \frac{1-\delta}{\delta}$ and $\beta = \frac{1-\delta}{\delta}$, which are satisfied with $\delta = \frac{1}{1+\beta}$ and $p = \delta (\alpha + \frac{1-\delta}{\delta}) = \frac{\alpha+\beta}{1+\beta}$. Proposition 18 gives the $A_1$-regularity of $Z$, which by implication 1 $\Rightarrow$ 2 of Theorem 2 implies the $A_p$-regularity of $Y' = X^{\delta}$. Thus, $Y'$ is $F^{1}_p$-regular, and so it is $F^{1}_p$-regular, and $X' = Y'^{\frac{\delta}{\gamma}} = Y'^{1+\beta}$ is $F^{\frac{\beta+1}{\alpha-1}}$-regular, as claimed.

**References**


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