Алгебра и анализ
Том 27 (2015), № 5

St. Petersburg Math. J.
Vol. 27 (2016), No. 5, Pages 825–840
http://dx.doi.org/10.1090/spmj/1419
Article electronically published on July 26, 2016

ON NECESSARY AND SUFFICIENT CONDITIONS FOR CONVERGENCE OF SINC-APPROXIMATIONS

A. YU. TRYNIN

ABSTRACT. Necessary and sufficient conditions are obtained for the convergence of sinc-approximations on a segment. Approximation properties of various modifications of sinc-approximations for continuous functions on a segment are studied. The Gibbs phenomenon near the ends of the segment is analyzed.

In connection with developments in signals coding theory, E. Borel and E. T. Whittaker introduced the notions of a cardinal function and a truncated cardinal function, the restriction of which to $[0, \pi]$ looks like this:

$$
\begin{aligned}
L_n(f,x) &= \sum_{k=0}^{n} \frac{\sin(nx-k\pi)}{nx-k\pi} f\Big(\frac{k\pi}{n}\Big) = \sum_{k=0}^{n} \frac{(-1)^k \sin nx}{nx-k\pi} f\Big(\frac{k\pi}{n}\Big) \\
&= \sum_{k=0}^{n} \operatorname{sinc}(nx-k\pi) f\Big(\frac{k\pi}{n}\Big) = \sum_{k=0}^{n} l_{k,n}(x) f\Big(\frac{k\pi}{n}\Big),
\end{aligned} \tag{1}
$$

where $\operatorname{sinc} t := \frac{\sin t}{t}$ by definition.

By the present time, the problem of sinc-approximation is studied rather deeply for functions analytic in a strip that contains the real axis and decaying exponentially at infinity (see, e.g., [1, 2, 3, 4, 5, 6]). The most complete survey of the results obtained in this direction before 1993, together with numerous important applications of sinc-approximations, can be found in [7]. Interesting historical surveys of investigations on this topic are contained in [8, 9].

Sinc-approximations are widely applied for constructing various numerical methods of mathematical physics and approximation theory for functions of one or several variables [7, 10, 2], in the theory of quadrature formulas [7] and Wavelet transformations, see [11, Chapter 7, §4,2], [12, Chapter 2], and [13, 14].

An interesting criterion for uniform convergence on the axis of the Whittaker cardinal functions was presented in [15]. For all functions of class

$$F^p = \big\{f \,:\, f \in L^p(\mathbb{R}) \cap C(\mathbb{R});\ \widehat{f} \in L^1(\mathbb{R}) \cap L^q(\mathbb{R})\big\}$$

(here $\frac{1}{p} + \frac{1}{q} = 1$, $1 \le p < \infty$, and $\widehat{f}$ denotes the Fourier transform of f), the following estimate is true on the real axis:

$$\Big| f(x) - \sum_{k=-\infty}^{+\infty} f\Big(\frac{k}{w}\Big) \operatorname{sinc}(wx-k) \Big| \le \sqrt{\frac{2}{\pi}} \int_{|t|\ge \pi w} \big|\widehat{f}(t)\big|\, dt, \quad x \in \mathbb{R}.$$

2010 *Mathematics Subject Classification.* Primary 41A30.
Key words and phrases. Sinc-approximations, interpolation formulas, approximation properties.
Supported by the Ministry of Education of RF (project no. 1.15.20.2014/K).

In the same paper, the case of approximation of Riemann locally integrable functions on $\mathbb{R}$ was treated and sufficient conditions for L^p-convergence of Whittaker cardinal functions were found.

A no less important condition sufficient for convergence of sinc-approximations was obtained in [16], where the authors established that, for some subclasses of functions absolutely continuous together with their derivatives on $(0,\pi)$ and having bounded variation on $\mathbb{R}$, the Kotel′nikov series (or Whittaker cardinal functions) converge uniformly inside the interval $(0,\pi)$. The author of [17] applied methods similar to those of [16], but, apparently, acted independently. He studied the convergence in $L_p(\mathbb{R})$, $1 < p < \infty$, of the Whittaker cardinal functions for various classes of functions. In [18], an estimate was obtained for the best approximation of the continuous functions that vanish at the endpoints of $[0,\pi]$ by linear combinations of sines. In the interesting papers [19, 20, 21], the discrepancy was estimated for uniform approximation of uniformly continuous and bounded functions on the entire axis by various operators presented by combinations of sines.

Unfortunately, approximation of continuous functions on a segment with the help of (1) or many other operators leads to the Gibbs phenomenon near the ends of the segment (see, e.g., [22, 23].

In [23, 24, 25, 26], various error estimates were obtained for approximation of functions analytic in the disk by the sinc expressions (1). As far as I know, before the papers [23, 24, 25, 26], approximation by the Whittaker cardinal functions on a segment or on a bounded interval was only studied for some classes of analytic functions via reduction to the case of the axis with the help of conformal mapping.

In [26], an analog of the Nevai formula suitable for the study of the operator (1) was obtained. The paper [27, 28] were devoted to necessary and sufficient conditions of the pointwise and uniform (inside $(0,\pi)$) convergence of the sinc-approximations (1) for functions continuous on $(0,\pi)$). The authors of the interesting paper [29] applied the results of [27] to the study of convergence in the case of many-level algorithms of sinc-approximations of functions with minimal smoothness.

The example constructed in [30] presents a continuous function vanishing at the ends of $[0,\pi]$ and such that the sequence of the values of the operators (1) diverges unboundedly everywhere on the interval $(0,\pi)$. The results of [30] show that, when we attempt to approximate nonsmooth continuous functions by the values of the operators (1), a resonance may arise, leading to the unbounded growth of the approximation error on the entire interval $(0,\pi)$. In the same paper, the absence of equiconvergence was established for the values of the operators (1) and the Fourier series or integrals on the class of continuous functions.

The paper [31] was devoted to the study of approximation properties of the interpolation operators constructed by solutions of Cauchy problems with second order differential expressions and generalizing the operators of the form (1). In [32], the results of [31] were applied to the study of approximation properties of the classical Lagrange interpolation processes such that each row of the matrix of interpolation nodes consists of the zeros of the Jacobi polynomials $P_n^{\alpha_n,\beta_n}$ with parameters depending on n. In the papers [33, 34], the operators considered in [31] were applied to the study of the Lagrange–Sturm–Liouville interpolation processes.

We shall denote by $C_0[0,\pi]$ the space of continuous functions that vanish at the ends of the segment, endowed with the Chebyshev norm, i.e., $C_0[0,\pi] = \{f : f \in C[0,\pi], f(0) = f(\pi) = 0\}$. In the case of the problem [31, (1.5)] with $\lambda_n = n^2$, $q_{\lambda_n} \equiv 0$, $h(\lambda_n) \neq 0$, Theorem 2 in [31] yields necessary and sufficient conditions for the uniform convergence of the sinc approximations (1) on the entire segment $[0,\pi]$ for the functions of class $C_0[0,\pi]$.

In [31, Theorem 1], a criterion of convergence at a point was obtained for the operators of the form (7); the proof of that theorem involved construction of a new operator possessing the Lagrange interpolation property and allowing one to approximate an arbitrary element of $C_0[0,\pi]$. However, the value of this operator at a node of interpolation may fail to be differentiable.

Let $V_{\rho_\lambda}[0,\pi]$ denote the ball of radius $\rho_\lambda \ge 0$ in the space of functions on $[0,\pi]$ that have bounded variation and vanish at the point 0. For every nonnegative λ, we assume that the function q_λ is such that

$$V_0^\pi[q_\lambda] \le \rho_\lambda, \quad \rho_\lambda = o\Big(\frac{\sqrt{\lambda}}{\ln\lambda}\Big) \text{ as } \lambda \to \infty, \quad q_\lambda(0) = 0. \tag{2}$$

Then, for any potential $q_\lambda \in V_{\rho_\lambda}[0,\pi]$, we consider the zeros of the solution of the Cauchy problem

$$\begin{cases} y'' + \big(\lambda - q_\lambda(x)\big)y = 0, \\ y(0,\lambda) = 1, \\ y'(0,\lambda) = h(\lambda), \end{cases} \tag{3}$$

or, under the additional condition $h(\lambda) \ne 0$, of the Cauchy problem

$$\begin{cases} y'' + \big(\lambda - q_\lambda(x)\big)y = 0, \\ y(0,\lambda) = 0, \\ y'(0,\lambda) = h(\lambda), \end{cases} \tag{4}$$

that lie in $[0,\pi]$. Enumerating these zeros in the increasing order, we denote them as follows:

$$0 \le x_{0,\lambda} < x_{1,\lambda} < \cdots < x_{n(\lambda),\lambda} \le \pi \quad (x_{-1,\lambda} < 0, x_{n(\lambda)+1,\lambda} > \pi). \tag{5}$$

(Here $x_{-1,\lambda} < 0$ and $x_{n(\lambda)+1,\lambda} > \pi$ are the zeros of the extension of the solution of (3) or (4), after extending the functions q_λ outside $[0,\pi]$ arbitrarily, with preservation of the boundedness of variation.) In the case of the Cauchy problem (4), we supplement conditions (2) with the requirement that the function $h(\lambda)$ be nonzero, i.e.,

$$V_0^\pi[q_\lambda] \le \rho_\lambda, \quad \rho_\lambda = o\Big(\frac{\sqrt{\lambda}}{\ln\lambda}\Big) \text{ as } \lambda \to \infty, \quad q_\lambda(0) = 0, \quad h(\lambda) \ne 0. \tag{6}$$

In [31], the approximation properties were studied for Lagrange type operators constructed by the solutions of Cauchy problems of the form (3) or (4); with any function f taking finite values on $[0,\pi]$, such an operator associates a continuous function that interpolates f at the nodes $\{x_{k,\lambda}\}_{k=0}^n$ and is expressed as

$$S_\lambda(f,x) = \sum_{k=0}^n \frac{y(x,\lambda)}{y'(x_{k,\lambda},\lambda)(x - x_{k,\lambda})} f(x_{k,\lambda}) = \sum_{k=0}^n s_{k,\lambda}(x) f(x_{k,\lambda}). \tag{7}$$

Excluding the trivial case of $f \equiv 0$ from consideration, we fix a positive function $\vartheta(\lambda)$ that satisfies the conditions

$$\vartheta(\lambda) = o(1), \ \lim_{\lambda\to\infty} \frac{\vartheta(\lambda)}{\omega\big(f, \frac{\pi}{\sqrt{\lambda}}\big)} = \infty; \quad \text{and put } \ \varepsilon(\lambda) = \exp\Big\{-\frac{\vartheta(\lambda)}{\omega\big(f, \frac{\pi}{\sqrt{\lambda}}\big)}\Big\}.$$

For example, for the role of $\vartheta(\lambda)$ we may take $\vartheta(\lambda) = \sqrt{\omega(f, \frac{\pi}{\sqrt{\lambda}})}$; then

$$\varepsilon(\lambda) = \exp\Big\{-\frac{1}{\sqrt{\omega\big(f, \frac{\pi}{\sqrt{\lambda}}\big)}}\Big\}.$$

For any positive λ and $x \in [0,\pi]$, we denote by p, m_1, and m_2 the integers satisfying

$$m_1 = \left[\frac{k_1}{2}\right] + 1, \quad m_2 = \left[\frac{k_2}{2}\right], \quad x_{p,\lambda} \le x < x_{p+1,\lambda}, \tag{8}$$

where k_1 and k_2 are determined by the inequalities

$$x_{k_1-1,\lambda} < x - \varepsilon(\lambda) \le x_{k_1,\lambda}, \quad x_{k_2,\lambda} \le x + \varepsilon(\lambda) < x_{k_2+1,\lambda}.$$

Here, for definiteness, it is assumed that, for an arbitrary nonnegative λ outside the segment $[0,\pi]$, the function $y(x,\lambda)$ is extended as the solution of the corresponding Cauchy problem ((3) or (4)) with the potential

$$q_\lambda(x) := \begin{cases} 0 & \text{if } x < 0, \\ q_\lambda(x) & \text{if } x \in [0,\pi], \\ q_\lambda(\pi) & \text{if } x > \pi, \end{cases}$$

and the nodes on the entire axis are enumerated as in (5).

Theorem 1 (see [31, Theorem 1])**.** *Let $f \in C_0[0,\pi]$, and suppose that q_λ and $h(\lambda)$ satisfy condition* (2) *in the case of the Cauchy problem* (3)*, or condition* (6) *in the case of problem* (4)*. Put $f(x) = 0$ for all $x \notin [0,\pi]$. Then for all operators of the form* (7) *constructed with the help of solutions of* (3)*, we have*

$$\lim_{\lambda\to\infty}\left| f(x) - S_\lambda(f,x) - \frac{\sqrt{\lambda}y(x,\lambda)}{2\pi\sqrt{\lambda+h^2(\lambda)}} \sum_{m=m_1}^{m_2}{}' \frac{f(x_{2m+1,\lambda}) - 2f(x_{2m,\lambda}) + f(x_{2m-1,\lambda})}{p-2m}\right| = 0. \tag{9}$$

uniformly in $x \in [0,\pi]$ and also uniformly with respect to $q_\lambda \in V_{\rho_\lambda}[0,\pi]$ and $h(\lambda) \in \mathbb{R}$. For the operators of the form (7) *constructed with the help of solutions of the Cauchy problem* (4)*, we have*

$$\lim_{\lambda\to\infty}\left| f(x) - S_\lambda(f,x) - \frac{\sqrt{\lambda}y(x,\lambda)}{2\pi h(\lambda)} \sum_{m=m_1}^{m_2}{}' \frac{f(x_{2m+1,\lambda}) - 2f(x_{2m,\lambda}) + f(x_{2m-1,\lambda})}{p-2m}\right| = 0, \tag{10}$$

uniformly in in$[0,\pi]$ and also uniformly with respect to $q_\lambda \in V_{\rho_\lambda}[0,\pi]$ and $h(\lambda) \in \mathbb{R}\setminus\{0\}$; the prime sign at the sums (9) *and* (10) *means that the term with zero denominator is absent. If $m_2 < m_1$* (*see* (8))*, then the sums in* (9) *and* (10) *are equal to zero.*

The brief historical information presented above can by no means be viewed as a complete survey of all publications devoted to the counting theorem or to discretization and its generalizations. All the more so, here we do not cite the papers from the hardly outlined range of works that deal with numerous applications of these topics to various fields of science.

In the present paper we use the ideas of [35, 36, 37, 38, 39, 40, 41, 42] to establish some necessary and sufficient conditions for the possibility to approximate a function continuous on $[0,\pi]$ by the sinc-approximations (1) and their modifications. The only information allowed to be used about the function in question is the set of its values at the interpolation nodes $x_{k,n} = \frac{\pi k}{n}$, $0 \le k \le n$, $n = 1,2,3,\ldots$.

Theorem 2. *For an arbitrary function f continuous on $[0,\pi]$, uniform approximation of f on the entire segment $[0,\pi]$ by the operators* (1)*,*

$$\lim_{n\to\infty} \|f - L_n(f,\cdot)\|_{C[0,\pi]} = 0,$$

is possible if and only if $f \in C_0[0,\pi]$ and the following relation is true:

$$\lim_{n\to\infty}\max_{x\in[0,\pi]}\left|\sum_{k=1}^{n-1} f(x_{k,n})\frac{(-1)^k \sin nx}{nx-k\pi}\left\{\frac{4(nx-k\pi)^2 - 2\pi^2\sin^2\left(\frac{nx-\pi k}{2}\right)}{\left(\pi^2 - 4(nx-k\pi)^2\right)}\right\}\right| = 0.$$

This statement will be proved in §2.

The results of the present paper allow us also to judge whether the system $\{l_{k,n}\}_{k=0,n=1}^{n,\ \infty}$ is complete in the spaces $C[0,\pi]$ and $C_0[0,\pi]$ (in accordance with the definition of the completeness of a system of functions as given in [43, Chapter 3, §3, item 2]).

Corollary 1. *The system* $\{l_{k,n}\}_{k=0,n=1}^{n,\ \infty}$ *is complete in* $C_0[0,\pi]$*, which agrees with the results of* [18]*. The system* $\{1,x\}\cup\{l_{k,n}\}_{k=0,n=1}^{n,\ \infty}$ *is complete in* $C[0,\pi]$.

Moreover, no linear combinations of the elements of the system $\{l_{k,n}\}_{k=0,n=1}^{n,\ \infty}$ can approximate an arbitrary function of class $C[0,\pi]$.

Theorem 3. *The linear hulls of the systems*

$$\{l_{k,n}\}_{k=0}^{n},\quad n\in\mathbb{N}, \tag{11}$$

are not dense in $C[0,\pi]$.

The proofs of these claims are also presented in §2.

We introduce the operator acting by the formula

$$\widetilde{L_n(f,x)} = L_n(f,x)+\frac{\sin nx}{2\pi}\sum_{m=m_1}^{m_2}{}' \frac{f(x_{2m+1,n})-2f(x_{2m,n})+f(x_{2m-1,n})}{[\frac{nx}{\pi}]-2m}, \tag{12}$$

where f is an arbitrary function that takes finite values on the set $\{\pi k/n\}_{k=0,n=1}^{n,\ \infty}$, $L_n(f,x)$ is the operator (1), $[\frac{nx}{\pi}]$ is the integral part of the number $\frac{nx}{\pi}$, and the prime at the sum sign means that the term with zero denominator is absent. The indices m_1 and m_2 are defined by (8), as in the case of the Cauchy problem (4), for $\lambda_n=n^2$, $h(\lambda_n)\neq 0$, $q_{\lambda_n}\equiv 0$, $x_{k,\lambda_n}=\pi k/n$. If $m_2<m_1$, then the sum in (12) equals zero.

The operator (12) possesses the interpolation property: $\widetilde{L_n(f,x_{k,n})}=f(x_{k,n})$ for all $0\le k\le n$, $n\in\mathbb{N}$. The first term on the right in (12) is a sinc-approximation operator (1). The second term in (12) compensates for the undesirable resonance if it arises when we approximate nonsmooth functions. However, the value taken by the operator (12) may fail to be differentiable at the interpolation notes $x_{k,n}=\pi k/n$.

Theorem 1 implies the following statement.

Corollary 2. *The values of the operator* (12) *approximate* $f\in C_0[0,\pi]$ *uniformly on* $[0,\pi]$*. In the case where* $f\in C[0,\pi]$*, approximation will be uniform inside the interval* $(0,\pi)$.

For the proof, see §1.

Also, we introduce the following operator, which acts on all functions with finite values at the points $\{\pi k/n\}_{k=0,n=1}^{n,\ \infty}$:

$$\begin{aligned} AT_n(f,x) &= \sum_{k=1}^{n}\frac{l_{k,n}(x)+l_{k-1,n}(x)}{2}\left\{f(x_{k,n})-\frac{(f(\pi)-f(0))k}{n}-f(0)\right\} \\ &\qquad +\frac{f(\pi)-f(0)}{\pi}x+f(0) \\ &= \sum_{k=0}^{n-1}\left\{\frac{f(x_{k+1,n})+f(x_{k,n})}{2}-\frac{(f(\pi)-f(0))(2k+1)}{2n}-f(0)\right\}l_{k,n}(x) \\ &\qquad +\frac{f(\pi)-f(0)}{\pi}x+f(0). \end{aligned} \tag{13}$$

In Lemma 3 it will be established that this operator makes it possible to approximate an arbitrary element of the space $C[0,\pi]$.

In what follows, we propose yet another modification of the operator of sinc-approximation (1). On that basis, we shall answer the following questions.

First, how to compensate for the arising of a undesirable resonance when we approximate nonsmooth functions of fractal type? An example of this phenomenon was presented in [30].

Second, can we find operators that do not have the Gibbs phenomenon near the ends of the segment $[0,\pi]$ (see, e.g., [22, 23])? For such operators, is it possible to preserve simultaneously the sufficient smoothness of their values and the interpolation property?

As before, let f be function that takes finite values on the set $x_{k,n} = \frac{\pi k}{n}$, $0 \le k \le n$, $n \in \mathbb{N}$. With any such f, we associate a function LT_n (of course, extending it by continuity to the points where removable singularities occur) defined by the formula

$$LT_n(f,x) = \frac{\pi}{2}\sum_{k=1}^{n-1}\left\{f(x_{k,n}) - \frac{\big(f(\pi)-f(0)\big)k}{n} - f(0)\right\} l_{k,n}(x) r_0(nx-k\pi) + \frac{f(\pi)-f(0)}{\pi}x + f(0) \tag{14}$$

where $r_0(t) := \frac{2\pi\cos t}{\pi^2-4t^2}$ is the Rogosinki kernel (for the definition of that kernel, see, e.g., [19]). We can use the equivalent representation

$$LT_n(f,x) \equiv \sum_{k=1}^{n-1}\left\{f(x_{k,n}) - \frac{(f(\pi)-f(0))k}{n} - f(0)\right\}\frac{(-1)^k\sin nx}{nx-k\pi}$$
$$+\sum_{k=1}^{n-1}\left\{f(x_{k,n}) - \frac{(f(\pi)-f(0))k}{n} - f(0)\right\}\frac{(-1)^k\sin nx}{nx-k\pi}\left\{\frac{4(nx-k\pi)^2-2\pi^2\sin^2\left(\frac{nx-\pi k}{2}\right)}{\pi^2-4(nx-k\pi)^2}\right\}$$
$$+\frac{f(\pi)-f(0)}{\pi}x + f(0).$$

Like in the case of the operator (12), the first term in the definition of $LT_n(f,\cdot)$ is the sinc-approximation operator (1), and the second term compensates for the undesirable resonance in case it arises when we approximate nonsmooth functions. The operator (14) keeps the advantages provided by the constructions of the operators (12) and (13), despite the fact that the value of that operator is a sufficiently smooth function (in contrast to (12)) and, simultaneously, interpolates the function to be approximated, i.e., for any $n \in \mathbb{N}$ and $0 \le k \le n$ we have $f(x_{k,n}) = LT_n(f, x_{k,n})$ (in contrast to (13)).

It should be noted that, for the operator (14), the role of information about the function f is played by its values exclusively at the modes $x_{k,n} = \frac{\pi k}{n}$, $n \in \mathbb{N}$, $k \in \mathbb{Z}$.

Now we define an operator constructed by the solutions of the Cauchy problems of the form (3) or (4). With every finite-valued function f on $[0,\pi]$, this operator associates the continuous function that interpolates f at the nodes $\{x_{k,\lambda}\}_{k=0}^{n}$ and is given by the formula

$$T_\lambda(f,x) = \sum_{k=0}^{n}\frac{y(x,\lambda)}{y'(x_{k,\lambda})(x-x_{k,\lambda})}\left\{f(x_{k,\lambda}) - \frac{f(\pi)-f(0)}{\pi}x_{k,\lambda} - f(0)\right\} + \frac{f(\pi)-f(0)}{\pi}x + f(0).$$

The trick used in the construction of the operator $T_\lambda(f,\cdot)$ (see [31, (1.9)]) allows us to get rid of the Gibbs phenomenon near the ends of $[0,\pi]$ also for approximation with the help of the operator (14).

Theorem 4. *For any function f continuous on $[0,\pi]$ we have*

$$\lim_{n\to\infty}\|f - LT_n(f,\cdot)\|_{C[0,\pi]} = 0.$$

The proof of this claim is given in §2.

§1. Approximation of functions of classes $C[0,\pi]$ and $C_0[0,\pi]$ by linear combinations

In this section we study the approximation properties of various modifications of operators of the form (1) and (7).

Proof of Corollary 2. In the case of the Cauchy problem (4) with $\lambda_n = n^2$, $h(\lambda_n) \neq 0$, $q_{\lambda_n} \equiv 0$, the operator (7) turns into (1). Therefore, relation (10) implies the uniform approximability on the entire segment $[0,\pi]$ of the elements of the space $C_0[0,\pi]$. In [27, Theorem 1] it was established that the values of the operator (12) converge uniformly inside the interval $[0,\pi]$ to the corresponding $f \in ([0,\pi]$. This proves Corollary 2. $\square$

Lemma 1 (see [26, Theorem 2]). *If f a function is continuous on $[0,\pi]$ then for all $x \in [0,\pi]$ we have*

$$\lim_{n\to\infty}\left(f(x) - L_n(f,x) - \frac{1}{2}\sum_{k=0}^{n-1}\bigl(f(x_{k+1,n}) - f(x_{k,n})\bigr)l_{k,n}(x)\right) = 0, \tag{15}$$

where

$$l_{k,n}(x) = \frac{(-1)^k \sin nx}{nx - k\pi}.$$

Convergence in (15) is pointwise on $[0,\pi]$ and uniform inside $(0,\pi)$, i.e., uniform on each compact set lying in $(0,\pi)$.

The following statement was proved in [31].

Lemma 2 (see [31, Proposition 9]). *Let $y(x,\lambda)$ be the solution of the Cauchy problem (3) or (4). For problem (3), relations (2) are fulfilled. In the case of problem (4)–(6), if $f \in C_0[0,\pi]$, then the following is true uniformly in $x \in [0,\pi]$ and uniformly will respect to all $q_\lambda \in V_{\rho_\lambda}[0,\pi]$:*

$$\lim_{\lambda\to\infty}\left(f(x) - S_\lambda(f,x) - \frac{1}{2}\sum_{k=0}^{n-1}\{f(x_{k+1,\lambda}) - f(x_{k,\lambda})\}s_{k,\lambda}(x)\right) = 0,$$

$$\lim_{\lambda\to\infty}\left(f(x) - S_\lambda(f,x) - \frac{1}{2}\sum_{k=1}^{n}\{f(x_{k-1,\lambda}) - f(x_{k,\lambda})\}s_{k,\lambda}(x)\right) = 0,$$

$$\lim_{\lambda\to\infty}\left(f(x) - S_\lambda(f,x) - \frac{1}{4}\sum_{k=1}^{n-1}\{f(x_{k+1,\lambda}) - 2f(x_{k,\lambda}) + f(x_{k-1,\lambda})\}s_{k,\lambda}(x)\right) = 0,$$

where $s_{k,\lambda}(x) = \frac{y(x,\lambda)}{y'(x_{k,\lambda},\lambda)(x - x_{k,\lambda})}$.

Corollary 3. *If $f \in C_0[0,\pi]$, then the statement of Lemma 2 with $S_{\lambda_n}(f,x) \equiv L_n(f,x)$ and $s_{k,\lambda_n}(x) \equiv l_{k,n}(x)$ is valid uniformly in $x \in [0,\pi]$.*

Proof of Corolary 3. In the case of the Cauchy problem (4) with $\lambda_n = n^2$, $h(\lambda_n) \neq 0$, $q_{\lambda_n} \equiv 0$, the operator (7) turns into (1), $S_{\lambda_n}(f,x) \equiv L_n(f,x)$, and $s_{k,\lambda_n}(x) \equiv l_{k,n}(x)$, which yields the claim. $\square$

To approximate nonsmooth continuous functions, e.g., functions f having fractal nature, we introduce the new operators $A_n(f,x)$ and $\widetilde{A}_n(f,x)$ that act by the rules

$$A_n(f,x) = \sum_{k=1}^{n}\frac{l_{k,n}(x) + l_{k-1,n}(x)}{2}f(x_{k,n}), \tag{16}$$

$$\widetilde{A}_n(f,x) = \sum_{k=0}^{n-1}\frac{f(x_{k,n}) + f(x_{k+1,n})}{2}l_{k,n}(x). \tag{17}$$

where f is an arbitrary finite-valued function on $[0,\pi]$. Note that these two operators coincide on $C_0[0,\pi]$. The construction (13) is a modification of the operators (16) and (17) after application of the trick that made it possible to get rid of the Gibbs phenomenon near the ends of $[0,\pi]$.

Lemma 3. *If $f \in C[0,\pi]$, then*

$$\lim_{n\to\infty} AT_n(f,x) = f(x) \tag{18}$$

uniformly on $[0,\pi]$.

Proof of Lemma 3. First, observe that if $f \in C_0[0,\pi]$, then, by Corollary 3 to Lemma 2, we have

$$\lim_{n\to\infty}\Big(f(x) - L_n(f,x) - \frac{1}{2}\sum_{k=0}^{n-1}\big(f(x_{k+1,n}) - f(x_{k,n})\big)l_{k,n}(x)\Big)$$
$$= \lim_{n\to\infty} f(x) - A_n(f,x) = \lim_{n\to\infty} f(x) - \widetilde{A}_n(f,x) = 0$$

uniformly on $[0,\pi]$.

To prove (18), we note that the function

$$f(x) - \frac{f(\pi)-f(0)}{\pi}x - f(0)$$

belongs to $C_0[0,\pi]$. Consequently,

$$\lim_{n\to\infty}\sum_{k=0}^{n-1}\Big\{\frac{f(x_{k+1,n})+f(x_{k,n})}{2} - \frac{\big(f(\pi)-f(0)\big)(2k+1)}{2n} - f(0)\Big\}l_{k,n}(x)$$
$$= f(x) - \frac{f(\pi)-f(0)}{\pi}x - f(0).$$

uniformly on $[0,\pi]$, i.e., (18) is true. □

The following operators, similar to (16), (17), and (13), can also be considered:

$$B_n(f,x) = \sum_{k=0}^{n-1}\frac{l_{k,n}(x)+l_{k+1,n}(x)}{2}f(x_{k,n}),$$
$$\widetilde{B}_n(f,x) = \sum_{k=1}^{n}\frac{f(x_{k-1,n})+f(x_{k,n})}{2}l_{k,n}(x),$$
$$BT_n(f,x) = \sum_{k=0}^{n-1}\frac{l_{k,n}(x)+l_{k+1,n}(x)}{2}\Big\{f(x_{k,n}) - \frac{(f(\pi)-f(0))k}{n} - f(0)\Big\}$$
$$+ \frac{f(\pi)-f(0)}{\pi}x + f(0)$$
$$= \sum_{k=1}^{n}\Big\{\frac{f(x_{k-1,n})+f(x_{k,n})}{2} - \frac{(f(\pi)-f(0))(2k-1)}{2n} - f(0)\Big\}l_{k,n}(x)$$
$$+ \frac{f(\pi)-f(0)}{\pi}x + f(0).$$

Finally, to get rid of asymmetry in the above constructions, we put

$$C_n(f,x)=\sum_{k=1}^{n-1}\frac{l_{k+1,n}(x)+2l_{k,n}(x)+l_{k-1,n}(x)}{4}f(x_{k,n}), \tag{19}$$

$$\widetilde{C}_n(f,x)=\sum_{k=1}^{n-1}\frac{f(x_{k+1,n})+2f(x_{k,n})+f(x_{k-1,n})}{4}l_{k,n}(x). \tag{20}$$

Modifying these operators so as to avoid the Gibbs phenomenon near the ends of the segment, we obtain the operators

$$CT_n(f,x)=\sum_{k=1}^{n-1}\frac{l_{k+1,n}(x)+2l_{k,n}(x)+l_{k-1,n}(x)}{4}\left\{f(x_{k,n})-\frac{(f(\pi)-f(0))k}{n}-f(0)\right\}l_{k,n}(x)+\frac{f(\pi)-f(0)}{\pi}x+f(0),$$

$$\widetilde{CT}_n(f,x)=\sum_{k=1}^{n-1}\left\{\frac{f(x_{k+1,n})+2f(x_{k,n})+f(x_{k-1,n})}{4}-\frac{(f(\pi)-f(0))k}{n}-f(0)\right\}l_{k,n}(x)+\frac{f(\pi)-f(0)}{\pi}x+f(0).$$

Remark 1. The following statement is established by analogy with Lemma 3. If $f\in C[0,\pi]$, then

$$\lim_{n\to\infty}BT_n(f,x)=f(x),$$
$$\lim_{n\to\infty}CT_n(f,x)=\lim_{n\to\infty}\widetilde{CT}_n(f,x)=f(x)$$

uniformly on $[0,\pi]$.

Unfortunately, the operators suggested above do not possess the same interpolation properties as L_n, i.e., in general, the values of the operators A_n, AT_n, B_n, BT_n, C_n, CT_n, $\widetilde{A}_n$, $\widetilde{B}_n$, $\widetilde{C}_n$, and $\widetilde{CT}_n$ may fail to coincide with the initial function at the points $x_{k,n}=\frac{k\pi}{n}$, $0\le k\le n$, $n\in\mathbb{N}$. However, their approximation properties are substantially less sensible to the smoothness of the function to be approximated; they allow us to approximate an arbitrary element of the space $C[0,\pi]$.

Remark 2. In the theory of approximation of functions by the classical algebraic polynomials, the Bernstein processes are well known. They involve the matrix of Chebyshev nodes (see [48, formula (11) and its immediate predecessor]), and, in some sense, are identical to the construction of $\widetilde{A}_n$ and $\widetilde{C}_n$ (see (17) and (20)). It should also be noted that an operator similar to C_n (see [19] was used by Sklyarov in the proof of Theorem 1 in [18], and that, after extension to the entire axis, this operator turns into the Blackman–Harris operator with $m=1$, $a_0=a_1=0,5$, see [19, formula (9)]. The methods used for the study of the approximation properties of the constructions listed above by Bernstein, Sklyarov, and the authors of [19] differ much from each other and from the approach exploited in the present paper.

§2. Proofs of the main results

The results of §1 make it possible to judge whether the system $\{l_{k,n}\}_{k=0,n=1}^{n,\ \infty}$ is complete in the normed spaces $C[0,\pi]$ and $C_0[0,\pi]$ (for the definition of the completeness of a system of functions, see [43, Chapter 3, §3, item 2]).

Proof of Corollary 1. Corollary 1 follows from Corollary 3 and Lemma 1. □

Proof of Theorem 3. We shall show that the linear hulls of the systems (11) are not dense in $C[0,\pi]$. Any system of the form (11) is a Chebyshev one (see [44, 45], i.e., the linear hulls of the functions (11) are Chebyshev spaces (see [44, Chapter 1, §2]). Indeed, first, they consist of continuous functions. Second, every generalized polynomial

$$\sum_{k=0}^{n} a_{k,n} l_{k,n}(x) = \frac{\sin nx}{\omega_n(x)} \sum_{k=0}^{n} \frac{a_{k,n}\omega_n'(x_{k,n})}{(-1)^k n} \frac{\omega_n(x)}{\omega_n'(x_{k,n})(x-x_{k,n})},$$

where $\omega_n(x) = \prod_{k=0}^{n}(x-x_{k,n})$, may have at most n zeros, because it is the product of a polynomial of degree n and the entire function $\frac{\sin nx}{\omega_n(x)}$, which does not vanish on the segment $[0,\pi]$. By the Haar theorem [44, Chapter 1, §2] or the Bernstein theorem [45, Chapter IX, §1], for any $f \in C[0,\pi]$ there exists a unique element of best approximation,

$$\Big\| f - \sum_{k=0}^{n} p_{k,n} l_{k,n} \Big\|_{C[0,\pi]} = \inf_{a_{k,n}\in\mathbb{R}} \Big\| f - \sum_{k=0}^{n} a_{k,n} l_{k,n} \Big\|_{C[0,\pi]} = E_n(f).$$

Consider the function $f \equiv 1$; then, for $n \geq 2$, we have

$$\Big| \sum_{k=0}^{n} p_{k,n} l_{k,n}\Big(\frac{\pi}{2n}\Big) - \sum_{k=0}^{n} p_{k,n} l_{k,n}\Big(\frac{\pi}{2n} + \frac{2\pi}{n}\Big) \Big| \leq 2E_n(1).$$

Since the systems (11) and $\{x_{k,n}\}_{k=0}^{n}$, $n \in \mathbb{N}$, are biorthogonal, we have $1 - E_n(1) \leq p_{k,n} \leq 1 + E_n(1)$ for all $0 \leq k \leq n$, $n \in \mathbb{N}$. If there exists a sequence $n_i \nearrow \infty$ $(i \to \infty)$ such that $E_{n_i}(1) \geq 1$, then Theorem 3 is proved. Otherwise, we estimate the difference

$$\begin{aligned}
2E_n(1) &\geq \sum_{k=0}^{n} p_{k,n} l_{k,n}\Big(\frac{\pi}{2n}\Big) - \sum_{k=0}^{n} p_{k,n} l_{k,n}\Big(\frac{\pi}{2n} + \frac{2\pi}{n}\Big) \\
&= \frac{8}{\pi}\Big\{ p_{0,n}\frac{1}{5} + p_{1,n}\frac{1}{3} - p_{2,n}\frac{1}{3} - \sum_{j=0}^{n-3} \frac{(-1)^j p_{j+3,n}}{(2j+5)(2j+1)} \Big\} \\
&\geq \frac{8}{\pi}\Big\{ \big(1 - E_n(1)\big)\frac{1}{5} + \big(1 - E_n(1)\big)\frac{1}{3} - \big(1 + E_n(1)\big)\frac{1}{3} \\
&\quad + \big(1 - E_n(1)\big) \sum_{m=0}^{[\frac{n-3}{2}]} \frac{1}{(4m+7)(4m+3)} - \big(1 + E_n(1)\big) \sum_{m=0}^{[\frac{n-3}{2}]+1} \frac{1}{(4m+1)(4m+5)} \Big\}.
\end{aligned}$$

Suppose that

$$E_n(1) \to 0 \text{ as } n \to \infty. \tag{21}$$

Since

$$\sum_{m=0}^{\infty} \frac{1}{(4m+1)(4m+5)} = \frac{1}{4}, \qquad \sum_{m=0}^{\infty} \frac{1}{(4m+3)(4m+7)} = \frac{1}{12},$$

(see [46, §5.1.11, items 4, 14]), the limit passage as $n \to \infty$ yields a contradiction with (21). Consequently, no linear combination of functions of the form (11) can approximate even the function $f \equiv 1$ uniformly on $[0,\pi]$. Theorem 3 is proved. □

Lemma 4 (see [27, Lemma 1]). *For all $x \in [0,\pi]$ and $n \in \mathbb{N}$ we have*

$$\sum_{k=1}^{n} \big| l_{k,n}(x) + l_{k-1,n}(x) \big| \leq 4\Big(1 + \frac{1}{\pi}\Big),$$

where

$$l_{k,n}(x) = \frac{(-1)^k \sin nx}{nx - k\pi}.$$

Lemma 4 shows that the sequence of the Lebesgue constants of the operators A_n as in (16) is bounded:

$$\|A_n\|_{C[0,\pi]\to C[0,\pi]} \le 2\Big(1+\frac{1}{\pi}\Big) \quad \text{for all} \quad n\in\mathbb{N}.$$

Unfortunately, this fact does not imply that, e.g., relation (22) is true. Indeed, the Banach–Steinhaus theorem (see [47, Chapter 4, Theorem 2]) says that we also need to find a subset M_0 in $C_0[0,\pi]$ such that its linear hull is dense in $C_0[0,\pi]$ and

$$\lim_{n\to\infty} A_n(f,x) = f(x)$$

uniformly on $[0,\pi]$ for all $f\in M_0$.

Nevertheless, the following statement is true.

Lemma 5. *Let $f\in C[0,\pi]$. Then*

$$\lim_{n\to\infty} A_n(f,x) = \lim_{n\to\infty} \widetilde{A}_n(f,x) = f(x). \tag{22}$$

uniformly inside $(0,\pi)$, i.e., uniformly on each compact set lying in the interval $(0,\pi)$. This convergence is uniform on $(0,\pi)$ if and only if $f\in C_0[0,\pi]$.

Proof of Lemma 5. We prove (22) for an arbitrary function continuous on $[0,\pi]$. Using the definitions (16) and (17), we reshape the left-hand side of (15) as follows:

$$\begin{aligned}
&\lim_{n\to\infty}\bigg(f(x) - L_n(f,x) - \frac{1}{2}\sum_{k=0}^{n-1}\Big(f(x_{k+1,n}) - f(x_{k,n})\Big) l_{k,n}(x)\bigg)\\
&\quad= \lim_{n\to\infty}\big(f(x) - \widetilde{A}_n(f,x) - f(\pi) l_{n,n}(x)\big)\\
&\quad= \lim_{n\to\infty}\Big(f(x) - A_n(f,x) - \frac{f(\pi)}{2} l_{n,n}(x) - \frac{f(0)}{2} l_{0,n}(x)\Big).
\end{aligned} \tag{23}$$

Let $[a,b]\subset(0,\pi)$. By Lemma 1, relation (15) is true uniformly on $[a,b]$, i.e., the limits on $[a,b]$ in (23) are uniform and equal to zero. It remains to observe that for all $x\in[a,b]$ we have

$$\big|f(\pi) l_{n,n}(x)\big| \le \|f\|_{C[0,\pi]}\frac{1}{n(\pi-b)} \to 0 \text{ as } n\to\infty,$$

$$\big|f(0) l_{0,n}(x)\big| \le \|f\|_{C[0,\pi]}\frac{1}{na} \to 0 \text{ as } n\to\infty.$$

Now, let $f\in C_0[0,\pi]$. Then, by Lemma 2, we have

$$\begin{aligned}
&\lim_{n\to\infty}\bigg(f(x) - L_n(f,x) - \frac{1}{2}\sum_{k=0}^{n-1}\Big(f(x_{k+1,n}) - f(x_{k,n})\Big) l_{k,n}(x)\bigg)\\
&\qquad\qquad = \lim_{n\to\infty} f(x) - A_n(f,x) = \lim_{n\to\infty} f(x) - \widetilde{A}_n(f,x) = 0.
\end{aligned}$$

uniformly on $[0,\pi]$. We have proved that convergence in (22) will be uniform whenever $f\in C_0[0,\pi]$.

The converse statement is a consequence of Theorem 3. Lemma 5 is proved. □

Remark 3. Similarly we can show that if $f\in C[0,\pi]$, then

$$\lim_{n\to\infty} B_n(f,x) = \lim_{n\to\infty} \widetilde{B}_n(f,x) = f(x), \tag{24}$$

$$\lim_{n\to\infty} C_n(f,x) = \lim_{n\to\infty} \widetilde{C}_n(f,x) = f(x) \tag{25}$$

uniformly inside $[0,\pi]$. Convergence in (24) and (25) is uniform on $[0,\pi]$ if and only if $f\in C_0[0,\pi]$.

Remark 4. In the construction of the operators AT_n, BT_n, CT_n, $\widetilde{CT}_n$, instead of the pair of functions $\{1, x\}$, the system $\{l_{k,n}\}_{k=0,n=1}^{n,\ \infty}$ can be supplemented by another convenient pair of linearly independent functions, e.g., by $\{l_{0,1}, l_{1,1}\}$.

Remark 5. Beside the operators (16), (17), (19), (20), we can consider, e.g., operators with "symmetric terms" of the form

$$\sum_{k=1}^{n-1} \frac{f(x_{k+1,n}) + f(x_{k-1,n})}{2} l_{k,n}(x) \quad \text{or} \quad \sum_{k=1}^{n-1} \frac{l_{k+1,n}(x) + l_{k-1,n}(x)}{2} f(x_{k,n}).$$

Then, to ensure that their values converge to the function f in question, we shall need some adequate necessary and sufficient conditions (e.g., those stated in [27, Theorems 1 and 2]).

Proof of Theorem 4. For any natural n, we estimate the modulus of the deviation from f of the value of the operator (14). Observe that for $k = 0$ and $k = n$ we have

$$f(x_{k,n}) - \frac{(f(\pi) - f(0))k}{n} - f(0) = 0.$$

Therefore, we can write

$$\begin{aligned}
&|f(x) - LT_n(f,x)| \\
&= \left| f(x) - \left[\frac{\pi}{2} \sum_{k=1}^{n-1} \left\{ f(x_{k,n}) - \frac{(f(\pi) - f(0))k}{n} - f(0) \right\} l_{k,n}(x) r_0(nx - k\pi) \right.\right. \\
&\qquad \left.\left. + \frac{f(\pi) - f(0)}{\pi} x + f(0)) \right] \right| \\
&= \left| f(x) - \left[\sum_{k=1}^{n-1} \Big(f(x_{k,n}) - \frac{(f(\pi) - f(0))k}{n} - f(0) \Big) \left\{ \frac{\pi^2 \sin 2nx}{2(nx - k\pi)\big(\pi^2 - 4(nx - k\pi)^2\big)} \right\} \right.\right. \\
&\qquad \left.\left. + \frac{f(\pi) - f(0)}{\pi} x + f(0) \right] \right|.
\end{aligned}$$

We reshape the ratio in braces:

$$\begin{aligned}
&|f(x) - LT_n(f,x)| \\
&= \left| f(x) - \left[\sum_{k=1}^{n-1} \frac{\big(f(x_{k,n}) - \frac{(f(\pi)-f(0))k}{n} - f(0) \big)}{2} \left\{ \frac{2\cos nx \sin nx}{nx - k\pi} - \frac{\sin nx \cos nx}{nx - (k + \frac{1}{2})\pi} \right\} \right.\right. \\
&\qquad \left.\left. - \sum_{k=1}^{n-1} \frac{\big(f(x_{k,n}) - \frac{(f(\pi)-f(0))k}{n} - f(0) \big)}{2} \frac{\sin nx \cos nx}{nx - (k - \frac{1}{2})\pi} + \frac{f(\pi) - f(0)}{\pi} x + f(0) \right] \right|.
\end{aligned}$$

Clanging the summation index in the second sum and rearranging the terms, we get the representation

$$\begin{aligned}
&|f(x) - LT_n(f,x)| \\
&= \left| f(x) - \left[\cos nx \sum_{k=0}^{n-1} \left\{ f(x_{2k,2n}) - \frac{(f(\pi) - f(0))2k}{2n} - f(0) \right\} \frac{(-1)^{2k} \sin nx}{nx - k\pi} \right.\right. \\
&\qquad + \sin nx \sum_{k=0}^{n-1} \left\{ \frac{f(x_{2k+2,2n}) - \frac{(f(\pi)-f(0))(2k+2)}{2n} + f(x_{2k,2n}) - \frac{(f(\pi)-f(0))2k}{2n}}{2} - f(0) \right\} \\
&\qquad \left.\left. \times \frac{(-1)^{2k+1} \cos nx}{nx - (k + \frac{1}{2})\pi} + \frac{f(\pi) - f(0)}{\pi} x + f(0) \right] \right|.
\end{aligned}$$

To continue calculation, we add and subtract the following expression:

$$\sum_{k=0}^{n-1}\left\{\frac{f(x_{2k+1,2n})-\frac{(f(\pi)-f(0))(2k+1)}{2n}-f(x_{2k,2n})+\frac{(f(\pi)-f(0))2k}{2n}}{2}\right\} \times\left(\frac{(-1)^{2k}\sin 2nx}{2nx-2k\pi}+\frac{(-1)^{2k+1}\sin 2nx}{2nx-(2k+1)\pi}\right).$$

After further transformations, we get the formula

$$\begin{aligned}
&|f(x)-LT_n(f,x)|\\
&=\Bigg|f(x)-\Bigg[\sum_{k=0}^{n-1}\left\{f(x_{2k,2n})-\frac{(f(\pi)-f(0))2k}{2n}-f(0)\right\}\frac{(-1)^{2k}\sin 2nx}{2nx-2k\pi}\\
&\quad+\sum_{k=0}^{n-1}\left\{\frac{f(x_{2k+1,2n})-\frac{(f(\pi)-f(0))(2k+1)}{2n}-f(x_{2k,2n})+\frac{(f(\pi)-f(0))2k}{2n}}{2}\right\}\\
&\qquad\times\left(\frac{(-1)^{2k}\sin 2nx}{2nx-2k\pi}+\frac{(-1)^{2k+1}\sin 2nx}{2nx-(2k+1)\pi}\right)\\
&\quad+\sum_{k=0}^{n-1}\left\{\frac{f(x_{2k+2,2n})-\frac{(f(\pi)-f(0))(2k+2)}{2n}+f(x_{2k,2n})-\frac{(f(\pi)-f(0))2k}{2n}}{2}-f(0)\right\}\\
&\qquad\times\frac{(-1)^{2k+1}\sin 2nx}{2nx-(2k+1)\pi}+\frac{f(\pi)-f(0)}{\pi}x+f(0)\Bigg]\\
&\quad+\sum_{k=0}^{n-1}\left\{\frac{f(x_{2k+1,2n})-f(x_{2k,2n})}{2}-\frac{(f(\pi)-f(0))}{4n}\right\}\\
&\qquad\times\left(\frac{(-1)^{2k}\sin 2nx}{2nx-2k\pi}+\frac{(-1)^{2k+1}\sin 2nx}{2nx-(2k+1)\pi}\right)\Bigg|.
\end{aligned}$$

Now we reshape the second term of the sum in brackets and also add and subtract the expression

$$f(x_{2k+1,2n})-\frac{(f(\pi)-f(0))(2k+1)}{2n}-f(0)$$

in the numerator of the first factor in the third term of the same sum. After rearranging, we obtain

$$\begin{aligned}
&|f(x)-LT_n(f,x)|\\
&=\Bigg|f(x)-\Bigg[\sum_{k=0}^{n-1}\left\{\frac{f(x_{2k+1,2n})-\frac{(f(\pi)-f(0))(2k+1)}{2n}+f(x_{2k,2n})-\frac{(f(\pi)-f(0))2k}{2n}}{2}-f(0)\right\}\\
&\qquad\times\frac{(-1)^{2k}\sin 2nx}{2nx-2k\pi}\\
&+\sum_{k=0}^{n-1}\left\{\frac{f(x_{2k+2,2n})-\frac{(f(\pi)-f(0))(2k+2)}{2n}+f(x_{2k+1,2n})-\frac{(f(\pi)-f(0))(2k+1)}{2n}}{2}-f(0)\right\}\\
&\qquad\times\frac{(-1)^{2k+1}\sin 2nx}{2nx-(2k+1)\pi}+\frac{f(\pi)-f(0)}{\pi}x+f(0)\Bigg]\\
&+\sum_{k=0}^{n-1}\left\{\frac{f(x_{2k+1,2n})-f(x_{2k,2n})}{2}-\frac{f(\pi)-f(0)}{4n}\right\}\left(\frac{(-1)^{2k}\sin 2nx}{2nx-2k\pi}+\frac{(-1)^{2k+1}\sin 2nx}{2nx-(2k+1)\pi}\right)\Bigg|.
\end{aligned}$$

Uniting the first and second sums in brackets, we get the identity

$$|f(x)-LT_n(f,x)|$$
$$=\Bigg|f(x)-\Bigg[\sum_{j=0}^{2n-1}\left\{\frac{f(x_{j+1,2n})+f(x_{j,2n})}{2}-\frac{(f(\pi)-f(0))(2j+1)}{4n}-f(0)\right\}$$
$$\times\frac{(-1)^j\sin 2nx}{2nx-j\pi}+\frac{f(\pi)-f(0)}{\pi}x+f(0)\Bigg]$$
$$+\sum_{k=0}^{n-1}\left\{\frac{f(x_{2k+1,2n})-f(x_{2k,2n})}{2}-\frac{(f(\pi)-f(0))}{4n}\right\}$$
$$\times\left(\frac{(-1)^{2k}\sin 2nx}{2nx-2k\pi}+\frac{(-1)^{2k+1}\sin 2nx}{2nx-(2k+1)\pi}\right)\Bigg|.$$

By the definition of the operator (13), we have

$$|f(x)-LT_n(f,x)|$$
$$=\Bigg|f(x)-AT_{2n}(f,x)+\sum_{k=0}^{n-1}\left\{\frac{f(x_{2k+1,2n})-f(x_{2k,2n})}{2}-\frac{(f(\pi)-f(0))}{4n}\right\}$$
$$\times\left(\frac{(-1)^{2k}\sin 2nx}{2nx-2k\pi}+\frac{(-1)^{2k+1}\sin 2nx}{2nx-(2k+1)\pi}\right)\Bigg|.$$

Now we use the triangle inequality to estimate the error of approximation of an arbitrary continuous function f by the values of the operator (14):

$$|f(x)-LT_n(f,x)|$$
$$\leq|f(x)-AT_{2n}(f,x)|$$
$$+\left|\sum_{k=0}^{n-1}\left\{\frac{f(x_{2k+1,2n})-f(x_{2k,2n})}{2}-\frac{f(\pi)-f(0)}{4n}\right\}\left(\frac{(-1)^{2k}\sin 2nx}{2nx-2k\pi}+\frac{(-1)^{2k+1}\sin 2nx}{2nx-(2k+1)\pi}\right)\right|$$
$$\leq|f(x)-AT_{2n}(f,x)|$$
$$+\left\{\frac{1}{2}\omega\left(f,\frac{\pi}{2n}\right)+\left|\frac{f(\pi)-f(0)}{4n}\right|\right\}\sum_{k=0}^{2n-1}\left|\frac{(-1)^{k+1}\sin 2nx}{2nx-(k+1)\pi}+\frac{(-1)^k\sin 2nx}{2nx-k\pi}\right|.$$

Using Lemmas 3 and 4, we get the following estimate uniform on the entire segment $[0,\pi]$:

$$|f(x)-LT_n(f,x)|\leq|f(x)-AT_{2n}(f,x)|+\left\{\frac{1}{2}\omega\left(f,\frac{\pi}{2n}\right)+\left|\frac{f(\pi)-f(0)}{4n}\right|\right\}4\left(1+\frac{1}{\pi}\right)=o(1).$$

Theorem 4 is proved. □

Proof of Theorem 2. Now Theorem 2 follows from Theorems 4 and 3. □

References

[1] A. S. Zhuk and V. V. Zhuk, *Some orthogonalities in approximation theory*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **314** (2004), 83–123; English transl., J. Math. Sci. (N.Y.) **133** (2006), no. 6, 1652–1675. MR2119736

[2] F. Stenger, S. A. Gustafson, B. Keyes, M. O'Reilly, and K. Parker, *ODE-IVP-PACK via sinc indefinite integration and Newton's method*, Numer. Algorithms **20** (1999), no. 2–3, 241–268. MR1709542

[3] A. Ignjatovic, *Local approximations based on orthogonal differential operators*, J. Fourier Anal. Appl. **13** (2007), no. 3, 309–330. MR2334612

[4] G. Schmeisser and F. Stenger, *Sinc approximation with a Gaussian multiplier*, Sampl. Theory Signal Image Process. **6** (2007), no. 2, 199–221. MR2343406

[5] A. Gelb, *Reconstruction of piecewise smooth functions from non-uniform grid point data*, J. Sci. Comput. **30** (2007), no. 3, 409–440. MR2295478

[6] M. H. Annaby and M. M. Tharwat, *Sinc-based computations of eigenvalues of Dirac systems*, BIT **47** (2007), no. 4, 699–713. MR2358366

[7] F. Stenger, *Numerical methods based on sinc and analytic functions*, Springer Ser. Comput. Math., vol. 20, Springer-Verlag, New York, 1993. MR1226236

[8] P. L. Butzer, *A retrospective on 60 years of approximation theory and associated fields*, J. Approx. Theory **160** (2009), no. 1–2, 3–18.
MR2558010

[9] J. R. Higgins, *Five short stories about the cardinal series*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), no. 1, 45–89. MR766960

[10] M. T. Alquran and K. Al-Khaled, *Numerical comparison of methods for solving systems of conservation laws of mixed type*, Int. J. Math. Anal. **5** (2011), no. 1, 35–47. MR2776542

[11] B. S. Kashin and A. A. Saakyan, *Orthogonal series*, 2nd ed., Izdat. Nauchno-Issledov. Actuarno-Finansov. Tsentra (AFTs), Moscow, 1999; English transl., Transl. Math. Monogr., vol. 75, Amer. Math. Soc., Providence, RI, 1989. MR1007141

[12] I. Daubechies, *Ten lectures on wavelets*, CBMS-NSF Reg. Conf. Ser. Appl. Math., vol. 61, SIAM, Philadelphia, 1992. MR1162107

[13] I. Ya. Novikov and S. B. Stechkin, *Basis constructions of wavelets*, Fundam. Prikl. Mat. **3** (1997), no. 4, 999–1028. (Russian) MR1794498

[14] ———, *Fundamentals of wavelet theory*, Uspekhi Mat. Nauk **53** (1998), no. 6, 53–128; English transl., Russian Math. Surveys **53** (1998), no. 6, 1159–1231. MR1702694

[15] P. L. Butzer, J. R. Higgins, and R. L. Stens, *Classical and approximate sampling theorems*: *studies in the $L^p(\mathbb{R})$ and the uniform norm*, J. Approx. Theory **137** (2005), no. 2, 250–263. MR2186950

[16] A. I. Smukler and T. A. Sulman, *Certain properties of Kotek'nikov series*, Izv. Vyssh. Uchebn. Laved. Mat. **1974**, no. 3, 93–103; Enngliish transl., Soviet Math. (zv. VUZ. Math.) **18** (1974), no. 3, 81–90. MR0361591

[17] F. Gensun, *Whittaker–Kotelnikov–Shannon sampling theorem and Aliasing error*, J. Approx. Theory **85** (1996), no. 2, 115–131. MR1385811

[18] V. P. Sklyarov, *On the best uniform sinc-approximation on a finite interval*, East J. Approx. **14** (2008), no. 2, 183–192. MR2422905

[19] A. Kivinukk and G. Tamberg, *On Blackman–Harris windows for Shannon sampling series*, Sampl. Theory Signal Image Process. **6** (2007), no. 1, 87–108. MR2296795

[20] ———, *Interpolating generalized Shannon sampling operators, their norms and approximation properties*, Sampl. Theory Signal Image Process. **8** (2009), no. 1, 77–95. MR2532205

[21] G. Schmeisser, *Interconnections between multiplier methods and window methods in generalized sampling*, Sampl. Theory Signal Image Process. **9** (2010), no. 1–3, 1–24. MR2814339

[22] A. J. Jerri, *Lanczos-like σ-factors for reducing the Gibbs phenomenon in general orthogonal expansions and other representations*, J. Computat. Anal. Appl. **2**(2000), no. 2, 111–127. MR1765299

[23] A. Yu. Trynin and V. P. Sklyarov, *Error of sinc approximation of analytic functions on an interval*, Sampl. Theory Signal Image Process. **7** (2008), no. 3, 263–270. MR2493858

[24] A. Yu. Trynin, *Approximation of analytic functions by Lagrange–Sturn-Liouville operators*, Tenth Saratov Winter School (27 Denuary 2 february, 2000), Abstracts, Saratov. Univ., Saratov, 2000, pp. 140–141. (Russian)

[25] ———, *The estimate of approximation of analytic functions by the sinc-interpolation operator*, Matematika. Mekhanika, vyp. 7, Saratov. Univ., Saratov., 2005, pp. 124–127. (Russian)

[26] ———, *Estimates for Lebesgue functions and the Nevai formula for the sinc approximation of continuous functions on an interval*, Sibirsk. Mat. Zh. **48** (2007), no. 4, 1155–1166; English transl., Sib. Math. J. **48** (2007), no. 5, 929–938. MR2364635

[27] ———, *Criteria for pointwise and uniform convergence of sine approximations of continuous functions on an interval*, Mat. Sb. **198** (2007), no. 10, 141–158; English transl., Sb. Math. **198** (2007), no. 9–10, 1517–1534. MR2362826

[28] ———, *A criterion for the uniform convergence of sine-approximations on an interval*, Izv. Vyssh. Uchebn. Laved. Mat. **2008**, no. 6, 66–78; English transl., Russian Math. (Iz. VUZ) **52** (2008), no. 6, 58–69. MR2467410

[29] O. E. Livne and A. E. Brandt, *MuST: The multilevel sinc transform*, SIAM J. Sci. Comput. **33** (2011), no. 4, 1726–1738. MR2821266

[30] A. Yu. Trynin, *On the divergence of sinc approximations everywhere on* $(0, \pi)$, Algebra i Analiz **22** (2010), no. 4, 232–256; English transl., St. Petersburg Math. J. **22** (2011), no. 4, 683–701. MR2768966

[31] ———, *A generalization of the Whittaker–Kotel'nikov–Shannon sampling theorem for continuous functions on an interval*, Mat. Sb. **200** (2009), no. 11, 61–108; English transl., Sb. Math. **200** (2009), no. 11–12, 1633–1679. MR2589998

[32] ———, *On operators of interpolation with respect to solutions of the Cauchy problem and Lagrange–Jacobi polynomials*, Izv. Ross. Akad. Nauk Ser. Mat. **75** (2011), no. 6, 129–162; English transl., Izv. Math. **75** (2011), no. 6, 1215–1248. MR2918896

[33] ———, *On the absence of stability of interpolation in eigenfunctions of the Sturm–Liouville problem*, Izv. Vyssh. Uchebn. Zaved. Mat. **2000**, no. 9, 60–73; English transl., Russian Math. (Iz. VUZ) **44** (2000), no. 9, 58–71. MR1801962

[34] ———, *On the divergence of Lagrange interpolation processes in eigenfunctions of the Sturm–Liouville problem*, Izv. Vyssh. Uchebn. Zaved. Mat. **2010**, no. 11, 74–85; Russian Math. (Iz. VUZ) **54** (2010), no. ‘, 66–76. MR2814566

[35] B. I. Golubov, *Spherical jump of a function and the Bochner–Riesz means of conjugate multiple Fourier series and Fourier integrals*, **91** (2012), no. 4, 506–514; English transl., Math. Notes **91** (2012), no. 3–4, 479–486. MR3201450

[36] ———, *Absolute convergence of multiple Fourier series*, Mat. Zametki **37** (1985), no. 1, 13–24; English transl., Math. Notes **37** (1985), no. 1, 8–15. MR792229

[37] M. I. D′yachenko, *On a class of methods for the summation of multiple Fourier series*, Mat. Sb. **204** (2013), no. 3, 3–18; English transl., Sb. Math. **204** (2013), no. 3–4, 307–322. MR3088097

[38] ———, *Uniform convergence of hyperbolic partial sums of multiple Fourier series*, Mat. Zametki **76** (2004), no. 5, 723–731; English transl., Math. Notes **76** (2004), no. 5–6, 673–681. MR2129338

[39] E. S. Polovinkin, *The integration of multivalued mappings*, Dokl. Akad. Nauk SSSR **271** (1983), no. 5, 1069–1074; English transl., Soviet Math. Dokl. **28** (1983), no. 1, 223–228. MR722023

[40] ———, *On some properties of the derivatives of multivalued mappings*, Tr. Moscov. Fis.-Techn. Inst. **4** (2012), no. 4, 141–154. (Russian)

[41] M. A. Skopina, *Orthogonal polynomial Schauder bases in* $\mathrm{C}[-1, 1]$ *with optimal growth of degrees*, Mat. Sb. **192** (2001), no. 3, 115–136; English transl., Sb. Math. **192** (2001), no. 3–4, 433–454. MR1836309

[42] M. A. Skopina and I. E. Maksimenko, *Multidimensional periodic wavelets*, Algebra i Analiz **15** (2003), no. 2, 1–39; English transl., St. Petersburg Math. J. **15** (2004), no. 2, 165–190. MR2052130

[43] A. N. Kilmogorov and S. V. Fomin, *Elements of the theory of functions and functional analysis*, Nauka, Moscow, 1981; English transl., First ed., Graylock Press, Rochester, NY, 1957. MR630899

[44] I. K. Daugavet, *Introduction to the theory of approximation of functions*, Leningrad. Univ., Leningrad, 1977. (Russian) MR0470560

[45] S. Karlin and W. Studden, *Tchebycheff systems: with applications in analysis and statistics*, Pure Appl. Mat., vol. 15, Intersei. Publ., New York, 1966. MR0204922

[46] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and series. Elementary functions*, Nauka, Moscow, 1981; English transl., Gordon & Breach Sci. Publ., New York, 1986. MR0874986 (88f:00013)

[47] L. A. Lusternik and V. I. Sobolev, *Elements of functional analysis*, Nauka, Moscow, 1965; English transl., Halsted Press, New York, 1974. MR0350359

[48] S. N. Bernshtein, *On a modification of the Largange interpolation formula*, Collected works, vol. 2, Izdat. Akad. Nauk SSSR, Moscow, 1954, pp. 130–140. (Russian)

Saratov State University, Astrakhanskaya str. 83, Saratov 410012, Russia
E-mail address: atrynin@gmail.com

Received 25/JUL/2014

Translated by A. PLOTKIN