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SMOOTHNESS OF A CONFORMAL MAPPING ON A SUBSET OF THE BOUNDARY

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Abstract. A conformal mapping f of the unit disk onto a Jordan domain G is considered. The boundary of G has the following structure. Another Jordan domain H is fixed whose boundary has Hölder smoothness $a > 1$, and a countable family of open arcs dense in the boundary is specified. G is obtained by replacement of each of these distinguished arcs with a Hölder arc of smoothness b, $1 < b < a$, having the same end-points. Thus, G has Hölder smoothness b. It is shown that if the lengths of the distinguished arcs decay sufficiently fast (depending on a and b), the function f still has Hölder smoothness a on a set of positive measure on the unit circle. The numbers a and b are assumed to be nonintegers.

Kellog was the first who considered (in 1912) the smoothness of a conformal mapping of the unit disk D onto a Jordan domain G with smooth boundary. He proved that if the boundary of G satisfies the Hölder condition of order $1+\beta$, $0<\beta<1$ (i.e., the angle between the tangent and the x-axis satisfies the Hölder condition of order β relative to the arc length on ∂G), then f also satisfies the Hölder condition of order $1+\beta$ in the closed disk D (in other words, f' is β-Hölder in D).

Much later, in the middle of the 1970s, Tamrazov and his followers extended this result to higher smoothness, specifically, to the Hölder classes $H^{r+\alpha}$ with arbitrary r and α. Subsequently, the proofs of these results were simplified by Dyn′kin to a great extent. A short version of the proof of the fact that, for any domain whose boundary is Hölder smooth of order $r+\alpha$, the conformal mapping of D onto this domain also satisfies the Hölder condition of order $r+\alpha$ in D, can be found presently in [1, Chapter 3].

All statements mentioned above involve uniform smoothness of the domain's boundary or of a function on the closed disk. It is interesting to find out how the nature of the smoothness of a conformal mapping changes if the smoothness of the boundary varies from point to point. For instance, under construction of various examples, a sufficiently smooth boundary is replaced fairly often by less smooth arcs on a subset. Precisely this case is treated here.

It turns out that a high smoothness of the conformal mapping can be traced on a fairly thick subset of the unit circle. Our approach to the proof of this statement is based on Dyn′kin's pseudocontinuation concept, which was successively employed by Dyn′kin in various difficult problems of mathematical analysis.

We pass to definitions. In what follows, G is a Jordan domain bounded by a closed curve $\Gamma = \partial G$ that does not pass through the point at infinity.

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For a closed interval I, we denote by $C^\alpha(I)$, $0<\alpha<1$, the Banach space of functions F with the Hölder norm

$$\|F\|_{C^\alpha(I)} = |D(x_I)| + \sup_{\substack{x_1,x_2\in I,\\ x_1\neq x_2}} \frac{|F(x_2)-F(x_1)|}{|x_2-x_1|^\alpha},$$

where x_I is the center of I; for a positive integer r, the symbol $C^{r+\alpha}$ stands for the Banach space of functions F with the norm

$$\|F\|_{C^{r+\alpha}(I)} = |F(x_I)| + \sum_{\nu=1}^{r} |F^{(\nu)}(x_I)| + \|F^{(r)}\|_{C^\alpha(I)}.$$

For a closed set $E\subset I$, $E\neq\varnothing$, we denote by $C^{r+\alpha}(E)$ the Banach space of functions φ defined on E and such that there exists $F\in C^{r+\alpha}(I)$ with $F|_E=\varphi$; we put

$$\|\varphi\|_{c^{r+\alpha}(E)} = \inf_{\substack{F\in C^{r+\alpha}(I)\\ F|E=\varphi}} \|F\|_{C^{r+\alpha}(I)}.$$

A set $J\subset C$ is called a $C^{r+\alpha}$-arc if there is a bijection $w\colon I\to J$, where $I\subset R$ is a closed interval, such that $w\in C^{r+\alpha}(I)$ and $|w'(t)|\geq c>0$, $t\in I$. If the injectivity of w is violated precisely at the end-points a and b of I, then $w(I)$ is a Jordan curve Γ, and we say that Γ is $C^{r+\alpha}$-smooth if w, when extended to $[2a-b,a]$ and $[b,2b-a]$ with period $b-a$, belongs to $C^{r+\alpha}([2a-b,2b-a])$. We describe the structure of the boundary Γ of the domain G in question. It is assumed that Γ is a $C^{1+\beta}$-smooth Jordan curve, $0<\beta<1$, and a countable collection of mutually disjoint open arcs $\lambda_j\subset\Gamma_\infty$ is distinguished. With each arc λ_j, we associate a certain arc $\widetilde{\lambda}_j$ with the same end-points in such way that the set

$$\widetilde{\Gamma} = \Big(\Gamma\setminus\bigcup_{j=1}^{\infty}\lambda_j\Big)\cup\bigcup_{j=1}^{\infty}\widetilde{\lambda}_j$$

is a Jordan $C^{r+\alpha}$-curve with $r\geq 1$, $0<\alpha<1$, $r+\alpha>1+\beta$, which bounds a domain $\widetilde{G}$. Fix a point $\xi_0\in G\cap\widetilde{G}$. We use the symbol $|\lambda|$ to denote the length of a smooth curve λ. Put $u=\frac{2+\beta}{r+\alpha+2}$. Suppose that

$$\sum_{j=1}^{\infty}|\lambda_j|^u<\infty. \tag{1}$$

We denote by $\xi=f(z)$ a conformal mapping of the unit disk onto G with $f(0)=\xi_0$, $f'(0)>0$, and let $z=\varphi(\xi)$ be the inverse mapping. Put $\delta_j=\varphi(\lambda_j)$, $j\geq 1$, and let $\Delta=\sum_{j=1}^{\infty}|\delta_j|$. We assume that $\Delta<2\pi$.

Theorem. *For every $\varepsilon>0$ there exists a set $E_\varepsilon\subset\mathbb{T}\overset{\text{def}}{=}\partial\mathbb{D}$ such that*

$$mE_\varepsilon>2\pi-(1+\varepsilon)\Delta \quad\textit{and}\quad f\in C^{r+\alpha}(E_\varepsilon).$$

Here mE_ε is the measure of E_ε on the unit circle.

Proof. Since Γ is a $C^{1+\beta}$-smooth curve, the Kellog theorem (see [1, Chapter 3]) shows that the conformal mapping f satisfies $f\in C^{1+\beta}(\overline{\mathbb{D}})$ and also $0<c_0\leq|f'(z)|\leq c_1$, $z\in\overline{\mathbb{D}}$, with some c_0,c_1. Therefore, by (1) we have

$$\sum_{j=1}^{\infty}|\delta_j|^u<\infty. \tag{2}$$

We exhibit the set E_ε. Choose N in such a way that $\delta_j \le 1$ for $j \ge N+1$ and

$$\sum_{j=N+1}^{\infty} |\delta_j|^u < \frac{\varepsilon}{2}\Delta. \tag{3}$$

Next, given an arc $s \subset \mathbb{T}$ and $a > 0$ with $a \cdot |s| \le 2\pi$, we denote by a^*s the arc on $\mathbb{T}$ with the same center and such that $|a^*s| = a \cdot |s|$. Put

$$\tau_j = \begin{cases} (1+\frac{\varepsilon}{2})^*\delta_j, & 1 \le j \le N, \\ |\delta_j|^{u-1*}\delta_j, & j \ge N+1, \end{cases} \tag{4}$$

and $E_\varepsilon = \mathbb{T} \setminus \bigcup_{j=1}^{\infty} \tau_j$. Relations (3) and (4) imply $|E_\varepsilon| > 2\pi - (1+\varepsilon)\Delta$. We split the annulus $\{z : \frac{1}{2} \le |z| < 1\}$ into the domains

$$Q_{m\nu} = \bigl\{z = re^{i\theta} : 2^{-m-1} \le 1 - r \le 2^{-m}, 2\pi \cdot (\nu - 1) \cdot 2^{-m} \le \theta \le 2\pi\nu \cdot 2^{-m}\bigr\},$$

and let $\sigma(Q_{m\nu})$ be the arc $\{z = e^{i\theta} : 2\pi(\nu-1)\cdot 2^{-m} \le \theta \le 2\pi\nu \cdot 2^{-m}\}$. We impose two conditions on n_0. The first of them is

$$2^{-n_0} < \frac{\varepsilon}{8} \min_{1 \le j \le N} |\delta_j|. \tag{5}$$

Let $S = \bigcup_{j=N+1}^{\infty} \delta_j$. We denote by B_0 the set of all $Q_{m\nu}$ with $m \ge n_0$ and

$$\sigma(Q_{m\nu}) \cap \Bigl(1 + \frac{2}{4}\Bigr)^* \delta_j \ne \varnothing, \quad 1 \le j \le N;$$

next, B_1 is the set of all $Q_{m\nu}$ not belonging to B_0 and satisfying $m \ge n_0$ and

$$|\sigma(Q_{m\nu}) \cap S| \ge \frac{1}{2}|\sigma Q_{m\nu})|.$$

Now, put $B = B_0 \cup B_1$, and A let be the set of all $Q_{m\nu}$ with $m \ge n_0$ that do not belong to B. We shall write the relations $Q_{m\nu} \in A$, $Q_{m\nu} \in B_0$, ..., etc. in the form $Q^A_{m\nu}$, $Q^{B_0}_{m\nu}$, ..., etc. Put $G^A_{m\nu} = f(Q^A_{m\nu})$, $G^{B_0}_{m\nu} = f(Q^{B_0}_{m\nu})$, ..., etc. Since $0 < c_0 \le |f'(z)| \le c_1$, $z \in \mathbb{D}$, we have

$$\operatorname{diam} G_{m\nu} \asymp 2^{-m}, \quad \operatorname{dist}(G_{m\nu}, \Gamma) \asymp 2^{-m}. \tag{6}$$

To prove the theorem, we employ Dyn′kin's analytic pseudocontinuation concept, see [2, 3, 4].

First, we consider a pseudoreflection of the domains $G^A_{m\nu}$ with respect to $\widetilde{\Gamma}$, and of the domains $G^B_{m\nu}$ with respect to Γ. Let $\widetilde{\Gamma} = \widetilde{F}([0, 2\pi])$, where $\widetilde{F}$ has period 2π, $\widetilde{F} \in C^{r+\alpha}([-2\pi, 4\pi])$, $\Gamma = F([0, 2\pi])$, and F also has period 2π,

$$F \in C^{1+\beta}([-2\pi, 4\pi]).$$

For $t \in \mathbb{R}$, we put $\widetilde{P}_t(\tau) = \widetilde{F}(t) + \sum_{j=1}^{i} \frac{\widetilde{F}^{(j)}(t)}{j!}(\tau - t)^j$, then

$$|\widetilde{F}(\tau) - \widetilde{P}_t(\tau)| \le c|t-\tau|^{r+\alpha}, \quad |\widetilde{F}^{(\nu)}(\tau) - \widetilde{P}^{(\nu)}_t(\tau)| \le c|t-\tau|^{r+\alpha-\nu}, \quad \tau \in \mathbb{R}, \quad 1 \le \nu \le r.$$

Here and below, $c, c_0, c_1, \dots$ are some constants, not always the same. Choose $\delta > 0$ such that the polynomial $\widetilde{P}_t(\tau)$ is univalent in the rectangle

$$q_\delta(t) = \bigl\{\tau = \sigma + i\lambda : |\sigma - t| \le 2\pi\delta, \ |\lambda| \le \delta\bigr\},$$

and let $c_3 > 0$ be such that $|\widetilde{P}'_t(\tau)| \ge c_3$, $\tau \in q_\delta(t)$. Now, we impose the second restriction on n_0: let

$$2^{-n_0} < \frac{1}{8\pi}(1 - e^{-\delta}).$$

Let $\widetilde{U}_\delta(t) = \widetilde{P}_t(q_\delta(t))$, and let $\widetilde{R}_t(w)$ be the mapping inverse to $\widetilde{P}_t$ and defined on $\widetilde{U}_\delta(t)$. We denote by $\check{Q}_{m\nu}$ the inversion of $Q_{m\nu}$ with respect to $\mathbb{T}$, i.e.,

$$\check{Q}_{m\nu} = \Big\{ z = Re^{i\theta} \,:\, 2\pi(\nu-1)2^{-m} \le \theta \le 2\pi\nu \cdot 2^{-m},\, 2^{-m-1} \le 1 - \frac{1}{R} \le 2^{-m} \Big\},$$

where $t_{m\nu} = 2\pi(\nu - \frac{1}{2}) \cdot 2^{-m}$, $1 \le \nu \le 2^m$. Next, let $\check{B}_0$, $\check{B}$, and $\check{A}$ stand for the sets of inversions of all all elements of B_0, B, and A, respectively. Now, putting $G_{m\nu} = G^A_{m\nu}$, we set

$$F(w) = \widetilde{P}_{t_{m\nu}}\big(\overline{\widetilde{R}_{t_{m\nu}(w)}}\big) \tag{7}$$

for $w \in G_{m\nu}$. Consider the curve $\Gamma = F([0, 2\pi])$, $F \in C^{1+\beta}([-2\pi, 4\pi])$. For $t \in \mathbb{R}$, we set

$$P_t(\tau) = F(t) + F'(t)(\tau - t),$$

then for $\tau \in \mathbb{R}$ we have

$$|F(\tau) - P_t(\tau)| \le c|t-\tau|^{1+\beta}, \quad |F'(\tau) - P_t'(\tau)| \le c|t-\tau|^{\beta}.$$

Since $\min_{t\in[0,2\pi]} |F'(t)| > 0$, for every $t \in \mathbb{R}$ there exists a linear mapping $P_t(w)$ inverse to R_t. For $w \in G^B_{m\nu}$, we set

$$F(w) {=} P_{t_{m\nu}}\left(\overline{R_{t_{m\nu}(w)}}\right). \tag{8}$$

The mapping F is defined on each $G_{m\nu}$ separately, depending on whether $Q_{m\nu} \in A$ or $Q_{m\nu} \in B$. Now, for $z \in \check{Q}_{m\nu}$ we define

$$f_0(z) = F\Big(f\Big(\frac{1}{\bar z}\Big)\Big).$$

Denoting $c_{m\nu j} = \frac{\widetilde{F}^{(j)}(t_{m\nu})}{j!}$, $0 \le j \le r$, by (7) we see that

$$F(w) = \sum_{j=0}^{r} c_{m\nu j}\left(\overline{R_{t_{m\nu}(w)}}\right)^j, \quad w \in G^A_{m\nu}.$$

Then

$$F'_w(w) \equiv 0, \quad w \in G^A_{m\nu}, \tag{9}$$

and similarly,

$$F'_w(w) \equiv 0, \quad w \in G^B_{mn}. \tag{9$'$}$$

Now, for $z \in \check{Q}_{n\nu}$, $Q_{m\nu} = Q^A_{m\nu}$, by using (9) and (9$'$), we obtain

$$f'_{0\bar z}(z) = \sum_{j=1}^{r} j c_{m\nu j}\left(\overline{\widetilde{R}_{t_{m\nu}}\Big(f\Big(\frac{1}{\bar z}\Big)\Big)}\right)^{j-1} \cdot \overline{\left(\widetilde{R}_{t_{m\nu}}\Big(f\Big(\frac{1}{\bar z}\Big)\Big)\right)}'_{\bar z},$$

$$\left(\overline{\widetilde{R}_{t_{m\nu}}\Big(f\Big(\frac{1}{\bar z}\Big)\Big)}\right)'_{\bar z} = \overline{\left(\widetilde{R}_{t_{m\nu}}\Big(f\Big(\frac{1}{\bar z}\Big)\Big)\right)'_{z}} = 0.$$

Therefore, $f'_{0\bar z}(z) \equiv 0$, $z \in \check{Q}_{m\nu}$, $Q_{m\nu} \in A$, and similarly,

$$f'_{0\bar z}(z) \equiv 0, \quad z \in \check{Q}_{m\nu}, \quad Q_{m\nu} \in B.$$

Put

$$B_\varepsilon(z) = \Big\{\xi \in \mathbb{C} \setminus \overline{\mathbb{D}} \,:\, |\xi - z| < \frac{\varepsilon}{32}(|z| - 1)\Big\}.$$

As Dyn′kin did in [2], we put

$$f_1(z) = \frac{1}{|B_\varepsilon(z)|}\int_{B_\varepsilon(z)} f_0(\xi)\, dA(\xi), \tag{10}$$

where A is the plane Lebesgue measure, $|B_\varepsilon(z)| = A(B_\varepsilon(z))$. Now, we introduce two sets $Q^*_{m\nu}$ and $\check{Q}^*_{m\nu}$, assuming for the informality of notation that ν runs through $\mathbb{Z}$ in the definition of $Q_{m\nu}$. So, we put

$$Q^*_{m,2\nu+1} = \bigcup_{\mu=2\nu}^{2\nu+2} Q_{m\mu} \cup \bigcup_{\mu=\nu-1}^{\nu} Q_{m-1,\mu} \cup \bigcup_{\mu=4\nu}^{4\nu+3} Q_{m+1,\mu}, \quad \nu \geq 0, \tag{11}$$

$$Q^*_{m,2\nu} = \bigcup_{\mu=2\nu-1}^{2\nu+1} Q_{m\mu} \cup \bigcup_{\mu=\nu}^{\nu+1} Q_{m-1,\mu} \cup \bigcup_{\mu=4\nu-1}^{4\nu+2} Q_{m+1,\mu}, \quad \nu \geq 1. \tag{12}$$

The set $\check{Q}^*_{m\nu}$ is obtained from $Q^*_{m\nu}$ (see (11) and (12)) by inversion with respect to $\mathbb{T}$.

Let $G^*_{m\nu} = f(Q^*_{m\nu})$, and let a mapping $F^*_{m\nu}$ be constructed for $w \in G^*_{m\nu}$ in accordance with (7) or (8) depending on whether $Q_{m\nu} \in A$ or $Q_{m\nu} \in B$. For $z \in \check{Q}^*_{m\nu}$, we put

$$f_{0m\nu}(z) = F^*_{m\nu}\Big(f\Big(\frac{1}{\bar{z}}\Big)\Big). \tag{13}$$

Repeating (9) for $f_{0m\nu}$, we deduce from (13) that

$$f'_{0m\nu\bar{z}}(z) = 0, \quad z \in \check{Q}^*_{m\nu}. \tag{14}$$

The choice of $B_\varepsilon(z)$ shows that $B_\varepsilon(z) \subset \check{Q}^*_{m\nu}$ for $z \in \check{Q}^*_{m\nu}$, therefore for $z \in \check{Q}_{m\nu}$ we have

$$\begin{aligned} f'_{1\bar{z}}(z) &= \big(f_1(z) - f_{0m\nu}(z)\big)'_{\bar{z}} \\ &= \Big(\frac{1}{|B_\varepsilon(z)|}\int_{B_\varepsilon(z)} f_0(\xi)\,dA(\xi)\Big)'_{\bar{z}} - \Big(\frac{1}{|B_\varepsilon(z)|}\int_{B_\varepsilon(z)} f_{0m\nu}(\xi)\,dA(\xi)\Big)'_{\bar{z}} \\ &= \Big(\frac{1}{|B_\varepsilon(z)|}\int_{B_\varepsilon(z)} (f_0(\xi) - f_{0m\nu}(\xi))\,dA(\xi)\Big)'_{\bar{z}} \end{aligned} \tag{15}$$

by (14). Now, using (15) we find (see Dyn′kins papers [2, 3, 4]):

$$\Big|\Big(\frac{1}{|B_\varepsilon(z)|}\int_{B_\varepsilon(z)} f_0(\xi) - f_{0m\nu}(\xi)\,dA(\xi)\Big)'_{\bar{z}}\Big| \leq \frac{c}{\varepsilon(|z|-1)} \sup_{\xi\in B_\varepsilon(z)} |f_0(\xi) - f_{0m\nu}(\xi)|. \tag{16}$$

By the choice of $B_\varepsilon(z)$, we have $B_\varepsilon(z) \subset \check{Q}_{m\nu}$ whenever $z \in \check{Q}^*_{m\nu}$, whence it follows that

$$f\Big(\frac{1}{\bar{z}}\Big) \in G_{m\nu}, \quad f\Big(\frac{1}{\bar{\xi}}\Big) \in G_{p\mu}, \quad \xi \in B_\varepsilon(z),$$

where p and μ are related to m and ν as in (11) and (12). Thus, if $w \in f(\frac{1}{\xi}) \in G^*_{m\nu}$, then the definitions of f_0 and $f_{0m\nu}$ imply

$$f_{0m\nu}(\xi) = P_{t_{m\nu}}(\overline{R_{t_{m\nu}}(w)}), \quad f_0(\xi) = P_{t_{p\mu}}(\overline{R_{t_{p\mu}}(w)}) \tag{17}$$

with p and μ specified above. Consequently, we must estimate the quantity

$$P_{t_{m\nu}}(\overline{R_{t_{m\nu}}(w)}) - P_{t_{p\mu}}(\overline{R_{t_{p\mu}}(w)}) \overset{\text{def}}{=} K_{m\nu,p\mu}(w). \tag{18}$$

We have

$$\begin{aligned} |K_{m\nu,p\mu}(w)| &\leq |P_{t_{m\nu}}(\overline{R_{t_{m\mu}}(w)}) - P_{t_{p\mu}}(\overline{R_{t_{t_\mu}}(w)})| \\ &\quad + |P_{t_{p\mu}}(\overline{R_{t_{m\mu}}(w)}) - P_{t_{p\mu}}(\overline{R_{t_{p\mu}}(w)})| \overset{\text{def}}{=} I_{m\nu,p\mu} + J_{m\nu,p\mu}. \end{aligned} \tag{19}$$

Let $\sigma_{m\nu} = \overline{R_{t_{m\nu}}(w)}$ and $Q_{m\nu} = Q^A_{m\nu}$; if $Q_{m\nu} = Q^B_{m\nu}$, the arguments are even simpler. We put

$$q_{m\nu}(\tau) = \widetilde{F}(\tau) - P_{t_{m\nu}}(\tau), \quad \tau \in [0, 2\pi],$$

then

$$|q_{m\nu}(\tau)| \le c|\tau - t_{m\nu}|^{r+\alpha}; \tag{20}$$

$$|q^{(l)}_{m\nu}(\tau)| \le c|\tau - t_{m\nu}|^{r+\alpha-l}, \quad 1 \le l \le r. \tag{21}$$

In particular, since $|t_{p\mu} - t_{m\nu} \le 2^{-m+1}$, inequalities (20) and (21) show that

$$|q_{m\nu}(t_{p\mu})| \le c \cdot 2^{-m(r+\alpha)}; \tag{22}$$

$$|q^{(l)}_{m\nu}(t_{p\mu})| \le c \cdot 2^{-m(r+\alpha-l)}, \quad 1 \le l \le r. \tag{23}$$

Now,

$$\begin{aligned} P_{t_{p\mu}}(\sigma_{m\nu}) &= \widetilde{F}(t_{p\mu}) + \sum_{l=1}^{r} \frac{\widetilde{F}^{(l)}(t_{p\mu})}{l!}(\sigma_{m\nu} - t_{p\mu})^l \\ &= P_{t_{m\nu}}(t_{p\mu}) + q_{m\nu}(t_{p\mu}) + \sum_{l=1}^{r}\left(\frac{P^{(l)}_{t_{m\nu}}(t_{p\mu})}{l!} + \frac{q^{(l)}_{m\nu}(t_{p\mu})}{l!}\right)(\sigma_{m\nu} - t_{p\mu})^l \\ &= P_{t_{m\nu}}(\sigma_{m\nu}) + q_{m\nu}(t_{p\mu}) + \sum_{l=1}^{r} \frac{q^{(l)}_{m\nu}(t_{p\mu})}{l!}(\sigma_{m\nu} - t_{p\mu})^l. \end{aligned}$$

Therefore, from (22) and (23) it follows that

$$\begin{aligned} I_{m\nu,p\mu} &\le |q_{m\nu}(t_{p\mu})| + \sum_{l=1}^{r} \frac{|q^{(l)}_{m\nu}(t_{p\mu})|}{l!}|\sigma_{m\nu} - t_{p\mu}|^l \\ &\le c\left(2^{-m(r+\alpha)} + \sum_{l=1}^{r} 2^{-m(r+\alpha-l)}|\sigma_{m\nu} - t_{m\nu}|^l\right). \end{aligned} \tag{24}$$

By the choice of n_0 and the fact that $m \ge n_0$, we have $\sigma_{m\nu} \in q_\delta(t_{m\nu})$, $t_{p\mu} \in q_\delta(t_{m\nu})$. Using the estimates

$$\begin{gathered} 0 < c_3 \le |P'_{t_{m\nu}}(\sigma)| \le c_4 < \infty, \quad \sigma \in q_\delta(t_{m\nu}), \\ \frac{1}{c_4} \le |R'_{t_{m\nu}}(w)| \le \frac{1}{c_3}, \quad w \in P_{t_{m\nu}}(q_\delta(t_{m\nu})), \end{gathered}$$

we see that

$$|\sigma_{m\nu} - t_{p\mu}| \le c \cdot 2^{-m}, \tag{25}$$

and then

$$I_{m\nu,p\mu} \le c\left(2^{-m(r+\alpha)} + \sum_{l=1}^{r} 2^{-m(r+\alpha-l)} \cdot 2^{-ml}\right) = c \cdot 2^{-m(r+\alpha)} \tag{26}$$

by (24) and (25). Next, $|P'_{t_{p\mu}}(0)| \le c$ for $|\sigma - t_{p\mu}| \le 2\pi$, consequently,

$$J_{m\nu,p\mu} \le \max_{|\sigma - t_{p\mu}| \le 2\pi} |P'_{t_{p\mu}}(\sigma)| \cdot |\overline{R_{t_{m\nu}(w)}} - \overline{R_{t_{p\nu}(w)}}| \le c|R_{t_{m\nu}}(w) - R_{t_{p\mu}}(w)|, \tag{27}$$

where $w = f\big(\frac{1}{\xi}\big) \in G^*_{m\nu}$. Let $s_{m\nu} = R_{t_{m\nu}}(w) = \overline{\sigma}_{m\nu}$ and $s_{p\mu} = R_{t_{p\mu}}(w)$, then $P_{t_{m\nu}}(s_{m\nu}) = w$. It follows that

$$\begin{aligned} P_{t_{m\nu}}(s_{m\nu}) - P_{t_{m\nu}}(s_{p\mu}) &= P_{t_{m\nu}}(s_{m\nu}) - P_{t_{p\mu}}(s_{p\mu}) + P_{t_{p\mu}}(s_{p\mu}) - P_{t_{m\nu}}(s_{p\mu}) \\ &= P_{t_{p\mu}}(s_{p\mu}) - P_{t_{m\nu}}(s_{p\mu}). \end{aligned} \tag{28}$$

By (25) and (28), we obtain

$$|P_{t_{m\nu}}(s_{m\nu}) - P_{t_{m\nu}}(s_{p\mu})| \le c \cdot 2^{-m(r+\alpha)}. \tag{29}$$

By the choice of δ and n_0, and the fact that $m \geq n_0$, we have $s_{m\nu} \in q_\delta(t_{m\nu})$. Therefore,

$$|P_{t_{m\nu}}(s_{m\nu}) - P_{t_{m\nu}}(s_{p\mu})| \geq c_1|s_{m\nu} - s_{p\mu}| \tag{30}$$

with some constant $c_1 > 0$, and (27), (29), and (30) yield

$$J_{m\nu,pq} \leq c|s_{m\nu} - s_{pq}| \leq \frac{c}{c_1} \cdot 2^{-m(r+\alpha)} = c_2 \cdot 2^{-m(r+\alpha)}. \tag{31}$$

Since $2^{-m} \asymp |z| - 1$ for $z \in \check{Q}_{m\nu}$, from (15)–(19) and (31) we deduce that

$$|f'_{1\bar{z}}(z)| \leq \frac{c}{\varepsilon}(|z| - 1)^{r+\alpha-1}, \quad z \in \check{Q}^A_{m\nu}. \tag{32}$$

Similar arguments yield

$$|f'_{1\bar{z}}(z)| \leq \frac{c}{\varepsilon}(|z| - 1)^{\beta}, \quad z \in \check{Q}^{B^*}_{m\nu}. \tag{33}$$

Let $Q^* = \bigcup_{Q_{m\nu} \in B} Q^*_{m\nu}$. In our case, the Green formula has the form

$$\begin{aligned} f(z) &= -\frac{1}{\pi}\int_{\mathbb{C}\setminus\mathbb{D}} \frac{f'_{1\bar{\xi}}(\xi)}{\xi - z}\, dA(\xi) \\ &= -\frac{1}{\pi}\int_{Q^*} \frac{f'_{1\bar{\xi}}(\xi)}{\xi - z}\, dA(\xi) + \sum_{\substack{Q_{m\nu}\in A \\ Q_{m\nu}\not\subset Q^*}} \left(-\frac{1}{\pi}\int_{Q_{\mu\nu}} \frac{f'_{1\bar{\xi}}(\xi)}{\xi - z}\, dA(\xi)\right) \\ &\stackrel{\text{def}}{=} f_{(1+\beta)}(z) + f_{(r+\alpha)}(z). \end{aligned} \tag{34}$$

The integrands in (34) are summable because $\check{Q}_{m\nu} \subset \{\xi : 1 < |\xi| \leq 2\}$, and by (32) and (33) we have

$$|f'_{1\bar{\xi}}(\xi)| \leq \frac{c}{\varepsilon}\begin{cases} (|\xi| - 1)^{\beta}, & \xi \in Q^*; \\ (|\xi| - 1)^{z+\alpha-1}, & \xi \in \check{Q}^A_{m\nu}, \quad Q^A_{m\nu} \not\subset Q^*. \end{cases} \tag{35}$$

By Dyn′kin's theorem (see [2]), from (32) it follows that

$$f_{(r+\alpha)} \in C^{r+\alpha}(\mathbb{T}). \tag{36}$$

Consider the function $f_{(1+\beta)}(z)$. Let $\sigma(Q^*_{m\nu}) = \bigcup_{Q_{p\mu}\subset Q^*_{m\nu}} \sigma(Q_{p\mu})$; we get $\sigma(Q^*_{m\nu}) \subset s^*\sigma(Q_{m\nu})$. Among the $Q^{B_1}_{m\nu}$, we choose some set $Q^{B_1}_{m_1\nu_1}$ with m_1 minimal and put

$$V_1 = \bigcup_{\sigma(Q^{B_1}_{m\nu})\cap\sigma(Q^{B_1}_{m_1\nu_1})\neq\varnothing} \check{Q}^{B_1^*}_{m\nu} \setminus \check{B}_0, \quad \check{B}_{1,2} \stackrel{\text{def}}{=} \check{B}_1 \setminus V_1. \tag{37}$$

In $\check{B}_{1,2}$, we find some $\check{Q}^{B_1}_{m_2\nu_2}$ with m_2 minimal and put

$$V_2 = \bigcup_{\sigma(Q^{B_1}_{m\nu})\cap\sigma(Q^{B_1}_{m_2\nu_2})\neq\varnothing} \check{Q}^{B_1^*}_{m\nu} \setminus (V_1 \cup \check{B}_0), \quad \check{B}_{1,3} = \check{B}_{1,2} \setminus (V_1 \cup V_2) \tag{38}$$

and so on: if $\check{Q}^{B_1}_{m_k\nu_k}$, $1 \leq k \leq l - 1$, and $V_1, \dots, V_{l-1}$, $\check{B}_{lk}$, $l \leq k \leq l$, have already been defined, we choose an element $\check{Q}^{B_1}_{m_l\nu_l}$ with minimal m_l in $\check{B}_{1,l}$ and put

$$V_l = \bigcup_{\sigma(Q^{B_1}_{m\nu})\cap\sigma(Q^{B_1}_{m_l\nu_l})\neq\varnothing} \check{Q}^{B_1^*}_{m\nu} \setminus (V_1 \cup \cdots \cup V_{l-1}) \cup \check{B}_0, \ \check{B}_{l,l+1} = \check{B}_{1,l} \setminus (V_1 \cup \cdots \cup V_l). \tag{39}$$

Next, we denote

$$f_{1+\beta,l}(z) = -\frac{1}{\pi}\int_{V_l} \frac{f'_{1\bar\xi}(\xi)}{\xi - z}\, dA(\xi), \tag{40}$$

$$f_{1+\beta}^{B_0}(z) = \sum_{Q_{m\nu}\subset B_0} \left(\frac{1}{\pi}\int_{Q^*_{m\nu}} \frac{f'_{1\bar\xi}(\xi)}{\xi - z}\, dA(\xi)\right), \tag{41}$$

then

$$f_{(1+\beta)}(z) = f_{1+\beta}^{B_0}(z) + \sum_{l=1}^{\infty} f_{1+\beta,l}(z). \tag{42}$$

Clearly,

$$f_{(1+\beta)}^{B_0} \in C^{r+\alpha}\left(\mathbb{T}\setminus \bigcup_{j=1}^{N}\left(1+\frac{\varepsilon}{2}\right)^{*}\delta_j\right) \tag{43}$$

by (41) and the definition of B_0. We extend this function to $\bigcup_{j=1}^{N}\left(1+\frac{\varepsilon}{2}\right)^{*}\delta_j$ preserving smoothness and retaining the same notation for the extension. Now, we turn to the functions $f_{1+\beta,l}$. Let

$$\widetilde{\delta}_l = \sigma(Q^{B_1}_{m_l\nu_l}), \quad \Delta_l = 2R_l^*\delta_l, \quad R_l = |\sigma(Q^{B_1}_{m_l\nu_l})|^{u-l} = 2^{m_l(1-u)},$$

where $u = \frac{\beta+2}{r+\alpha+2}$. By (40) and (35), for $z \in \mathbb{T}\setminus\Delta_l$ we have

$$\begin{aligned} |f^{(r+1)}_{1+\beta,l}(z)| &\le c\int_{\check{Q}^*_{m_l\nu_l}} \frac{|f'_{1\bar\xi}(\xi)|}{|\xi - z|^{r+2}}\, dA(\xi) \le c\frac{1}{|\Delta_l|^{r+2}}\int_{\check{Q}^*_{m_l\nu_l}} (|\xi|-1)^{\beta}\, dA(\xi) \\ &= c\frac{1}{|\Delta_l|^{r+2}}\cdot |\widetilde{\delta}_l|^{\beta+2}, \end{aligned} \tag{44}$$

From (44) it follows that

$$\begin{aligned} \|f_{1+\beta,l}\|_{C^{r+\alpha}(\mathbb{T}\setminus\Delta_l)} &\le c|\Delta_l|^{1-\alpha}\cdot\frac{1}{|\Delta_l|^{r+2}}|\widetilde{\delta}_l|^{\beta+2} \\ &= c\frac{|\widetilde{\delta}_l|^{\beta+2}}{|\Delta_l|^{r+\alpha+1}} = c\cdot\frac{2^{-m_l(\beta+2)}}{2^{-m_l(r+\alpha+1)u}}. \end{aligned} \tag{45}$$

But

$$\beta + 2 - (r+\alpha+1)u = \beta + 2 - \frac{(r+\alpha+1)(\beta+2)}{r+\alpha+2} = \frac{\beta+2}{r+\alpha+2} = u,$$

so

$$\|f_{1+\beta,l}\|_{c^{r+\alpha}(\mathbb{T}\setminus\Delta_l)} \le c\cdot 2^{-m_l u} \tag{46}$$

by (44) and (45). We extend $f_{1+\beta,l}$ to Δ_l preserving smoothness and retaining the same notation. The definition of V_l shows that

$$\begin{aligned} \sum_{l=1}^{\infty} 2^{-m_l u} &\le c\sum_{l=1}^{\infty}\Bigg(\sum_{\delta_j\subset\sigma(Q^*_{m_l\nu_l})} |\delta_j|\Bigg)^{u} \\ &\le c\sum_{l=1}^{\infty}\sum_{\delta_j\subset\sigma(Q^*_{m_l\nu_l})} |\delta_j|^{u} \le \sum_{j\ge N+1} |\delta_j|^u < c\varepsilon. \end{aligned} \tag{47}$$

The definition of Δ_l and estimate (47) imply

$$\sum_{l=1}^{\infty} |\Delta_l| \le c\sum_{l=1}^{\infty} 2^{-m_l u} \le c_4\varepsilon. \tag{48}$$

Putting $\widetilde{\Delta}_l = \frac{1^*}{2c_4}\Delta_l$, we arrive at $\sum_{l=1}^{\infty} |\widetilde{\Delta}_l| < \frac{\varepsilon}{2}$, and

$$\|f_{1+\beta,l}\|_{C^{r+\alpha}(\mathbb{T}\setminus\widetilde{\Delta}_l)} \le c_5 \|f_{1+\beta,l}\|_{C^{r+\alpha}(\mathbb{T}\setminus\Delta_l)}. \tag{49}$$

Extending $f_{1+\beta,l}$ to $\widetilde{\Delta}_l$ with preservation of the $C^{r+\alpha}$-norm, we deduce from (45) and (47)–(49) that

$$\sum_{l=1}^{\infty} \|f_{1+\beta,l}\|_{C^{r+\alpha}(\mathbb{T})} \le c \cdot c_5 \sum_{l=1}^{\infty} 2^{-m_l u} < c_6\varepsilon. \tag{50}$$

But, by (34), for

$$z \in \mathbb{T} \setminus \left(\bigcup_{j=1}^{N} \left(1 + \frac{\varepsilon}{2}\right)^* \delta_j \cup \bigcup_{l=1}^{\infty} \widetilde{\Delta}_l \right)$$

we have

$$f(z) = f_{(r+\alpha)}(z) + f_{1+\beta}^{B_0}(z) + \sum_{l=1}^{\infty} f_{1+\beta,l}(z),$$

and the proof is finished with the help of (36), (43), and (50). □

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