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LOGARITHMS OF FORMAL A-MODULES IN THE CASE OF SMALL RAMIFICATION

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*To Sergeĭ Vladimirovich Vostokov
on the occasion of his 70th anniversary*

Abstract. Formal $\mathcal{O}_0$-modules over the ring of integers $\mathcal{O}$ of a local field, i.e., formal groups over $\mathcal{O}$ with endomorphism ring including a fixed ring $\mathcal{O}_0$ are studied. A complete description of the logarithms of all such modules is obtained in the case of small ramification. Earlier it was shown that in the case of small ramification ($e(\mathcal{O}/\mathcal{O}_0) < q$), any $\mathcal{O}_0$-module is strictly isomorphic to an $\mathcal{O}_0$-module the logarithm of which can be represented in the form $vu^{-1}(X)$, where u and v are certain matrices over the ring of operators described in the paper. The result obtained in the present paper enables one to determine the type (u and v) of a formal $\mathcal{O}_0$-module by the form of its logarithm, and provides a way for constructing all formal $\mathcal{O}_0$-modules.

§1. Introduction

Our aim in the present paper is to give a complete description of the logarithms of formal modules over the ring of integers of a local field in the case of small ramification.

The result obtained in this paper enables one to determine the type of a formal $\mathcal{O}_0$-module (see the definition below) by the form of its logarithm in the case of small ramification ($v_K(\pi_0) < q$); moreover, it provides a way for construction all formal $\mathcal{O}_0$-modules.

Throughout, we shall use the following notation, which is also necessary for the statement of the main result. Let K_0 be a local field (a finite extension of $\mathbb{Q}_p$) with the ring of integers $\mathcal{O}_0$ and a prime element π_0; let K be a finite extension of the field K_0 with the ring of integers $\mathcal{O}$ and a prime element π; let N be the subfield of inertia in K/K_0, let $\mathcal{O}_N$ be its ring of integers, and let e_0 be the ramification index of K/K_0; finally, let $X = (X_1, \dots, X_m)$. As in [1], $M_m(\mathfrak{A})$ denotes the ring of $(m \times m)$ matrices over the ring $\mathfrak{A}$, I_m is the identity $(m \times m)$ matrix. The main result of the paper is the following theorem.

Theorem 1. *Suppose* $\lambda(X) \in K[[X]]^m$, $\lambda(X) \equiv X \bmod \deg 2$, *and*

$$\lambda(X) = \sum_{k=0}^{e_0-1} \pi^k \lambda_k(X),$$

where $\lambda_k(X) \in N[[X]]^m$. *Then* $\lambda(X)$ *is the logarithm of an* m*-dimensional formal* $\mathcal{O}_0$*-module over* $\mathcal{O}$ *if and only if for some* $u \in M_m(\mathcal{O}_N[[\Delta]]')$ *and* $v \in M_m(\mathcal{O}[[\Delta]])$, *where* $u \equiv \pi_0 I_m \bmod \Delta$ *and* $v = \pi_0 I_m - \pi r_1 - \dots - \pi^{e_0-1} r_{e_0-1}$, $r_i \in M_m(\mathcal{O}_N[[\Delta]]'\Delta)$,

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$1 \le i \le e_0 - 1$, *the congruences*

$$\begin{cases} u\lambda_0(X) \equiv 0 \bmod \pi_0, \\ r_i\lambda_0(X) + \pi_0\lambda_i(X) \equiv 0 \bmod \pi_0, \quad 1 \le i \le e_0 - 1, \end{cases} \tag{1}$$

are valid.

Here $\mathcal{O}_N[[\Delta]]'$ is a ring acting on the series in $N[[X]]$, which will be defined below in Subsection 3.1. Congruence to 0 modulo π_0 means that the corresponding column lies in $\pi_0\mathcal{O}_N[[X]]^m$.

In Subsection 3.3 (see Proposition 1), in the case of small ramification we present a system of congruences that explicitly describes the formal Cartier module corresponding to an $\mathcal{O}_0$-module. We recall that the description of the classes of isomorphic formal groups with the help of Cartier modules, known from [1, 2], was given in terms of the right action of the ring $\mathcal{O}_N[[\Delta]]'$. Theorem 1 describes all the logarithms of formal $\mathcal{O}_0$-modules with the help of the left action.

For the first time, a similar description of formal groups was obtained by T. Honda (in a more special case) in the paper [3], in which formal groups (of arbitrary dimension) were constructed over the ring of p-adic integers. Now these groups are called *formal Honda groups*. Over the ring of Witt vectors of a perfect field of positive characteristic, the formal Honda groups exhaust all possible formal groups, thus providing a complete classification of formal groups in this case. We recall that a formal groups F is called a formal Honda group of type $u \in M_m(\mathcal{O}_N[[\Delta]]')$, where $u \equiv \pi_0 I_m \bmod \Delta$, if its logarithm λ satisfies the congruence

$$u\lambda(X) \equiv 0 \bmod \pi_0. \tag{2}$$

The formal Honda groups of types u and u_1 are isomorphic if and only if $u = cu_1$ for a series c such that $c \equiv 1 \bmod \Delta$.

The description of logarithms with the help of congruences of the form (2) has turned out to be convenient in dealing with formal groups. In particular, constructions with distinguished isogenies were successfully found for formal Honda groups, which made it possible to obtain an explicit formula of the Hilbert pairing for this class of formal groups (see [4, 5]).

§2. Notation

In addition to the notation given in the Introduction, we also use the following:

- v_K denotes a discrete valuation on K for which $v_K(\pi) = 1$;
- q is the order of the residue field of the field K_0;
- σ is the Frobenius automorphism of N/K_0.

§3. Classification of $\mathcal{O}_0$-modules in small ramification

The paper [1] was devoted to the description of a method for classifying all formal groups over the ring of integers of an arbitrary local field with the help of module invariants, and this method was illustrated in several cases, including the case of small ramification. In [7], small ramification results were generalized to the case of $\mathcal{O}_0$-modules, and their description was obtained without using module invariants. In this section, we recall and generalize these results.

3.1. Preliminary remarks and lemmas. Below we briefly list some known classification statements and deduce necessary technical results.

In the same way as in [6, Propositions 15.2.6, 15.2.8, 15.2.9, and Theorem 21.5.6], it can be shown that each formal $\mathcal{O}_0$-module over the ring $\mathcal{O}$ is strictly isomorphic to an $\mathcal{O}_0$-typical one the logarithm of which can be represented in the form

$$\lambda(X) = \Lambda(\Delta)(X) = X + b_1 X^q + b_2 X^{q^2} + \dots,$$

where $b_i \in M_m(K)$, $X^{q^i} = \begin{pmatrix} X_1^{q^i} \\ \vdots \\ X_m^{q^i} \end{pmatrix}$, $\Lambda(\Delta) = (\Lambda_j)$ lies in $M_m(K[[\Delta]])$, and $\Delta(X_i^n) = X_i^{qn}$ (the action of Δ on a column is determined by its action on each element of the column).

Let $\mathcal{O}_N[[\Delta]]'$ be the noncommutative ring of series that coincides with $\mathcal{O}_N[[\Delta]]$ as a left $\mathcal{O}_N$-module and satisfies the relation $\Delta a = \sigma(a)\Delta$ for all $a \in \mathcal{O}_N$. This relation also determines the structure of a right $\mathcal{O}_N[[\Delta]]'$-module on $K[[\Delta]]$. We define an action of the operator Δ on the series from $N[[X]]$: for a series $A(X) = \sum a_{i_1\dots i_m} X_1^{i_1} \dots X_m^{i_m}$ we set

$$\Delta(A(X)) := \sum a^{\sigma}_{i_1\dots i_m} X_1^{qi_1} \dots X_m^{qi_m}.$$

Thus, we have an action of the ring $\mathcal{O}_N[[\Delta]]'$ on $N[[X]]$.

Lemma 1. *Let $\Lambda = \sum_{i=0}^{\infty} C_i \Delta^i$, $C_i \in M_m(K)$, and let the series $\Lambda(\Delta)(X)$ be the logarithm of a certain formal $\mathcal{O}_0$-module. Then $\pi^s C_i \in M_m(\mathcal{O})$, where*

$$s = \max(ie_0, (i+l)e_0 - q^l), \quad l = \left[\log_q \frac{e_0}{q-1}\right].$$

Proof. By Theorem 2 in [8], Λ can be represented in the form vu^{-1}; moreover, $v\pi_0^l \pi^{-q^l} \in M_m(\mathcal{O}[[\Delta]])$, $u \in M_m(\mathcal{O}_N[[\Delta]]')$, and $u \equiv \pi_0 I_m \bmod \Delta$. Let

$$u = \pi_0 I_m - A(\Delta)\Delta,\ v = \pi_0 I_m - \pi_0^{-l}\pi^{q^l} B(\Delta)\Delta,\ A, B \in M_m(\mathcal{O}[[\Delta]]).$$

Then

$$vu^{-1} = \sum_{j=0}^{\infty} \frac{(A(\Delta)\Delta)^j}{\pi_0^j} - \frac{\pi^{q^l}}{\pi_0^l} B(\Delta)\Delta \sum_{j=0}^{\infty} \frac{(A(\Delta)\Delta)^j}{\pi_0^{j+1}}.$$

Therefore,

$$\Lambda \equiv \sum_{j=0}^{i} \frac{(A(\Delta)\Delta)^j}{\pi_0^j} - \frac{\pi^{q^l}}{\pi_0^l} B(\Delta)\Delta \sum_{j=0}^{i-1} \frac{(A(\Delta)\Delta)^j}{\pi_0^{j+1}} \bmod \Delta^i,$$

which implies the lemma. □

Lemma 1 can be proved differently by applying the universal $\mathcal{O}_0$-typical $\mathcal{O}_0$-module (for the definition, see, e.g., [6, 21.5]).

Throughout this section, we fix elements $u \in M_m(\mathcal{O}_N[[\Delta]]')$ and $v \in M_m(\mathcal{O}[[\Delta]])$ such that

$$\begin{aligned} &u \equiv \pi_0 I_m \bmod \Delta, \\ &v = \pi_0 I_m - \pi r_1 - \dots - \pi^{e_0-1} r_{e_0-1}, \qquad (3) \\ &r_i \in M_m(\mathcal{O}_N[[\Delta]]'\Delta), \quad 1 \le i \le e_0 - 1. \end{aligned}$$

For each $\alpha \in \mathcal{O}$, we define an operator $\langle\alpha\rangle$ on $K[[\Delta]]^m$ in the following way:

$$\langle\alpha\rangle\left(\sum_{i=0}^{\infty} c_i \Delta^i\right) = \sum_{i=0}^{\infty} c_i \alpha^{q^i} \Delta^i.$$

Let $D_F = \{f \in K[[\Delta]]^m : \lambda_F^{-1}(f(\Delta)(X)) \in \mathcal{O}[[X]]^m\}$ be the Cartier module corresponding to the formal $\mathcal{O}_0$-module F with the logarithm $\lambda_F(X) = \Lambda(\Delta)(X)$, where $\Lambda = vu^{-1}$. We recall (see [1, Definition 5.2.1]) that as the right $\mathcal{O}_N[[\Delta]]'$-submodule of $K[[\Delta]]^m$ (see [1] for the details) D_F is generated by the elements $\langle\pi^i\rangle\Lambda_j$.

As in [1], D_{F_π} denotes the module corresponding to the formal law $F_\pi(X,Y) = \pi^{-1}F(\pi X, \pi Y)$.

In what follows we assume that

$$e_0 < q. \tag{4}$$

Lemma 2. $D_{F_\pi} = \mathcal{O}[[\Delta]]^m$.

Proof. Since $v_K(\pi_0) < q$ and, after passage from F to F_π, the coefficients of the logarithm at $X_j^{q^i}$ are multiplied by π^{q^i-1}, Lemma 1 shows that the claim can be proved in the same way as a similar item of Theorem 6.3.1 in [1]. □

3.2. Classification theorem. In the same way as in [1], using the preliminary remarks one can obtain the following result.

Theorem 2.

1) $\lambda(X) = \Lambda(\Delta)(X)$ *is the logarithm of an* $\mathcal{O}_0$*-typical formal* $\mathcal{O}_0$*-module over the ring* $\mathcal{O}$ *if and only if* $\Lambda = vu^{-1}$ *for some elements* $u \in M_m(\mathcal{O}_N[[\Delta]]')$ *and* $v \in M_m(\mathcal{O}[[\Delta]])$ *such that* $u \equiv \pi_0 I_m \bmod \Delta$, $v = \pi_0 I_m - \pi r_1 - \dots - \pi^{e_0-1} r_{e_0-1}$, $r_i \in M_m(\mathcal{O}_N[[\Delta]]'\Delta)$, $1 \le i \le e_0 - 1$.

2) *Any formal* $\mathcal{O}_0$*-module over* $\mathcal{O}$ *is isomorphic to an* $\mathcal{O}_0$*-typical module with such logarithm.*

3) *Two* $\mathcal{O}_0$*-typical formal* $\mathcal{O}_0$*-modules* $F(X,Y)$ *and* $F'(X,Y)$ *with logarithms* $\lambda(X) = vu^{-1}(X)$ *and* $\lambda'(X) = v'u'^{-1}(X)$ *are strictly isomorphic over* $\mathcal{O}$ *if and only if* $u' = \varepsilon u$, $v' = v + gu$ *for some* $\varepsilon \in M_m(\mathcal{O}_N[[\Delta]]')$, $\varepsilon \equiv I_m \bmod \Delta$, *and* $g \in \pi M_m(\mathcal{O}[[\Delta]]\Delta)$.

In the one-dimensional case, this theorem was proved in [7, Theorems 1, 2].

In the case where a formal $\mathcal{O}_0$-module F is isomorphic to an $\mathcal{O}_0$-module with logarithm of the form $vu^{-1}(X)$, we shall say that F is of type (u, v).

3.3. The form of the module D_F in the case of small ramification. We describe the Cartier module for a formal $\mathcal{O}_0$-module in the case of small ramification (i.e., under the assumption (4)).

Proposition 1. *Let* $\eta(\Delta) \in K[[\Delta]]^m$ *and* $\eta(\Delta) = \sum_{i=0}^{e_0-1} \pi^i \eta_i(\Delta)$, *where* $\eta_i(\Delta) \in N[[\Delta]]^m$, $0 \le i \le e_0 - 1$. *Then* $\eta(\Delta) \in D_F$ *if and only if the following congruences are valid*:

$$\begin{cases} u\eta_0 \equiv 0 \bmod \pi_0, \\ r_i\eta_0 + \pi_0\eta_i \equiv 0 \bmod \pi_0, \quad 1 \le i \le e_0 - 1. \end{cases} \tag{5}$$

Proof. Let $\eta \in D_F$. As has been mentioned above, D_F is generated as an $\mathcal{O}_N[[\Delta]]'$-module by the elements $\langle\pi^i\rangle\Lambda_j$, $0 \le i \le e_0 - 1$, $1 \le j \le m$. It is clear that the Λ_j satisfy system (5), because $\Lambda = vu^{-1}$. From condition (4) it follows that π^{q^i} is divisible by $\pi_0^i\pi$. Therefore, the $\langle\pi^s\rangle\Lambda_j$ lie in $\pi\mathcal{O}[[\Delta]]^m$ for all $s > 0$, and thus the $\langle\pi^s\rangle\Lambda_j$ also satisfy congruences (5). Consequently, all the elements of the module D_F satisfy system (5). We show that any element of $K[[\Delta]]^m$ satisfying this system lies in the module D_F.

Congruences (5) mean that

$$\begin{cases} u\eta_0 = \pi_0 A(\Delta), \\ r_i\eta_0 + \pi_0\eta_i = \pi_0 B_i(\Delta), \quad 1 \le i \le e_0 - 1, \end{cases}$$

for some series A, $B_i \in \mathcal{O}_N[[\Delta]]^m$. This implies

$$\eta = \sum_{i=0}^{e_0-1} \pi^i \eta_i = \pi_0 u^{-1} + \sum_{i=1}^{e_0-1} \pi^i (B_i - r_i u^{-1} A) = \Lambda A + \sum_{i=1}^{e_0-1} \pi^i B_i.$$

Clearly, $\sum_{i=1}^{e_0-1} \pi^i B_i \in \pi\mathcal{O}[[\Delta]]^m$. By Lemma 2, we have $\pi\mathcal{O}[[\Delta]]^m = \pi D_{F_\pi} \subset D_F$. Thus, $\eta \in D_F$. □

§4. Proof of the main theorem

Proof of Theorem 1. Let

$$\lambda(X) = \sum a_{i_1,\dots,i_m} X_1^{i_1} \dots X_m^{i_m},\ a_{i_1,\dots,i_m} \in K^m.$$

By [1, Proposition 5.5.1], the series $\lambda(X)$ is the logarithm of a formal $\mathcal{O}_0$-module of type (u, v) if and only if the series

$$\lambda^{(i_1 \dots i_m)}(\Delta) = \sum_{s=0}^{\infty} a_{i_1 q^s,\dots,i_m q^s} \Delta^s$$

lie in D_F for all multi-indices $(i_1 \dots i_m)$ such that the i_j are not all divisible by q. Let J denote the set of all such multi-indices. For $I = (i_1 \dots i_m)$ we put $X^I = X_1^{i_1} \dots X_m^{i_m}$.

It is easily seen that $\lambda(X) = \sum_{I\in J} \lambda^I(\Delta) X^I$. Let

$$\lambda^I(\Delta) = \sum_{k=0}^{e_0-1} \pi^k \lambda_k^I(\Delta), \quad \lambda_k^I(\Delta) \in N[[\Delta]]^m.$$

Then the validity of the system of congruences (1) is equivalent to the fact that for all $I \in J$ we have

$$\begin{cases} u\lambda_0^I \equiv 0 \bmod \pi_0, \\ r_k \lambda_0^I + \pi_0 \lambda_k^I \equiv 0 \bmod \pi_0, \quad 1 \le k \le e_0 - 1. \end{cases} \tag{6}$$

System 1 and Proposition 1 yield the required statement. □

Conceivably, condition (4) can be relaxed, but not drastically. In any case, we note the following.

Remark 1. If $v(\pi_0) > q$, then there are u and v satisfying conditions (3) for which there exists a (nonunique) $\lambda(X) \in K[[X]]^m$ such that $\lambda(X) \equiv X \bmod \deg 2$ and $\lambda(X)$ satisfies the system of congruences (1); moreover, $\lambda(X)$ is not the logarithm of a formal group.

We give an example of such a series $\lambda(X)$. For simplicity, assume that $m = 1$. Let $\lambda_{(q)}(X)$ be the logarithm of a formal Lubin–Tate group over the ring $\mathcal{O}_0$. For example,

$$\lambda_{(q)}(X) = \pi_0(\pi_0 - \Delta)^{-1}(X) = X + \frac{X^q}{\pi_0} + \dots.$$

We set $\lambda(X) = \lambda_{(q)}(X) + \pi X^2$. It is easy to verify that the series $\lambda(X)$ satisfies system (1) for $v = \pi_0$ and $u = \pi_0 - \Delta$. We show that the series is not the logarithm of a formal $\mathcal{O}_0$-module. Indeed, assume that $\lambda(X)$ is the logarithm of a formal $\mathcal{O}_0$-module F; then F is isomorphic to a formal module $F_{(q)}$ with logarithm $\lambda_{(q)}(X)$, because $\lambda_{(q)}(X)$ is the q-typical part of the series $\lambda(X)$. Then, by [1, Proposition 5.5.1], the series $\lambda^{(2)}(\Delta) = \pi$ lies in $D_{F_{(q)}}$. Consequently, $\lambda_{(q)}^{-1}(\pi X) \in \mathcal{O}[[X]]$, which is impossible because $\lambda_{(q)}^{-1}(\pi X) = \pi X + \frac{\pi^q}{\pi_0} X^q + \dots$, but $\frac{\pi^q}{\pi_0} \notin \mathcal{O}$.

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