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ON CHOW WEIGHT STRUCTURES FOR *cdh*-MOTIVES WITH INTEGRAL COEFFICIENTS

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Dedicated to S. V. Vostokov,
our Teacher in mathematics and in life

Abstract. The paper is aimed at defining a certain *Chow weight structure* w_{Chow} on the category $\mathcal{DM}_c(S)$ of (constructible) *cdh*-motives over an equicharacteristic scheme S. In contrast to the previous papers of D. Hébert and the first author on weights for relative motives (with rational coefficients), this goal is achieved for motives with integral coefficients (if $\operatorname{char} S = 0$; if $\operatorname{char} S = p > 0$, then motives with $\mathbb{Z}[\frac{1}{p}]$-coefficients are considered). It is proved that the properties of the Chow weight structures that were previously established for $\mathbb{Q}$-linear motives can be carried over to this "integral" context (and some of them are generalized using certain new methods). Mostly, the version of w_{Chow} defined via "gluing from strata" is studied; this makes it possible to define Chow weight structures for a wide class of base schemes.

As a consequence, certain (Chow)-weight spectral sequences and filtrations are obtained on any (co)homology of motives.

Introduction

In this paper we construct certain "weights" for R-linear motives over a scheme S. Here S is an excellent finite-dimensional Noetherian scheme of characteristic p (that can be 0) and R is a unital commutative associative coefficient ring; in the case where $p > 0$, we require p to be invertible in R. These weights are compatible with Deligne's weights for constructible complexes of étale sheaves (see Remark 3.2.2(4) below).

Now we explain this in more detail. Deligne's weights for étale sheaves and mixed Hodge structures (and for the corresponding derived categories) are very important for modern algebraic geometry. So, lifting these weights to motives is an important part of the so-called Beilinson's (motivic) dream. The "classical" approach (due to Beilinson) to do this is to define a filtration on motives that would split Chow motives into their components corresponding to single (co)homology groups (i.e., it should yield the so-called Chow–Kunneth decompositions). Since the existence of Chow–Kunneth decompositions is very much conjectural, it is no wonder that this approach was not really successful (up to now); moreover, it cannot work for R-linear motives if R is not a $\mathbb{Q}$-algebra.

In [3] an alternative method for defining weights for motives was proposed and successfully implemented. The so-called Chow weight structure on the triangulated category of Voevodsky motives $\mathcal{DM}_{gm}$ (with integral coefficients) over a characteristic 0 field was defined; the *heart* of this weight structure is the "classical" category of Chow motives.

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Now, arbitrary weight structures yield functorial weight filtrations and weight spectral sequences for any (co)homological functor (from $\mathcal{DM}_{gm}$). These weight filtrations and spectral sequences generalize Deligne's ones; note that they are also well defined for any (co)homology with integral coefficients (this is a vast extension of the earlier results of [14] on cohomology with compact support)!

The next paper in this direction was [4], where the Chow weight structure for $\mathbb{Z}[\frac{1}{p}]$-linear motives over a perfect field of characteristic p was defined. At the same time, the theory of Voevodsky triangulated motivic categories over any "more or less general" base scheme S was (in [10]) developed to the stage that the Chow weight structure for Beilinson motives over S (i.e., for S-motives with rational coefficients) could be defined; independently, this was done in [15] and in [7] (see Remark 2.3.3(2) below).

Our main goal in the current paper is to define the Chow weight structure on $\mathbb{Z}[\frac{1}{p}]$-linear motives (and more generally, on R-linear motives for any $\mathbb{Z}[\frac{1}{p}]$-algebra R) over any excellent finite-dimensional Noetherian base scheme S of characteristic p (here we set $\mathbb{Z}[\frac{1}{p}] = \mathbb{Z}$ if $p = 0$, and consider *cdh*-motives with R-coefficients that were denoted by $\mathcal{DM}_{cdh}(S, R)$ in [12]). To achieve this we use the "gluing construction" of the Chow weight structure; this construction was described (for $\mathbb{Q}$-linear motives) in [7, §2.3] (whereas the method was first proposed in [3, §8.2]). This requires some new methods for studying morphisms between relative motives (in §1.3). We also note that all the properties and applications of the Chow weight structure described in [7] carry over to our "integral" context. We apply some new arguments for studying the weight-exactness of motivic functors (in §2.2; following [8], we use *Borel–Moore motives*); they allow us to extend the corresponding results to not necessarily quasiprojective morphisms of base schemes.

Thus, this paper gives convenient tools for studying "integral" (and torsion) weight phenomena for (equicharacteristic) schemes and motives. In particular, we obtain functorial "Chow-weight" filtrations and spectral sequences (see §3.2). Note still that we are able to prove that "explicit Chow motives" over S yield a weight structure for S-motives (as in [15] and [7, §2.1]) only if S is a "pro-smooth limit" of schemes of finite type over a field (see §2.3 and [17]).

Finally, we note that one can (certainly) consider motivic categories corresponding to Grothendieck topologies distinct from the *cdh* one. In particular, a (not really "successful") attempt was made in [6] to construct certain Chow weight structures on relative Nisnevich motivic categories (using their properties established in [10]). Note yet that the Nisnevich motives are isomorphic to the *cdh*-ones over regular bases (see [12, Corollary 5.9]); this is also expected to be true in general. On the other hand, though (R-linear) étale motivic categories (that were thoroughfully studied in [11]) enjoy several "nice" properties, there is no chance to define w_{Chow} for them unless $\mathbb{Q} \subset R$, whereas in the latter case the relative motivic categories mentioned "do not depend on the choice of a topology" (if we compare *cdh*, Nisnevich, étale, and h-motives).

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§1. Preliminaries

This section is mostly a recollection of basics on (relative *cdh*)-motives and weight structures; yet the results of §1.3 and the methods of their proofs are (more or less) new.

1.1. Notation.

- For categories C, D we write $D \subset C$ if D is a full subcategory of C.

- For a category C and $X, Y \in \operatorname{Obj} C$, we denote by $C(X, Y)$ the set of C-morphisms from X to Y.
- An additive subcategory D of C is said to be *Karoubi-closed* in it if it contains all retracts of its objects in C. The full subcategory of C whose objects are all retracts of objects of D (in C) will be called the *Karoubi-closure* of D in C.
- $\underline{C}$ will always denote some triangulated category; usually it will be endowed with a weight structure w (see Definition 1.4.1 below).
- For a set of objects $C_i \in \operatorname{Obj} \underline{C}$, $i \in I$, we denote by $\langle C_i \,|\, i \in I \rangle$ the smallest strictly full triangulated subcategory of $\underline{C}$ containing all C_i; for $D \subset \underline{C}$ we write $\langle D \rangle$ instead of $\langle \operatorname{Obj} D \rangle$. We call the Karoubi-closure of $\langle C_i \,|\, i \in I \rangle$ in $\underline{C}$ the *triangulated category generated by* C_i (recall that it is triangulated indeed).
- For $X, Y \in \operatorname{Obj} \underline{C}$ we write $X \perp Y$ if $\underline{C}(X, Y) = \{0\}$. For $D, E \subset \operatorname{Obj} \underline{C}$ we write $D \perp E$ if $X \perp Y$ for all $X \in D$, $Y \in E$. For $D \subset \underline{C}$ we denote by $D^{\perp}$ the class
$$\{Y \in \operatorname{Obj} \underline{C} \,:\, X \perp Y \ \forall X \in D\}.$$
Dually, ${}^{\perp}D$ is the class $\{Y \in \operatorname{Obj} \underline{C} \,:\, Y \perp X \ \forall X \in D\}$.
- We say that some $C_i \in \operatorname{Obj} \underline{C}$, $i \in I$, *weakly generate* $\underline{C}$ if for $X \in \operatorname{Obj} \underline{C}$ the condition $\underline{C}(C_i[j], X) = \{0\}$ for all $i \in I$, $j \in \mathbb{Z}$ implies $X = 0$ (i.e., if $\{C_i[j] \,:\, j \in \mathbb{Z}\}^{\perp}$ contains only zero objects).
- $M \in \operatorname{Obj} \underline{C}$ is said to be *compact* if the functor $\underline{C}(M, -)$ commutes with all small coproducts that exist in $\underline{C}$ (we shall only consider compact objects in those categories that are closed with respect to arbitrary small coproducts).
- $D \subset \operatorname{Obj} \underline{C}$ is said to be *extension-stable* if $0 \in D$ and for any distinguished triangle $A \to B \to C$ in $\underline{C}$ we have $A, C \in D \implies B \in D$.
- The smallest Karoubi-closed extension-stable subclass of $\operatorname{Obj} \underline{C}$ containing D is called the *envelope* of D.
- Sometimes, we shall need certain stratifications of a scheme S. Recall that a stratification α is a presentation of S as $\bigcup S_\ell^\alpha$, where S_ℓ^α, $1 \le \ell \le n$, are pairwise disjoint locally closed subschemes of S. Omitting α, we shall denote by $j_\ell \colon S_\ell^\alpha \to S$ the corresponding immersions. We do not demand the closure of each S_ℓ^α to be the union of strata (though we could do this); we only assume that each S_ℓ^α is open in $\bigcup_{i \ge \ell} S_i^\alpha$.
- Below we identify a Zariski point (of a scheme S) with the spectrum of its residue field.
- k is a prime field, $p = \operatorname{char} k$ (p may be 0).
- All the schemes we consider will be excellent, separated, Noetherian k-schemes (i.e., characteristic p schemes) of finite Krull dimension (so, a "scheme" will always mean a scheme of this sort).
- A variety over a field F/k is a (separated) reduced scheme of finite type over $\operatorname{Spec} F$.
- S_{red} will denote the reduced scheme associated with S.
- All morphisms of schemes considered below will be separated. They will also mostly be of finite type.
- Throughout the paper R will be some fixed unital associative commutative algebra over $\mathbb{Z}[\frac{1}{p}]$ (we set $\mathbb{Z}[\frac{1}{p}] = \mathbb{Z}$ if $p = 0$).

1.2. On *cdh*-motives (after Cisinski and Deglise). We list some of the properties of the triangulated categories of *cdh*-motives (those are certain relative Voevodsky motives with R-coefficients described by Cisinski and Deglise). They are very much similar to the properties of Beilinson motives (i.e., of $\mathbb{Q}$-linear ones) that were established in [10] (and applied in [15] and [7] for the construction of the corresponding Chow weight structures).

Theorem 1.2.1. *Let X, Y be (Noetherian finite-dimensional excellent characteristic p) schemes, and let $f\colon X \to Y$ be a (separated) morphism of finite type.*

(1) *For any X, there exists a tensor triangulated R-linear category $\mathcal{DM}(X)$ with a (well-defined) unit object R_X (in* [12, Definition 1.5] *this category was denoted by $DM_{cdh}(X,R)$); it is closed with respect to arbitrary small coproducts.*
(2) *The (full) subcategory $\mathcal{DM}_c(X) \subset \mathcal{DM}(X)$ of compact objects is tensor triangulated, and $R_X \in \operatorname{Obj} \mathcal{DM}_c(S)$. $\mathcal{DM}_c(X)$ weakly generates $\mathcal{DM}(X)$.*
(3) *The following functors are defined:*

$$f^*\colon \mathcal{DM}(Y) \leftrightarrows \mathcal{DM}(X) : f_* \quad \text{and} \quad f_!\colon \mathcal{DM}(X) \leftrightarrows \mathcal{DM}(Y) : f^!$$

for any f; f^* is left adjoint to f_* and $f_!$ is left adjoint to $f^!$.
We call these **the motivic image functors**. *Any of them (when f varies) yields a 2-functor from the category of (separated finite-dimensional excellent characteristic p) schemes with morphisms of finite type to the 2-category of triangulated categories. Moreover, all motivic image functors preserve compact objects (i.e., they can be restricted to the subcategories $\mathcal{DM}_c(-)$); they also commute with arbitrary (small) coproducts.*
(4) *For a Cartesian square of morphisms of finite type*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

we have $g^*f_! \cong f'_!g'^*$ and $g'_*f'^! \cong f^!g_*$.
(5) *For any X there exists a Tate object $R(1) \in \operatorname{Obj} \mathcal{DM}_c(X)$; tensoring by it yields an exact Tate twist functor $-(1)$ on $\mathcal{DM}(X)$. This functor is an autoequivalence of $\mathcal{DM}(X)$; we will denote the inverse functor by $-(-1)$. Tate twists commute with all motivic image functors mentioned (up to an isomorphism of functors). Moreover, for $X = \mathbb{P}^1(Y)$ there is a functorial isomorphism $f_!(R_{\mathbb{P}^1(Y)}) \cong R_Y \bigoplus R_Y(-1)[-2]$.*
(6) *f^* is symmetric monoidal; $f^*(R_Y) = R_X$.*
(7) *$f_* \cong f_!$ if f is proper; $f^!(-) \cong f^*(-)(s)[2s]$ if f is smooth (everywhere) of relative dimension s. If f is an open immersion, we have $f^! = f^*$.*
(8) *If $i\colon S' \to S$ is an immersion of regular schemes everywhere of codimension d, then $R_{S'}(-d)[-2d] \cong i^!(R_S)$.*
(9) *If $i\colon Z \to X$ is a closed immersion, $U = X \setminus Z$, and $j\colon U \to X$ is the complementary open immersion, then the motivic image functors yield a gluing datum for $\mathcal{DM}(-)$ (in the sense of* [1, §1.4.3]; *see also* [3, Definition 8.2.1]). *This means that (in addition to the adjunctions given by assertion* 3) *the following statements are valid.*
  (a) *$i_* \cong i_!$ is a full embedding; $j^* = j^!$ is isomorphic to the localization (functor) of $\mathcal{DM}(X)$ by $i_*(\mathcal{DM}(Z))$.*
  (b) *For any $M \in \operatorname{Obj} \mathcal{DM}(X)$, the pairs of morphisms $j_!j^!(M) \to M \to i_*i^*(M)$ and $i_!i^!(M) \to M \to j_*j^*(M)$ can be uniquely completed to distinguished triangles (here the connecting morphisms come from the adjunctions of assertion* 3).
  (c) *$i^*j_! = 0$; $i^!j_* = 0$.*

(d) *All the adjunction transformations* $i^*i_* \to 1_{\mathcal{DM}(Z)} \to i^!i_!$ *and* $j^*j_* \to 1_{\mathcal{DM}(U)} \to j^!j_!$ *are isomorphisms of functors.*

(10) *For the subcategories* $\mathcal{DM}_c(-) \subset \mathcal{DM}(-)$ *an obvious analog of the previous assertion is fulfilled.*

(11) *If* f *is a finite universal homeomorphism, then* f^*, f_*, $f^!$, *and* $f_!$ *are equivalences of categories. Moreover,* $f^!R_Y \cong f^*R_Y = R_X$ *and* $f_*R_X = f_!R_X \cong R_Y$.

(12) *If* S *is of finite type over a field, then* $\mathcal{DM}_c(S)$ (*as a triangulated category*) *is generated by* $\{g_*(R_X)(r)\}$, *where* $g\colon X \to S$ *runs through all projective morphisms such that* X *is regular,* $r \in \mathbb{Z}$.

(13) *Let a scheme* S *be the limit of an essentially affine* (*filtering*) *projective system of schemes* S_β (*for* $\beta \in B$). *Then* $\mathcal{DM}_c(S)$ *is isomorphic to the* 2*-colimit of the categories* $\mathcal{DM}_c(S_\beta)$; *in this isomorphism all the connecting functors are given by the corresponding motivic inverse image functors* (*cf. Remark* 1.2.2(1) *below*).

(14) *If* S *is smooth over* k (*or over any other perfect field*), *then for* $b, c, r \in \mathbb{Z}$ *with* $r \geq 0$ *we have* $R_S(b)(2b) \perp R_S(c)[2c + r]$.

Proof. These statements can (mostly) be found in [12]. Specifically:

(1) [12, §1.6].

(2) Immediate from the definition of $\mathcal{DM}(-)$ given in [12, §1.5].

(3) [12, §1.6, Theorem 6.4].

(4,5,6,7,9) [12, Proposition 4.3, Theorem 5.1] show that $\mathcal{DM}(-)$ is a *motivic triangulated category*; see [10, 2.4.45] for the definition and [10, 2.4.50] for a list of properties that includes the desired assertions.

(8) [12, Proposition 6.2].

(10) This is an easy consequence of assertions 3 and 9; cf. (the proof of) Proposition 1.1.2(11) in [7].

(11) By [12, Proposition 7.1], the functors $f^* \cong f^!$ are equivalences of categories. Hence, their (right and left) adjoints f_* and $f_!$ are also equivalences.

(12) [12, Proposition 7.2].

(13) [12, Theorem 5.11].

(14) [12, Corollary 8.6, putting $X = \operatorname{Spec} k$]. □

Remark 1.2.2. 1. In [12] the functor g^* was constructed for any morphism $g\colon Y' \to Y$ not necessarily of finite type; it preserves compact objects (see §6.1(ii) of ibid.) and unit objects R_- (i.e., $g^*(R_Y) = R_{Y'}$). Moreover, for any such g and any morphism $f\colon X \to Y$ of finite type we have an isomorphism $g^*f_! \cong f'_!g'^*$ (for the corresponding f' and g'; cf. part 4 of our theorem).

We also note: if f is a pro-open limit of immersions, then one can define $f^! = f^*$ (in particular, one can define $j_K^!$ for the natural morphism $j_K\colon K \to S$ if K is a Zariski point of a scheme S; cf. [1, §2.2.12]); $f^!$ preserves compact objects. For the functors of this type we also have $g'_*f'^! \cong f^!g_*$ (for g of finite type; see part 4 of our theorem once again); see [10, Proposition 4.3.14].

2. For any morphism $f\colon X \to Y$ of finite type we set $\mathcal{M}_Y^{BM}(X) = f_!R_X$ (this is a certain *Borel–Moore* motif of X; cf. [8], [13], and [18, §I.IV.2.4]). Note that Theorem 1.2.1(11) shows that $\mathcal{M}_Y^{BM}(X) \cong \mathcal{M}_Y^{BM}(X_{\mathrm{red}})$ (recall that X_{red} is the reduced scheme associated with X). Moreover, for any (separated) morphism $g\colon Y' \to Y$ the previous part of this remark yields $g^*(\mathcal{M}_Y^{BM}(X)) \cong \mathcal{M}_{Y'}^{BM}(X \times_Y Y')$.

Lemma 1.2.3. *Let* $S = \bigcup S_\ell^\alpha$ *be a stratification. Then the following statements are valid.*

(1) *For any* $M, N \in \operatorname{Obj} \mathcal{DM}(S)$ *there exists a filtration of* $\mathcal{DM}(S)(M, N)$ *whose factors are certain subquotients of* $\mathcal{DM}(S_\ell^\alpha)(j_\ell^*(M), j_\ell^!(N))$. *If* $M = R_S(a)[2a]$,

$N = R_S(b)[2b+r]$ for $a, b, r \in \mathbb{Z}$, S is regular, and all S_ℓ^α are regular and connected, then the factors of this filtration on $\mathcal{DM}(S)(M, N)$ are certain subquotients of the groups $\mathcal{DM}(S_\ell^\alpha)(R_{S_\ell^\alpha}, R_{S_\ell^\alpha}(b-a-c_\ell)[2b+r-2a-2c_\ell])$, where c_ℓ is the codimension of S_ℓ^α in S.

(2) *If $g\colon S \to Y$ is a morphism of finite type, then $\mathcal{M}_Y^{BM}(S)$ belongs to the envelope of $\mathcal{M}_Y^{BM}(S_{\ell,\mathrm{red}}^\alpha)$.*

(3) *Let $Z \subset X$ be a closed subscheme, where X is a scheme of finite type over S; denote by $U \subset X$ the complementary open subscheme. If Z and X are regular, and Z is everywhere of codimension c in X, then there is a distinguished triangle*

$$z_*R_Z(-c)[-2c] \to x_*R_X \to u_*R_U \tag{1.1}$$

in $\mathcal{DM}(S)$, where z, x, u are the corresponding structure morphisms (to S).

Proof. 1. The second part of the assertion is a simple consequence of the first (see Theorem 1.2.1(6,8)).

We prove the first statement by induction on the number of strata. By definition (see §1.1), S_1^α is open in S, and the remaining S_ℓ^α yield a stratification of $S \setminus S_1^\alpha$. We denote $S \setminus S_1^\alpha$ by Z, the (open) immersion $S_1^\alpha \to S$ by j and the (closed) immersion $Z \to S$ by i.

Now, Theorem 1.2.1(9) yields a distinguished triangle

$$j_!j^!(M) \to M \to i_*i^*(M). \tag{1.2}$$

Hence, there exists a (long) exact sequence

$$\cdots \to \mathcal{DM}(S)(i_*i^*(M), N) \to \mathcal{DM}(S)(M, N) \to \mathcal{DM}(S)(j_!j^!(M), N) \to \dots .$$

The corresponding adjunctions of functors yield:

$$\mathcal{DM}(S)(i_*i^*(M), N) \cong \mathcal{DM}(Z)(i^*(M), i^!(N)),$$
$$\mathcal{DM}(S)(j_!j^!(M), N) \cong \mathcal{DM}(S_1^\alpha)(j^*(M), j^!(N)).$$

Now, by the inductive assumption the group $\mathcal{DM}(Z)(i^*(M), i^!(N))$ has a filtration whose factors are certain subquotients of $\mathcal{DM}(S_\ell^\alpha)(j_\ell^*(M), j^!(N))$ (for $\ell \neq 1$). This concludes the proof.

2. We use the same induction and notation as in the previous proof. Considering the distinguished triangle (1.2) for $M = R_S$ and applying $g_!$ to it, we obtain a distinguished triangle

$$\mathcal{M}_Y^{BM}(S_1^\alpha) \to \mathcal{M}_Y^{BM}(S) \to \mathcal{M}_Y^{BM}(Z). \tag{1.3}$$

In order to complete the inductive step, it suffices to apply (the first statement in) Remark 1.2.2(2).

3. We may assume that X is connected. Theorem 1.2.1(9) yields a distinguished triangle $i_!i^!R_X \to R_X \to j_*j^*R_X (\cong j_*R_U)$ (see also part 6 of the theorem). If Z and X are regular, then $i_!i^!R_X \cong i_*R_Z(-c)[-2c]$ (see part 8 of the theorem). Hence, the application of x_* to this distinguished triangle yields (1.1). □

1.3. Some orthogonality lemmas. The following motivic statements are very important for the current paper.

Lemma 1.3.1. *If S is a regular scheme, then for any $a, b, r \in \mathbb{Z}$ with $r > 0$ we have $R_S(a)[2a] \perp R_S(b)[2b+r]$.*

Proof. We stratify S as $\bigcup_{1\le\ell\le n} S_\ell$ with all S_ℓ regular, affine, and connected. For such a stratification (by Lemma 1.2.3(1)) it suffices to prove that

$$\mathcal{DM}(S_\ell)\big(R_{S_\ell}(a-\operatorname{codim}_S S_\ell)[2a-2\operatorname{codim}_S S_\ell],\\ R_{S_\ell}(b-\operatorname{codim}_S S_\ell)[2b-2\operatorname{codim}_S S_\ell+r]\big)=\{0\}.$$

Therefore, it suffices to prove the statement for strata. Thus, we may assume that S is regular and affine. Such a scheme S can be presented as the inverse limit of regular schemes of finite type over k (by the Popescu–Spivakovsky theorem; see [10, Theorem 4.1.5]). If $S=\varprojlim S_\beta$, then

$$\mathcal{DM}_c(S)(R_S(b)[2b],R_S(c)[2c+r])=\varinjlim \mathcal{DM}_c(S_\beta)(R_{S_\beta}(b)[2b],R_{S_\beta}(c)[2c+r])$$

by Theorem 1.2.1(13). In conclusion, we refer to part (14) of that theorem. □

Remark 1.3.2. This continuity argument along with [12, Corollary 8.6] also easily shows that $\mathcal{DM}(R_S,R_S(b)[2b+r])$ is isomorphic to the corresponding higher Chow group of S, i.e., it can be computed by using the Bloch or Suslin complex (of codimension b cycles in $S\times\Delta^{-*}$) with R-coefficients if S is a regular affine scheme. This result cannot be generalized automatically to arbitrary (regular excellent finite-dimensional equicharacteristic) schemes because (to the knowledge of the authors) the Mayer–Vietoris property is not known for the higher Chow groups in this generality.

Lemma 1.3.3. *Let X and Y be regular schemes, let $x\colon X\to S$ and $y\colon Y\to S$ be quasiprojective morphisms, and let $r,b,c\in\mathbb{Z}$.*

Then $x_!(R_X)(b)[2b]\perp y_*(R_Y)(c)[2c+r]$ if $r>0$.

Proof. Theorem 1.2.1(5) allows us to assume that $b=0$. Next, we have

$$\mathcal{DM}_c(S)(x_!(R_X),y_*(R_Y)(c)[2c+r])\cong\mathcal{DM}_c(X)(R_X,x^!y_*(R_Y)(c)[2c+r])$$

because $x^!$ is left adjoint to $x_!$.

Thus, we must prove that

$$\mathcal{DM}_c(X)(R_X,x^!y_*(R_Y)(c)[2c+r])=\{0\}.$$

We argue somewhat similarly to [8, §2.1]. Let us make certain reduction steps.

Consider a factorization of x as $X\xrightarrow{f}S'\xrightarrow{h}S$, where h is smooth of dimension q, f is an embedding, and S' is connected, and consider the corresponding diagram

$$\begin{array}{ccccc} Z & \xrightarrow{f_Y} & Y' & \xrightarrow{h_Y} & Y\\ \big\downarrow{\scriptstyle z_X} & & \big\downarrow{\scriptstyle y'} & & \big\downarrow{\scriptstyle y}\\ X & \xrightarrow{f} & S' & \xrightarrow{h} & S \end{array}$$

(the upper row is the base change of the lower one to Y). Then we have

$$x^!y_*(R_Y)(c)[2c+r]=f^!h^!y_*(R_Y)(c)[2c+r]\cong f^!y'_*h_Y^!(R_Y)(c)[2c+r]$$

(by Theorem 1.2.1(4)). Parts (7) and (6) of Theorem 1.2.1 allow us to transform this into $f^!y'_*(R_{Y'})(q{+}c)[2q{+}2c{+}r]$. Hence, below we may assume that x is an embedding (because we can replace S by S' in the assertion). Furthermore, the isomorphism $x^!y_*\cong z_{X*}z_Y^!$ for $z_Y=h_Y\circ f_Y$ shows that the group in question is zero if Y lies over $S\setminus X$ (viewed as a set); see Theorem 1.2.1(1).

Applying Lemma 1.2.3(1), we see that it suffices to verify the statement for Y replaced by the components of some regular connected stratification.

Now, we can choose a stratification of this sort so that each Y_ℓ lies either over X or over $S\setminus X$. Therefore, it suffices to verify our assertion in the case where y factors through x. Moreover, since $x^!x_*$ is the identity functor on $\mathcal{DM}_c(X)$ (in this case; see Theorem 1.2.1(9)), we may also assume that $X = S$. Next, applying the adjunction $y^* \dashv y_*$ we transform $\mathcal{DM}(S)(R_S, y_*(R_Y)(c)[2c+r])$ into

$$\mathcal{DM}(Y)(y^*(R_S), R_Y(c)[2c+r]) = \mathcal{DM}(Y)(R_Y, R_Y(c)[2c+r]).$$

Thus it remains to apply the previous lemma. □

1.4. Weight structures: short reminder. We recall some basics of the theory of weight structures.

Definition 1.4.1. (1) A pair of subclasses $\underline{C}_{w\le 0}, \underline{C}_{w\ge 0} \subset \operatorname{Obj} \underline{C}$ is said to define a *weight structure* w for $\underline{C}$ if these subclasses satisfy the following conditions:

(a) $\underline{C}_{w\ge 0}, \underline{C}_{w\le 0}$ are Karoubi-closed in $\underline{C}$ (i.e., contain all $\underline{C}$-retracts of their objects);

(b) ***semiinvariance with respect to translations:*** $\underline{C}_{w\le 0} \subset \underline{C}_{w\le 0}[1]$, $\underline{C}_{w\ge 0}[1] \subset \underline{C}_{w\ge 0}$;

(c) ***orthogonality:*** $\underline{C}_{w\le 0} \perp \underline{C}_{w\ge 0}[1]$;

(d) ***weight decompositions:*** for any $M \in \operatorname{Obj} \underline{C}$ there exists a distinguished triangle

$$B \to M \to A \xrightarrow{f} B[1]$$

such that $A \in \underline{C}_{w\ge 0}[1]$, $B \in \underline{C}_{w\le 0}$.

(2) The category $\underline{Hw} \subset \underline{C}$ whose objects are $\underline{C}_{w=0} = \underline{C}_{w\ge 0} \cap \underline{C}_{w\le 0}$, $\underline{Hw}(Z,T) = \underline{C}(Z,T)$ for $Z, T \in \underline{C}_{w=0}$, will be called the *heart* of w.

(3) $\underline{C}_{w\ge i}$ (respectively, $\underline{C}_{w\le i}$, respectively, $\underline{C}_{w=i}$) will denote the class $\underline{C}_{w\ge 0}[i]$ (respectively, $\underline{C}_{w\le 0}[i]$, respectively, $\underline{C}_{w=0}[i]$). We denote $\underline{C}_{w\ge i} \cap \underline{C}_{w\le j}$ by $\underline{C}_{[i,j]}$ (so it equals $\{0\}$ for $i > j$).

(4) We say that $(\underline{C}, w)$ is *bounded* if $\bigcup_{i\in\mathbb{Z}} \underline{C}_{w\le i} = \operatorname{Obj} \underline{C} = \bigcup_{i\in\mathbb{Z}} \underline{C}_{w\ge i}$.

(5) Let $\underline{C}$ and $\underline{C}'$ be triangulated categories endowed with weight structures w and w', respectively; let $F\colon \underline{C} \to \underline{C}'$ be an exact functor. We say that F is *left weight-exact* (with respect to w, w') if it maps $\underline{C}_{w\le 0}$ into $\underline{C}'_{w'\le 0}$; it is *right weight-exact* if it maps $\underline{C}_{w\ge 0}$ into $\underline{C}'_{w'\ge 0}$. F is *weight-exact* if it is both left and right weight-exact.

(6) Let H be a full subcategory of a triangulated $\underline{C}$. We say that H is *negative* if $\operatorname{Obj} H \perp \left(\bigcup_{i>0} \operatorname{Obj}(H[i])\right)$.

(7) We call a category $\frac{A}{B}$ a *factor* of an additive category A by its (full) additive subcategory B if $\operatorname{Obj}\left(\frac{A}{B}\right) = \operatorname{Obj} A$ and

$$\frac{A}{B}(M,N) = A(M,N) \Big/ \Bigg(\sum_{O\in \operatorname{Obj} B} A(O,N) \circ A(M,O)\Bigg).$$

Now we recall the properties of weight structures that will be needed below (and can easily be formulated). See [7] for the references to the proofs (whereas the fact that $\underline{Hw}' \cong \frac{\underline{Hw}}{\underline{Hw}_D}$ in the setting of assertion 8 is given by Proposition 3.3.4(1) of [21]).

Proposition 1.4.2. *Let $\underline{C}$ be a triangulated category.*

(1) *A pair (C_1, C_2) $(C_1, C_2 \subset \operatorname{Obj} \underline{C})$ defines a weight structure for $\underline{C}$ if and only if $(C_2^{\mathrm{op}}, C_1^{\mathrm{op}})$ defines a weight structure for $\underline{C}^{\mathrm{op}}$.*

(2) *Let w be a weight structure for $\underline{C}$. Then $\underline{C}_{w\ge 0} = (\underline{C}_{w\le -1})^{\perp}$ and $\underline{C}_{w\le 0} = {}^{\perp}\underline{C}_{w\ge 1}$ (see §1.1).*

(3) *Let w be a weight structure on $\underline{C}$. Then $\underline{C}_{w\le 0}$, $\underline{C}_{w\ge 0}$, and $\underline{C}_{w=0}$ are extension-stable.*

(4) *Suppose that v, w are weight structures for $\underline{C}$; let $\underline{C}_{v\le 0}\subset \underline{C}_{w\le 0}$ and $\underline{C}_{v\ge 0}\subset \underline{C}_{w\ge 0}$. Then $v=w$ (i.e., the inclusions are identities).*

(5) *Assume that $H\subset \operatorname{Obj}\underline{C}$ is negative and $\underline{C}$ is idempotent complete. Then there exists a unique weight structure w on the triangulated subcategory T of $\underline{C}$ generated by H such that $H\subset T_{w=0}$. Its heart is the envelope (see §1.1) of H in $\underline{C}$; it is the idempotent completion of H if H is additive.*

(6) *For the weight structure mentioned in the previous assertion, $T_{w\le 0}$ is the envelope of $\bigcup_{i\le 0} H[i]$; $T_{w\ge 0}$ is the envelope of $\bigcup_{i\ge 0} H[i]$.*

(7) *Let $\underline{C}$ and $\underline{D}$ be triangulated categories endowed with weight structures w and v, respectively. Let $F\colon \underline{C}\leftrightarrows \underline{D} : G$ be adjoint functors. Then F is left weight-exact if and only if G is right weight-exact.*

(8) *Let w be a weight structure for $\underline{C}$; let $\underline{D}\subset\underline{C}$ be a triangulated subcategory of $\underline{C}$. Suppose that w yields a weight structure for $\underline{D}$ (i.e., $\operatorname{Obj}\underline{D}\cap\underline{C}_{w\le 0}$ and $\operatorname{Obj}\underline{D}\cap\underline{C}_{w\ge 0}$ give a weight structure for $\underline{D}$).*
Then w also induces a weight structure on $\underline{C}/\underline{D}$ (the localization, i.e., the Verdier quotient of $\underline{C}$ by $\underline{D}$) in the following sense: the Karoubi-closures of $\underline{C}_{w\le 0}$ and $\underline{C}_{w\ge 0}$ (viewed as classes of objects of $\underline{C}/\underline{D}$) give a weight structure w' for $\underline{C}/\underline{D}$ (note that $\operatorname{Obj}\underline{C}=\operatorname{Obj}\underline{C}/\underline{D}$). Moreover, $\underline{Hw'}$ is naturally equivalent to $\frac{\underline{Hw}}{\underline{Hw}_{\underline{D}}}$ if w is bounded.

(9) *Suppose that $\underline{D}\subset\underline{C}$ is a full subcategory of compact objects endowed with a bounded weight structure w'. Suppose that $\underline{D}$ weakly generates $\underline{C}$; let $\underline{C}$ admit arbitrary (small) coproducts. Then w' can be extended to a certain weight structure w for $\underline{C}$.*

(10) *Let $\underline{D}\stackrel{i_*}{\to}\underline{C}\stackrel{j^*}{\to}\underline{E}$ be a part of a gluing datum (see Theorem 1.2.1(9)). Then for any pair of weight structures on $\underline{D}$ and $\underline{E}$ (we will denote them by $w_{\underline{D}}$ and $w_{\underline{E}}$, respectively) there exists a weight structure w on $\underline{C}$ such that both i_* and j^* are weight-exact (with respect to the corresponding weight structures). Furthermore, the functors $i^!$ and j_* are right weight-exact (with respect to the corresponding weight structures); i^* and $j_!$ are left weight-exact. Moreover,*

$$\underline{C}_{w\ge 0}=C_1=\big\{M\in\operatorname{Obj}\underline{C}\ :\ i^!(M)\in\underline{D}_{w_{\underline{D}}\ge 0},\ j^*(M)\in\underline{E}_{w_{\underline{E}}\ge 0}\big\},$$
$$\underline{C}_{w\le 0}=C_2=\big\{M\in\operatorname{Obj}\underline{C}\ :,\ i^*(M)\in\underline{D}_{w_{\underline{D}}\le 0},\ j^*(M)\in\underline{E}_{w_{\underline{E}}\le 0}\big\}.$$

Finally, C_1 (respectively, C_2) is the envelope of $j_!(\underline{E}_{w\le 0})\cup i_*(\underline{D}_{w\le 0})$ (respectively, of $j_*(\underline{E}_{w\ge 0})\cup i_*(\underline{D}_{w\ge 0})$).

(11) *In the setting of the previous assertion, if $w_{\underline{D}}$ and $w_{\underline{E}}$ are bounded, then w is also bounded. Next, $\underline{C}_{w\le 0}$ (respectively, $\underline{C}_{w\ge 0}$) is the envelope of*

$$\big\{i_*(\underline{D}_{w_{\underline{D}}=l}),\ j_!(\underline{E}_{w_{\underline{E}}=l}),\ l\le 0\big\}$$

(respectively, $\{i_*(\underline{D}_{w_{\underline{D}}=l}),\ j_*(\underline{E}_{w_{\underline{E}}=l}),\ l\ge 0\}$).

(12) *In the setting of assertion 10, the weight structure w described is the only weight structure for $\underline{C}$ such that both i_* and j^* are weight-exact.*

Remark 1.4.3. Part 8 of the proposition can be reformulated as follows. If $i_*\colon\underline{D}\to\underline{C}$ is an embedding of triangulated categories that is weight-exact (with respect to certain weight structures for $\underline{D}$ and $\underline{C}$), and an exact functor $j^*\colon\underline{C}\to\underline{E}$ is equivalent to the localization of $\underline{C}$ by $i_*(\underline{D})$, then there exists a unique weight structure w' for $\underline{E}$ such that the functor j^* is weight-exact. If w is bounded, then $\underline{Hw}_{\underline{E}}$ is equivalent to $\frac{\underline{Hw}}{i_*(\underline{Hw}_{\underline{D}})}$ (with respect to the natural functor $\frac{\underline{Hw}}{i_*(\underline{Hw}_{\underline{D}})} \to \underline{E}$).

§2. On the Chow weight structures for relative motives

This is the main section of our paper. We define the Chow weight structures for relative motives by using the "gluing construction" and study their properties. We also prove that the heart of $w_{\mathrm{Chow}}(S)$ consists of certain "Chow" motives if S is a variety over a field (or if it is "pro-smooth affine" over a variety).

A substantial part of this section is merely a "recombination" of (the corresponding parts of) [7, §2]; yet some of the arguments used in §2.2 are quite new (and rather interesting).

2.1. The construction of the Chow weight structure. First, we describe certain candidates for $\mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\geq 0}$ and $\mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\leq 0}$; next we shall prove that they yield a weight structure for $\mathcal{DM}_c(S)$ indeed. A reader interested in certain "motivation" for this construction is strongly recommended to look at (the remarks in) [7, §2.3].

For a scheme X we denote by $\mathcal{OP}(X)$ (respectively, $\mathcal{ON}(X)$) the envelope (see §1.1) of $p_*(R_P)(s)[i+2s](\cong \mathcal{M}_X^{BM}(P)(s)[i+2s]$; see Remark 1.2.2(2)) in $\mathcal{DM}_c(X)$; here $p\colon P \to X$ runs through all morphisms to X that can be factorized as $g \circ h$, where $h\colon P \to X'$ is a smooth projective morphism, X' is a regular scheme, $g\colon X' \to X$ is a finite universal homeomorphism, $s \in \mathbb{Z}$, whereas $i \geq 0$ (respectively, $i \leq 0$). We denote $\mathcal{OP}(X) \cap \mathcal{ON}(X)$ by $\mathcal{OZ}(X)$.

Remark 2.1.1. 1. Recall that for any morphism of finite type $f\colon Y \to X$ we have $f^*\mathcal{M}_X^{BM}(P) \cong \mathcal{M}_Y^{BM}(P_Y)$ (see Remark 1.2.2(2)). Next, suppose that for X'/X as above the scheme Y'_{red} associated with $Y' = X'_Y$ is regular. Then the remark cited above immediately yields $f^*\mathcal{M}_X^{BM}(P)(s)[2s] \in \mathcal{OZ}(Y)$.

Moreover, Theorem 1.2.1 shows that $f^!\mathcal{M}_X^{BM}(P)(s)[2s] \in \mathcal{OZ}(Y)$ if f induces an immersion $Y'_{\mathrm{red}} \to X'$ of regular schemes. Indeed, consider the diagram

$$\begin{array}{ccccc} P_{Y,red} & \xrightarrow{p_r} & P_Y & \xrightarrow{f_P} & P \\ \downarrow{\scriptstyle h_{Y,red}} & & \downarrow{\scriptstyle h_Y} & & \downarrow{\scriptstyle h} \\ Y'_{\mathrm{red}} & \xrightarrow{y'_r} & Y' & \xrightarrow{f'} & X' \\ & & \downarrow{\scriptstyle g_Y} & & \downarrow{\scriptstyle g} \\ & & Y & \xrightarrow{f} & X \end{array} \tag{2.1}$$

where the p_r and y'_r are the corresponding nil-immersions. We may assume that Y and P are connected; hence Y'_{red} and $P_{Y,red}$ are also connected. Denote the codimension of Y'_{red} in X' by c; then $p_r \circ f_P\colon P_{Y,\mathrm{red}} \to P$ is an immersion of regular schemes of codimension c. Then $f^!\mathcal{M}_X^{BM}(P)(s)[2s] = f^!p_*(R_P)(s)[2s] \cong (g_Y \circ y'_r)_* h_{Y,\mathrm{red}_*}(p_r \circ f_P)^! R_P(s)[2s]$. Using part 8 of the theorem, we transform this into $(g_Y \circ y'_r)_* h_{Y,\mathrm{red}_*} R_{P_{Y_{\mathrm{red}}}}(s-c)[2s-2c] \in \mathcal{OZ}(Y)$.

2. Certainly, for $\operatorname{char} X = 0$ the universal homeomorphisms mentioned are isomorphisms.

For a stratification α: $S = \bigcup S_\ell^\alpha$, $1 \leq \ell \leq n$, we denote by $\mathcal{OP}(\alpha)$ (respectively, $\mathcal{ON}(\alpha)$) the class $\{M \in \operatorname{Obj} \mathcal{DM}_c(S) : j_\ell^!(M) \in \mathcal{OP}(S_\ell^\alpha), 1 \leq \ell \leq n\}$ (respectively, $\{M \in \operatorname{Obj} \mathcal{DM}_c(S) : j_\ell^*(M) \in \mathcal{ON}(S_\ell^\alpha), 1 \leq \ell \leq n\}$).

We define the Chow weight structure for $\mathcal{DM}_c(S)$: $\mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\geq 0} = \bigcup_\alpha \mathcal{OP}(\alpha)$, $\mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\leq 0} = \bigcup_\alpha \mathcal{ON}(\alpha)$; here α runs through all stratifications of S.

Lemma 2.1.2.

1. *Let δ be a (not necessarily regular) stratification of S; we denote the corresponding immersions $S^\delta_\ell \to S$ by j_ℓ. Let M be an object of $\mathcal{DM}_c(S)$. Suppose that $j^!_\ell(M) \in \mathcal{DM}_c(S^\delta_\ell)_{w_{\mathrm{Chow}}\geq 0}$ (respectively, $j^*_\ell(M) \in \mathcal{DM}_c(S^\delta_\ell)_{w_{\mathrm{Chow}}\leq 0}$) for all ℓ.*
   Then $M \in \mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\geq 0}$ (respectively, $M \in \mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\leq 0}$).
2. *For any immersion $j\colon V \to S$ we have*
$$j_*(\mathcal{DM}_c(V)_{w_{\mathrm{Chow}}\geq 0}) \subset \mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\geq 0},$$
$$j_!(\mathcal{DM}_c(V)_{w_{\mathrm{Chow}}\leq 0}) \subset \mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\leq 0}.$$
3. *For any $M \in \mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\geq 1}$ and $N \in \mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\leq 0}$, there exists a stratification α of S such that $M \in \mathcal{OP}(\alpha)[1]$, $N \in \mathcal{ON}(\alpha)$.*

Proof. 1. We use induction on the number of strata in δ. The 2-functoriality of motivic upper image functors yields that it suffices to prove the statement for δ consisting of two strata.

So, suppose $S = U \cup Z$, U and Z are disjoint, and $U \neq \{0\}$ is open in S; we denote the immersions $U \to S$ and $Z \to S$ by j and i, respectively. By the assumptions on M, there exist stratifications β of Z and γ of U such that $i^!(M) \in \mathcal{OP}(\beta)$ and $j^!(M) \in \mathcal{OP}(\gamma)$ (respectively, $i^*(M) \in \mathcal{ON}(\beta)$ and $j^*(M) \in \mathcal{ON}(\gamma)$).

We take the union of β and γ and denote by α the stratification of S obtained (for $\#\gamma = \Gamma$ we put $S^\alpha_\ell = U^\gamma_\ell$ if $1 \leq \ell \leq \Gamma$ and $S^\alpha_\ell = Z^\beta_{\ell-\Gamma}$ if $\ell > \Gamma$; note that in this way we indeed obtain a stratification in our weak sense of this notion; see §1.1). Then the 2-functoriality of $-^!$ (respectively, of $-^*$) shows that $M \in \mathcal{OP}(\alpha)$ (respectively, $M \in \mathcal{ON}(\alpha)$).

2. We choose a stratification δ containing V (as one of the strata). So, we assume that $V = S^\delta_v$ for some index v. Then it can easily be seen that $j^!_u j_{v*} = 0 = j^*_u j_{v!}$ for any $u \neq v$ and $j^!_v j_{v*} \cong 1_{\mathcal{DM}(V)} \cong j^*_v j_{v!}$ (see Theorem 1.2.1(9)). Hence the result follows from assertion 1.

3. By Remark 2.1.1(1), it suffices to verify the following: if β, γ are stratifications of S, and $S_{i\ell} \to S^\beta_\ell$, $S'_{i\ell} \to S^\gamma_\ell$ are (finite) sets of finite universal homomorphisms, then there exists a common subdivision α of β and γ such that all the (reduced) schemes $(S_{i\ell} \times_S S^\alpha_m)_{\mathrm{red}}, (S'_{i\ell} \times_S S^\alpha_m)_{\mathrm{red}}$ are regular. For this, it obviously suffices to prove the following: if $f\colon Z \to S$ is an immersion and $g_i\colon T_i \to Z$ are some finite universal homeomorphisms, then there exists a stratification δ of Z such that the schemes $T_{i\ell} = (T_i \times_Z Z^\delta_\ell)_{\mathrm{red}}$ are regular for all i and ℓ.

We prove this by Noetherian induction. Suppose that the claim is true for any proper closed subscheme Z' of Z. Since all $(T_i)_{\mathrm{red}}$ are generically regular, we can choose a (sufficiently small) open nonempty subscheme Z_1 of Z such that all of $(T_i \times_Z Z_1)_{\mathrm{red}}$ are regular.

Next, apply the inductive assumption to the scheme $Z' = Z \setminus Z_1$ and the morphisms $g'_i = g_i \times_Z Z'$; we choose a stratification α' of Z' such that all $T'_{i\ell} = \big(T_i \times_Z {Z'}^{\alpha'}_\ell\big)_{\mathrm{red}}$ are regular. Then it remains to take the union of Z_1 with α', i.e., we consider the following stratification α: $Z^\alpha_1 = Z_1$, and $Z^\alpha_\ell = {Z'}^{\alpha'}_{\ell-1}$ for all $\ell > 1$. □

Theorem 2.1.3.

(1) *The couple $(\mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\geq 0}, \mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\leq 0})$ yields a bounded weight structure w_{Chow} for $\mathcal{DM}_c(S)$.*

(2) *In addition, $\mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\ge 0}$ (respectively, $\mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\le 0}$) is the envelope of $p_*(R_P)(s)[2s+i]$ (respectively, of $\mathcal{M}_S^{BM}(P)(s)[2s-i]$) for $s\in\mathbb{Z}$, $i\ge 0$, and $p\colon P\to S$ being the composition of a smooth projective morphism with a finite universal homeomorphism whose domain is regular and with an immersion.*

(3) *w_{Chow} can be extended to a weight structure w_{Chow}^{big} for the whole $\mathcal{DM}(S)$.*

Proof. 1–2. We prove the statement by Noetherian induction. So, we suppose that assertions 1 and 2 are fulfilled for all proper closed subschemes of S. We prove them for S.

We denote by $(\mathcal{DM}_c(S)_{w'_{\mathrm{Chow}}\ge 0}, \mathcal{DM}_c(S)_{w'_{\mathrm{Chow}}\le 0})$ the envelopes mentioned in assertion 2. We must prove that w_{Chow} and w'_{Chow} yield coinciding weight structures for $\mathcal{DM}_c(S)$.

Obviously, $\mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\le 0}$, $\mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\ge 0}$, $\mathcal{DM}_c(S)_{w'_{\mathrm{Chow}}\le 0}$, and $\mathcal{DM}_c(S)_{w'_{\mathrm{Chow}}\ge 0}$ are Karoubi-closed in $\mathcal{DM}_c(S)$, and are semiinvariant with respect to translations (in the corresponding sense).

Now, using Lemma 2.1.2(2), we see that $\mathcal{DM}_c(S)_{w'_{\mathrm{Chow}}\le 0}\subset \mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\le 0}$ and $\mathcal{DM}_c(S)_{w'_{\mathrm{Chow}}\ge 0}\subset \mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\ge 0}$. Hence, in order to check that w_{Chow} and w'_{Chow} are indeed weight structures, it suffices to verify that

(i) the orthogonality axiom for w_{Chow} is fulfilled;

(ii) any $M\in \operatorname{Obj}\mathcal{DM}_c(S)$ possesses a weight decomposition with respect to w'_{Chow}.

Thus, these statements along with the boundedness of w_{Chow} imply assertion 1. Next, Proposition 1.4.2(4) shows that these two statements imply assertion 2 also, whereas in order to prove assertion 1 it suffices to verify the boundedness of w'_{Chow} (instead of that for w_{Chow}).

Now we verify (i). For some $M\in\mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\le 0}$ and $N\in\mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\ge 1}$ we check that $M\perp N$. By Lemma 2.1.2(3), we may assume that $M\in\mathcal{ON}(\alpha)$, $N\in\mathcal{OP}(\alpha)[1]$ for some stratification α of S. Hence, it suffices to prove that $\mathcal{ON}(\alpha)\perp\mathcal{OP}(\alpha)[1]$ for any α, which is an easy consequence of Lemmas 1.3.3 and 1.2.3(1).

Now we verify (ii) along with the boundedness of w'_{Chow}. We choose some generic point K of S, denoting by K^p its perfect closure, and by $j_{K^p}\colon K^p\to S$ the corresponding morphism. We fix some M. By Theorem 1.2.1(12), there exist smooth projective varieties P_i/K^p, $1\le i\le n$ (we denote the corresponding morphisms $P_i\to K^p$ by p_i), and some $s\in\mathbb{Z}$ such that $j^*_{K^p}(M)$ belongs to the triangulated subcategory of $\mathcal{DM}_c(K^p)$ generated by $\{p_{i*}(R_{P_i})(s)[2s]\}$. Now we choose a finite universal homeomorphism $K'\to K$ (i.e., a morphism of spectra of fields corresponding to a finite purely inseparable extension) such that the P_i are defined (and are smooth projective) over K'. By Theorem 1.2.1(13,11), for the corresponding morphisms $j_{K'}\colon K'\to S$ and $p'_i\colon P_{K',i}\to K'$ we have the following: $j^*_{K'}(M)$ belongs to the triangulated subcategory of $\mathcal{DM}_c(K')$ generated by $\{p'_{i*}(R_{P_{K',i}})(s)[2s]\}$. Applying Zariski's main theorem in Grothendieck's form, we can choose a finite universal homeomorphism g from a regular scheme U' whose generic fiber is K' to an open $U\subset S$ ($j\colon U\to S$ will denote the corresponding immersion) and smooth projective $h_i\colon P_{U',i}\to U'$ such that the fibers of $P_{U',i}$ over K' are isomorphic to $P_{K',i}$. Moreover, by Theorem 1.2.1(13), we can also assume that $(j\circ g)^*(M)$ belongs to the triangulated subcategory of $\mathcal{DM}_c(U')$ generated by $\{h_{i*}(R_{P_{U',i}})(s)[2s]\}$. Then Theorem 1.2.1(11) shows that $j^*(M)$ belongs to the triangulated subcategory D of $\mathcal{DM}_c(U)$ generated by $\{(g\circ h_i)_*(R_{P_{U',i}})(s)[2s]\}$.

Since id_U yields a stratification of U, the set $\{(g\circ h_i)_*(R_{P_{U',i}})(s)[2s]\}$ is negative in $\mathcal{DM}_c(U)$ (because $\mathcal{ON}(\alpha)\perp\mathcal{OP}(\alpha)[1]$ for any α, as we have proved). Therefore, by Proposition 1.4.2(5–6), there exists a weight structure d for D such that $D_{d\ge 0}$ (respectively, $D_{d\le 0}$) is the envelope of $\bigcup_{n\ge 0}\{(g\circ h_i)_*(R_{P_{U',i}})(s)[2s+n]\}$ (respectively, of

$\bigcup_{n\geq 0}\{(g\circ h_i)_*(R_{P_{U',i}})(s)[2s-n]\})$. We also obtain that $D_{d\geq 0}\subset \mathcal{DM}_c(U)_{w'_{\mathrm{Chow}}\geq 0}$ and $D_{d\leq 0}\subset \mathcal{DM}_c(U)_{w'_{\mathrm{Chow}}\leq 0}$.

We denote $S\setminus U$ by Z (Z may be empty); $i\colon Z\to S$ is the corresponding closed immersion. By the inductive assumption, w_{Chow} and w'_{Chow} yield coinciding bounded weight structures for $\mathcal{DM}_c(Z)$.

We have a gluing datum $\mathcal{DM}_c(Z)\stackrel{i_*}{\to}\mathcal{DM}_c(S)\stackrel{j^*}{\to}\mathcal{DM}_c(U)$. We can "restrict it" to a gluing datum

$$\mathcal{DM}_c(Z)\stackrel{i_*}{\to} j^{*-1}(D)\stackrel{j_0^*}{\to} D$$

(see Proposition 1.4.2(10)), whereas $M\in \operatorname{Obj}(j^{*-1}(D))$; here j_0^* is the corresponding restriction of j^*. Hence, by Proposition 1.4.2(10) there exists a weight structure w' for $j^{*-1}(D)$ such that the functors i_* and j_0^* are weight-exact (with respect to the weight structures mentioned). Therefore, there exists a weight decomposition $B\to M\to A$ of M with respect to w'. Moreover, there exist $m,n\in\mathbb{Z}$ such that $j_0^*(M)\in\mathcal{DM}_c(U)_{w'_{\mathrm{Chow}}\geq m}$, $j_0^*(M)\in\mathcal{DM}_c(U)_{w'_{\mathrm{Chow}}\leq n}$, $i^!(M)\in\mathcal{DM}_c(Z)_{w'_{\mathrm{Chow}}\geq m}$, and $i^*(M)\in\mathcal{DM}_c(Z)_{w'_{\mathrm{Chow}}\leq n}$. Hence $A[-1], M[-m]\in\mathcal{DM}_c(S)_{w'_{\mathrm{Chow}}\geq 0}$; $B, M[-n]\in\mathcal{DM}_c(S)_{w'_{\mathrm{Chow}}\leq 0}$; here we apply Proposition 1.4.2(11). So, we have verified (ii) and the boundedness of w'_{Chow}. As was shown above, this finishes the proof of assertions 1-2.

3. Since $\underline{Hw}_{\mathrm{Chow}}$ generates $\mathcal{DM}_c(S)$, and, by Theorem 1.2.1(2)), $\mathcal{DM}_c(S)$ weakly generates $\mathcal{DM}(S)$, we see that $\underline{Hw}_{\mathrm{Chow}}$ weakly generates $\mathcal{DM}(S)$. Hence the assertion follows immediately from assertion 1 and Proposition 1.4.2(9). □

2.2. The main properties of $w_{\mathrm{Chow}}(-)$. Now we study the (left and right) weight-exactness of the motivic image functors.

Theorem 2.2.1.

(1) *The functors* $-(b)[2b](=-\otimes R_S(b)[2b])$ *are weight-exact with respect to* w_{Chow} *for all* S *and all* $b\in\mathbb{Z}$.

(2) *Let* $f\colon X\to Y$ *be a (separated) morphism of finite type.*
   (a) *The functors* $f^!$ *and* f_* *are right weight-exact;* f^* *and* $f_!$ *are left weight-exact.*
   (b) *Suppose moreover that* f *is smooth. Then* f^* *and* $f^!$ *are also weight-exact.*
   (c) *If* S_{red} *is regular then* $R_S\in\mathcal{DM}_c(S)_{w_{\mathrm{Chow}}=0}$.
   (d) $\mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\leq 0}$ *is the envelope of* $\mathcal{M}_S^{BM}(T)(b)[2b-r]$ *for* T *running through all schemes of finite type over* S, $b\in\mathbb{Z}$, $r\geq 0$; $\mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\geq 0}$ *is the envelope of* $t_*(R_T)(b)[2b+r]$ *for* $t\colon T\to S$ *running through all morphisms of finite type with* **regular** *domains,* $b\in\mathbb{Z}$, $r\geq 0$.
   (e) *Next, the functor* g^* *is right weight-exact if* g *is an arbitrary (separated, not necessarily of finite type) morphism of schemes. It is weight-exact if* g *is either* (i) *a (filtering) projective limit of smooth morphisms such that the corresponding connecting morphisms are smooth affine or* (ii) *a finite universal homeomorphism. In the latter case* $g^!$ *is weight-exact also.*

(3) *Let* $i\colon Z\to X$ *be a closed immersion; let* $j\colon U\to X$ *be the complementary open immersion.*
   (a) $\underline{Hw}_{\mathrm{Chow}}(U)$ *is equivalent to the factor (in the sense of Definition* 1.4.1(7)) *of* $\underline{Hw}_{\mathrm{Chow}}(X)$ *by* $i_*(\underline{Hw}_{\mathrm{Chow}}(Z))$.
   (b) *For* $M\in\operatorname{Obj}\mathcal{DM}_c(X)$ *the following is true:* $M\in\mathcal{DM}_c(X)_{w_{\mathrm{Chow}}\geq 0}$ *(respectively,* $M\in\mathcal{DM}_c(X)_{w_{\mathrm{Chow}}\leq 0}$) *if and only if* $j^!(M)\in\mathcal{DM}_c(U)_{w_{\mathrm{Chow}}\geq 0}$ *and* $i^!(M)\in\mathcal{DM}_c(Z)_{w_{\mathrm{Chow}}\geq 0}$ *(respectively,* $j^*(M)\in\mathcal{DM}_c(U)_{w_{\mathrm{Chow}}\leq 0}$ *and* $i^*(M)\in\mathcal{DM}_c(Z)_{w_{\mathrm{Chow}}\leq 0}$).

(4) *Let $S = \bigcup S_\ell^\alpha$ be a stratification, and let $j_\ell\colon S_\ell^\alpha \to S$ be the corresponding immersions. Then for $M \in \operatorname{Obj} \mathcal{DM}_c(S)$ the following is true: $M \in \mathcal{DM}_c(S)_{w_{\mathrm{Chow}} \geq 0}$ (respectively, $M \in \mathcal{DM}_c(S)_{w_{\mathrm{Chow}} \leq 0}$) if and only if $j_\ell^!(M) \in \mathcal{DM}_c(S_\ell^\alpha)_{w_{\mathrm{Chow}} \geq 0}$ (respectively, $j_\ell^*(M) \in \mathcal{DM}_c(S_\ell^\alpha)_{w_{\mathrm{Chow}} \leq 0}$) for all ℓ.*

(5) *For any S we have $R_S \in \mathcal{DM}_c(S)_{w_{\mathrm{Chow}} \leq 0}$.*

Proof. 1. Immediate from Theorem 2.1.3(2).

2. Let f be an immersion. Then the description of $w_{\mathrm{Chow}}(-)$ given by Theorem 2.1.3(2) implies that f_* is left weight-exact and $f_!$ is right weight-exact. Hence, the corresponding adjunctions show (by Proposition 1.4.2(7)) that $f^!$ is left weight-exact and f^* is right weight-exact.

If f is smooth, using Theorem 2.1.3(2) (along with Theorem 1.2.1(4)), we easily see that f^* is left weight-exact and $f^!$ is right weight-exact (because schemes that are smooth over regular bases are regular themselves). Hence, part 7 of the theorem (along with assertion 1) implies that both of these functors are weight-exact (so, we obtain assertion 2b). Next, adjunctions show (by part (7) of Proposition 1.4.2) that $f_!$ is left weight-exact and f_* is right weight-exact.

Thus assertion 2a is valid for any quasiprojective f (because such an f can be presented as the composition of a closed immersion with a smooth morphism).

Now we verify assertion 2c. Let S_{red} be a regular scheme; denote by v the canonical immersion $S_{\mathrm{red}} \to S$. Then $v_*(R_{S_{\mathrm{red}}}) \in \mathcal{DM}_c(S)_{w_{\mathrm{Chow}}=0}$ by Theorem 2.1.3(2). Since $v_*(R_{S_{\mathrm{red}}}) \cong R_S$ by Theorem 1.2.1(11), we obtain the result.

Now we are able to prove assertion 2d. First, we observe (using Theorem 2.1.3(2)) that $\mathcal{DM}_c(S)_{w_{\mathrm{Chow}} \leq 0}$ and $\mathcal{DM}_c(S)_{w_{\mathrm{Chow}} \geq 0}$ are subclasses of the corresponding envelopes. So, we must verify the reverse inclusions. Note that any excellent Noetherian scheme admits a stratification the reductions of whose components are regular. Hence, by Lemma 1.2.3(2), it suffices to check the following: if T is a regular scheme of finite type over S, $b \in \mathbb{Z}$, and $r \geq 0$, then $\mathcal{M}_S^{BM}(T)(b)[2b-r] \in \mathcal{DM}_c(S)_{w_{\mathrm{Chow}} \leq 0}$ and $t_*(R_T)(b)[2b+r] \in \mathcal{DM}_c(S)_{w_{\mathrm{Chow}} \geq 0}$. Applying the lemma once again, we reduce the first of these inclusion statements to the case where T is quasiprojective over S (because any scheme of finite type over S possesses a stratification whose components are quasiprojective over S). Similarly, part 3 of the lemma allows us to assume that T is (regular and) quasiprojective over S in the second of these inclusion statements. Hence it suffices to note that $R_T \in \mathcal{DM}_c(T)_{w_{\mathrm{Chow}}=0}$ (by assertion 2c of our Theorem), and apply our assertion 1 along with assertion 2a (for the quasiprojective morphism t).

Now we return to the proof of assertion 2a for a general f (of finite type). Assertion 2d immediately yields the left weight-exactness of $f_!$. Along with Theorem 1.2.1(4) it also easily yields the left weight-exactness of f^*. Finally, $f^!$ and f_* are right weight-exact by Proposition 1.4.2(7).

The first statement in assertion 2e (also) easily follows from assertion 2d (along with Remark 1.2.2(2)). Assertion 2d (along with Remark 1.2.2(1)) also implies the weight-exactness of g^* in case (i) (because pro-smooth limits of regular schemes are regular). Also, $g^! \cong g^*$ if g is a finite universal homeomorphism (see Theorem 1.2.1(11)); this finishes the proof of the assertion.

3. Since $i_* \cong i_!$ in this case, the functor i_* is weight-exact by assertion 2a. The functor j^* is weight-exact by assertion 2b.

a) $\mathcal{DM}_c(U)$ is the localization of $\mathcal{DM}_c(X)$ by $i_*(\mathcal{DM}_c(Z))$ by Theorem 1.2.1(10). Hence, Proposition 1.4.2(8) yields the result (see Remark 1.4.3; cf. also [22, Theorem 1.7]).

b) Theorem 1.2.1(10) shows that $w_{\mathrm{Chow}}(X)$ is exactly the weight structure obtained by "gluing $w_{\mathrm{Chow}}(Z)$ with $w_{\mathrm{Chow}}(U)$" via Proposition 1.4.2(10) (here we use Theorem 1.2.1(12)). So, we obtain the desired assertion (note that $j^* = j^!$).

4. The assertion can easily be proved by induction on the number of strata, by using assertion 3b.

5. Immediate from assertion 2c. □

Remark 2.2.2. 1. Theorem 2.1.3(2) and assertion 2d of the previous theorem give two distinct descriptions of $(\mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\geq 0}, \mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\leq 0})$ as certain envelopes. It follows that, instead of all T considered in the assertion mentioned, it suffices to take only those T that are quasiprojective over S.

2. One may apply the argument used in the proof of [17, Lemma 2.23] to show that $\mathcal{M}_S^{BM}(X) \otimes \mathcal{M}_S^{BM}(Y) \cong \mathcal{M}_S^{BM}(X \times Y)$, where X and Y are any schemes of finite type over S (note that loc. cit. itself gives this statement for $R = \mathbb{Q}$). It certainly follows that $\mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\leq 0} \otimes \mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\leq 0} \subset \mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\leq 0}$.

Now we prove that positivity and negativity of objects of $\mathcal{DM}_c(S)$ (with respect to w_{Chow}) can be "checked at points"; this is a motivic analog of [1, §5.1.8].

Proposition 2.2.3. *Let $\mathcal{S}$ denote the set of (Zariski) points of S; for $K \in \mathcal{S}$ we denote the corresponding morphism $K \to S$ by j_K.*

Then $M \in \mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\leq 0}$ (respectively, $M \in \mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\geq 0}$) if and only if for any $K \in \mathcal{S}$ we have $j_K^*(M) \in \mathcal{DM}_c(K)_{w_{\mathrm{Chow}}\leq 0}$ (respectively, $j_K^!(M) \in \mathcal{DM}_c(K)_{w_{\mathrm{Chow}}\geq 0}$); see Remark 1.2.2(1).

Proof. If $M \in \mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\leq 0}$ (respectively, $M \in \mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\geq 0}$) then parts 2a and 2e of Theorem 2.2.1 imply that

$$j_K^*(M) \in \mathcal{DM}_c(K)_{w_{\mathrm{Chow}}\leq 0}$$

(respectively, $j_K^!(M) \in \mathcal{DM}_c(K)_{w_{\mathrm{Chow}}\geq 0}$) indeed.

We prove the reverse implication by Noetherian induction. So, suppose that our assumption is true for motives over any closed subscheme of S, and that for some $M \in \operatorname{Obj} \mathcal{DM}_c(S)$ we have

$$j_K^*(M) \in \mathcal{DM}_c(K)_{w_{\mathrm{Chow}}\leq 0}$$

(respectively, $j_K^!(M) \in \mathcal{DM}_c(K)_{w_{\mathrm{Chow}}\geq 0}$) for any $K \in \mathcal{S}$.

We show that $M \in \mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\leq 0}$ (respectively, $M \in \mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\geq 0}$). By Proposition 1.4.2(2) it suffices to verify that for any $N \in \mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\geq 1}$ (respectively, for any $N \in \mathcal{DM}_c(S)_{w_{\mathrm{Chow}}\leq -1}$), and any $h \in \mathcal{DM}_c(S)(M, N)$ (respectively, any $h \in \mathcal{DM}_c(S)(N, M)$) we have $h = 0$. We fix some N and h.

By the "only if" part of our assertion (which we have already proved) we have $j_K^*(N) \in \mathcal{DM}_c(K)_{w_{\mathrm{Chow}}\geq 1}$ (respectively, $j_K^*(N) \in \mathcal{DM}_c(K)_{w_{\mathrm{Chow}}\leq -1}$); hence $j_K^*(h) = 0$. By Theorem 1.2.1(13) we see that $j^*(h) = 0$ for some open embedding $j\colon U \to S$, where K is a generic point of U.

Now suppose that $h \neq 0$; let $i\colon Z \to S$ be the closed embedding that is complementary to j. Then Lemma 1.2.3(2) shows that $\mathcal{DM}_c(S)(i^*(M), i^!(N)) \neq \{0\}$ (respectively, $\mathcal{DM}_c(S)(i^*(N), i^!(M)) \neq \{0\}$). Yet $i^!(N) \in \mathcal{DM}_c(Z)_{w_{\mathrm{Chow}}\geq 1}$ (respectively, $i^*(N) \in \mathcal{DM}_c(Z)_{w_{\mathrm{Chow}}\leq -1}$) by Theorem 2.2.1(2b), whereas $i^*(M) \in \mathcal{DM}_c(Z)_{w_{\mathrm{Chow}}\leq 0}$ (respectively, $i^!(M) \in \mathcal{DM}_c(Z)_{w_{\mathrm{Chow}}\geq 0}$) by the inductive assumption. The contradiction obtained proves our assertion. □

Finally, we prove that "weights are continuous".

Lemma 2.2.4. *Let K be a generic point of S; denote the morphism $K \to S$ by j_K. Let M be an object of $\mathcal{DM}_c(S)$, and suppose that $j_K^*M \in \mathcal{DM}_c(K)_{w_{\mathrm{Chow}}\geq 0}$ (respectively, $j_K^*M \in \mathcal{DM}_c(K)_{w_{\mathrm{Chow}}\leq 0}$). Then there exists an open immersion $j\colon U \to S$, $K \in U$, such that $j^*M \in \mathcal{DM}_c(U)_{w_{\mathrm{Chow}}\geq 0}$ (respectively, $j^*M \in \mathcal{DM}_c(U)_{w_{\mathrm{Chow}}\leq 0}$).*

Proof. First we treat the case where $j_K^*(M) \in \mathcal{DM}_c(K)_{w_{\mathrm{Chow}}\geq 0}$. We consider a weight decomposition of $M[1]$: $B \xrightarrow{g} M[1] \to A \to B[1]$. We see that $j_K^*(g) = 0$ (because $\mathcal{DM}_c(K)_{w_{\mathrm{Chow}}\leq 0} \perp j_K^*(M)[1]$); hence (by Theorem 1.2.1(13)), there exists an open immersion $j\colon U \to S$ ($K \in U$) such that $j^*(g) = 0$. Thus $j^*M[1]$ is a retract of j^*A. Since $j^*A[-1] \in \mathcal{DM}_c(U)_{w_{\mathrm{Chow}}\geq 0}$ (see Theorem 2.2.1(2b)), and $\mathcal{DM}_c(U)_{w_{\mathrm{Chow}}\geq 0}$ is Karoubi-closed in $\mathcal{DM}_c(U)$, we obtain the result.

The second part of our statement (i.e., that for the case $j_K^*(M) \in \mathcal{DM}_c(K)_{w_{\mathrm{Chow}}\leq 0}$) can easily be verified by using the dual argument (see Proposition 1.4.2(1)). □

2.3. Describing $\underline{Hw}_{\mathrm{Chow}}$ via Chow motives. Now we prove that for some S the heart of $w_{\mathrm{Chow}}(S)$ has quite an "explicit" description in terms of certain Chow motives over S (whence the name).

Remark 2.3.1. For a scheme S, define the category $\mathrm{Chow}(S)$ of Chow motives over S as the Karoubi-closure of $\{\mathcal{M}_S^{BM}(X)(r)[2r]\} = \{f_*(R_X)(r)[2r]\}$ in $\mathcal{DM}_c(S)$; here $f\colon X \to S$ runs through all proper morphisms such that X is regular, $r \in \mathbb{Z}$.

Then Theorem 2.2.1(2c,2a) yields $\mathrm{Chow}(S) \subset \underline{Hw}_{\mathrm{Chow}}(S)$.

Now we prove that in some cases the last embedding is an equivalence of categories.

Proposition 2.3.2. *Assume that S can be presented as a filtered (projective) limit of varieties over some (not necessarily prime) field with smooth and affine transition morphisms. Then* $\mathrm{Chow}(S) = \underline{Hw}_{\mathrm{Chow}}(S)$.

Proof. Since $\mathrm{Chow}(S) \subset \underline{Hw}_{\mathrm{Chow}}(S)$, it is negative (by the orthogonality axiom for weight structures). Hence, Proposition 1.4.2(5) yields the existence of a weight structure w on the triangulated subcategory T of $\mathcal{DM}_c(S)$ generated by $\mathrm{Chow}(S)$ such that $\mathrm{Chow}(S) \cong \underline{Hw}$. Next, parts 3 and 6 of the proposition show that the embedding of T into $\mathcal{DM}_c(S)$ is weight-exact (with respect to w and w_{Chow}). Hence, part 4 of the proposition reduces the assertion to the fact that $T = \mathcal{DM}_c(S)$, i.e., that $\mathrm{Chow}(S)$ generates $\mathcal{DM}_c(S)$.

By Theorem 1.2.1(12) the last statement is true if S is a variety itself. Hence, it remains to pass to the ("pro-smooth affine") limit in this statement. By continuity (Theorem 1.2.1(13)), to achieve this, it suffices to note that pro-smooth affine morphisms respect Chow motives, which is immediate from Remark 1.2.2(2) (along with the fact that pro-smooth base change preserves the regularity of schemes). □

Remark 2.3.3. 1. This argument also shows that we could have considered only projective (regular) X/S in the definition of $\mathrm{Chow}(S)$; we would have obtained the same category $\mathrm{Chow}(S)$ (at least) when S is as in Proposition 2.3.2.

2. Actually, the negativity of $\mathrm{Chow}(S)$ in $\mathcal{DM}_c(S)$ (for the "projective version" of the definition) follows immediately from Lemma 1.3.3. Thus, we could have defined the corresponding "restriction" of w_{Chow} (on a full subcategory of $\mathcal{DM}_c(S)$) without any gluing arguments. Next, restricting ourselves to the case where $\mathcal{DM}_c(S)$ is "Chow-generated", we could easily apply the arguments used in the proof of Theorem 2.2.1 to this version of the Chow weight structure. This is (basically) the approach to the study of the Chow weight structures used in [15] and [7, §2.1, §2.2] (yet some of the methods used in the proof of Theorem 2.2.1 are "newer"; they were developed in [8] and in the current paper).

We chose not to apply this approach in the current paper because the class of base schemes for which we can use it is too small. The reason for this is that for $\mathbb{Q}$-linear motives one only needs certain alterations for the corresponding analog of Theorem 1.2.1(12) (cf. [10, §4.1]), whereas for $\mathbb{Z}[\frac{1}{p}]$-linear motives one requires the so-called prime-to-l alterations (for all primes $l \neq p$; cf. the proof of [12, Proposition 7.2]), whose existence is only known in the context the cited assertion.

3. Certainly, our proposition does not give a "full description" of $\underline{Hw}_{\mathrm{Chow}}(S)$ because we have not "computed the morphisms" in $\mathrm{Chow}(S)$. We note that the argument used in (the proof of) [7, Lemma 1.1.4(I.1)] makes it possible to express the morphism group $\mathcal{DM}_c(S)(\mathcal{M}_S^{BM}(X)(r)[s], \mathcal{M}_S^{BM}(X')(r')[s'])$ in terms of certain (Borel–Moore) motivic homology groups; here $r, s, r', s' \in \mathbb{Z}$, while X and X' are regular schemes that are projective over S (whereas S is "pro-smooth affine" over a variety). Thus, we can compute a "substantial part" of morphism groups in $\mathrm{Chow}(S)$. Yet computing the composition operation for $\mathrm{Chow}(S)$-morphisms is a much more difficult problem; for $R = \mathbb{Q}$ it was recently solved in [17].

§3. Applications to (co)homology of motives and other matters

In this section we describe some immediate applications of our results (following [7]; so a reader well acquainted with [7] may skip this section completely or merely have a look at Proposition 3.2.3). Most of the statements below easily follow from the results of [3]; there is absolutely no problem to apply the corresponding arguments from [7, §3] (where the case of $R = \mathbb{Q}$ was considered) for their proofs. The results seem to be "more interesting" in the case where S is a pro-smooth affine limit of varieties (cf. Proposition 2.3.2).

3.1. The weight complex functor and the Grothendieck group for $\mathcal{DM}_c(S)$. We note that the weight complex functor (whose "first ancestor" was defined in [14]) can be defined for $\mathcal{DM}_c(S)$.

Proposition 3.1.1.

(1) *The embedding $\underline{Hw}_{\mathrm{Chow}}(S) \to K^b(\underline{Hw}_{\mathrm{Chow}}(S))$ factors through a certain weight complex functor $t_S\colon \mathcal{DM}_c(S) \to K^b(\underline{Hw}_{\mathrm{Chow}}(S))$ which is exact and conservative.*

(2) *Suppose $M \in \mathrm{Obj}\,\mathcal{DM}_c(S)$, $i, j \in \mathbb{Z}$, then $M \in \mathcal{DM}_c(S)_{[i,j]}$ (see Definition 1.4.1(4)) if and only if $t_S(M) \in K(\underline{Hw}_{\mathrm{Chow}}(S))_{[i,j]}$.*

Now we calculate $K_0(\mathcal{DM}_c(S))$ and define a certain Euler characteristic for schemes that are of finite type (and separated) over S.

Proposition 3.1.2.

(1) *We define $K_0(\underline{Hw}_{\mathrm{Chow}}(S))$ as the Abelian group whose generators are $[M]$, $M \in \mathcal{DM}_c(S)_{w_{\mathrm{Chow}}=0}$, and the relations are*

$$[B] = [A] + [C] \quad \textit{if} \quad A, B, C \in \mathcal{DM}_c(S)_{w_{\mathrm{Chow}}=0} \quad \textit{and} \quad B \cong A \oplus C.$$

For $K_0(\mathcal{DM}_c(S))$ we take similar generators and set $[B] = [A] + [C]$ if $A \to B \to C \to A[1]$ is a distinguished triangle.

Then the embedding $\underline{Hw}_{\mathrm{Chow}}(S) \to \mathcal{DM}_c(S)$ yields isomorphism

$$K_0(\underline{Hw}_{\mathrm{Chow}}(S)) \cong K_0(\mathcal{DM}_c(S)).$$

(2) *For the correspondence $\chi\colon X \mapsto [\mathcal{M}_S^{BM}(X)]$ from the class of schemes separated of finite type over S to $K_0(\mathcal{DM}_c(S)) \cong K_0(\underline{Hw}_{\mathrm{Chow}}(S))$ we have $\chi(X \setminus Z) = \chi(X) - \chi(Z)$ if Z is a closed subscheme of X.*

3.2. On Chow-weight spectral sequences and filtrations. Now we discuss (Chow)-weight spectral sequences and filtrations for cohomology of motives. Certainly, here one can pass to homology via obvious dualization (see Proposition 1.4.2(1)). We note that any weight structure yields certain weight spectral sequences for any (co)homology theory; the main distinction of the result below from the general case (i.e., from [3, Theorem 2.4.2]) is that $T(H, M)$ always converges (because w_{Chow} is bounded).

Proposition 3.2.1. *Let $\underline{A}$ be an Abelian category.*

(1) *Let $H\colon \mathcal{DM}_c(S) \to \underline{A}$ be a cohomological functor; for any $r \in \mathbb{Z}$ denote $H \circ [-r]$ by H^r. For $M \in \operatorname{Obj} \mathcal{DM}_c(S)$, we denote by (M^i) the terms of $t_S(M)$ (so, $M^i \in \mathcal{DM}_c(S)_{w_{\mathrm{Chow}}=0}$; here we can take any possible choice of $t_S(M)$).*
Then the following statements are valid.
  (a) *There exists a (Chow-weight) spectral sequence $T = T(H, M)$ with $E_1^{pq} = H^q(M^{-p}) \implies H^{p+q}(M)$; the differentials for $E_1(T(H, M))$ come from $t_S(M)$.*
  (b) *$T(H, M)$ is $\mathcal{DM}_c(S)$-functorial in M (and does not depend on any choices) starting with E_2.*

(2) *Let $G\colon \mathcal{DM}_c(S) \to \underline{A}$ be any contravariant functor. Then for any $m \in \mathbb{Z}$ the object $(W^m G)(M) = \operatorname{Im}(G(w_{\mathrm{Chow}, \geq m} M) \to G(M))$ does not depend on the choice of $w_{\mathrm{Chow}, \geq m} M$; it is functorial in M.*
We call the filtration of $G(M)$ by $(W^m G)(M)$ its Chow-weight filtration. If G is cohomological, it coincides with the filtration given by $T(G, M)$.

Remark 3.2.2. 1. We obtain certain ("motivically functorial") *Chow-weight* spectral sequences and filtrations for any (co)homology of motives. In particular, we have them for étale and motivic cohomology of S-motives (with coefficients in an R-algebra). The corresponding functoriality results cannot be proved by using "classical" (i.e., Deligne's) methods, because they heavily rely on the degeneration of (an analog of) T at E_2.

On the other hand, we probably do not have the "strict functoriality" property for this filtration unless $T(H, M)$ degenerates for any $M \in \operatorname{Obj} \mathcal{DM}_c(S)$ (see [5, Proposition 3.1.2(II)]), whereas this degeneration is only known for "more or less classical" $\mathbb{Q}$-linear cohomology theories.

2. $T(H, M)$ can be described naturally in terms of the virtual t-truncations of H (starting from E_2); see [3, §2.5] and [7].

3. For S being a pro-smooth affine limit of varieties (cf. §2.3), we see that the (co)homology of any $M \in \operatorname{Obj} \mathcal{DM}_c(S)$ possesses a filtration by subfactors of (co)homology of regular projective S-schemes.

4. The arguments in [11, §7.2] easily yield (for $R \subset \mathbb{Q}$, $p \neq l \in \mathbb{P}$) the existence of a (covariant) étale realization functor from $\mathcal{DM}_c(S)$ into the corresponding category of Ekedahl's (étale, constructible) $\mathbb{Q}_l$-adic systems.

Note next that if S is a variety over k, then for $p > 0$ the target category is endowed with a certain weight filtration (that was defined in [1, §5]); for $p = 0$ one can factor the realization functor mentioned through a certain category endowed with certain "weights" also (as was defined by A. Huber; see [9, Proposition 2.5.1]). Moreover (see loc. cit.), in both cases the corresponding functors "respect weights". So, we see that w_{Chow} is closely related to Deligne's weights for constructible complexes of sheaves (that were also crucial for defining "weights" in [16]). Yet (as was explained in [9, Remark 2.5.2]) the weight filtrations on the "étale categories" mentioned do not yield weight structures for them.

The functoriality of Chow-weight filtrations has quite interesting consequences.

Proposition 3.2.3. *For a scheme X, let $H\colon \mathcal{DM}_c(X) \to \underline{A}$ be a contravariant functor. For all $N \in \mathcal{DM}_c(U)_{w_{\mathrm{Chow}} \le 0}$ consider $G(N) = (W^0 H)(j_!(N)) \subset H(j_!(N))$.*

(1) *The following statements are valid*:
   (a) *$G(N)$ is $\mathcal{DM}_c(U)$-functorial in N;*
   (b) *$G(N)$ is a quotient of $H(M)$ for some $M \in \mathcal{DM}_c(X)_{w_{\mathrm{Chow}}=0}$.*

(2) *Let now $N = j^*(M)(= j^!(M))$ for $M \in \mathcal{DM}_c(X)_{w_{\mathrm{Chow}}=0}$. Then also the following is true.*
   (a) *$G(N) = \operatorname{Im}(H(M) \to H(j_!j^!(M)))$ (here we apply H to the morphism $j_!j^!(M) \to M$ coming from the adjunction $j_! \dashv j^! = j^*$).*
   (b) *Let $H = \mathcal{DM}_c(X)(-, \mathbb{H})$ for some $\mathbb{H} \in \operatorname{Obj} \mathcal{DM}(X)$. Then $G(N) \cong \operatorname{Im}(\mathcal{DM}(X)(M, \mathbb{H}) \to \mathcal{DM}(U)(N, j^*(\mathbb{H})))$.*

Remark 3.2.4. 1. Thus, one may say that $(W^0H)(j_!(N))$ yields the "integral part" of $H(j_!(N))$: we obtain the subobject of $H^*(j_!(N))$ that "comes from any nice X-lift" of N if such a lift exists, and a factor of $H(M)$ for some $M \in \mathcal{DM}_c(X)_{w_{\mathrm{Chow}}=0}$ in general; cf. [2] and [20]. If $N = \mathcal{M}^{BM}(T_U)$ for a regular T_U proper over U, then any regular proper "X-model" T_X for T_U yields such a "nice X-lift" for N. Next, in the setting of assertion 2b one can take $\mathbb{H}$ representing étale or motivic cohomology (for an appropriate choice of coefficients and some fixed degree, so that $\mathcal{DM}(S)(R_Y, f^*(\mathbb{H}))$ is the corresponding cohomology of Y for any $f\colon Y \to X$); then $G(N)$ will be the image of this cohomology of T_X in the one of T_U.

In this case one can also study the cohomology of an object N_K of $\mathrm{Chow}(K)$ (i.e., of an element of $\mathcal{DM}_c(K)_{w_{\mathrm{Chow}}=0}$), where K is some generic point of X. Indeed, any such N_K can be lifted to a Chow motif N over some U ($K \in U$, U is open in X) by Remark 1.2.2(2) combined with Theorem 1.2.1(13) (because if N_K is a retract of $\mathcal{M}_K^{BM}(T_K)$ for some regular variety T_K/K, then T_K along with this retraction can be lifted to a regular T_U over some open $U \subset X$ that contains K). Note that this construction enjoys "the usual" $\mathrm{Chow}(K)$-functoriality; this is an easy consequence of Theorem 1.2.1(13).

2. It could also be interesting to consider $W^l H^*(j_!(N))$ for $l < 0$.

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