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A NULLSTELLENSATZ FOR TRIANGULATED CATEGORIES

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To Sergei Vladimirovich Vostokov with our best wishes

Abstract. The paper is aimed at proving the following: for a triangulated category $\underline{C}$ and $E \subset \operatorname{Obj}\underline{C}$, there exists a cohomological functor F (with values in some Abelian category) such that E is its set of zeros if (and only if) E is closed with respect to retracts and extensions (so, a certain Nullstellensatz is obtained for functors of this type). Moreover, if $\underline{C}$ is an R-linear category (where R is a commutative ring), this is also equivalent to the existence of an R-linear functor $F : \underline{C}^{\mathrm{oop}} \to R-\mathrm{mod}$ with this property. As a corollary, it is proved that an object Y belongs to the corresponding "envelope" of some $D \subset \operatorname{Obj}\underline{C}$ whenever the same is true for the images of Y and D in all the categories $\underline{C}_p$ obtained from $\underline{C}$ via "localizing the coefficients" at maximal ideals $p \triangleleft R$. Moreover, certain new methods are developed for relating triangulated categories to their (nonfull) countable triangulated subcategories.

The results of this paper can be applied to weight structures and triangulated categories of motives.

Introduction

Certainly, for any class of cohomological functors $\{F_i\}$ from a triangulated category $\underline{C}$ (with values in some Abelian categories $\underline{A}_i$), the class $E = \{c \in \operatorname{Obj}\underline{C},\ F_i(c) = 0\ \forall i\}$ is extension-closed and Karoubi-closed in $\underline{C}$ (i.e., it is closed with respect to retracts and "extensions"). In this paper we prove that (somewhat surprisingly) the converse statement is also true (assuming that $\operatorname{Obj}\underline{C}$ is a set, to avoid set-theoretical difficulties).

Being more precise, we prove the following statement.

Theorem 0.1. *Let $\underline{C}$ be a small R-linear triangulated category* (*where R is a commutative unital ring*); *let E be an extension-closed Karoubi-closed subset of* $\operatorname{Obj}\underline{C}$. *Then for any $Y \in \operatorname{Obj}\underline{C} \setminus E$ there exists an R-linear* ("*separating*") *cohomological functor $F^Y : \underline{C}^{\mathrm{op}} \to R-\mathrm{mod}$ such that $F^Y|_E = 0$ and $F^Y(Y) \neq \{0\}$.*

Certainly, this result can be applied in the (most general) case where $R = \mathbb{Z}$. Thus (cf. Corollary 3.11 below), $E \subset \operatorname{Obj}\underline{C}$ is the set of zeros of some collection of cohomological functors if and only if it is extension-closed and Karoubi-closed; this is a certain Nullstellensatz for functors of this type (whence the name of the paper).

Now we explain our motivation for studying this question. For various triangulated categories, the existence of certain distinguished classes of objects is very important; in particular, one is often interested in sets of zeros for certain collections of

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(co)representable functors (because such classes can yield t-structures, weight structures, and more generally Hom-orthogonal pairs; see Definition 3.1 in [9], as well as [10, 8] and [3]). Yet it is hardly possible to describe in general all classes of objects that can be characterized this way (for arbitrary families of representable functors; note still that quite important results related to this question were obtained in the papers mentioned above). This led the authors to modifying this description problem (and obtaining Theorem 0.1 as a complete answer to the corresponding question).

This theorem has a corollary that seems to be interesting for itself. For $D \subset \operatorname{Obj} \underline{C}$, we call a class $E \subset \operatorname{Obj} \underline{C}$ the *envelope* of D if it is the smallest subclass of $\operatorname{Obj} \underline{C}$ that contains $D \cup \{0\}$ and is closed with respect to retracts and extensions.

We note the following: if $F\colon \underline{C}^{\mathrm{op}} \to R-\mathrm{mod}$ is a cohomological functor (where R is a commutative unital ring) and M is a flat R-module, then the ("naive tensor product") functor $F(-) \otimes M\colon \underline{C}^{\mathrm{op}} \to R-\mathrm{mod}$ is also cohomological. Considering these functors for $M = R_p$, where p runs through all maximal ideas of R, we easily deduce the following statement.

Corollary 0.2. *Let $\underline{C}$ be an R-linear triangulated category. For any maximal ideal $p \lhd R$, denote by R_p the localization of R at p, and by $\underline{C}_p$ the triangulated category whose objects are those of $\underline{C}$ and whose morphisms are obtained by tensoring the $\underline{C}$-ones by R_p (over R; see Proposition* 4.1 *below). Denote by L_p the obvious functor $\underline{C} \to \underline{C}_p$.*

Let $D \subset \operatorname{Obj} \underline{C}$, and let $Y \in \operatorname{Obj} \underline{C}$. Then Y belongs to the envelope E of D in $\underline{C}$ if and only if $L_p(Y)$ belongs to the envelope of $L_p(D)$ (in $\underline{C}_p$) for every maximal ideal p of R.

Note that L_p (for various $\underline{C}$ any prime ideal p of R) is exactly "the natural localization of coefficients functor" (corresponding to passing from R-linear categories to R_p-linear ones); see Appendix A.2 in [5] for a thorough study of this construction in the case of $R = \mathbb{Z}$ and Proposition B.1.5 in [4] for the general case.

We would also like to note that our motivation for considering the functors of type L_p along with the corollary stated was the following "motivic" one: certain facts about resolution of singularities needed for the study of various motivic categories (that were proved by Gabber) can only be applied "directly" to various R_p-linear categories of motivic origin (R is usually a localization of $\mathbb{Z}$). Whereas the ("triangulated") properties of L_p were sufficient for the "globalization of coefficients" arguments in [4] and in [5] (because the study of the "triangulated" envelope of $\cup_{i\in\mathbb{Z}} D[i]$ for certain D was sufficient for those papers, and in this setting it suffices to apply the theory of Verdier quotients), for some other problems (including that studied in [2]) the consideration of "general" envelopes is quite relevant.

Now we describe the content of the paper in some more detail.

In §1 we introduce some basic (and more or less common) categorical definitions.

In §2 we prove Theorem 0.1 in the case where both E and all the hom-sets in $\underline{C}$ are countable. Our idea is to construct F^Y as a (direct) limit of representable functors, whereas the construction of the corresponding objects is closely related to the "approximation" methods introduced in § III.2 of [1] (which were also applied in the proof of the main statement of [8]; the obvious duals of those arguments were used in the proof of Theorem 2.2.6 in [3]).

In §3 we introduce certain (new) techniques that relate general triangulated categories to their subcategories with countable sets of objects and morphisms. This easily yields the proof of the theorem in the case where $R = \mathbb{Z}$; this proof also works for the R-linear case if R is at most countable. Finally, in Subsection 3.2 we prove the theorem in the general case (via approximating the ring R by its countable subrings). Actually, the corresponding argument (that may possibly be interesting by itself, especially in conjunction with other methods of the section) was our main reason for formulating the main results for R-linear categories; yet the reader may certainly restrict himself throughout to the case of $R = \mathbb{Z}$ (which seems to be sufficient for the motivic applications the authors have in mind).

In §4 we prove Corollary 0.2.

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§1. Notation and conventions

- Given a category C and $X, Y \in \operatorname{Obj} C$, we denote by $C(X, Y)$ the set of morphisms from X to Y in C.
- R is a commutative unital ring.
- $\underline{C}$ will always denote an R-linear triangulated category; we assume it to be small till §4;
- $\underline{A}$ will be an Abelian category.
- For categories C and C', we write $H: C \to C'$ only if H is a covariant functor. So, we say that F is a cohomological functor from $\underline{C}$ to $\underline{A}$ and write $F: \underline{C}^{\mathrm{op}} \to \underline{A}$ if F is contravariant on $\underline{C}$ and sends $\underline{C}$-distinguished triangles into long exact sequences. Note yet that an alternative convention is often used for this matter; then such an F is called a cohomological functor from $\underline{C}^{\mathrm{op}}$ into $\underline{A}$ (cf. [6]).
- For any $A, B, C \in \operatorname{Obj} \underline{C}$, we say that C is an *extension* of B by A if there exists a distinguished triangle $A \to C \to B \to A[1]$ (in $\underline{C}$). A class $E \subset \operatorname{Obj} \underline{C}$ is said to be *extension-closed* if it is closed with respect to extensions and contains 0.
- For $X, Y \in \operatorname{Obj} \underline{C}$, we say that X is a *retract* of Y if id_X can be factored through Y (since $\underline{C}$ is triangulated, X is a retract of Y if and only if X is its direct summand). A class $E \subset \operatorname{Obj} \underline{C}$ is said to be *Karoubi-closed* in $\underline{C}$ if E contains all $\underline{C}$-retracts of its elements.
- The smallest extension-closed Karoubi-closed subclass of $\operatorname{Obj} \underline{C}$ that contains a given class $D \subset \operatorname{Obj} \underline{C}$ is called the *envelope* of D (in $\underline{C}$).
- Given $f \in \underline{C}(X, Y)$ (where $X, Y \in \operatorname{Obj} \underline{C}$), we call the third vertex of (any) distinguished triangle $X \xrightarrow{f} Y \to Z$ a *cone* of f (recall that various choices of cones are isomorphic, but the choice of these isomorphisms is not canonical).
- A directed set is a set with a partial order such that every pair of elements has an upper bound.
- Let $(I, \succeq)$ be a directed set, and let $x \in I$. By I_x we denote the set of $i \in I$ such that $i \succeq x$. We say that a certain property of $i \in I$ is fulfilled for sufficiently large i if it is fulfilled for all $i \in I_x$ with some $x \in I$.
- We assume that the set $\mathbb{N}$ of natural numbers starts with 1.

§2. Proof of the theorem in the countable case

In this section we prove the claim of Theorem 0.1 under certain (rather strong) countability restrictions.

Proposition 2.1. *Suppose that all hom-sets in* $\underline{C}$ *are at most countable*; *let* $E = \{e_i\}_{i \in \mathbb{N}}$ *be* $D = \{e_i\}_{i \in \mathbb{N}} \subset \operatorname{Obj} \underline{C}$. *a countable extension-closed and Karoubi-closed subset of* $\underline{C}$. *Then for any* $Y \in \operatorname{Obj} \underline{C} \setminus E$ *there exists an R-linear cohomological functor* $F^Y : \underline{C}^{\mathrm{op}} \to R - \mathrm{mod}$ *such that* $F^Y|_E = 0$ *and* $F^Y(Y) \neq \{0\}$.

Proof. For every $Z \in \operatorname{Obj} \underline{C}$, $i \in \mathbb{N}$, we fix some surjection $\mathbb{N} \to \underline{C}(e_i, Z)$. We shall denote by $f_{Z,i,j}$ the image of j under this map.

Now we construct a certain inductive system $\{Y_i \in \operatorname{Obj} \underline{C}\}_{i \geq 0}$ along with the corresponding connecting morphisms $\alpha_i: Y_i \to Y_{i+1}$ (for all $i \geq 0$). We start with $Y_0 = Y$.

Then we use induction to choose Y_i and the corresponding morphisms. For $n > 0$ assume we have chosen $Y_k \in \operatorname{Obj} \underline{C}$ for all $0 \le k \le n-1$ along with $\alpha_k \in \underline{C}(Y_k, Y_{k+1})$ for $0 \le k \le n-2$. For $0 \le k \le n-1$, denote by $\alpha_{k,n-1}$ the composition $\alpha_{n-2} \circ \dots \circ \alpha_k$ (so, $\alpha_{n-1,n-1} = \mathrm{id}_{Y_{n-1}}$). We put

$$Y_n = \operatorname{Cone}\Big(\bigoplus_{0 \le k \le n-1} \bigoplus_{1 \le j \le n} \bigoplus_{1 \le i \le n} e_i \xrightarrow{p_{n-1} \circ (\oplus(\alpha_{k,n-1} \circ f_{Y_k,i,j}))} Y_{n-1}\Big),$$

where p_{n-1} is the projection morphism $\bigoplus_{0 \le k \le n-1} \bigoplus_{1 \le j \le n} \bigoplus_{1 \le i \le n} Y_{n-1} \to Y_{n-1}$. To finish the inductive step, we take for α_{n-1} the morphism $Y_{n-1} \to Y_n$ coming from the definition of a cone.

We define $F^Y(-) = \varinjlim_{n \in \mathbb{N}} \underline{C}(-, Y_n)$ (with the connecting morphisms induced by the corresponding $\alpha_{k,n}$). Certainly, F^Y is an R-linear cohomological functor.

Now we verify that F^Y fulfills the (remaining) conditions required. First, we check that any element x of $\underline{C}(e_i, Y_n)$ (for some $i, n \in \mathbb{N}$) dies in $\underline{C}(e_i, Y_N)$ for some $N > n$. Indeed, there is a number k such that $x = f_{Y_n,i,k}$. Consider $N = \max\{k, n+1\}$. By the definition of Y_N, there exists a distinguished triangle

$$e_i \oplus X \xrightarrow{(\alpha_{n,N-1} \circ x\; x')} Y_{N-1} \xrightarrow{\alpha_{N-1}} Y_N \longrightarrow (e_i \oplus X)[1],$$

where X belongs to E and x' is a certain element of $\underline{C}(X, Y_{N-1})$. Hence, the sequence

$$\dots \to \underline{C}(e_i, e_i \oplus X) \to \underline{C}(e_i, Y_{N-1}) \xrightarrow{(\alpha_{N-1})_*} \underline{C}(e_i, Y_N) \to \dots$$

is exact. Since the morphism $e_i \xrightarrow{\alpha_{n,N-1} \circ x} Y_{N-1}$ factors through $e_i \oplus X$, we have

$$\alpha_{n,N} \circ x = (\alpha_{N-1})_*(\alpha_{n,N-1} \circ x) = \alpha_{N-1} \circ \alpha_{n,N-1} \circ x = 0.$$

By the definition of the direct limit of functors, we obtain $F^Y(e_i) = \{0\}$ (for all $i \in \mathbb{N}$), i.e., $F^Y|_E = 0$.

It remains to figure out whether $F^Y(Y) = \{0\}$. If this is the case, then $\mathrm{id}_Y \in \underline{C}(Y, Y) = \underline{C}(Y, Y_0)$ goes to zero in $\underline{C}(Y, Y_N)$ for some $N \in \mathbb{N}$. Now, $\operatorname{Cone}(\alpha_k)[-1]$ belongs to E for any $k \ge 0$ by construction. Hence, $\operatorname{Cone}(\alpha_{k,n})[-1] \in E$ for any $0 \le k \le n$ (and $\alpha_{k,n}$ defined as above; here we apply the octahedral axiom). We denote $\operatorname{Cone}(\alpha_{0,N})[-1]$ by X'; applying the functor $\underline{C}(Y, -)$ to the distinguished triangle $X' \to Y \to Y_N \to X'[1]$, we obtain a long exact sequence

$$\dots \to \underline{C}(Y, X') \to \underline{C}(Y, Y) \xrightarrow{(\alpha_{0,N})_*} \underline{C}(Y, Y_N) \to \dots .$$

Since $(\alpha_{0,N})_*(\mathrm{id}_Y)$ is zero, id_Y factors through X'. So, Y is a retract of an element of E; hence it also belongs to E. This contradicts our assumption on Y. □

Remark 2.2. The above proof does not work without countability restrictions, because it does not seem possible to construct a similar inductive system. At each step of our construction we only need to construct a certain Y_i from Y_{i-1} via considering a certain cone. However, constructing the inductive system in the uncountable case would certainly require "completing" such a sequence of Y_i by a certain "transfinite cone". The authors do not know how to achieve this unless $\underline{C}$ possesses some sort of "enhancement" (for example, a differential graded one is certainly sufficient for these purposes because it can be used to construct certain "canonical cones" of morphisms that permit passing to "transfinite limits"). Possibly, this difficulty is related to the reason that persuaded the authors of [9] to consider derivators (in their Theorem 3.7).

§3. Proof of the theorem in the general case

In this subsection we prove Theorem 0.1 in the general case (though we start with considering the intermediate case of an at most countable R).

3.1. "Approximating" categories by countable subcategories. In this subsection we introduce several constructions and techniques to be used in the proof of the main theorem.

Till the end of Subsection 3.1, R will be an at most countable ring.

We start with the following very easy "additive" lemma; next we apply it in the proof of a certain closely related (yet somewhat more complicated) "triangulated" statement.

Lemma 3.1. *For any at most countable set of objects O and at most countable set of morphisms M between elements of O in an (additive) R-linear category $\underline{B}$, there exists a (nonfull!) R-linear subcategory $\underline{B}(M;O)$ of $\underline{B}$ whose set of objects and set of morphisms are both countable, such that* $\operatorname{Obj}\underline{B}(M;O)$ *contains O and* $\operatorname{Mor}\underline{B}(M;O)$ *contains M.*

Proof. For any finite set $o_1,\dots,o_n$ of elements of O $(n \geq 0)$, choose a representative in the isomorphism class of $\bigoplus_{i=1}^n o_i$ in $\operatorname{Obj}\underline{B}$. Denote it by $S(o_1,\dots,o_n)$; and denote the countable and set $\{S(o_1,\dots,o_n)|n\geq 0, o_i\in O\}$ by O'.

Now for the role of $\underline{B}(M;O)$ we take the subcategory of $\underline{B}$ whose set of objects is O' and the sets of morphisms are as follows:

$$\underline{B}(M;O)(o_1,o_2)=\Big\{\sum_{i=1}^{n} r_i f_i^{n_i}\circ\cdots\circ f_i^1\Big\};$$

here we consider all composable chains (of lengths $n_i \geq 0$) of elements of M such that the domain of f_i^1 is o_1, the codomain of $f_i^{n_i}$ is o_2, and $r_i \in R$. By definition, this is an R-submodule of $\underline{B}(o_1,o_2)$. For any $o_1,\dots,o_m,o_1',\dots,o_n' \in O$ (where $m,n\geq 0$) we define $\underline{B}(M;O)\big(\bigoplus_{i=1}^m o_i, \bigoplus_{i=1}^n o_i'\big)$ to be the set of matrices whose (i,j)-entry belongs to $\underline{B}(M;O)(o_i,o_j')$. The composition of morphisms in $\underline{B}$ certainly restricts to these sets; so we obtain an R-linear (additive) subcategory of $\underline{B}$. □

Proposition 3.2. *Let $\underline{C}$ be a small R-linear triangulated category. Then for any countable $O\subset \operatorname{Obj}\underline{C}$ and any countable set of morphisms M between elements of O, there exists an R-linear triangulated subcategory $\underline{C}_{\mathrm{tr}}(M;O)$ of $\underline{C}$ whose set of objects and set of morphisms are both countable and such that* $\operatorname{Obj}\underline{C}_{\mathrm{tr}}(M;O)$ *contains O and the group* $\operatorname{Mor}\underline{C}_{\mathrm{tr}}(M;O)$ *contains M.*

Proof. First, we construct some countable sets $O^{(n)}, M^{(n)}$ of $\underline{C}$-objects and morphisms for $n\geq 0$.

The construction is inductive. We start with $O^{(0)}=O$ and $M^{(0)}=M$.

To make the inductive step, for $n\geq 1$ we consider the corresponding $(O^{(n-1)},M^{(n-1)})$ and choose a countable R-linear subcategory $\underline{C}(O^{(n-1)};M^{(n-1)})$ in $\underline{C}$ (see Lemma 3.1). We denote the sets of objects and morphisms in $\underline{C}(O^{(n-1)},M^{(n-1)})$ by O' and M', respectively. Next, for every $f\in M'$ we choose a distinguished triangle containing it and add the corresponding (countable and shift-stable) data to (O',M'), $T^{(n-1)}$, obtaining certain sets O'' and M'', respectively. Finally, for every lower cap of a $\underline{C}$-octahedral diagram whose morphisms belong to M'', we choose an upper cap for it in $\underline{C}$ and add all the "new" objects and morphisms to O'' and M'', respectively. We denote the sets obtained as a result of the above series of operations by $O^{(n)}$ and $M^{(n)}$, respectively.

Obviously, the sets $\bigcup_{n\in\mathbb{N}} O^{(n)}$ and $\bigcup_{n\in\mathbb{N}} M^{(n)}$ yield a certain R-linear subcategory $\underline{C}_{\mathrm{tr}}(M;O)$ of $\underline{C}$; the shift functors for $\underline{C}$ (i.e., the functors $[n]$ for $n\in\mathbb{Z}$) can surely be restricted to it. We define the distinguished triangles in $\underline{C}_{\mathrm{tr}}(M;O)$ as those triangles of

$\underline{C}_{\mathrm{tr}}(M;O)$-morphisms that are distinguished in $\underline{C}$. By construction, all the axioms of triangulated categories are fulfilled in $\underline{C}_{\mathrm{tr}}(M;O)$ (we need not ensure the validity of the axiom TR3 in $\underline{C}_{\mathrm{tr}}(M;O)$ by Lemma 2.2 of [7]). □

Remark 3.3. 1. Certainly, the proofs of Lemma 3.1 and Proposition 3.2 are closely related to the proof of the seminal Löwenheim–Skolem theorem.

2. Also, we could have chosen to ensure the validity of the axiom TR3 in $\underline{C}_{\mathrm{tr}}(M;O)$ "directly" (i.e., by adding certain morphisms to $M^{(n)}$ at each step).

Now we consider some versions of "filtered products" of objects, categories, and functors. First, note that any (small) product of triangulated categories has the natural structure of a triangulated category (respectively, any small product of Abelian categories is Abelian). So we can introduce the following definition.

Definition 3.4. Let $(I, \succeq)$ be a directed set; let $\underline{A}_i$, $i \in I$, be triangulated (respectively, Abelian) categories.

We define the reduced product $\prod_{(\to, i\in I)} \underline{A}_i$ as the Verdier localization (respectively, the Serre localization) of the product $\prod_{i\in I} \underline{A}_i$ by the full triangulated subcategory (respectively, the full Serre subcategory) whose objects are the families (a_i), $a_i \in \operatorname{Obj} \underline{A}_i$, such that $a_n = 0$ for sufficiently large n.

Clearly, if all $\underline{A}_i$ are A-linear (for some commutative unital ring A), then their reduced product is also A-linear.

In case no confusion can arise, we shall simply write $\prod_{\to} \underline{A}_i$.

Remark 3.5. 1. This reduced product can be characterized as the direct categorical limit $\varinjlim_{n\in I} \prod_{i\in I_n} \underline{A}_i$ (where I_n is defined in §1); here for $n' \succeq n$ the corresponding functor $\prod_{i\in I_n} \underline{A}_i \to \prod_{i\in I_{n'}} \underline{A}_i$ is the natural projection.

2. A family $(f_i) \in \prod \underline{A}_i(X, Y)$ yields a zero morphism in $\prod_{\to} \underline{A}_i$ if and only if such that f_n is zero for every $n \succeq N$. $f_n = 0$ for sufficiently large n (see §1).

In the case of triangulated $\underline{A}_i$, we have the following characterization of distinguished triangles in $\prod_{\to} \underline{A}_i$: a family of morphisms $(X_n) \xrightarrow{f_n} (Y_n) \xrightarrow{g_n} (Z_n) \to (X_n[1])$ yields a distinguished triangle if and only if $X_n \to Y_n \to Z_n \to X_n[1]$ are distinguished triangles for sufficiently large n. Similarly, in the case where the $\underline{A}_i$ are Abelian, a family of morphisms $(X_n) \xrightarrow{f_n} (Y_n) \xrightarrow{g_n} (Z_n)$ yields an exact sequence in $\prod_{\to} \underline{A}_i$ if and only if $g_n \circ f_n = 0$ for sufficienty large n and the corresponding morphisms $\operatorname{Im} f_n \to \operatorname{Ker} g_n$ are isomorphisms for sufficiently large n. The latter is equivalent to $X_n \to Y_n \to Z_n$ being exact sequences for sufficiently large n.

3. Let A be a (unital, commutative) ring, and let $F_i \colon \underline{C}_i^{\mathrm{op}} \to A-\mathrm{mod}$ be a family of A-linear cohomological functors (for some A-linear triangulated $\underline{C}_i$, $i \in I$). Denote by G the (A-linear) functor from the "reduced power" $\prod_{(\to, i\in I)} A-\mathrm{mod}$ into $A-\mathrm{mod}$ that maps a family ($M_i \in \operatorname{Obj} A-\mathrm{mod}$) into $\varinjlim_{n\in I} \prod_{i\in I_n} M_i$. Certainly, $G((M_i)) = \{0\}$ if and only if $M_i = \{0\}$ for sufficiently large i. Denote by F' the composition

$$\prod_{\to} \underline{C}_i^{\mathrm{op}} \xrightarrow{\prod_{\to} F_i} \prod_{\to} A-\mathrm{mod} \xrightarrow{G} A-\mathrm{mod}.$$

Then we define the reduced product of F_i as the functor $\prod_{\to} F_i \colon \prod_{\to} \underline{C}_i^{\mathrm{op}} \to A-\mathrm{mod}$ such that the composition $\prod \underline{C}_i^{\mathrm{op}} \to \prod_{\to} \underline{C}_i^{\mathrm{op}} \to A-\mathrm{mod}$ is equal to F' (which exists and is unique by the universal property of localization).

4. If I has the greatest element i^{max}, then certainly $\prod_{\to} \underline{A}_i \cong \underline{A}_{i^{max}}$. Thus, the category $\mathfrak{C}$ defined in Lemma 3.6 is isomorphic to $\underline{C}$ if $\underline{C}$ is countable; if R is countable, then the functor F^Y that we construct in the proof of Theorem 0.1 (in §3.2 below) is isomorphic to the corresponding $\widehat{F}_R^Y$.

The following lemma is crucial for the proof of Theorem 0.1.

Denote by I be the set of countable R-linear triangulated subcategories (i.e., of the subcategories whose set of objects and set of morphisms are both countable) of $\underline{C}$ ordered by (nonfull) exact inclusions (it is a directed set thanks to Proposition 3.2; recall that we assume $\underline{C}$ to be a small category).

Lemma 3.6. *Denote by $\mathfrak{C}$ the reduced product $\prod_{(\to,C\in I)} C$. Then there exists a faithful exact R-linear functor $F\colon \underline{C} \to \mathfrak{C}$ sending $X \in \operatorname{Obj}\underline{C}$ to the family $(X_C)_{C\in I}$, where we set $X_C = X$ if $X \in \operatorname{Obj} C$ and $X_C = 0$ otherwise.*

Proof. For any morphism $f \in \underline{C}(X, Y)$, we define $F(f)$ as the class of the family $(F_C(f))$, where we set $F_C(f) = f$ if $f \in \operatorname{Mor} C$ (for $C \in I$) and $F_C(f) = 0$ otherwise. Now we check that this correspondence yields a functor. Take some composable morphisms f and g in $\underline{C}$. By Remark 3.5(2), it suffices to show that $F_C(f) \circ F_C(g) = F_C(f \circ g)$ if C is sufficiently large. By Proposition 3.2, there exists a category $C' \in I$ containing f and g (and so, also $f \circ g$). Obviously, $F_C(f) \circ F_C(g) = F_C(f \circ g)$ for any $C \in I$ such that C' is a (triangulated) subcategory of C.

Next, clearly, F is faithful, R-linear, and respects shifts.

It remains to prove that F sends distinguished triangles to distinguished triangles. as triples of arrows). Let $X \to Y \to Z \to X[1]$ be a distinguished triangle in $\underline{C}$. By Remark 3.5(2), it suffices to show that the triangles $F_C(X) \to F_C(Y) \to F_C(Z) \to F_C(X)[1]$ are distinguished if C is sufficiently large. By Proposition 3.2, there exists a category $C' \in I$ containing the triangle $X \to Y \to Z \to X[1]$. Obviously, the triangle $F_C(X) \to F_C(Y) \to F_C(Z) \to F_C(X)[1]$ is distinguished for any $C \in I$ such that C' is a (triangulated) subcategory of C. □

Corollary 3.7. *Assume that for any $C \in I$ there exists an R-linear cohomological functor $\mathcal{F}_C\colon C^{\mathrm{op}} \to R-\mathrm{mod}$. Then there exists an R-linear cohomological functor $\mathcal{F}\colon \underline{C}^{\mathrm{op}} \to R-\mathrm{mod}$ such that $\mathcal{F}(X)$ is zero if and only if $\mathcal{F}_C(X) = \{0\}$ for sufficiently large C satisfying $X \in \operatorname{Obj} C$.*

Proof. Denote by Q the faithful embedding of $\underline{C}$ into the category $\mathfrak{C} = \prod_{(\to,C\in I)} C$ (which exists by Lemma 3.6). Denote by $\mathcal{F}'\colon \mathfrak{C}^{\mathrm{op}} \to R-\mathrm{mod}$ the reduced product of the cohomological functors $\mathcal{F}_C$ (see Remark 3.5(3)). Now we take $\mathcal{F} = \mathcal{F}' \circ Q$. By the definition of $\mathcal{F}'$, if $\mathcal{F}_C(X) = \{0\}$ for C sufficiently large, then $\mathcal{F}'(Q(X)) = \{0\}$. Conversely, assume that $\mathcal{F}'(Q(X)) = \{0\}$. Then $\mathcal{F}_C(Q(X)) = \{0\}$ for C sufficiently large. By the definition of Q, we have $\mathcal{F}_C(Q(X)) = \mathcal{F}_C(X) = \{0\}$ for C sufficiently large (and satisfying $X \in \operatorname{Obj} C$). □

Now we can prove Theorem 0.1 under less restrictive countability assumptions. Already the partial case $R = \mathbb{Z}$ of this statement is quite interesting and nontrivial.

Proposition 3.8. *Theorem 0.1 is valid in the case where R is at most countable.*

Proof. We define certain R-linear cohomological functors F_C^Y from C to $R-\mathrm{mod}$ for $C \in I$. In the case where Y is not an object of C we set $F_C^Y = 0$. If $Y \in \operatorname{Obj} C$, for the role of F_C^Y we take a cohomological functor such that $F_C^Y|_{E\cap \operatorname{Obj} C} = 0$ and $F_C^Y(Y) \neq \{0\}$ (as constructed in Proposition 2.1).

Applying Corollary 3.7, we obtain an R-linear cohomological functor $F^Y : \underline{C}^{\mathrm{op}} \to R-\mathrm{mod}$ such that $F^Y(X)$ is zero if and only if $F_C^Y(X) = \{0\}$ for C sufficiently large (and $X \in \operatorname{Obj} C$). Certainly, it follows that $F^Y|_E = 0$. Moreover, if $F^Y(Y) = \{0\}$, then $F_C^Y(Y) = \{0\}$ for some C containing Y. Thus, Y belongs to $E \cap \operatorname{Obj} C \subset E$, which contradicts our assumption on Y. □

3.2. The case of an uncountable R.

Definition 3.9. For a directed set I and a family $(M_i : \ i \in I)$ of objects of some (closed with respect to arbitrary filtered limits and colimits) Abelian category we define the reduced product $\prod_{(\to, i\in I)} M_i$ as $\varinjlim_{k\in I}\prod_{i\in I_k} M_i$

Certainly, the reduced product can also be characterized as the cokernel of the natural morphism $\coprod_{k\in I}(\prod_{k\not\succeq n} M_n) \to \prod_{n\in I} M_n$ because both of these have the same universal property.

Proposition 3.10. *Let $M_i \in \operatorname{Obj}\underline{A}$ for $i \in I$, where I is a directed set and $\underline{A}$ is an Abelian category. Assume that for any $i \in I$ the object M_i is equipped with an action of a commutative unital ring R_i, where $(R_i, \phi_{j,i})$ is an inductive system of (unital) rings (with connecting homomorphisms $\phi_{j,i}\colon R_j \to R_i$ corresponding to $\preceq$). Then the following statements hold.*

1. *There is a natural action of $\varinjlim_{i\in I} R_i$ on $\prod_{(\to, i\in I)} M_i$.*
2. *Let $(L_i)_{i\in I}$ be objects of $\underline{A}$ that are equipped also with R_i-actions (for all $i \in I$), and let $f_i\colon M_i \to L_i$ be a family of morphisms that respect the corresponding actions. Then the corresponding morphism $f\colon \prod_\to M_i \to \prod_\to L_i$ respects the $\varinjlim_{i\in I} R_i$-action.*

Proof. 1. It suffices to construct ring homomorphisms $R_i \xrightarrow{g_i} \operatorname{End}_{\underline{A}}(\prod_{(\to,n\in I)} M_n)$ such that the composition $R_j \xrightarrow{\phi_{j,i}} R_i \xrightarrow{g_i} \operatorname{End}_{\underline{A}}(\prod_\to M_n)$ equals g_j for any $i \succeq j \in I$ (i.e., to verify that these actions are compatible).

For any $i \succeq j \in I$ we have an action of R_j on M_i via the composition $R_j \to R_i \to \operatorname{End}_{\underline{A}}(M_i)$. This yields a diagonal action of R_j on $\prod_{k\in I_n} M_k$ for any $n \succeq j$. Moreover, if $n \succeq i$, this action is compatible with the corresponding action of R_i. So, we obtain an action g_j of R_j on $\varinjlim_{n\in I_j}\prod_{I_n} M_k \cong \prod_\to M_n$. By construction, the action of R_j on $\varinjlim_{n\in I_i}\prod_{I_n} M_k \cong \varinjlim_{n\in I_j}\prod_{I_n} M_k \cong \prod_\to M_n$ is compatible with the corresponding action of R_i.

2. To prove the claim it suffices to verify that f respects the action of R_i for any $i \in I$. Certainly, the morphism $\varinjlim_{n\in I_i}\prod_{I_n} M_k \to \varinjlim_{n\in I_i}\prod_{I_n} N_k$ respects the action of R_i because it is induced by the R_i-linear morphisms $\prod_{I_n} M_k \to \prod_{I_n} N_k$ (for $n \in I_i$). $\square$

Now we are able to finish the proof of Theorem 0.1.

Proof of Theorem 0.1 *for an arbitrary R.* Note that R is equal to the inductive limit of its (at most) countable unital subrings; thus, $R = \varinjlim R_i$ for some directed set of at most countable (commutative unital) rings R_i. Next, $\underline{C}$ is R_i-linear for each of these R_i. Hence (by Proposition 3.8), there exist R_i-linear cohomological functors $\widehat{F}^Y_{R_i}\colon \underline{C}^{\mathrm{op}} \to R_i-\mathrm{mod}$ such that $\widehat{F}^Y_{R_i}|_E = 0$ and $\widehat{F}^Y_{R_i}(Y) \neq \{0\}$. We take $F^Y_{R_i} = G_{R_i}\circ\widehat{F}^Y_{R_i}\colon \underline{C}^{\mathrm{op}} \to Ab$, where G_{R_i} is the corresponding forgetful functor.

Now we denote by $\widehat{F}^Y$ the reduced product of cohomological functors $\prod_\to F^Y_{R_i}$ (see Remark 3.5(3)). By Proposition 3.10, this functor factors through $R-\mathrm{mod}$. Denote the corresponding functor $\underline{C}^{\mathrm{op}} \to R-\mathrm{mod}$ by F^Y. Since F^Y is R_i-linear for any R_i, F^Y is also R-linear. By Remark 3.5(3), $F^Y(X)$ is zero if and only if $\widehat{F}^Y_{R_i}(X) = \{0\}$ for R_i sufficiently large. Hence, $F^Y|_E = 0$ and $F^Y(Y) \neq \{0\}$. $\square$

As a corollary we present a list of criteria for E to be Karoubi-closed and extension-closed in $\underline{C}$.

Corollary 3.11. *Let E be a subset of $\operatorname{Obj}\underline{C}$. The following conditions are equivalent.*

1. *There exists a set* $\{F_i\}$, $i \in I$, *of cohomological functors from* $\underline{C}$ *with values in some Abelian categories* $\underline{A}_i$ *such that the set* $\{c \in \operatorname{Obj}\underline{C} \,:\, F_i(c) = 0\ \forall i \in I\}$ *equals* E.

2. E *is an extension-closed Karoubi-closed subset of* $\operatorname{Obj}\underline{C}$.

3. *There exists a set* $\{F_i\}$ *of* R*-linear cohomological functors on* $\underline{C}$ *with values in* $R-\mathrm{mod}$ *such that the set* $\{c \in \operatorname{Obj}\underline{C} \,:\, F_i(c) = \{0\}\}$ *equals* E.

4. *There exists a single* R*-linear cohomological functor* $F\colon \underline{C}^{\mathrm{op}} \to R-\mathrm{mod}$ *such that the set* $\{c \in \operatorname{Obj}\underline{C} \,:\, F(c) = \{0\}\}$ *equals* E.

5. *Denote by* $H\colon \underline{C} \to \underline{A}(\underline{C})$ *the universal homological functor for* $\underline{C}$ ($\underline{A}(\underline{C})$ *is the Abelianization of* $\underline{C}$, *and we call* H *a homological functor because it is covariant*; *see Appendix A in* [6]). *Denote by* $\underline{A}(E)$ *the Serre subcategory of* $\underline{A}(\underline{C})$ *generated by* $H(E)$. *Then* $H(\operatorname{Obj}\underline{C}) \cap \operatorname{Obj}\underline{A}(E) = H(E)$.

Proof. Obviously, (1) ⇒ (2), (3) ⇒ (1), and (4) ⇒ (3). The implication (2) ⇒ (3) follows from Theorem 0.1.

The composition functor $H_E\colon \underline{C} \to \underline{A}(\underline{C}) \to \underline{A}(\underline{C})/\underline{A}(E)$ (the last category is the corresponding Serre localization) yields a homological functor on $\underline{C}$ whose set of zeros is $H(\operatorname{Obj}\underline{C}) \cap \operatorname{Obj}\underline{A}(E)$. Consider the opposite to this functor (i.e., the corresponding $H_E^{\mathrm{op}}\colon \underline{C}^{\mathrm{op}} \to (\underline{A}(\underline{C})/\underline{A}(E))^{\mathrm{op}}$; this is a cohomological functor with values in $(\underline{A}(\underline{C})/\underline{A}(E))^{\mathrm{op}}$), we obtain (5) ⇒ (1). Conversely, for any set of cohomological functors $\{F_i\}$ with values in $\underline{A}_i$ such that the set $\{c \in \operatorname{Obj}\underline{C} \,:\, F_i(c) = 0\ \forall i \in I\}$ equals E, there exist exact functors $G_i \,:\, (\underline{A}(\underline{C})/\underline{A}(E))^{\mathrm{op}} \to \underline{A}_i$ such that $G_i \circ H_E^{\mathrm{op}} = F_i$ (by Lemma A.2 in [6]). Hence, the set

$$\{c \in \operatorname{Obj}\underline{C} \,:\, H_E(c) = 0\} = H(\operatorname{Obj}\underline{C}^{\mathrm{op}}) \cap \operatorname{Obj}\underline{A}(E)$$

is a subset of $\{c \in \operatorname{Obj}\underline{C} \,:\, F_i(c) = 0\ \forall i \in I\} = E$. Thus $H(\operatorname{Obj}\underline{C}^{\mathrm{op}}) \cap \operatorname{Obj}\underline{A}(E)$ is equal to $H(E)$, and we obtain condition (5).

So it remains to prove that (3) ⇒ (4). Assume (3) to be fulfilled. Certainly, the product functor $F(-) = \prod_{Y \in (\operatorname{Obj}\underline{C})\setminus E} F^Y(-)$ is R-linear and cohomological. Moreover, $F(X) = \{0\}$ if and only if $F^Y(X) = \{0\}$ for every $Y \in (\operatorname{Obj}\underline{C}) \setminus E$. Thus, $F(X) = \{0\}$ if and only if $X \in E$. □

§4. An application: computing envelopes via "localizing the coefficients"

For any prime ideal $p \triangleleft R$, denote the localization of R at p by R_p. Consider the category $\underline{C}_p$ whose objects are those of $\underline{C}$ and whose morphisms are obtained by tensoring the $\underline{C}$-ones by R_p (over R). Denote by L_p the obvious functor $\underline{C} \to \underline{C}_p$. We recall some well-known properties of this construction (which were studied in detail in Appendix A.2 of [5] in the case where $R = \mathbb{Z}$).

Proposition 4.1. $\underline{C}_p$ *is a triangulated category, whereas* L_p *is isomorphic to the Verdier quotient functor for the localization of* $\underline{C}$ *by its triangulated subcategory generated by* $\{\operatorname{Cone}(X \xrightarrow{s\,\mathrm{id}_X} X) | s \in R \setminus p, X \in \operatorname{Obj}\underline{C}\}$.

Proof. See Proposition B.1.5 in [4]. □

Now we are able to prove Corollary 0.2.

The "only if" part of the statement is obvious.

We prove the reverse implication. Denote by E_p the envelope of $L_p(D)$ in $\underline{C}_p$ (and recall that E denotes the $\underline{C}$-envelope of D). Assume that $L_p(Y)$ belongs to E_p for every maximal ideal p of R.

First we assume that $\underline{C}$ is small. By Corollary 3.11, there exists an R-linear cohomological functor $F\colon \underline{C}^{\mathrm{op}} \to R-\mathrm{mod}$ such that $F(X) = \{0\}$ if and only if $X \in E$. Certainly, for any p (that is a maximal ideal of R) the correspondence $F_p\colon X \mapsto F(X) \otimes_R R_p$ yields a cohomological functor on $\underline{C}_p$. Since $F_p|_{L_p(D)} = 0$, we also have $F_p|_{E_p} = 0$. Since for any maximal ideal p of R the object $L_p(Y)$ belongs to E_p, we have $F_p(Y) = F(Y) \otimes_R R_p = \{0\}$ (for any p). Thus, $F(Y) = \{0\}$; hence Y belongs to E.

It remains to deduce the general case of the corollary from the "small" one. Certainly, for p running through the maximal ideals of R, there exist finite sets $D_p \subset D$ such that $L_p(Y)$ belongs to the $\underline{C}_p$-envelope of $L_p(D_p)$. Next, there exists a small full triangulated R-linear subcategory $\underline{C}'$ of $\underline{C}$ whose set of objects contains $\left(\bigcup_p D_p\right) \cup \{Y\}$. Denote $E \cap \operatorname{Obj} \underline{C}'$ by E'; for any maximal $p \trianglelefteq R$ denote by L'_p the corresponding functor $\underline{C}' \to \underline{C}'_p$ of localization of coefficients at p. Since $\underline{C}'_p$ is a full subcategory of $\underline{C}_p$ for any p, $L'_p(Y)$ belongs to the $\underline{C}'_p$-envelope of $L'_p(E)$ (note that the latter set contains the $\underline{C}'_p$-envelope of $L'_p(D_p)$). Since our corollary is valid for the triple $(\underline{C}', E', Y)$, we see that Y belongs to $E' \subset E$.

Remark 4.2. In the "motivic" applications the authors have in mind, the usage of Gabber's resolution of singularities (see the Introduction) does not "naturally" yield a single D for all p. Instead, for any maximal ideal p of R we can find certain $D_p \subset \operatorname{Obj} \underline{C}$ such that $L_p(Y)$ belongs to the $\underline{C}_p$-envelope of $L_p(D_p)$. Yet, surely, this implies (in a way somewhat similar to the above argument) that Y belongs to the $\underline{C}$-envelope of $\cup_p D_p$.

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