

NORMALIZER OF THE CHEVALLEY GROUP OF TYPE E_7

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a teacher, a colleague, and a friend*

ABSTRACT. The simply connected Chevalley group $G(E_7, R)$ of type E_7 is considered in the 56-dimensional representation. The main objective is to prove that the following four groups coincide: the normalizer of the elementary Chevalley group $E(E_7, R)$, the normalizer of the Chevalley group $G(E_7, R)$ itself, the transporter of $E(E_7, R)$ into $G(E_7, R)$, and the extended Chevalley group $\tilde{G}(E_7, R)$. This holds over an arbitrary commutative ring R , with all normalizers and transporters being calculated in $GL(56, R)$. Moreover, $\tilde{G}(E_7, R)$ is characterized as the stabilizer of a system of quadrics. This last result is classically known over algebraically closed fields, here it is proved that the corresponding group scheme is smooth over \mathbb{Z} , which implies that it holds over arbitrary commutative rings. These results are a key step in a subsequent paper, devoted to overgroups of exceptional groups in minimal representations.

The most natural way to study the general orthogonal group is to represent it as the stabilizer of a quadric. In the present paper, we establish a similar geometric characterization of the normalizer of the simply connected Chevalley group $G_{sc}(E_7, R)$ as the stabilizer of the intersection of 133 quadrics in a 56-dimensional space, and prove that the above normalizer coincides with the normalizer of the elementary Chevalley group $E_{sc}(E_7, R)$.

The present work is a direct sequel of our papers [10, 17], where a similar exercise was carried through for the groups of types E_6 and F_4 .

§1. INTRODUCTION

In the paper [16] (see also [17, 18]), the second author started to carry over the results by the first author and Victor Petrov [13, 14, 39] on overgroups of classical groups in vector representations, to the exceptional groups $E(E_6, R)$ and $E(E_7, R)$, in minimal representations. From the very start, it became apparent that a key step necessary to carry through a reduction proof in the spirit of the cited papers, would be an *explicit* calculation of the normalizer of the above groups in the corresponding general linear group, $GL(27, R)$ or $GL(56, R)$, respectively.

In our previous paper [10] we solved this problem completely for the group $E(E_6, R)$, whereas in [17] this problem was solved for the group $E(F_4, R)$. In the present paper, in the same spirit we consider the group of type E_7 .

More precisely, in §4, in the ring of integer polynomials $\mathbb{Z}[x_1, \dots, x_{56}]$ we explicitly construct an ideal I that is generated by 133 quadratic forms f_1, \dots, f_{133} and has the

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following property. Denote by $\text{Fix}_R(I)$ the set of R -linear transformations preserving the ideal I , see §4 for the precise definitions.

Our first main objective in the present paper is to prove the following result. Here, G_I denotes the affine group scheme such that $G_I(R) = \text{Fix}_R(I)$.

Theorem 1. *There is an isomorphism $G_I \cong \bar{G}_{\text{sc}}(\mathbb{E}_7, -)$ of affine groups schemes over \mathbb{Z} .*

This result can be viewed as an explicit description by equations of the *extended* simply connected Chevalley–Demazure group scheme $\bar{G}_{\text{sc}}(\Phi, -)$ of type $\Phi = \mathbb{E}_7$. This scheme was constructed in [23], see also [4, 5, 7] and §2 below. For $\Phi = \mathbb{E}_7$ the most straightforward way to visualize the scheme $\bar{G}_{\text{sc}}(\Phi, -)$ is to view it as the Levi factor of the parabolic subscheme of type P_8 in $G_{\text{sc}}(\mathbb{E}_8, -)$, where $G_{\text{sc}}(\Phi, -)$ — is the usual simply connected Chevalley–Demazure groups scheme of type Φ . We refer the reader to [38] for the scheme-theoretic definition of parabolic subgroups and their Levi factors, see also [7] for the above identification itself.

Our results are intimately related to the description of $G_{\text{sc}}(\mathbb{E}_7, R)$ as the stabilizer of a system of four-linear forms on $V = V(\varpi_7)$. Namely, in [19] we gave a new construction of a four-linear form $f: V \times V \times V \times V \rightarrow R$ and a symplectic form $h: V \times V \rightarrow R$, invariant under the action of the group $G_{\text{sc}}(\mathbb{E}_7, R)$. We reproduce the construction of the form f in §3. The bulk of our system of quadratic forms consists of the second partial derivatives of the [regular part of] the form f .

It turns out that $G_{\text{sc}}(\mathbb{E}_7, R)$ is precisely the group of linear transformations preserving both f and h :

$$G_{(f,h)}(R) = \left\{ g \in \text{GL}(56, R) \mid \begin{aligned} f(gu, gv, gw, gz) &= f(u, v, w, z), \\ h(gu, gv) &= h(u, v) \text{ for all } u, v, w, z \in V \end{aligned} \right\}.$$

It is only marginally more complicated to describe the extended group $\bar{G}_{\text{sc}}(\mathbb{E}_7, R)$ in terms of the forms f and h . Namely, let

$$\begin{aligned} \bar{G}_{(f,h)}(R) &= \{ g \in \text{GL}(56, R) \mid \text{there exist } \varepsilon, \varepsilon' \in R^*, c_2, c_3, c_4 \in R \text{ such that} \\ f(gu, gv, gw, gz) &= \varepsilon f(u, v, w, z) + c_2 h(u, v)h(w, z) + c_3 h(u, w)h(v, z) + c_4 h(u, z)h(v, w) \\ &\text{and } h(gu, gv) = \varepsilon' h(u, v) \text{ for all } u, v, w, z \in V \}. \end{aligned}$$

Theorem 2. *There are isomorphisms $G_{(f,h)} \cong G_{\text{sc}}(\mathbb{E}_7, -)$, $\bar{G}_{(f,h)} \cong \bar{G}_{\text{sc}}(\mathbb{E}_7, -)$ of affine groups schemes over \mathbb{Z} .*

This theorem readily implies that the above definition of the extended group $\bar{G}_{\text{sc}}(\mathbb{E}_7, R)$ can be simplified as follows. Namely, Lemma 10 asserts that

$$\begin{aligned} \bar{G}_{(f,h)}(R) &= \{ g \in \text{GL}(56, R) \mid \text{there exists } \varepsilon \in R^* \text{ such that} \\ f(gu, gv, gw, gz) &= \varepsilon f(u, v, w, z) \text{ and } h(gu, gv) = \varepsilon^2 h(u, v) \text{ for all } u, v, w, z \in V \}. \end{aligned}$$

Now, let E, F be two subgroups of a group G . Recall that the *transporter* of the subgroup E to the subgroup F is the set

$$\text{Tran}_G(E, F) = \{ g \in G \mid E^g \leq F \}.$$

Actually, we mostly use this notation in the case where $E \leq F$, and then

$$\text{Tran}_G(E, F) = \{ g \in G \mid [g, E] \leq F \}.$$

In the sequel, we only work with simply connected groups and omit the subscript in the notation $G_{\text{sc}}(\Phi, R)$. By $E(\Phi, R) \leq G(\Phi, R)$ we denote the elementary Chevalley group. Now we are all set to state the main result of the present paper. Observe that all normalizers and transporters here are taken in the general linear group $\text{GL}(56, R)$.

Theorem 3. *Let R be an arbitrary commutative ring. Then*

$$N(E(E_7, R)) = N(G(E_7, R)) = \text{Tran}(E(E_7, R), G(E_7, R)) = G_I(R).$$

The interrelation of Theorems 1 and 3 and the general outline of their proof are exactly the same as in our previous paper [10], and some familiarity with [10] (at least with the Introduction and §5) would be *extremely* useful to facilitate understanding the proofs in the present paper.

Observe that after the publication of [10] its subject matter became *unexpectedly* pertinent. Namely, recently Elena Bunina reconsidered one of the central classical problems of the whole theory, description of [abstract] automorphisms of Chevalley groups, without any simplifying assumptions such as R being Noetherian, or 2 being invertible in R . For local rings she *almost* succeeded in proving that all automorphisms of the group $E(\Phi, R)$ are standard, see [2]. Namely, she established that an arbitrary automorphism of the adjoint elementary Chevalley group is the product of ring, inner, and graph automorphisms. There is a catch, though, that with her approach the inner automorphisms are taken not in the adjoint Chevalley group $G_{\text{ad}}(\Phi, R)$ itself, but rather in the corresponding general linear group $\text{GL}(n, R)$. In this context, the fact that the abstract and algebraic normalizers coincide, means precisely that all such conjugations are genuine inner automorphisms.

This means that, modulo the results of [2], an analog of the results of [10] and the present paper for adjoint *representations* would then imply that all automorphisms of Chevalley groups of types E_l over local rings — and thus also arbitrary commutative rings — are standard in the usual sense. We are convinced that our results on the equations in adjoint representations [37, 36] make it possible to obtain the requisite results for the adjoint case. In cooperation with Elena Bunina, we hope to work out the details shortly.

The paper is organized as follows. In §2 we recall the basic notation pertaining to the extended Chevalley group of type E_7 . In §3 we discuss the invariant four-linear forms, and in §4 we construct an invariant system of quadrics, which in this case is *significantly* trickier than in the case of E_6 . In §5 we prove that this system of quadrics is indeed invariant. The technical core of the paper are §§6–10, which are directly devoted to the proof of Theorems 1, 2, and 3. Due to the limited space, we do not explicitly list the resulting equations here, this will be done in a subsequent publication.

§2. EXTENDED CHEVALLEY GROUP OF TYPE E_7

We refer the reader to [10] for the general context of the present paper and further references. In the papers [38, 42, 44, 45, 47] one can find many further details pertaining to Chevalley groups over rings, and many related references.

Nevertheless, to fix the requisite notation, for the reader's convenience below we reproduce §1 of [10] with minor modifications.

Let Φ be a reduced irreducible root system of rank l (in the main body of the paper we assume that $\Phi = E_7$), and P a lattice intermediate between the root lattice $Q(\Phi)$ and the weight lattice $P(\Phi)$. We fix an order on Φ and denote by $\Pi = \{\alpha_1, \dots, \alpha_l\}$, Φ^+ , and Φ^- the corresponding sets of fundamental, positive, and negative roots. Our numbering of the fundamental roots is as in [3]. By δ we denote the maximal root of the system Φ with respect to this order. For instance, for $\Phi = E_7$ we have $\delta = \frac{234321}{2}$. Denote by $P(\Phi)_{++}$ the set of dominant weights with respect to this order. Recall that it consists of all nonnegative integral linear combinations of the fundamental weights $\varpi_1, \dots, \varpi_l$, for this order. Finally, $W = W(\Phi)$ denotes the Weyl group of the root system Φ .

Next, let R be a commutative ring with 1. It is classically known that, starting with this data, one can construct the *Chevalley group* $G_P(\Phi, R)$, which is the group of R -points of an affine group scheme $G_P(\Phi, -)$, known as the *Chevalley–Demazure scheme*. For the problems we consider, it suffices to limit ourselves with the simply connected (alias, universal) groups, for which $P = P(\Phi)$. For the simply connected groups we usually omit any reference to the lattice P and simply write $G(\Phi, R)$ or, when we wish to stress that the group in question is simply connected, $G_{sc}(\Phi, R)$. The adjoint group, for which $P = Q(\Phi)$, is denoted by $G_{ad}(\Phi, R)$.

Fix a split maximal torus $T(\Phi, R)$ in $G(\Phi, R)$ and a parametrization of the unipotent root subgroups X_α , $\alpha \in \Phi$, elementary with respect to this torus. Let $x_\alpha(\xi)$ be the elementary unipotent element corresponding to $\alpha \in \Phi$ and $\xi \in R$ in this parametrization. The group $X_\alpha = \{x_\alpha(\xi), \xi \in R\}$ is called an (elementary) *root subgroup*, and the group $E(\Phi, R) = \langle X_\alpha, \alpha \in \Phi \rangle$ generated by all elementary root subgroups is called the (absolute) *elementary subgroup* of the Chevalley group $G(\Phi, R)$.

As a matter of fact, apart from the usual Chevalley group, we also consider the corresponding *extended* Chevalley group $\bar{G}(\Phi, R)$, which plays the same role with respect to $G(\Phi, R)$ as the general linear group $GL(n, R)$ plays with respect to the special linear group $SL(n, R)$. *Adjoint* extended groups were constructed in the original paper [21] by Chevalley. It is somewhat harder to construct *simply connected* extended groups because, unlike the adjoint case, here one must increase the dimension of the maximal torus. A unified elementary construction was proposed by Berman and Moody in [23]. However, for the case of $\bar{G}_{sc}(E_7, R)$ that we consider in the present paper, this group can naturally be viewed as a subgroup of the usual Chevalley group $G_{sc}(E_8, R)$, and

$$\bar{G}_{sc}(E_7, R) = G_{sc}(E_7, R) \cdot T_{sc}(E_8, R).$$

In the majority of the existing constructions, the Chevalley group $G(\Phi, R)$ arises together with an action on the *Weyl module* $V = V(\omega)$ for some dominant weight ω . Denote by $\Lambda = \Lambda(\omega)$ the *multiset* of weights of the module $V = V(\omega)$ with *multiplicities*. In the present paper we consider the group $G(E_7, R)$ in the minimal representation with the highest weight ϖ_7 . This is a microweight representation, in particular the multiplicities of all weights are equal to 1. Fix an admissible base v^λ , $\lambda \in \Lambda$, of the module V . We conceive a vector $a \in V$, $a = \sum v^\lambda a_\lambda$, as a coordinate *column* $a = (a_\lambda)$, $\lambda \in \Lambda$.

In Figure 1 we reproduce the weight diagram of the representation (E_7, ϖ_7) , together with the *natural* numbering of weights, used in the sequel. In this numbering the weights are listed according to the order determined by the fundamental root system Π . In the picture, the highest weight is the leftmost one. The weight diagram of the representation (E_7, ϖ_7) is symmetric, and this symmetry is reflected in the numbering, the weights are numbered as $1, 2, \dots, 28, -28, \dots, 2, 1$. Often, to save space we write \bar{n} instead of $-n$. We refer the reader to [11] for lists of weight in the Dynkin form and in the hyperbolic form, as well as other common numberings.

Recall that in the weight diagram two weights are joined by an edge if their difference is a *fundamental* root. The weight graph is constructed similarly, only that now two weights are joined by an edge provided their difference is a *positive* root. In the sequel we denote by $d(\lambda, \mu)$ the distance between two weights λ and μ in the weight graph. In other words, $d(\lambda, \mu) = 0$ if $\lambda = \mu$; $d(\lambda, \mu) = 1$ if $\lambda - \mu \in \Phi$; $d(\lambda, \mu) = 2$ if $\lambda \neq \mu$, $\lambda - \mu \notin \Phi$, $\lambda - \mu$ is the sum of two roots of Φ ; and finally, $d(\lambda, \mu) = 3$ otherwise.

The above realization of the representation (E_7, ϖ_7) as an internal Chevalley module inside the Chevalley group of type E_8 provides a natural identification of the set of weights Λ with the set of roots of the root system E_8 , in whose expansion with respect to the fundamental roots the root α_8 occurs with the coefficient 1. Obviously, the roots of the

§3. THE INVARIANTS OF DEGREE 4

In our paper [10], the simply connected Chevalley group of type E_6 acting on the 27-dimensional module $V = V(\varpi_1)$ was identified with the isometry group of a three-linear form $T: V \times V \times V \rightarrow R$. There is a similar, but much more complicated description of the simply connected Chevalley group of type E_7 acting on the 56-dimensional module $V = V(\varpi_7)$. In this case, to determine the group one needs two invariants, one of degree 2, and another of degree 4. First of all, the module V is self-dual and carries a unimodular symplectic form h . Next, there exists a four-linear form $f: V \times V \times V \times V \rightarrow R$ such that G can be identified with the full isometry group of the pair h, f , in other words, with the group of all $g \in \text{GL}(V)$ such that $h(gu, gv) = h(u, v)$ and $f(gu, gv, gw, gz) = f(u, v, w, z)$ for all $u, v, w, z \in V$. The similarities of this pair of forms determine the *extended* Chevalley group $\tilde{G}(E_7, R)$ (see Theorem 2).

It is obvious how to construct h . Construction of the fourth degree invariant is considerably more complicated, and classically one constructs not the four-linear form f , but rather the corresponding quartic¹. The fact that the group G preserves a form of degree 4 in 56 variables, was first observed by E. Cartan, at least in characteristic 0, but his explicit construction of this form was flawed (probably, it was just a misprint). A very elegant construction of such an invariant over a field K of characteristic distinct from 2 was given by H. Freudenthal. Namely, he identified the module V with the space $A(8, K)^2$, where $A(8, K)$ is the set of antisymmetric (8×8) -matrices, and considered the following symplectic inner product and form of degree 4:

$$h((a_1, b_1), (a_2, b_2)) = \frac{1}{2}(\text{tr}(a_1 b_2^t) - \text{tr}(a_2 b_1^t)),$$

$$Q((a, b)) = \text{pf}(a) + \text{pf}(b) - \frac{1}{4} \text{tr}((ab)^2) + \frac{1}{16} \text{tr}(ab)^2.$$

Now, for all characteristics distinct from 2, one can identify the isometry group of this pair with the simply-connected Chevalley group G of type E_7 over K (see [22, 25]). The constructions of the above form in the papers by M. Aschbacher and B. Cooperstein is somewhat different. Actually, in [22] the form was constructed in terms of A_6 (the gist of this construction is expressed by the partition $56 = 7 + 21 + 21 + 7$), whereas the construction in [25] is closer to Freudenthal's original construction, and is phrased in terms of A_7 (where $56 = 28 + 28$). The isometry group of the form Q is generated by G and a diagonal element of order 2 (see [25]). There are no serious complications in characteristic $p \geq 5$, whereas characteristic 3 requires some extra-care.

However, in characteristic 2 this approach is almost immediately blocked by serious obstructions. Obviously, the above construction fails. Apparently, in characteristic 2 there are whatsoever no nontrivial symmetric G -invariant four-linear forms on V , (see [22]). This is related to the fact that in characteristic 2 the four-linear form

$$f_0(u, v, x, y) = h(u, v)h(x, y) + h(u, x)h(v, y) + h(u, y)h(v, x),$$

obtained by the squaring of the symplectic form, becomes symmetric, which is not the case in characteristics at least 3. Actually, in characteristic 2 Aschbacher [22] constructed a four-linear G -invariant form F , which is symmetric with respect to the *even* permutations.

¹This form of degree 4 first occurred in a 1901 paper by L. E. Dickson in the context of the 28 bitangents, and thus, of the Weyl group $W(E_7)$. Apparently, Dickson did not notice an explicit relationship with the group of type E_7 itself. Otherwise, Chevalley groups could had been discovered some 50 years earlier!

There are other constructions of the form Q , most notably a construction by R. Brown [24], which works in characteristics different from 2, 3. Let V be a space with a nondegenerate inner product. Then to define a three-linear form on V is essentially the same as to define an algebra structure on V . By the same token, to define a four-linear form on V is essentially the same as to define on V a *ternary* algebra structure. Indeed, there exists a remarkable ternary algebra of dimension 56, constructed in terms of the exceptional 27-dimensional Jordan algebra \mathbb{J} (see [24, 27] and the references there). This algebra consists of (2×2) -matrices over \mathbb{J} with scalar diagonal entries, $56 = 1 + 27 + 27 + 1$.

The orbits of the group $G = G(E_7, K)$ on the 56-dimensional module were classified in [30] in the absolute case, in [34] for finite fields, and in [25] for arbitrary fields. Basically, these orbits are described in terms of the four-linear form. Again, the cases of characteristics 2 and 3 require separate analysis, and to take care of all details one has to use the notion of 4-forms, introduced by M. Aschbacher (see [22, 25]). Essentially, a 4-form is a form of degree 4 together with all of its polarizations. For simplicity, assume that $\text{char } K \neq 2, 3$. Then a vector $u \in V$ is said to be *singular* if $F(u, u, x, y) = 0$ for all $x, y \in V$; *brilliant*, if $F(u, u, u, x) = 0$ for all $x \in V$; and *luminous* if $F(u, u, u, u) = 0$. Otherwise, i.e., if $F(u, u, u, u) \neq 0$, the vector u is said to be *dark*. The orbits of the group G on V are as follows: 0, nonzero singular vectors, nonsingular brilliant vectors, luminous vectors that are not brilliant, and, finally, one or several orbits of dark vectors, parametrized by K^*/K^{*2} (these last orbits fuse to one orbit under the action of the *extended* Chevalley group \widetilde{G} of type E_7).

The feeling that E_7 stands in the same relationship to E_6 , as E_6 itself stands to D_5 , suggests the following definition of the form of degree 4 on V . Take a base vector v^λ . Then the vectors v^μ with $d(\lambda, \mu) = 2$ generate a 27-dimensional module U , which supports the cubic form related to E_6 . Let us define *tetrads* as quadruples $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ of pairwise orthogonal weights. Let Θ and Θ_0 be the sets of ordered and unordered tetrads, respectively. Clearly, $|\Theta| = 56 \cdot 27 \cdot 10$, whereas $|\Theta_0| = |\Theta|/24 = 630$. Now, we can tentatively define the form Q_{tent} of degree 4 by setting $Q_{\text{tent}}(x) = \sum \pm x_{\lambda_1} x_{\lambda_2} x_{\lambda_3} x_{\lambda_4}$, where the sum is taken over all $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \in \Theta_0$, while the signs are determined by the condition that the resulting form is invariant under the action of the extended Weyl group \widetilde{W} . Here, one should be slightly more cautious than in the case of E_6 , because now, in addition to the two possible cases that occurred there, the following possibility occurs: w_α moves all 4 weights of a tetrad, two of them in the positive and the other two in the negative direction, in which case the sign does not change. Nevertheless, an expression of the sign in terms of $h(\lambda_i, \mu_i)$ still works. This is essentially the same as to define the four-linear form F_{tent} by $F_{\text{tent}}(v^{\lambda_1}, v^{\lambda_2}, v^{\lambda_3}, v^{\lambda_4}) = (-1)^{h(\lambda_1, \lambda_2, \lambda_3, \lambda_4)}$ for a tetrad $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \Theta$ and by $F_{\text{tent}}(v^{\lambda_1}, v^{\lambda_2}, v^{\lambda_3}, v^{\lambda_4}) = 0$ otherwise. By construction, this form is invariant under the action of \widetilde{W} , and we only need to verify that it is invariant under the action of the root subgroup X_α for some root $\alpha \in \Phi$. Unfortunately, this is not the case. Namely, for any tetrad $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and any elementary root unipotent $g = x_\alpha(\xi)$ the following formula holds true:

$$F_{\text{tent}}(gv^{\lambda_1}, gv^{\lambda_2}, gv^{\lambda_3}, gv^{\lambda_4}) = F_{\text{tent}}(v^{\lambda_1}, v^{\lambda_2}, v^{\lambda_3}, v^{\lambda_4}).$$

As it happens, though, there exist quadruples of weights that are not tetrads, for which the right-hand side equals 0, whereas the left-hand side is distinct from 0. For instance, take four weights $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that $\lambda_1 + \alpha, \lambda_2 + \alpha, \lambda_3 + \alpha, \lambda_4 - \alpha$ are weights, and, together, the 8 above weights form a cube (in other words, the corresponding weight diagram is the tensor product of three copies of (A_1, ϖ_1) , see [25, 40]). Then one of the weights $\lambda_1, \lambda_2, \lambda_3$ will be adjacent to the other two, say, $d(\lambda_1, \lambda_2) = d(\lambda_1, \lambda_3) = 1$, so that $F_{\text{tent}}(v^{\lambda_1}, v^{\lambda_2}, v^{\lambda_3}, v^{\lambda_4}) = 0$.

At the same time, decomposing the expression $F_{\text{tent}}(gv^{\lambda_1}, gv^{\lambda_2}, gv^{\lambda_3}, gv^{\lambda_4})$ by linearity, we get 8 summands, of which exactly one, namely $F_{\text{tent}}(v^{\lambda_1+\alpha}, v^{\lambda_2}, v^{\lambda_3}, v^{\lambda_4})$, corresponds to a tetrad, and equals ± 1 . Thus, the form F_{tent} is not preserved by the action of X_α .

In itself, this is not yet critical, because one can hope to save the situation by throwing in another Weyl orbit of monomials. This is, however, exactly the point where real problems start. As a matter of fact, in the above counter-example, throwing in another orbit of monomials will produce *two* nonzero extra summands, so that the resulting correction will be a multiple of 2. This means that one cannot define an invariant form of degree 4 by setting its values on the tetrads to be equal to ± 1 , one should start with ± 2 instead. This is precisely where *serious* trouble starts. In characteristic different from 2 the above construction is *essentially* correct, in the sense that it tells how the *relevant* part of an invariant form of degree 4, responsible for the reduction to E_6 , looks like. Let us fix a vector v^λ . Then $F(v^\lambda, -, -, -)$ consists of two parts: the form F_{tent} , as defined above, and another part, introduced for the resulting form to be G -invariant. This second part has the form $F(v^\lambda, v^{\lambda^*}, -, -)$ and does not say anything beyond the fact that our group preserves the usual symplectic form.

In the works by Jacob Lurie [35] and the second author [19], these difficulties arising in characteristic 2 were sorted out in a systematic way, but the resulting four-linear forms were not symmetric anymore. Namely, let \mathfrak{g} be the Lie algebra of type E_8 ; recall that

$$\delta = \frac{2465432}{3}$$

is the highest root of E_8 . The coefficient with which α_8 occurs in the expansion of a root $\alpha \in E_8$ with respect to the fundamental roots, is called the α_8 -height of α and can only take the values $-2, -1, 0, 1, 2$. This gives rise to the following length 5 grading of the algebra \mathfrak{g} :

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

The 56-dimensional space \mathfrak{g}_1 has a base consisting of the elementary root elements e_α , where α runs over the roots of α_8 -height 1, i.e., the weights of $V(\varpi_7)$. Let λ, μ, ν, ρ be four weights of $V(\varpi_7)$. Clearly, the element

$$[[[[e_{-\delta}, e_\lambda], e_\mu], e_\nu], e_\rho]$$

has α_8 -weight 2, so that the resulting element is a multiple of e_δ . Denote the corresponding scalar coefficient by $c(\lambda, \mu, \nu, \rho)$ and consider the four-linear form

$$f(u, v, w, z) = \sum_{\lambda, \mu, \nu, \rho \in \Lambda} c(\lambda, \mu, \nu, \rho) u_\lambda v_\mu w_\nu z_\rho.$$

Obviously, this form is invariant under the action of the group $G(E_7, R)$ on the module $V(\varpi_7)$.

The orbit of the highest weight vector. It is well known that in any representation of the group G the orbit Gv^+ of the highest weight vector v^+ is an intersection of quadrics [33]. Here, as a motivation for the next section, we explicitly describe the equations defining the orbit of v^+ for the microweight representation of E_7 . For the microweight representation of E_6 this was done in [11]. Of course, for these cases the corresponding equations were found by H. Freudenthal and J. Tits more than 40 years ago (see also [11] and the references there), but again we wish to show how to recover the equations directly from the weight diagram.

Let $\omega = \varpi_1$ for E_6 or $\omega = \varpi_7$ for E_7 ; the case of $(\Phi, \omega) = (E_6, \varpi_6)$ is dual to the first case. We use the same interpretation of the modules as in § 1. In particular, $\Phi = E_l, l = 6, 7, \Delta = E_{l+1}$, and $\Sigma = \Sigma_{l+1}(1)$. The group $G = G(\Phi, R)$ acts on

$V = U_{l+1}(1)/U_{l+1}(2) \cong \prod X_\alpha$, $\alpha \in \Sigma$ by conjugation. Since we are only interested in the equations satisfied by the orbit Gv^+ , we may assume that $R = K$ is an (algebraically closed) field².

In both cases one can take $v^+ = v^\rho = x_\rho(1)$ as the highest weight vector, where

$$\rho = \frac{234321}{2} \quad \text{or} \quad \rho = \frac{2465431}{3}$$

are the maximal root of E_7 or a unique submaximal root of E_8 , respectively. Recall that the vector $a = (a_\alpha) \in V$ is now viewed as the product $x = \prod x_\alpha(a_\alpha) \in \prod X_\alpha$, $\alpha \in \Sigma$. In the case of E_7 this product is considered modulo $U_8(2) = X_{\rho+\alpha_8}$, the root subgroup corresponding to the maximal root of E_8 .

§4. CONSTRUCTION OF THE SYSTEM OF QUADRICS

The first set of quadrics defining the highest weight orbit consists of *square equations*; for large classes of representations, their construction and numerology were described by the first author in [8, 9]. Here, we recall some basic definitions of [8] in the context of the 56-dimensional representation of (E_7, ϖ_7) .

The set of weights $\Omega \subseteq \Lambda$ is called a *square* if $|\Omega| \geq 4$ and for all $\lambda \in \Omega$ its difference $\lambda - \mu$ with all weights $\mu \in \Omega$, except exactly one, denoted by λ^* , is a root, whereas the difference $\lambda - \lambda^*$ is not a root (and thus $\lambda \perp \lambda^*$). A square maximal with respect to inclusion is called a *maximal square*. In [8] it was proved that for our representation each maximal square Ω consists of 12 weights and the sum $\lambda + \lambda^*$ does not depend on the choice of $\lambda \in \Omega$. Moreover, a maximal square Ω is completely determined by this sum. Furthermore, in the case of a microweight representation of E_7 , the maximal squares are in bijective correspondence with the roots of E_7 , namely, a root $\alpha \in E_7$ is associated with the square

$$\Omega(\alpha) = \{\lambda \in \Lambda \mid \lambda - \alpha \in \Lambda\}.$$

Let, as above, Ω be some maximal square. Choose orthogonal weights $\rho, \rho^* \in \Omega$ and define a polynomial $f_{\rho, \rho^*} \in \mathbb{Z}[\{x_\lambda\}_{\lambda \in \Lambda}]$ by

$$f_{\rho, \rho^*} = x_\rho x_{\rho^*} - \sum N_{\rho, -\lambda} N_{\rho^*, -\lambda^*} x_\lambda x_{\lambda^*},$$

where the sum is taken over all orthogonal pairs of weights $\{\lambda, \lambda^*\}$ except $\{\rho, \rho^*\}$ itself. In [8], the equation $f_{\rho, \rho^*}(v) = 0$ on the components of a vector $v = (v_\lambda)_{\lambda \in \Lambda} \in V$ was called a *square equation* corresponding to the maximal square Ω . In particular, this equation depends only on the square Ω itself, and not on the arbitrary choice of a pair ρ, ρ^* of orthogonal weights: under passage to another such pair, the polynomial f_{ρ, ρ^*} is multiplied by ± 1 .

Fixing one such pair of orthogonal weights in each maximal square Ω gives us 126 polynomials, corresponding (up to the sign) to the 126 maximal squares (or, what is the same, to the 126 roots of E_7).

Finally, for a root $\alpha \in E_7$ we consider the polynomial $g_\alpha \in \mathbb{Z}[\{x_\lambda\}_{\lambda \in \Lambda}]$ defined by

$$g_\alpha = \sum_{\lambda \in \Omega(\alpha)} N_{\lambda, \bar{\lambda}} x_\lambda x_{\bar{\lambda}}.$$

Again, by definition, these polynomials are in bijective correspondence with the roots of E_7 (i.e., there are 126 of them), but the following lemma asserts that it suffices to consider only the g_α corresponding to $\alpha \in \Pi$.

²For rings, there are further obstructions, related to the fact that the lower K -functors, or their analogs, can be nontrivial, which we do not discuss here.

Lemma 1. *The ideal in $\mathbb{Z}[\{x_\lambda\}_{\lambda \in \Lambda}]$ generated by the polynomials $\{g_\alpha\}$, $\alpha \in E_7$, coincides with the ideal generated by the polynomials $\{g_\alpha\}_{\alpha \in \Pi}$.*

Proof. Let $\alpha \in E_7$. Observe that $\Omega(\alpha) = \{\lambda \in \Lambda \mid \lambda - \alpha \in \Lambda\} = \{\lambda \in \Lambda \mid (\lambda, \alpha) = 1/2\}$. If $\lambda \in \Omega(\alpha)$, then $(\bar{\lambda}, -\alpha) = (\delta - \lambda, -\alpha) = (\lambda, \alpha) - (\delta, \alpha) = (\lambda, \alpha)$, because $\delta \perp \alpha$ for all $\alpha \in E_7$. Thus, $\Omega(-\alpha) = \{\bar{\lambda} \mid \lambda \in \Omega(\alpha)\}$. We get

$$g_{-\alpha} = \sum_{\lambda \in \Omega(-\alpha)} N_{\lambda, \bar{\lambda}} x_\lambda x_{\bar{\lambda}} = \sum_{\lambda \in \Omega(\alpha)} N_{\bar{\lambda}, \lambda} x_\lambda x_{\bar{\lambda}} = -g_\alpha.$$

Now, let $\alpha, \beta, \alpha + \beta \in E_7$. We show that $g_{\alpha+\beta} = g_\alpha + g_\beta$. Observe that $(\lambda, \alpha + \beta) = (\lambda, \alpha) + (\lambda, \beta)$, and each of these inner products equals 0 or $\pm 1/2$. If $\lambda \in \Omega(\alpha + \beta)$, i.e., $(\lambda, \alpha + \beta) = 1/2$, then one of the expressions (λ, α) , (λ, β) equals $1/2$, while the other is 0. In this case the monomial $N_{\lambda, \bar{\lambda}} x_\lambda x_{\bar{\lambda}}$ is contained either in g_α , or in g_β , but not in both. Conversely, if $\lambda \in \Omega(\alpha)$ and $\lambda \notin \Omega(\alpha + \beta)$, then necessarily $(\lambda, \beta) = -1/2$, which implies that $(\bar{\lambda}, \beta) = 1/2$. It follows that g_α contains the monomial $N_{\lambda, \bar{\lambda}} x_\lambda x_{\bar{\lambda}}$, while g_β contains the monomial $N_{\bar{\lambda}, \lambda} x_\lambda x_{\bar{\lambda}} = -N_{\lambda, \bar{\lambda}} x_\lambda x_{\bar{\lambda}}$, which cancel. \square

Set $g_i = g_{\alpha_i}$ and let I be the ideal generated by the above quadratic polynomials f_{μ, μ^*} and the polynomials g_i , $i = 1, \dots, 7$ (altogether, this gives us $126 + 7 = 133$ polynomials).

Theorem 4. *Denote by $\text{Fix}_R(I)$ the set of R -linear transformations preserving the ideal I :*

$$\text{Fix}_R(I) = \{g \in \text{GL}(56, R) \mid f(gx) \in I \text{ for all } f \in I\}.$$

Then the elementary Chevalley group $E(E_7, R)$ is contained in $\text{Fix}_R(I)$.

§5. PROOF OF THEOREM 4

Since we realize the representation of the group of type E_7 inside the group E_8 , the calculations of this section mostly reproduce the calculations in [37] for the adjoint representation of E_8 . However, we cannot directly cite the results of [37], since we are only interested in the parts of these polynomials that correspond to the weights of E_7 inside E_8 .

To prove Theorem 4 it suffices to show that if $g = x_\gamma(\xi)$ (here $\gamma \in E_7$, $\xi \in R$), then $f_{\rho, \sigma}(gx) \in I$ for all $\rho, \sigma \in \Lambda$, $\rho \perp \sigma$, and $g_\alpha(gx) \in I$ for all $\alpha \in \Phi$.

The following special case of the Matsumoto lemma (see [38, Lemma 2.3]), describes the action of an elementary root unipotent $x_\gamma(\xi)$ on the vectors in V .

Lemma 2. *Let $\gamma \in E_7$, $v \in V$.*

- (1) *If $\lambda \in \Lambda$, $\lambda - \gamma \notin \Lambda$, then $(x_\gamma(\xi)v_\lambda) = v_\lambda$.*
- (2) *If $\lambda, \lambda - \gamma \in \Lambda$, then $(x_\gamma(\xi)v_\lambda) = v_\lambda + N_{\gamma, \lambda - \gamma} \xi v_{\lambda - \gamma}$.*

In particular, if $(\gamma, \lambda) \neq 1/2$, then $(x_\gamma(\xi)v_\lambda) = v_\lambda$.

Lemma 2 was stated in terms of the structure constants $N_{\alpha\beta}$ of the Lie algebras of type E_8 . It is classically known (see the references in [6]) that they satisfy the following relations:

$$\begin{aligned} N_{\alpha\beta} &= N_{-\beta, -\alpha} = -N_{-\alpha, -\beta} = -N_{\beta\alpha}, \\ N_{\alpha\beta} &= N_{\beta\gamma} = N_{\gamma\alpha}. \end{aligned}$$

Moreover, they are subject to the *cocycle identity*:

$$N_{\beta\gamma} N_{\alpha, \beta + \gamma} = N_{\alpha + \beta, \gamma} N_{\alpha\beta}.$$

In the sequel we use these identities without any specific reference.

Recall that $f_{\rho, \sigma}$ corresponds to the maximal square $\Omega = \Omega(\alpha)$ for some root $\alpha \in E_7$. The weights of the square Ω can be partitioned into pairs of orthogonal weights $\{\lambda, \mu\}$

with $\lambda + \mu = \delta + \alpha$; the polynomial $f_{\rho,\sigma}$ consists of monomials of the form $\pm x_\lambda x_\mu$ for all such pairs. Let us trace what happens with such monomials when x is mapped to $gx = x_\gamma(\xi)x$. Observe that (λ, γ) can only take values 0 and $\pm 1/2$. If $(\lambda, \gamma) \neq 1/2$, then by Lemma 2 we have $(gx)_\lambda = x_\lambda$. Calculating the inner product of $\lambda + \mu = \delta + \alpha$ by γ and recalling that $(\delta, \gamma) = 0$, we get

$$(\lambda, \gamma) + (\mu, \gamma) = (\alpha, \gamma).$$

The following cases can possibly occur:

- $(\alpha, \gamma) = -1$ or $-1/2$. Then none of the summands on the right-hand side equals $1/2$, and thus $f_{\rho,\sigma}(gx) = f_{\rho,\sigma}(x) \in I$.
- $(\alpha, \gamma) = 1/2$. Then exactly one summand on the right-hand side equals $1/2$, whereas the second is 0. Let, for instance, $(\lambda, \gamma) = 1/2$ and $(\mu, \gamma) = 0$. Similarly, suppose that $(\rho, \gamma) = 1/2$ and $(\sigma, \gamma) = 0$. Then $(gx)_\lambda(gx)_\mu = (x_\lambda + \xi N_{\gamma,\lambda-\gamma} x_{\lambda-\gamma}) x_\mu$. Expanding these identities and summing with signs over all pairs of orthogonal weights, we get

$$f_{\rho,\sigma}(gx) = f_{\rho,\sigma}(x) + \xi N_{\gamma,\rho-\gamma} x_{\rho-\gamma} x_\sigma - \xi \sum N_{\rho,-\lambda} N_{\sigma,-\mu} N_{\gamma,\lambda-\gamma} x_{\lambda-\gamma} x_\mu.$$

Observe that the sum $\lambda - \gamma + \mu = \delta + \alpha - \gamma$ does not depend on λ . Thus, the pairs of weights $\{\rho - \gamma, \sigma\}$ and $\{\lambda - \gamma, \mu\}$ appear in the maximal square $\Omega(\alpha - \gamma)$. We verify that

$$f_{\rho,\sigma}(gx) = f_{\rho,\sigma}(x) + \xi N_{\gamma,\rho-\gamma} f_{\rho-\gamma,\sigma}(x).$$

For this, it only remains to check that the signs coincide:

$$N_{\rho,-\lambda} N_{\sigma,-\mu} N_{\gamma,\lambda-\gamma} = N_{\gamma,\rho-\gamma} N_{\rho-\gamma,\gamma-\lambda} N_{\sigma,-\mu}.$$

But this immediately follows from the cocycle identity.

- $(\alpha, \gamma) = 0$. Then either the two summands on the left-hand side are 0, or one of them equals $1/2$, while the other equals $-1/2$. The summands for which $(\lambda, \gamma) = (\mu, \gamma) = 0$ do not contribute to the difference $f_{\rho,\sigma}(gx) - f_{\rho,\sigma}(x)$. Now, let $(\lambda, \gamma) = 1/2$ and $(\mu, \gamma) = -1/2$. Then $\lambda - \gamma$ and $\mu + \gamma$ are weights that sum to $\lambda + \mu = \delta + \alpha$; Moreover, $(\lambda - \gamma, \gamma) = -1/2$ and $(\mu + \gamma, \gamma) = 1/2$. Thus, the weights $\{\lambda - \gamma, \mu + \gamma\}$ are orthogonal and, so, belong to the same maximal square Ω . Next,

$$\begin{aligned} & N_{\rho,-\lambda} N_{\sigma,-\mu} (gx)_\lambda (gx)_\mu + N_{\rho,-\lambda+\gamma} N_{\sigma,-\mu-\gamma} (gx)_{\lambda-\gamma} (gx)_{\mu+\gamma} \\ &= N_{\rho,-\lambda} N_{\sigma,-\mu} x_\lambda x_\mu + N_{\rho,-\lambda+\gamma} N_{\sigma,-\mu-\gamma} x_{\lambda-\gamma} x_{\mu+\gamma} \\ & \quad + \xi N_{\rho,-\lambda} N_{\sigma,-\mu} N_{\gamma,\lambda-\gamma} x_{\lambda-\gamma} x_\mu + \xi N_{\rho,-\lambda+\gamma} N_{\sigma,-\mu-\gamma} N_{\gamma,\mu} x_{\lambda-\gamma} x_\mu. \end{aligned}$$

An easy calculation shows that the summands containing ξ cancel. Thus,

$$f_{\rho,\sigma}(gx) = f_{\rho,\sigma}(x).$$

- $(\alpha, \gamma) = 1$, i.e., $\alpha = \gamma$. In this case $(\lambda, \gamma) = (\mu, \gamma) = 1/2$. Thus,

$$\begin{aligned} (gx)_\lambda (gx)_\mu &= (x_\lambda + \xi N_{\gamma,\lambda-\gamma} x_{\lambda-\gamma})(x_\mu + \xi N_{\gamma,\mu-\gamma} x_{\mu-\gamma}) \\ &= x_\lambda x_\mu + \xi N_{\gamma,\lambda-\gamma} x_{\lambda-\gamma} x_\mu \\ & \quad + \xi N_{\gamma,\mu-\gamma} x_\lambda x_{\mu-\gamma} + \xi^2 N_{\gamma,\lambda-\gamma} N_{\gamma,\mu-\gamma} x_{\lambda-\gamma} x_{\mu-\gamma}. \end{aligned}$$

Altogether, we get 6 summands involving ξ^2 ; the corresponding monomials are of the form $\pm x_{\lambda-\gamma} x_{\mu-\gamma}$, where $(\lambda - \gamma) + (\mu - \gamma) = \lambda + \mu - 2\gamma = \delta - \alpha$. Thus, the 6 pairs of weights of the form $\{\lambda - \gamma, \mu - \gamma\}$ constitute the maximal square

$\Omega(-\alpha)$. It remains to verify that the signs of these summands coincide with the signs in the square equation corresponding to $\Omega(-\alpha)$. Indeed,

$$N_{\rho,-\lambda}N_{\sigma,-\mu}N_{\gamma,\lambda-\gamma}N_{\gamma,\mu-\gamma} = N_{\gamma,\rho-\gamma}N_{\gamma,\sigma-\gamma}N_{\rho-\gamma,-\lambda+\gamma}N_{\sigma-\gamma,-\mu+\gamma}.$$

Finally, 12 summands in the above sum involve ξ ; the corresponding monomials are of the form $\pm x_{\lambda-\gamma}x_{\mu}$ and $\pm x_{\lambda}x_{\mu-\gamma}$, where $(\lambda - \gamma) + \mu = \lambda + (\mu - \gamma) = \delta$. It is easily seen that these are precisely the monomials that occur in g_{α} . It only remains to verify that their signs agree:

$$N_{\rho,-\lambda}N_{\sigma,-\mu}N_{\gamma,\mu-\gamma} = N_{\lambda,\mu-\gamma}.$$

By summarising the above, we get

$$f_{\rho,\sigma}(gx) = f_{\rho,\sigma}(x) + \xi N_{\gamma,\sigma-\gamma}N_{\rho,\sigma-\gamma}g_{\alpha}(x) + \xi^2 N_{\gamma,\rho-\gamma}N_{\gamma,\sigma-\gamma}f_{\rho-\gamma,\sigma-\gamma}.$$

Next, we look at $g_{\alpha}(gx)$. The monomials that occur in g_{α} are of the form $x_{\lambda}x_{\bar{\lambda}}$, where λ runs over the maximal square $\Omega(\alpha)$, whereas $\bar{\lambda} = \delta - \lambda$. Taking the inner product of $\lambda + \bar{\lambda} = \delta$ with γ , we get

$$(\lambda, \gamma) + (\bar{\lambda}, \gamma) = (\delta, \gamma) = 0.$$

Observe that $(\lambda, \alpha) = 1/2$ and $(\bar{\lambda}, \alpha) = -1/2$. The inner product (α, γ) can take the following values:

- $(\alpha, \gamma) = -1$, i.e., $\gamma = -\alpha$. But then $(\lambda, \gamma) = -1/2$, and thus $(gx)_{\lambda} = x_{\lambda}$ for all $\lambda \in \Omega(\alpha)$. It follows that $g_{\alpha}(gx) = g_{\alpha}(x)$.
- $(\alpha, \gamma) = 1$, i.e., $\gamma = \alpha$. Then $(gx)_{\lambda}(gx)_{\bar{\lambda}} = (x_{\lambda} + \xi N_{\gamma,\lambda-\gamma}x_{\lambda-\gamma})x_{\bar{\lambda}}$, and thus

$$g_{\alpha}(gx) = g_{\alpha}(x) + \sum_{\lambda \in \Omega(\alpha)} \xi N_{\lambda,\bar{\lambda}}N_{\gamma,\lambda-\gamma}x_{\lambda-\gamma}x_{\bar{\lambda}}.$$

But $(\lambda - \gamma) + \bar{\lambda} = \delta - \gamma$. Observe that if $\lambda \in \Omega(\alpha)$, then $\lambda^* = \bar{\lambda} + \gamma \in \Omega(\alpha)$; thus, the weights $\{\lambda - \gamma, \bar{\lambda}\}$ form an orthogonal pair of weights and sit in $\Omega(-\alpha)$. Take an arbitrary $\lambda_0 \in \Omega(\alpha)$ and set $\rho = \lambda_0 - \gamma$, $\sigma = \bar{\lambda}_0$. The identity $N_{\lambda,\bar{\lambda}}N_{\gamma,\lambda-\gamma} = N_{\bar{\lambda}+\gamma,\lambda-\gamma}N_{\gamma,\bar{\lambda}}$ implies that the sum on the right-hand side equals $2\xi N_{\rho+\gamma,\sigma}N_{\gamma,\rho}f_{\rho,\sigma}(x)$. Therefore, $g_{\alpha}(gx) = g_{\alpha}(x) + 2\xi N_{\rho+\gamma,\sigma}N_{\gamma,\rho}f_{\rho,\sigma}(x)$.

- $(\alpha, \gamma) = 0$. Let $\lambda \in \Omega(\alpha)$, i.e., $(\lambda, \alpha) = 1/2$. If $(\lambda, \gamma) = 1/2$, then $\lambda - \gamma$ is a weight. Moreover, $(\lambda - \gamma, \alpha) = 1/2$, so that $\lambda - \gamma \in \Omega(\alpha)$. Furthermore, $(\lambda - \gamma, \gamma) = -1/2$, and thus $(\bar{\lambda} - \gamma, \gamma) = 1/2$.

Let us look at what happens with the monomials in g_{α} corresponding to the weights $\lambda, \lambda - \gamma$:

$$\begin{aligned} N_{\lambda,\bar{\lambda}}(gx)_{\lambda}(gx)_{\bar{\lambda}} + N_{\lambda-\gamma,\bar{\lambda}-\gamma}(gx)_{\lambda-\gamma}(gx)_{\bar{\lambda}-\gamma} \\ = N_{\lambda,\bar{\lambda}}x_{\lambda}x_{\bar{\lambda}} + \xi N_{\lambda,\bar{\lambda}}N_{\gamma,\lambda-\gamma}x_{\lambda-\gamma}x_{\bar{\lambda}} \\ + N_{\lambda-\gamma,\bar{\lambda}-\gamma}x_{\lambda-\gamma}x_{\bar{\lambda}-\gamma} + \xi N_{\lambda-\gamma,\bar{\lambda}-\gamma}N_{\gamma,\bar{\lambda}}x_{\lambda-\gamma}x_{\bar{\lambda}}. \end{aligned}$$

But $N_{\lambda,\bar{\lambda}}N_{\gamma,\lambda-\gamma} = -N_{\lambda-\gamma,\bar{\lambda}-\gamma}N_{\gamma,\bar{\lambda}}$, so that the summands involving ξ cancel. This shows that $g_{\alpha}(gx) = g_{\alpha}(x)$.

- $(\alpha, \gamma) = 1/2$. Let $\lambda \in \Omega(\alpha)$. Look at the weight $\lambda^* = \delta + \alpha - \lambda \in \Omega(\alpha)$. We have $\lambda + \lambda^* = \delta + \alpha$, and thus $(\lambda, \gamma) + (\lambda^*, \gamma) = (\alpha, \gamma) = 1/2$. This means that one of the summands on the right-hand side equals $1/2$, while the other equals 0. It follows that for 6 out of the 12 weights $\lambda \in \Omega(\alpha)$ we have $(\lambda, \gamma) = 1/2$. Denote the set of these weights by L . We see that

$$g_{\alpha}(gx) = \sum N_{\lambda,\bar{\lambda}}(gx)_{\lambda}(gx)_{\bar{\lambda}} = \sum_{\lambda \in \Omega(\alpha)} N_{\lambda,\bar{\lambda}}x_{\lambda}x_{\bar{\lambda}} + \sum_{\lambda \in L} \xi N_{\lambda,\bar{\lambda}}N_{\gamma,\lambda-\gamma}x_{\lambda-\gamma}x_{\bar{\lambda}}.$$

Observe that the sum $(\lambda - \gamma) + \bar{\lambda} = \delta - \gamma$ does not depend on λ . This means that the pairs of orthogonal weights $\{\lambda - \gamma, \bar{\lambda}\}$ in the second sum belong to the maximal

square $\Omega(-\gamma)$, and since there are 6 such pairs, they exhaust this square. We fix one such pair $\{\rho, \sigma\}$ and show that up to a sign the second sum equals $\xi f_{\rho, \sigma}(x)$. For this, it remains to notice that $N_{\lambda, \bar{\lambda}} N_{\gamma, \lambda - \gamma} = N_{\rho + \gamma, \sigma} N_{\gamma, \rho} N_{\rho, -\lambda + \gamma} N_{\sigma, -\bar{\lambda}}$. Finally, we get

$$g_\alpha(gx) = g_\alpha(x) + \xi N_{\rho + \gamma, \sigma} N_{\gamma, \rho} f_{\rho, \sigma}(x).$$

- $(\alpha, \gamma) = -1/2$. Observe that $g_{-\alpha} = -g_\alpha$, and thus, replacing α by $-\alpha$, we fall into the above case.

§6. PROOF OF THEOREM 1: AN OUTLINE

First, let f_1, \dots, f_s be arbitrary polynomials in t variables with coefficients in a commutative ring R (in the majority of the real world applications, $R = \mathbb{Z}$ or $R = \mathbb{Z}[1/2]$). We are interested in the linear changes of variables $g \in \text{GL}(t, R)$ that preserve the condition that all these polynomials vanish simultaneously. In other words, we consider all $g \in \text{GL}(t, R)$ preserving the ideal A of the ring $R[x_1, \dots, x_t]$ generated by f_1, \dots, f_s . This last condition means that for any polynomial $f \in A$ the polynomial $f \circ g$ obtained from f by the linear substitution g is again in A . It is well known (see, e.g., [26, Lemma 1] or [48, Proposition 1.4.1]) that the set $G_A(R) = \text{Fix}_R(A) = \text{Fix}_R(f_1, \dots, f_s)$ of all such linear variable changes g forms a group. For any R -algebra S with 1 we can view f_1, \dots, f_s as polynomials with coefficients in S and, thus, the group $G(S)$ is defined for all R -algebras. It is clear that $G(S)$ depends functorially on S . It is easy to provide examples showing that $S \mapsto G(S)$ may fail to be an affine group scheme over R . This is due to the fact that $G_A(R)$ is defined by *congruences*, rather than equations, for its matrix entries. However, in [48, Theorem 1.4.3 and further], a simple sufficient condition was found that guarantees that $S \mapsto G(S)$ is an affine group scheme. Denote by $R[x_1, \dots, x_t]_r$ the submodule of polynomials of degree at most r . For our purposes it suffices to invoke Corollary 1.4.6 of [48], pertaining to the case where $R = \mathbb{Z}$.

Lemma 3. *Let $f_1, \dots, f_s \in \mathbb{Z}[x_1, \dots, x_t]$ be polynomials of degree at most r , and let A be the ideal they generate. Then for the functor $S \mapsto \text{Fix}_S(f_1, \dots, f_s)$ to be an affine group scheme, it suffices that the rank of the intersection $A \cap R[x_1, \dots, x_t]_r$ does not change under reduction modulo any prime $p \in \mathbb{Z}$.*

We apply this lemma to the case of the ideal $A = I$ in $\mathbb{Z}[x_\lambda]$ constructed in §4. For any commutative ring R we set $G_I(R) = \text{Fix}_R(I)$.

Lemma 4. *The functor $R \mapsto G_I(R)$ is an affine group scheme defined over \mathbb{Z} .*

Proof. We show that for any prime p the 133 polynomials generating the ideal I are independent modulo p . Indeed, specializing x_λ appropriately, we can guarantee that one of these polynomials takes value 1, while all other vanish. Observe that the polynomials $f_{\lambda, \mu}$ only involve monomials $x_\nu x_\rho$ for $d(\nu, \rho) = 2$ and $\nu + \rho = \lambda + \mu$, and that for all 126 polynomials of our generating set the sum $\lambda + \mu$ takes distinct values. Furthermore, the polynomials g_α only involve monomials $x_\nu x_\rho$ for $d(\nu, \rho) = 3$. Thus, for $f_{\lambda, \mu}$ is suffices to set $x_\lambda = x_\mu = 1$ and $x_\nu = 0$ for all other weights, the monomial $x_\lambda x_\mu$ only occurs in $f_{\lambda, \mu}$. Finally, for g_i , $i = 1, \dots, 7$, one can set $x_{\lambda_i} = x_{\bar{\lambda}_i} = 1$ and $x_\nu = 0$ for all other ν , where λ_i has the following property: α_i is a unique fundamental root such that the difference $\lambda_i - \alpha_i$ is a weight. As $\lambda_1, \dots, \lambda_7$ we can take, for instance, the weights $\bar{8}, 8, 6, 4, 3, 2, 1$ (Figure 2). \square

To prove the main results of the present paper, we need to recall some further well known facts. The following lemma is Theorem 1.6.1 of [48].

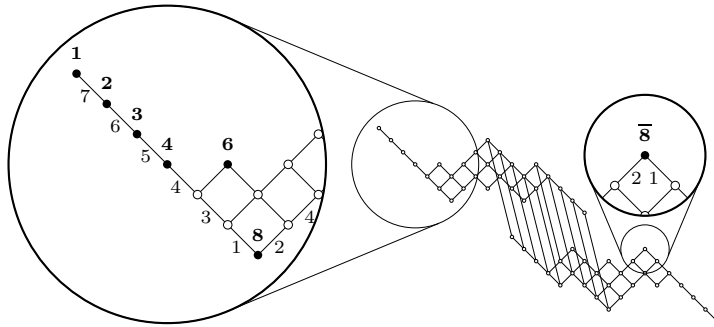


FIGURE 2

Lemma 5. *Let G and H be affine group schemes of finite type over \mathbb{Z} , where G is flat, and let $\phi: G \rightarrow H$ be a morphism of group schemes. Assume that the following conditions are satisfied for an algebraically closed field K :*

- (1) $\dim(G_K) \geq \dim_K(\text{Lie}(H_K))$,
- (2) ϕ induces monomorphisms on the groups of points $G(K) \rightarrow H(K)$ and $G(K[\delta]) \rightarrow H(K[\delta])$,
- (3) the normalizer of $\phi(G^0(K))$ in $H(K)$ is contained in $\phi(G(K))$.

Then ϕ is an isomorphism of group schemes over \mathbb{Z} .

Here G_0 denotes the connected component of the identity in G , G_K denotes the scheme obtained from G by a change of scalars, and $\text{Lie}(H_K)$ denotes the Lie algebra of the scheme H_K . Recall that $K[\delta] = K[x]/(x^2)$ is the algebra of dual numbers over K .

Observe that in our case the preliminary assumptions on the schemes are satisfied automatically. All schemes considered are of finite type, being subschemes of appropriate GL_n . Flatness follows from the fact that G is connected and after the change of base to an algebraically closed field, we will get smooth schemes of the same dimension. Thus, we only need to verify the three conditions of the above lemma.

§7. THE CASE OF AN ALGEBRAICALLY CLOSED FIELD

The following lemma summarizes obvious properties of the minimal representation $\pi: \bar{G}(E_7, -) \rightarrow \text{GL}_{56}$ of the simply connected Chevalley group of type E_7 . The fact that $\pi(\bar{G}(E_7, -))$ is irreducible and tensor indecomposable immediately follows from the fact that π is microweight. Its faithfulness follows from equality of weight lattices $\Lambda(\pi) = P(\Phi)$. The claim about normalizers follows from the classical description of the abstract automorphisms of Chevalley groups over fields (see [42]). Recall that this description asserts that any algebraic automorphism of an extended Chevalley group is the product of an inner automorphism, a central automorphism, and a graph automorphism. The usual Chevalley group may have diagonal automorphisms, but they become inner in the extended group. Modulo the algebraic ones, the only nonalgebraic automorphisms are field automorphisms. Clearly, the groups of type E_7 do not have any nontrivial graph automorphisms.

Lemma 6. *Viewed as a subgroup of $\text{GL}(56, K)$, the algebraic group $\bar{G}(E_7, K)$ is irreducible and tensor indecomposable. Moreover, it is equal to its own normalizer.*

Let us recall the general outline of the proof of the following lemma. It is almost the same as the proof of Lemma 10 in [10], but there is a minor difference, due to the fact

that now the extended Chevalley group of type E_7 is not maximal in $GL(n, K)$, but is contained in the general symplectic group $GSp(56, K)$. In a classical 1952 paper, Eugene Dynkin [15] described the maximal connected closed subgroups of simple algebraic groups over an algebraically closed field of characteristic 0. More precisely, he reduced their description to the representation theory of simple algebraic groups. Relying on earlier results by Seitz himself, and by Donna Testerman, Gary Seitz [41] generalized this description to subgroups of classical algebraic groups over an arbitrary algebraically closed field. Theorem 2 of [41] can be stated as follows. Let V be the vector representation of $SL(V)$, and let X be a proper simple algebraic subgroup of $SL(n, K)$ such that the restriction $V|_X$ of the module V to X is irreducible and tensor indecomposable. Next, let Y be a proper connected closed subgroup of $SL(n, K)$ that strictly contains X . Then either $Y = Sp(V)$ or $Y = SO(V)$, or else the pair (X, Y) is explicitly listed in [41, Table 1].

Lemma 7. *Theorem 1 holds true for any algebraically closed field.*

Proof. It suffices to prove that the connected components of the groups in question coincide. Since $\bar{G}(E_7, K)$ coincides with its normalizer, it will automatically follow that the group $G_I(K)$ is connected. The fact that $\bar{G}(E_7, K)$ stabilizes the requisite system of forms is a consequence of Theorem 4. The reverse inclusion can be established as follows. In Table 1 of [41] the group of type E_7 occurs in the column X four times³ but each time in the embedding $E_7 < C_{28}$. Formally, this only implies the maximality of $G(E_7, K)$ in $Sp(56, K)$, rather than the maximality of $\bar{G}(E_7, K)$ in $GSp(56, K)$. However, since $\det(h_{\varpi_7}(\eta)) = \eta^{-28}$, for every algebraically closed field the determinant of $h_{\varpi_7}(\eta)$ can be arbitrary. Therefore, any connected closed subgroup that properly contains $\bar{G}(E_7, K)$ contains both $Sp(56, K)$ and matrices of an arbitrary determinant, and thus coincides with $GSp(56, K)$. It only remains to observe that the group $GSp(56, K)$ does not preserve the ideal I . Therefore, $\bar{G}(E_7, K)$ is maximal among all such groups, and thus coincides with $G_I(K)$. □

Lemma 8. *Theorem 2 holds true for any algebraically closed field.*

Proof. Everything is completely similar to the proof of Lemma 7, with the only exception that instead of a reference to Theorem 4 one should invoke the results of [19], where it was proved that $E(E_7, R)$ stabilizes the pair (f, h) . Therefore, $G(E_7, K) = E(E_7, K)$ is contained in $G_{(f,h)}(K)$. Moreover, it is easy to show that the elements of the maximal torus $T(E_8, K)$ act as similarities of the pair (f, h) , and thus $\bar{G}(E_7, K)$ is contained in $\bar{G}_{(f,h)}(K)$. □

§8. DIMENSION OF THE LIE ALGEBRA

In the present section we proceed with the proofs of Theorems 1 and 2. Namely, here we prove that the affine group schemes $G_I, G_{(f,h)}, \bar{G}_{(f,h)}$ are smooth. This is one of the key calculations in the present paper.

First, we consider G_I . We should evaluate the dimension of the Lie algebra of this scheme. It is well known how to calculate the Lie algebra that stabilizes a system of forms, see, e.g., [32]. Of course, before the advent of the theory of group schemes in positive characteristic it was not possible to derive any information concerning the group stabilizing the same system of forms. Morally, our calculation faithfully imitates the works by William Waterhouse, especially [48], where similar calculations were performed

³Cases II₆–II₉ in the Dynkin notation [15], no further inclusions arise in positive characteristics.

in Lemmas 3.2, 5.3, and 6.3. Analogous calculations for the cases of polyvector representation of GL_n , and for the microweight representation of E_6 were carried through in [12, 10].

As above, let K be a field. The Lie algebra $\text{Lie}((G_I)_K)$ of an affine group scheme $(G_I)_K$ is most naturally interpreted as the kernel of the homomorphism $G_I(K[\delta]) \rightarrow G_I(K)$ that sends δ to 0, see [1, 20, 46]. Let G be a subscheme of GL_n . Then $\text{Lie}(G_K)$ consists of all matrices of the form $e + z\delta$, where $z \in M(n, K)$, that satisfy the equations defining $G(K)$. In the next lemma we specialize this statement in the case where G is the stabilizer of a system of polynomials.

Lemma 9. *Let $f_1, \dots, f_s \in K[x_1, \dots, x_t]$. Then a matrix $e + z\delta$, where $z \in M(t, K)$, belongs to $\text{Lie}(\text{Fix}_K(f_1, \dots, f_s))$ if and only if*

$$\sum_{1 \leq i, j \leq t} z_{ij} x_i \frac{\partial f_h}{\partial x_j} = 0$$

for all $h = 1, \dots, s$.

The following result is proved in exactly the same way as Lemma 5.3 in [48], and as Theorem 4 in [10]. Clearly, the dimension that arises in this proof is the dimension of the Lie algebra of type E_7 increased by 1. As also in [10], from the proof it will be clear which of the coefficients $z_{\lambda\mu}$ correspond to roots, and which correspond to the Cartan subalgebra. The extra 1 is accounted for by an additional toral summand, because the Lie algebra we consider is in fact the Lie algebra of the *extended* Chevalley group, whose dimension exceeds dimension of the usual Chevalley group by 1.

Theorem 5. *For any field K , the dimension of the Lie algebra $\text{Lie}(G_I(K))$ does not exceed 134.*

Proof. Our equations take the form

$$\sum_{\lambda, \mu} z_{\lambda\mu} x_\lambda \frac{\partial f_{\rho, \sigma}}{\partial x_\mu} = 0, \quad \rho, \sigma \in \Lambda, \quad \rho \perp \sigma;$$

$$\sum_{\lambda, \mu} z_{\lambda\mu} x_\lambda \frac{\partial g_\alpha}{\partial x_\mu} = 0, \quad \alpha \in \Phi.$$

Recall that the partial derivatives look like this:

$$\frac{\partial f_{\rho, \sigma}}{\partial x_\mu} = \begin{cases} \pm x_{\rho + \sigma - \mu} & \text{if } \rho + \sigma - \mu \in \Lambda, \\ 0 & \text{otherwise.} \end{cases}$$

$$\frac{\partial g_\alpha}{\partial x_\mu} = \begin{cases} \pm x_{\bar{\mu}} & \text{if } \mu + \alpha \in \Lambda \text{ or } \mu - \alpha \in \Lambda, \\ 0 & \text{otherwise} \end{cases}$$

- If $d(\lambda, \mu) = 3$, then $z_{\lambda\mu} = 0$. Indeed, in this case $\mu = \bar{\lambda}$. There exists a root $\alpha \in \Phi$ such that $\bar{\lambda} + \alpha \in \Lambda$ or $\bar{\lambda} - \alpha \in \Lambda$. Consider the equation corresponding to the polynomial g_α . It involves the monomial $z_{\lambda\bar{\lambda}} x_\lambda \frac{\partial g_\alpha}{\partial x_{\bar{\lambda}}} = \pm z_{\lambda\bar{\lambda}} x_\lambda^2$. However, the monomial x_λ^2 does not occur in any generator of the ideal I . It follows that the coefficient $\pm z_{\lambda\bar{\lambda}}$ must be 0.
- If $d(\lambda, \mu) = 2$, then $z_{\lambda\mu} = 0$. Choose a root $\alpha \in \Phi$ such that $\mu + \alpha \in \Lambda$ or $\mu - \alpha \in \Lambda$. Consider the equation corresponding to the polynomial g_α . It involves the monomial $z_{\lambda\mu} x_\lambda \frac{\partial g_\alpha}{\partial x_\mu} = \pm z_{\lambda\mu} x_\lambda x_{\bar{\mu}}$, where $d(\lambda, \bar{\mu}) = 1$. Thus, the monomial $x_\lambda x_{\bar{\mu}}$ does not occur in any generator of the ideal I . The nonzero summand $\pm z_{\lambda\mu} x_\lambda x_{\bar{\mu}}$ could only possibly cancel with the nonzero summand of the form $z_{\bar{\mu}\bar{\lambda}} x_{\bar{\mu}} \frac{\partial g_\alpha}{\partial x_{\bar{\lambda}}}$. However, varying α one can guarantee that $\bar{\lambda} + \alpha \notin \Lambda$ and

$\bar{\lambda} - \alpha \notin \Lambda$. For instance, by the transitivity of the Weyl groups on the pairs of weights at distance 2, one may assume that $\lambda = \varpi_7$ and $\mu = \bar{\varpi}_7 + \alpha_7$, in which case one can take $\alpha = \alpha_6$. For this choice of α the only summand $\pm z_{\lambda\mu} x_\lambda x_{\bar{\mu}}$ remains the summand involving $x_\lambda x_{\bar{\mu}}$, and thus $z_{\lambda\mu} = 0$.

- If $d(\lambda, \mu) = d(\nu, \rho) = 1$ and $\lambda - \mu = \nu - \rho$, then $z_{\lambda\mu} = \pm z_{\nu\rho}$. First, assume that $(\nu, \rho) \neq (\bar{\mu}, \bar{\lambda})$. In this case, $\lambda \perp \rho$, $\mu \perp \nu$, and $\lambda + \rho = \mu + \nu$. Consider the equation corresponding to the polynomial $f_{\lambda,\rho} = f_{\mu,\nu}$. It involves the monomials $z_{\lambda\mu} x_\lambda \frac{\partial f_{\lambda,\rho}}{\partial x_\mu} = \pm z_{\lambda\mu} x_\lambda x_\nu$ and $z_{\nu\rho} x_\nu \frac{\partial f_{\lambda,\rho}}{\partial x_\rho} = \pm z_{\nu\rho} x_\nu x_\lambda$. However, $d(\lambda, \nu) = 1$, so that the monomial $x_\lambda x_\nu$ does not occur in any generator of the ideal I . This means that these two monomials must sum to 0, so that $z_{\lambda\mu} = \pm z_{\nu\rho}$.
- If $d(\lambda, \mu) = d(\nu, \rho) = 1$ and $\lambda - \mu = \nu - \rho$, then $z_{\rho\rho} = \pm z_{\lambda\lambda} \pm z_{\mu\mu} \pm z_{\nu\nu}$. Indeed, the equation corresponding to the same polynomial $f_{\lambda,\rho} = f_{\mu,\nu}$ as in the preceding item involves the monomials $z_{\lambda\lambda} x_\lambda \frac{\partial f_{\lambda,\rho}}{\partial x_\lambda} = \pm z_{\lambda\lambda} x_\lambda x_\rho$, $z_{\mu\mu} x_\mu \frac{\partial f_{\lambda,\rho}}{\partial x_\mu} = \pm z_{\mu\mu} x_\mu x_\nu$, $z_{\nu\nu} x_\nu \frac{\partial f_{\lambda,\rho}}{\partial x_\nu} = \pm z_{\nu\nu} x_\nu x_\mu$, and $z_{\rho\rho} x_\rho \frac{\partial f_{\lambda,\rho}}{\partial x_\rho} = \pm z_{\rho\rho} x_\rho x_\lambda$. Observe that the monomial $x_\lambda x_\rho$ occurs in exactly one of the generators of I (namely, in $f_{\lambda,\rho}$), and $x_\mu x_\nu$ occurs in the same polynomial with the same coefficient, up to a sign. Equating the coefficients of the above monomials, we see that $z_{\lambda\lambda} \pm z_{\rho\rho} = \pm z_{\mu\mu} \pm z_{\nu\nu}$.

Let us summarize what we have established. The first two items show that the matrix entries $z_{\lambda\mu}$ with $d(\lambda, \mu) \geq 2$ do not contribute to the dimension of the Lie algebra, whereas the entries $z_{\lambda\mu}$ with $d(\lambda, \mu) = 1$ give the contribution equal to the number of roots of Φ , namely, 126. Finally, the last item allows us to express all entries $z_{\lambda\lambda}$ as linear combinations of the entries $z_{\mu\mu}$ for $\mu = \mu_1, \dots, \mu_t$ such that each fundamental root of Φ occurs among the pairwise differences of the weights μ_i . It is easily seen that the smallest number of such weights is 8, and that one can use the weights 1, 2, 3, 4, 5, 6, 7, 8 as such. Figure 3 shows their location in the weight diagram.

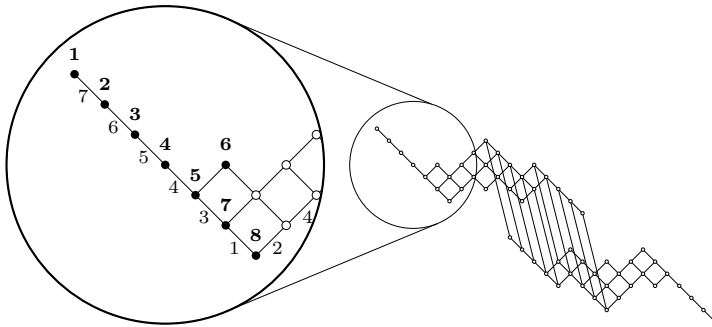


FIGURE 3

Thus, the dimension of the Lie algebra does not exceed $126 + 8 = 134$. □

Next, we pass to the schemes $G_{(f,h)}$ and $\bar{G}_{(f,h)}$. As above, we can identify the Lie algebras $\text{Lie}(G_{(f,h)}(K))$ and $\text{Lie}(\bar{G}_{(f,h)}(K))$ with the kernels of the homomorphisms obtained by specializing δ in the ring of dual numbers $K[\delta]$ to 0. Thus, $\text{Lie}(G_{(f,h)}(K))$ consists of the matrices $g = e + x\delta$, where $x \in M(n, K)$, that satisfy the following conditions: $f(gu, gv, gw, gz) = f(u, v, w, z)$ and $h(gu, gv) = h(u, v)$ for all $u, v, w, z \in V$. Similarly, $\text{Lie}(\bar{G}_{(f,h)}(K))$ consists of all matrices $g = e + x\delta$, where $x \in M(n, K)$, that

satisfy the conditions $f(gu, gv, gw, gz) = \varepsilon(g)f(u, v, w, z)$ and $h(gu, gv) = \varepsilon^2(g)h(u, v)$ for all $u, v, w, z \in V$.

Theorem 6. *For any field K , the dimension of the Lie algebra $\text{Lie}(G_{(f,h)}(K))$ does not exceed 133, while the dimension of the Lie algebra $\text{Lie}(\bar{G}_{(f,h)}(K))$ does not exceed 134.*

Proof. First, observe that the conditions that define elements of the Lie algebra

$$\text{Lie}(G_{(f,h)}(K))$$

are obtained from the corresponding conditions for elements of $\text{Lie}(\bar{G}_{(f,h)}(K))$ by substituting $\varepsilon(g) = \varepsilon'(g) = 1$ and $c_2(g) = c_3(g) = c_4(g) = 0$. Let g be a matrix satisfying the above conditions for all $u, v, w, z \in V$. Plugging in $g = e + x\delta$ and using the fact that the form f is four-linear, whereas the form h is bilinear, we get

$$\begin{aligned} & f(xu, v, w, z) + f(u, xv, w, z) + f(u, v, xw, z) + f(u, v, w, xz) \\ &= (\varepsilon(g) - 1)f(u, v, w, z) + c_2(g)h(u, v)h(w, z) + c_3(g)h(u, w)h(v, z) + c_4(g)h(u, z)h(v, w) \end{aligned}$$

and

$$h(xu, v) + h(u, xv) = (\varepsilon'(g) - 1)h(u, v).$$

Now we show that the entries of the matrix x are subject to exactly the same linear dependences as in the proof of Theorem 5.

- If $d(\lambda, \mu) = 3$, then $x_{\lambda\mu} = 0$. Indeed, in this case $\mu = \bar{\lambda}$. Let $(\lambda, \rho, \sigma, \tau)$ be a nondegenerate quadruple of weights containing λ . Set $u = e_\rho, v = e_\sigma, w = e_\tau, z = e_\mu$. Then $(\rho, \mu) = (\rho, \delta - \lambda) = (\rho, \delta) = 1/2$. Similarly, $(\sigma, \mu) = (\tau, \mu) = 1/2$. This means that μ is not orthogonal and not opposite to any of the weights ρ, σ, τ . Thus, $f(xu, v, w, z) = f(u, xv, w, z) = f(u, v, xw, z) = f(u, v, w, xz) = 0$. It follows that $f(u, v, w, xz) = \pm x_{\lambda\mu} = 0$.
- If $d(\lambda, \mu) = 2$, then $x_{\lambda\mu} = 0$. Let $\{\lambda\}^\perp \subseteq \Lambda$ be the set of weights orthogonal to λ . Observe that $\mu \in \{\lambda\}^\perp$. Moreover, $|\{\lambda\}^\perp| = 27$, and these weights are the weights of the 27-dimensional representation of the Chevalley group of type E_6 . Take in $\{\lambda\}^\perp$ three weight ρ, σ, τ forming a triad (i.e., pairwise orthogonal) in such a way that $\mu \notin \{\rho, \sigma, \tau\}$. Then $(\lambda, \rho, \sigma, \tau)$ is a nondegenerate quadruple of weights. Set $u = e_\rho, v = e_\sigma, w = e_\tau, z = e_\mu$. Observe that μ cannot be orthogonal to more than one of the weights ρ, σ, τ . Indeed, let μ be orthogonal to two of them, say $\mu \perp \rho$ and $\mu \perp \sigma$. Since $\mu \in \{\lambda\}^\perp$, and in $\{\lambda\}^\perp$ there is a unique weight orthogonal to both ρ and σ , namely, τ , it follows that $\mu = \tau$, a contradiction. Thus, $f(xu, v, w, z) = f(u, xv, w, z) = f(u, v, xw, z) = f(u, v, w, xz) = 0$. Therefore, $f(u, v, w, xz) = \pm x_{\lambda\mu} = 0$.
- If $d(\lambda, \mu) = 1$ and $\lambda - \mu = \nu - \rho$, then $x_{\lambda\mu} = \pm x_{\nu\rho}$. By the transitivity of the action of the Weyl group on the pairs of weights at distance 1, we can move the pair (λ, μ) to the pair $(1, 2)$. Then $\nu - \rho = \alpha_7$. In the weight diagram there are exactly 12 edges marked 7, which gives us exactly 12 possibilities for the pair (ν, ρ) . This leaves us with the following three cases to examine:
 - (1) $(\nu, \rho) = (\lambda, \mu)$;
 - (2) $d(\lambda, \rho) = 2$;
 - (3) $(\nu, \rho) = (\bar{\mu}, \bar{\lambda})$.

The first of these cases is trivial. Next, we observe that it suffices to prove that $x_{\lambda\mu} = \pm x_{\nu\rho}$ for the second case, and then to use the chain of identities $x_{\lambda\mu} = \pm x_{\nu\rho} = \pm x_{\bar{\mu}\bar{\lambda}}$ to derive the third case. Thus, we are left with the analysis of the situation where $d(\lambda, \rho) = 2$. There exists a nondegenerate quadruple of weights $(\lambda, \rho, \sigma, \tau)$. Furthermore, we can stipulate that $d(\mu, \sigma) = 2$. Indeed, in the case where $\lambda = \lambda_1, \mu = \lambda_2, \rho = \lambda_{1\bar{6}},$ the choice $\sigma = \lambda_{1\bar{5}}, \tau = \lambda_{1\bar{4}}$ would

do. Now, set $u = e_\mu$, $v = e_\rho$, $w = e_\sigma$, $z = e_\tau$. Then $d(\mu, \rho) = 1$, and thus $f(u, v, xw, z) = f(u, v, w, xz) = f(u, v, w, z) = 0$. It follows that $f(xu, v, w, z) + f(u, xv, w, z) = 0$. But $f(xu, v, w, z) = \pm x_{\lambda\mu}$ and $f(u, xv, w, z) = \pm x_{\nu\rho}$.

- If $d(\lambda, \mu) = 1$ and $\lambda - \mu = \nu - \rho$, then $x_{\lambda\lambda} - x_{\mu\mu} = x_{\nu\nu} - x_{\rho\rho}$. As in the proof of the preceding item, we can limit ourselves to the analysis of the case where $d(\lambda, \rho) = 2$. Again we can find a nondegenerate quadruple of weights $(\lambda, \rho, \sigma, \tau)$ such that $d(\mu, \sigma) = 2$. Setting $u = e_\lambda$, $v = e_\rho$, $w = e_\sigma$, $z = e_\tau$, we get $(x_{\lambda\lambda} + x_{\rho\rho} + x_{\sigma\sigma} + x_{\tau\tau} - \varepsilon(g) + 1)f(e_\lambda, e_\rho, e_\sigma, e_\tau) = 0$. It follows that $x_{\lambda\lambda} + x_{\rho\rho} + x_{\sigma\sigma} + x_{\tau\tau} = \varepsilon(g) - 1$. On the other hand, (μ, ν, σ, τ) is another nondegenerate quadruple. Setting $u = e_\mu$, $v = e_\nu$, $w = e_\sigma$, $z = e_\tau$, we get $(x_{\mu\mu} + x_{\nu\nu} + x_{\sigma\sigma} + x_{\tau\tau} - \varepsilon(g) + 1)f(e_\mu, e_\nu, e_\sigma, e_\tau) = 0$. It follows that $x_{\mu\mu} + x_{\nu\nu} + x_{\sigma\sigma} + x_{\tau\tau} = \varepsilon(g) - 1$. Comparing these expressions, we can conclude that $x_{\lambda\lambda} - x_{\mu\mu} = x_{\nu\nu} - x_{\rho\rho}$.

Thus, as in the proof of Theorem 5, it turns out that the dimension of the Lie algebra $\text{Lie}(\bar{G}_{(f,h)}(K))$ does not exceed $126 + 8 = 134$. The same arguments are also applicable for the case of $\text{Lie}(G_{(f,h)}(K))$. It suffices to set $\varepsilon(g) = \varepsilon'(g) = 1$ and $c_2(g) = c_3(g) = c_4(g) = 0$. Again, we can conclude that the dimension of $\text{Lie}(G_{(f,h)}(K))$ does not exceed 134: the entries $x_{\lambda\mu}$ do not contribute to the dimension when $d(\lambda, \mu) \geq 2$, they make a contribution of 126 when $d(\lambda, \mu) = 1$, and, finally, they make a contribution of at most 8 for $d(\lambda, \mu) = 0$.

To conclude the proof of the theorem, we need to find a nontrivial relation among these last entries. From the final paragraph of the proof of Theorem 5 we know that the entries $z_{\lambda\lambda}$ are linear combinations of 8 of them, namely $z_{\mu\mu}$, for $\mu = \mu_1, \dots, \mu_8$. Here, as μ_1, \dots, μ_8 one can takes the weights 1, 2, 3, 4, 5, 6, 7, 8, respectively. Now, set $u = e_{\mu_1}$, $v = e_{\bar{\mu}_1}$. Plugging these entries into the equation $h(xu, v) + h(u, xv) = 0$, we get $x_{\mu_1\mu_1} + x_{\bar{\mu}_1\bar{\mu}_1} = 0$. One the other hand, $\mu_1 - \bar{\mu}_1 = \delta = (2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7)$, and thus the relations $x_{\lambda\lambda} - x_{\mu\mu} = x_{\nu\nu} - x_{\rho\rho}$ for $\lambda - \mu = \nu - \rho \in \Phi$ imply that

$$\begin{aligned} x_{\mu_1\mu_1} - x_{\bar{\mu}_1\bar{\mu}_1} &= 2(x_{\mu_7\mu_7} - x_{\mu_8\mu_8}) + 3(x_{\mu_5\mu_5} - x_{\mu_6\mu_6}) + 4(x_{\mu_5\mu_5} - x_{\mu_7\mu_7}) \\ &+ 6(x_{\mu_4\mu_4} - x_{\mu_5\mu_5}) + 5(x_{\mu_3\mu_3} - x_{\mu_4\mu_4}) + 4(x_{\mu_2\mu_2} - x_{\mu_3\mu_3}) + 3(x_{\mu_1\mu_1} - x_{\mu_2\mu_2}) \\ &= 3x_{\mu_1\mu_1} + x_{\mu_2\mu_2} + x_{\mu_3\mu_3} + x_{\mu_4\mu_4} + x_{\mu_5\mu_5} - 3x_{\mu_6\mu_6} - 2x_{\mu_7\mu_7} - 2x_{\mu_8\mu_8}. \end{aligned}$$

Comparing this with the relation $x_{\mu_1\mu_1} + x_{\bar{\mu}_1\bar{\mu}_1} = 0$, we see that $x_{\mu_1\mu_1} + x_{\mu_2\mu_2} + x_{\mu_3\mu_3} + x_{\mu_4\mu_4} + x_{\mu_5\mu_5} - 3x_{\mu_6\mu_6} - 2x_{\mu_7\mu_7} - 2x_{\mu_8\mu_8} = 0$. This is precisely the desired nontrivial linear relation among the elements $x_{\mu_i\mu_i}$, which, over a field of any chacteristic, shows that the dimension of our Lie algebra is 1 smaller than the above bound. Thus, $\dim \text{Lie}(G_{(f,h)}(K)) \leq 133$, as claimed. \square

§9. PROOFS OF THEOREMS 1 AND 2

Now we are all set to finish the proofs of our main results.

Proof of Theorem 1. Consider the rational representation of algebraic groups

$$\pi: \bar{G}(\mathbb{E}_7, -) \rightarrow \text{GL}_{56}$$

with the highest weight ϖ_7 . This representation is faithful, and by Theorem 4 its image is contained in G_I . We wish to apply Lemma 5 to this morphism π .

Indeed, for an algebraically closed field K and for $K[\delta]$ the representation π is a monomorphism. This means that condition 2 of Lemma 5 is fulfilled. Clearly,

$$\dim(\bar{G}(\mathbb{E}_7, K)) = 134,$$

and Theorem 5 implies that also $\dim_K(\text{Lie}(G_K)) \leq 134$, so that condition 1 of Lemma 5 follows from the fact that by Lemma 6 already the normalizer of $\bar{G}(\mathbb{E}_7, K)$ in $\text{GL}(56, K)$

is contained in — and in fact coincides with — $G_I(K)$. This means that we can apply Lemma 5 to conclude that π establishes an isomorphism of $\bar{G}(E_7, -)$ and G_I , as affine group schemes over \mathbb{Z} . □

Proof of Theorem 2. Here, again, we can use Lemma 5. The situation is completely similar to the proof of Theorem 1, only that instead of a reference to Theorem 4, one should invoke the main theorem of [19], and instead of Theorem 5 one should use Theorem 6. □

Lemma 10. *In the definition of the group $\bar{G}_{(f,h)}(R)$ always $\varepsilon'(g) = (\varepsilon(g))^2$ and $c_2(g) = c_3(g) = c_4(g) = 0$. In other words,*

$$\bar{G}_{(f,h)}(R) = \{g \in \text{GL}(56, R) \mid \text{there exists } \varepsilon \in R^* \text{ such that}$$

$$f(gu, gv, gw, gz) = \varepsilon f(u, v, w, z) \text{ and } h(gu, gv) = \varepsilon^2 h(u, v) \text{ for all } u, v, w, z \in V\}.$$

Proof. We have already shown that the group $\bar{G}_{(f,h)}(R)$ coincides with the extended Chevalley group $\bar{G}(E_7, R)$. Clearly, both ε and ε' are homomorphisms from $\bar{G}(E_7, -)$ to \mathbb{G}_m , trivial on the commutator subgroup. Thus, their appropriate powers should coincide. Calculating their values on the semisimple element $h_{\varpi_7}(\eta)$, we see that ε takes the value η^2 , whereas ε' takes the value η . Thus, $\varepsilon = (\varepsilon')^2$.

On the other hand, it is easy to check that the c_i satisfy the 1-cocycle identity $c_i(gh) = \varepsilon(g)c_i(h) + c_i(g)\varepsilon(h)$ and vanish on both the commutator subgroup and the semisimple elements of the form $h_{\varpi_7}(\eta)$. As an algebraic group, the extended Chevalley group is generated by these two subgroups, so that the c_i are identically 0. □

§10. PROOF OF THEOREM 3

Proof of Theorem 3. Clearly, $\bar{G}(E_7, R) \leq N(G(E_7, R))$. It is well known (see, e.g., [31] and the references there), that for any irreducible root system Φ of rank strictly larger than 1, and for any commutative ring R , the elementary group $E(\Phi, R)$ is normal in the extended Chevalley group $\bar{G}(\Phi, R)$. Therefore, $\bar{G}(E_7, R) \leq N(E(E_7, R))$. On the other hand, obviously, the two normalizers $N(E(E_7, R))$ and $N(G(E_7, R))$ are contained in the transporter $\text{Tran}(E(E_7, R), G(E_7, R))$. Thus, to finish the proof of the theorem, it suffices to verify that $\text{Tran}(E(E_7, R), G(E_7, R))$ is contained in $\bar{G}(E_7, R)$.

Let $g \in \text{GL}(56, R)$ belong to $\text{Tran}(E(E_7, R), G(E_7, R))$. We pick any root $\alpha \in \Phi$ and any $\xi \in R$. Then $a = gx_\alpha(\xi)g^{-1}$ lies in $G(E_7, R)$, and thus $f(au, av, aw, az) = f(u, v, w, z)$ and $h(au, av) = h(u, v)$ for all $u, v, w, z \in V$. Substituting (gu, gv, gw, gz) for (u, v, w, z) , we get

$$f(gx_\alpha(\xi)u, gx_\alpha(\xi)v, gx_\alpha(\xi)w, gx_\alpha(\xi)z) = f(gu, gv, gw, gz) \quad \text{for all } u, v, w, z \in V.$$

Consider the form $F: V \times V \times V \times V \rightarrow R$ defined by

$$F(u, v, w, z) = f(gu, gv, gw, gz).$$

By our assumption, we have

$$F(x_\alpha(\xi)u, x_\alpha(\xi)v, x_\alpha(\xi)w, x_\alpha(\xi)z) = F(u, v, w, z)$$

for all $u, v, w, z \in V$ and for all $\alpha \in \Phi, \xi \in R$. The root unipotents $x_\alpha(\xi)$ generate the elementary group $E(E_7, R)$. It follows that the form F is invariant under the action of this group. Obviously, the form F is four-linear. Thus, we can apply the main result of [19, Theorem 2] to this form. It says that in this case the form F has the shape

$$F(u, v, w, z) = \varepsilon f(u, v, w, z) + c_2 h(u, v)h(w, z) + c_3 h(u, w)h(v, z) + c_4 h(u, z)h(v, w),$$

for some $\varepsilon, c_2, c_3, c_4 \in R$. Plugging in g^{-1} instead of g , we can conclude that $\varepsilon \in R^*$.

A similar calculation for h shows that $h(gu, gv) = \varepsilon' h(u, v)$ for some $\varepsilon' \in R$. Again, plugging in g^{-1} instead of g , we can conclude that $\varepsilon' \in R^*$.

This shows that g belongs to the group $\bar{G}_{(f,h)}(R)$, which by Theorem 2 coincides with $\bar{G}(E_7, R)$. \square

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