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APPROXIMATION APPROACH TO RAMIFICATION THEORY

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Abstract. A new approach is suggested to the theory of ramification in finite extensions of complete discrete valuation fields in the case of an imperfect residue field. It is based on the notion of a distance between extensions that shows the difference in ramification depths arising after a base change of a certain type. For two-dimensional local fields of prime characteristic, the following is proved. If the distance between two constant extensions (i.e., extensions defined over a given field with perfect residue field) is zero, then the corresponding Hasse–Herbrand functions coincide. The converse is verified only for extensions of degree p.

Introduction

Numerous papers devoted to the ramification theory of complete discrete valuation fields in the case of an imperfect residue field often contain examples illustrating the difference of this case from the classical one. In particular, in §5 of the survey paper [1] (= [2]) there is an example of a compositum of two Artin–Schreier extensions such that the Herbrand property and, consequently, nearly all properties constituting the classical "nice ramification theory" fail for it. A key observation here is as follows. If we have an extension L/K with the Galois group isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$ and we know the ramification number (break) for each subextension M/K of degree p, then in the classical case we can determine the ramification for each L/M out of these data. However, in the nonclassical case this is not so; moreover, in general we cannot determine even the type of L/M (i.e., whether they are wild or ferocious).

These examples show that "difficulties" in the nonclassical case start already at the level of elementary Abelian extensions. But maybe they also finish at this level?

We would be able to answer this question in the positive should be have managed, in a certain sense, to "approximate" an arbitrary finite p-extension L/K of a complete discrete valuation field of prime characteristic p by an elementary Abelian extension L'/K that is ramification-theoretically similar to the original extension (possibly after replacing K by a tame extension). This requirement means that L'/K and L/K must have close ramification invariants, also after any reasonable base change. Moreover, it is natural to expect that an appropriate elementary Abelian extension can be defined over a subfield of K that is a completion of $\mathbb{F}_p(X_1, \dots, X_n)$ for some n and ultimately over an n-dimensional local field $\mathbb{F}_p((X_1))\dots((X_n))$.

In the case of a complete discrete valuation field of characteristic 0 with residue field of characteristic $p > 0$, the ramification numbers of the elementary Abelian extensions are bounded from above and, therefore, we need to work not only with elementary Abelian but also with all Abelian extensions of fields like $\mathbb{Q}_p\{\{X_1\}\}\dots\{\{X_{n-1}\}\}$.

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Implementation of this idea would give way to a certain reduction of the ramification theory to the case of elementary Abelian (respectively, all Abelian) extensions of n-dimensional local fields of simplest type, i.e., to the study of subgroups in K_nF/pK_nF, where

$$F = \mathbb{F}_p((X_1))\dots((X_n)),$$

in the case of prime characteristic, or of subgroups in K_nF, where

$$F = \mathbb{Q}_p\big\{\{X_1\}\big\}\dots\big\{\{X_{n-1}\}\big\}$$

in the mixed characteristic case.

Note that in the classical case the suggested proximity of extensions is reduced to the closeness of their ramification numbers, taking multiplicities into account. The Hasse–Arf theorem allows only integers as the possible values of ramification breaks in upper numbering for Abelian extensions. However, as soon as we allow tame base changes, after an appropriate normalization these breaks are rationals with prime-to-p denominators and an analog of our (anticipated) theory is a simple observation that given positive rationals can be approximated by rationals with prime-to-p denominators.

It seems that in the nonclassical case full information on ramification (including the hidden invariants that reveal under base changes) cannot be expressed in terms of finitely many discrete invariants (see a discussion in [1]). On the other hand, for an extension given by the Artin–Schreier equation $x^p - x = a$, all necessary information is contained in the "principal part" of a, i.e., in $a \mod \deg 0$.

A certain evidence in favor of the idea that ramification invariants "manifest themselves" at the level of elementary Abelian extensions is given by the results of the papers [3, 4] devoted to the study of compositums of given extension with the maximal Abelian extension with a certain bound on ramification.

In the present paper we introduce the notion of the distance between extensions of a given field in the first nontrivial case of 2-dimensional local fields of prime characteristic. We prove an initial important property: if the distance between two constant (i.e., defined over a given subfield with perfect residue field extensions) is zero, then the respective Hasse–Herbrand functions coincide (Proposition 2.3). As yet, the converse is checked only for extensions of degree p (Corollary 2.2.1).

Notation

Everywhere in this paper:
- k is a perfect field of characteristic $p > 0$;
- $\widetilde{k}$ is a certain perfect overfield of k;
- $\lambda \in \widetilde{k}$ is an element transcendent over k.

The expression "$+\dots$" means adding terms of higher order with respect to all preceding terms. Next, $O(a)$ means an arbitrary element of the field under consideration with valuation at least that of a.

§1. Computations in the classical case

In this section we study ramification invariants for compositums of extensions of complete discrete valuation fields with perfect residue fields. The claim of the following lemma, refining Lemma 4.3.1 in [1, 2], is expected to be known, but we have not found it in the literature.

Lemma 1.1. *Let $F = k((\pi))$, and let L_i/F be an extension given by an Artin–Schreier equation*

$$x_i^p - x_i = a_i = \theta_i\pi^{-s_i} + \dots, \quad \theta_i \in k^*, \; s_i > 0, \; p \nmid s_i,$$

for $i = 1, 2$*. Assume that either* $s_1 \neq s_2$ *or* $\theta_1/\theta_2 \notin \mathbb{F}_p^*$*. Denote by* π_1 *any uniformizer in* L_1 *such that* $\pi = \pi_1^p + \dots$*. Then* L_2L_1/L_1 *can be defined by the Artin–Schreier equation*

$$x^p - x = \theta_2' \pi_1^{-s_2'} + \dots,$$

where

$$s_2' = \begin{cases} s_2 & \text{if} \quad s_2 \leq s_1, \\ ps_2 - (p-1)s_1 & \text{if} \quad s_2 > s_1, \end{cases}$$

and

$$\theta_2' = \begin{cases} \theta_2^{p^{-1}} & \text{if} \quad s_2 < s_1, \\ \theta_2^{p^{-1}} - \theta_2 \theta_1^{p^{-1}-1} & \text{if} \quad s_2 = s_1, \\ -s_2 s_1^{-1} \theta_2 \theta_1^{p^{-1}-1}, & \text{if} \quad s_2 > s_1. \end{cases}$$

Proof. The statement is independent of the choice of π_1. Choosing appropriate π_1, we can ensure that

$$\pi = \pi_1^p + \delta, \quad v_1(\delta) > p,\ p \nmid v_1(\delta),$$

where v_1 is the valuation on L_1. The equation

$$x_1^p - x_1 = \theta_1 \pi^{-s_1} + \dots \tag{1}$$

implies immediately

$$x_1 = \theta_1^{p^{-1}} \pi_1^{-s_1} + \dots$$

Substituting this expression in (1) again and identifying the terms with the minimal non-p-divisible exponent of π_1 on both sides, we obtain

$$-\theta_1^{p^{-1}} \pi_1^{-s_1} = -s_1 \theta_1 \pi_1^{p\cdot(-s_1-1)} \cdot \delta + \dots,$$

whence

$$\pi = \pi_1^p + s_1^{-1} \theta_1^{p_1^{-1}-1} \pi_1^{p+(p-1)s_1} + \dots \tag{2}$$

Let

$$a_2 = \sum_{i \geq -s_2} \varepsilon_i \pi^i, \quad \varepsilon_i \in k$$

(in particular, $\varepsilon_{-s_2} = \theta_2$). Now (2) implies

$$a_2 = \sum_{i \geq -s_2} \varepsilon_i \pi_1^{pi} - s_2 s_1^{-1} \theta_2 \theta_1^{p^{-1}-1} \pi_1^{-ps_2+(p-1)s_1} + O\big(\pi_1^{-ps_2+(p-1)s_1+1}\big).$$

It follows that L_1L_2/L_1 can be defined by the Artin–Schreier equation with the right-hand side

$$a_2' = \sum_{i \geq -s_2} \varepsilon_i^{p^{-1}} \pi_1^i - s_2 s_1^{-1} \theta_2 \theta_1^{p^{-1}-1} \pi_1^{-ps_2+(p-1)s_1} + O\big(\pi_1^{-ps_2+(p-1)s_1+1}\big),$$

and it is not difficult to show that

$$a_2' = \begin{cases} \theta_2^{p^{-1}} \pi_1^{-s_2} + \dots & \text{if} \quad s_2 < s_1, \\ (\theta_2^{p^{-1}} - \theta_2 \theta_1^{p^{-1}-1}) \pi_1^{-s_2} + \dots & \text{if} \quad s_2 = s_1, \\ -s_2 s_1^{-1} \theta_2 \theta_1^{p^{-1}-1} \pi_1^{-ps_2+(p-1)s_1} + \dots & \text{if} \quad s_2 > s_1, \end{cases}$$

which proves the claim. □

Remark 1.1.1. Formally, the condition $\theta_1/\theta_2 \notin \mathbb{F}_p^*$ (with $s_1 = s_2$) is not used in the proof. However, it is under this condition that in the second case we have $\theta_2^{p^{-1}} - \theta_2\theta_1^{p^{-1}-1} \neq 0$. Thus, in either case the valuation of a_2' is negative and not a multiple of p, and this allows us to determine the ramification number of L_1L_2/L_1. It is easily seen that in the case where $\theta_1/\theta_2 \in \mathbb{F}_p^*$ the ramification number cannot be determined (see the example at the beginning of the next section).

Remark 1.1.2. A similar statement can easily be proved also in the mixed characteristic case.

If any of the conditions $\theta_1/\theta_2 \notin \mathbb{F}_p^*$ or $s_1 \neq s_2$, is satisfied, we shall say that the extensions L_1/K and L_2/K are *not in touch*. This notion extends easily to p-extensions of arbitrary degree.

Let $F = k((\pi))$. We say that finite totally ramified Galois p-extensions L_1/F and L_2/F are *not in touch* if for any intermediate fields $F \subset S_i \subset T_i \subset L_i$, where S_i/F is normal and T_i/S_i is a Galois extension of degree p $(i = 1, 2)$, the extensions T_1S_2/S_1S_2 and S_1T_2/S_1S_2 are not in touch.

Corollary 1.1.3. *Let $F = \widetilde{k}((\pi))$, let E/F and L/F be finite totally ramified Galois p-extensions, and let E/F (respectively, L/F) be obtained by adjoining an element algebraic over $k((\pi))$ (respectively, over $k((\lambda\pi))$). Then E/F and L/F are not in touch.*

Proof. Fix towers $F \subset S_1 \subset T_1 \subset E$ and $F \subset S_2 \subset T_2 \subset L$, where S_i/F are normal and T_i/S_i is a Galois extension of degree p $(i = 1, 2)$. Consider two arbitrary towers

$$F = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

and

$$F = L_0 \subset L_1 \subset \cdots \subset L_m = L,$$

where all E_{i+1}/E_i L_{j+1}/L_j are Galois extensions of degree p, and S_1, T_1 are among the E_i, whereas S_2, T_2 are among the L_j. Put $E_{ij} = E_iL_j$ and use induction on $i + j = n$ to check the following statement.

The extension $E_{i+1,j}/E_{ij}$ (respectively, $E_{i,j+1}/E_{ij}$) is obtained by adjoining a root of $x^p - x = a$, where $v_{E_{ij}}(a) = -s < 0$, $p \nmid s$, and the reduction of $\pi^s a^{p^n}$ is of type $f_{ij}(\lambda)$ (respectively, $g_{ij}(\lambda)$), where $f_{ij}, g_{ij} \in k(X)$, with $v_X(f_{ij}) \geq 0$, $v_X(g_{ij}) < 0$, where v_X is the X-adic valuation on $k(X)$.

The last condition implies immediately that $f_{ij}(\lambda)/g_{ij}(\lambda) \notin \mathbb{F}_p^*$, so that $E_{i+1,j}/E_{ij}$ and $E_{i,j+1}/E_{ij}$ are not in touch.

As the base of induction, we consider $E_{i+1,0}/E_{i0}$ and $E_{0,j+1}/E_{0j}$. Choose uniformizers π_i in E_{i0} to be algebraic over $k((\pi))$; we may assume that

$$\pi = \pi_i^{p^i} + \dots$$

Then $E_{i+1,0}/E_{i0}$ is given by an Artin–Schreier equation with right-hand side in $k((\pi_i))$, and for f_{i0} one can take an element of k^*. As for the extensions E_{0j}, we choose uniformizers ρ_j algebraic over $k((\lambda\pi))$ there; we may assume that

$$\lambda\pi = \rho_j^{p^j} + \dots$$

Note that $E_{0,j+1}/E_{0j}$ can be given by $x^p - x = a$ with $a \in k((\rho_j))$, i.e., $a = c\rho_j^{-s} + \dots$, $s > 0$, $p \nmid s$, $c \in k^*$. Then the reduction of $\pi^s a^{p^j}$ is of type $c^{p^j}\lambda^{-s}$, i.e., $c^{p^j}X^{-s}$ fits as g_{0j}.

For the step of induction, we apply Lemma 1.1 (preserving the notation from the statement of the lemma) to $E_{i+1,j}/E_{ij}$ and $E_{i,j+1}/E_{ij}$; by π_{ij} we denote a uniformizer of E_{ij} such that $\pi = \pi_{ij}^{p^n} + \dots$

By the inductive assumption, $E_{i+1,j}/E_{ij}$ and $E_{i,j+1}/E_{ij}$ are given by the Artin–Schreier equation

$$x_\nu^p - x_\nu = \theta_\nu \pi_{ij}^{-s_\nu} + \dots, \quad \theta_\nu \in k^*, \ s_\nu > 0, \ p \nmid s_\nu$$

for $\nu = 1, 2$, and $\theta_1^{p^n} = f_{ij}(\lambda)$, $\theta_2^{p^n} = g_{ij}(\lambda)$, $v_X(f_{ij}) \geq 0$, $v_X(g_{ij}) < 0$. By Lemma 1.1, the extension $E_{i+1,j+1}/E_{i,j+1}$ can be given by an equation of the form

$$x^p - x = \theta_2' \pi_{i+1,j}^{-s_2'} + \dots,$$

where $\pi_{ij} = \pi_{i+1,j}^p + \dots$, and

$$\theta_2' = \alpha \theta_2^{p^{-1}} - \beta \theta_2 \theta_1^{p^{-1}-1},$$

where α, β belong to $\mathbb{F}_p$ and do not vanish simultaneously. Raising to the power p^{n+1}, we obtain

$$\begin{aligned}(\theta_2')^{p^{n+1}} &= \alpha \theta_2^{p^n} - \beta \theta_2^{p^{n+1}} \theta_1^{p^n(1-p)} \\ &= \alpha g_{ij}(\lambda) - \beta g_{ij}(\lambda)^p f_{ij}(\lambda)^{1-p} =: g_{i,j+1}(\lambda).\end{aligned}$$

It is easily seen that $v_X(g_{ij}^p f_{ij}^{1-p}) < v_X(g_{ij}) < 0$, whence $v_X(g_{i,j+1}) < 0$.

Similarly, $E_{i+1,j+1}/E_{i+1,j}$ can be defined by the equation

$$x^p - x = \theta_1' \pi_{i,j+1}^{-s_1'} + \dots$$

with $\pi_{ij} = \pi_{i,j+1}^p + \dots$, and

$$\theta_1' = \alpha' \theta_1^{p^{-1}} - \beta' \theta_1 \theta_2^{p^{-1}-1},$$

where α', β' belong to $\mathbb{F}_p$ and do not vanish simultaneously. Raising to the power p^{n+1}, we obtain

$$(\theta_1')^{p^{n+1}} = \alpha' \theta_1^{p^n} - \beta' \theta_1^{p^{n+1}} \theta_2^{p^n(1-p)} = \alpha' f_{ij}(\lambda) - \beta' f_{ij}(\lambda)^p g_{ij}(\lambda)^{1-p} =: f_{i+1,j}(\lambda),$$

and $v_X(f_{i+1,j}) \geq 0$. □

Up to the end of this section we shall assume that $F = k((\pi))$.

Let L/F be a finite Galois extension of degree p^n. We define its ith ramification break in upper numbering as

$$h_i = h_i(L/F) = \min \{ j_0 \, : \, |G(L/K)^j| \leq p^{n-i} \text{ for any } j > j_0 \}.$$

In other words, $h_1 \leq \dots \leq h_n$ are the usual ramification breaks in upper numbering, and a break of multiplicity m (cf. [1, 2.1]) is repeated m times in the sequence.

Observe the following obvious fact.

Lemma 1.2. *Let $h_1 \leq \dots \leq h_n$ be all ramification breaks of E/F in upper numbering. Then there exists a tower of intermediate fields*

$$F = E_0 \subset E_1 \subset \dots \subset E_n = E$$

such that for any $i \leq n-1$ the ramification breaks of E_i/F are equal to $h_1, \dots, h_i$, any E_{i+1}/E_i is a Galois extension of degree p, and a unique ramification break of E_{i+1}/E_i is equal to

$$H_{i+1} = \psi_{E_i/F}(h_{i+1}) = h_1 + p(h_2 - h_1) + \dots + p^i(h_{i+1} - h_i).$$

In this situation

$$d_F(E_{i+1}/E_i) = p^{-i} d_{E_i}(E_{i+1}/E_i) = \frac{p-1}{p^{i+1}}\big(h_1 + p(h_2 - h_1) + \cdots + p^i(h_{i+1} - h_i)\big),$$

and

$$\begin{aligned} d_F(E/F) &= d_F(E_1/E_0) + \cdots + d_F(E_n/E_{n-1}) \\ &= (p-1)\left(\left(\sum_{i=1}^{n} \frac{1}{p^i}\right)h_1 + \left(\sum_{i=1}^{n-1} \frac{1}{p^i}\right)(h_2 - h_1) + \cdots + \frac{1}{p}(h_n - h_{n-1})\right), \end{aligned}$$

whence

$$d_F(E/F) = (p-1)\left(\frac{h_1}{p^n} + \frac{h_2}{p^{n-1}} + \cdots + \frac{h_n}{p}\right). \tag{3}$$

The next proposition is somewhat stronger than [1, 4.3.2] for the situation under study.

Proposition 1.3. *Let L/F and T/F be finite Galois p-extensions that are not in touch, and let $[L : F] = p^n$; $h_i = h_i(L/F)$, $i = 1, \dots, n$. Then*

$$h_i(LT/T) = \psi_{T/F}(h_i), \quad i = 1, \dots, n.$$

Proof. This follows immediately from Lemma 1.1 by using induction on the degree of each of the extensions. □

§2. Distance between constant extensions

In this section the following notation will also be used.

- $K = k((t))((\pi_0))$;
- $\widetilde{K} = \widetilde{k}((t))((\pi_0))$;
- f_λ is a continuous automorphism of $\widetilde{K}$ that is trivial on $\widetilde{k}((t))$ and takes π_0 to $\lambda\pi_0$;
- $K_\lambda = f_\lambda(K)$.

Now we define the main object of study, a "distance" between finite extensions of a given field.

Definition 2.1. Let L_1/K, L_2/K be finite extensions. Put

$$D(L_1/K, L_2/K) = \frac{p}{p-1} \max_T(|d_K(L_1T/T) - d_K(L_2T/T)|),$$

where T runs over the finite totally ramified extensions of K_λ; here all compositums are taken inside a fixed algebraic closure of $\widetilde{K}$.

Remark 2.1.1. Should we have taken arbitrary finite extensions of K (rather than of K_λ) as T, this would have resulted in an inefficient definition, because some extensions of K are not "in general position" with respect to L_1/K or L_2/K.

For example, let a, b, c be positive integers that are prime to p, and let $a < b < c$. Denote by L_1, L_2, and T the extensions of K given by the following Artin–Schreier equations:

$$\begin{aligned} L_1/K &: x^p - x = \pi_0^{-c} + \pi_0^{-a}; \\ L_2/K &: x^p - x = \pi_0^{-c} + \pi_0^{-b}; \\ T/K &: x^p - x = \pi_0^{-c}. \end{aligned}$$

It is easily seen that $L_1T/T = L_1'T/T$, where L_1'/K is defined by an Artin–Schreier equation with the right-hand side π_0^{-a}. Then by Lemma 1.1 the extension L_1T/T can be defined by an Artin–Schreier equation with the right-hand side of the type $\pi^{-a} + \dots$, where π is a certain uniformizer in T. Similarly, L_2T/T can be defined by an Artin–Schreier equation with the right-hand side of the type $\pi^{-b}+\dots$. Thus,

$$\frac{p}{p-1}\Big|d_K(L_1T/T)-d_K(L_2T/T)\Big| = \frac{p}{p-1}\left|\frac{p-1}{pe(T/K)}a-\frac{p-1}{pe(T/K)}b\right| = \frac{1}{p}(b-a);$$

where we would like to treat L_1/K and L_2/K as having equal ramification and, accordingly, we would like the distance between them to be zero.

Remark 2.1.2. Consider a similar example where a, b, c are as above with $c/p < a$, and two extensions are given

$$L_1/K\ :\ x^p - x = \pi_0^{-c} + \pi_0^{-a}t;$$
$$L_2/K\ :\ x^p - x = \pi_0^{-c} + \pi_0^{-b}t.$$

In this case, as it is seen from an example in [1, §5], the extensions L_1/K and L_2/K, after certain base changes (even if imposing "general position" conditions), have distinct ramification invariants; we do not tend to treat such extensions as equally ramified. Indeed, if we define T/K_λ by $x^p - x = (\lambda\pi_0)^{-c}$, we see that L_1T/T and L_2T/T are defined by Artin–Schreier equations with right-hand sides of the type $\pi^{-pa}t+\dots$ and $\pi^{-pb}t+\dots$, respectively, where π is a certain prime in T. Therefore,

$$\begin{aligned} D(L_1/K, D_2/K) &\ge \frac{p}{p-1}\Big|d_K(L_1T/T)-d_K(L_2T/T)\Big| \\ &= \frac{p}{p-1}\left|\frac{p-1}{pe(T/K)}pa-\frac{p-1}{pe(T/K)}pb\right| \\ &= b-a>0. \end{aligned}$$

The difference between L_1/K and L_2/K can be also undisguised when using the "cutting-by-curves" method as in Example 9.1.1 of [1].

An extension L/K is said to be *constant* if there is an algebraic extension $E/k((\pi_0))$ such that $L = EK$. (This definition is compatible with that in [5, 6], if one takes $\mathbb{F}_p((\pi_0))$ as a ground subfield in K.)

We shall show that for two constant cyclic extensions of degree p with the same ramification number, their ramification invariants cannot be distinguished even after a "nonclassical" base change.

Proposition 2.2. *Let $F = k((\pi_0)) \subset K$. Fix a finite totally ramified extension T_λ/K_λ and put $\widetilde{T} = T_\lambda\widetilde{K}$.*

1. *Let L/F be a totally ramified cyclic extension of degree p. Then $d_K(L\widetilde{T}/\widetilde{T})$ depends only on $d_F(L/F)$.*

2. *Let $\delta(d) = \delta_{T_\lambda}(d)$ be the common value of $d_K(L\widetilde{T}/\widetilde{T})$ for $d_F(L/F) = d$. Then δ is a restriction of a strictly monotone increasing piecewise linear function with slopes at most* 1.

Proof. Denote by T/K an extension isomorphic to T_λ/K_λ; the extension of f_λ up to an isomorphism of T onto T_λ will be denoted also by f_λ. By π' and t' we denote some local parameters in T, i.e., $T = k((t'))((\pi'))$.

Then π_0 expands into a series

$$\pi_0 = \sum_{i>0,\, j\in\mathbb{Z}} \theta_{ij}(\pi')^i(t')^j, \quad \theta_{ij}\in k,$$

whence

$$\lambda\pi_0 = \sum_{i>0,\, j\in\mathbb{Z}} \theta_{ij}\pi^i t^j, \tag{4}$$

where $\pi = f_\lambda(\pi')$, $t = f_\lambda(t')$.

Let $v_T(\pi_0) = (p^n\beta, p^n\alpha)$, where $p \nmid (\alpha, \beta)$. Put $\lambda = \mu^{p^n}$ and rewrite (4) as

$$\pi_0 = \prod_{j=n}^{0} \varphi_j(\pi^{p^j}, t^{p^j}),$$

where

$$\varphi_n(X, Y) = \mu^{-p^n}\theta_n^{p^n} X^\alpha Y^\beta + \dots, \quad \theta_n \in k^*,$$

and for $0 \le j \le n-1$ we have either $\varphi_j = 1$, or

$$\varphi_j(X, Y) = 1 + \theta_j^{p^j} X^{\alpha_j} Y^{\beta_j} + \dots, \quad \theta_j \in k^*, \quad p \nmid (\alpha_j, \beta_j).$$

First, we assume that L/F is defined by an equation of the form

$$x^p - x = \theta\pi_0^{-m},$$

where $\theta \in k^*$, $m > 0$, $(m, p) = 1$.

The extension $L\widetilde{T}/\widetilde{T}$ is defined by adjoining a root of

$$x^p - x = \theta \prod_{j=n}^{0} \varphi_j(\pi^{p^j}, t^{p^j})^{-m} = b_n^{p^n} + b_{n-1}^{p^{n-1}} + \dots + b_0,$$

where $b_n = \mu^{-1}\theta_n\pi^{-m\alpha}t^{-m\beta} + \dots$, and

$$b_j = \begin{cases} 0, & \text{if } \varphi_j = 1, \\ -\mu^{-p^{n-j}}\theta_n^{p^{n-j}}\theta_j\pi^{-mp^{n-j}\alpha+\alpha_j}t^{-mp^{n-j}\beta+\beta_j} + \dots & \text{if } \varphi_j \ne 1, \end{cases}$$

for $j = n-1, \dots, 0$.

Obviously, $L\widetilde{T}/\widetilde{T}$ can also be defined by

$$x^p - x = b_n + b_{n-1} + \dots + b_0 =: B.$$

Assuming that $\alpha_n = \beta_n = 0$, put

$$(-\widetilde{\beta}_m, -\widetilde{\alpha}_m) = \min\big\{(-mp^{n-j}\beta + \beta_j, -mp^{n-j}\alpha + \alpha_j) \,|\, 0 \le j \le n, b_j \ne 0\big\}.$$

Let $j_1, \dots, j_s$ be all values of j such that

$$(-\widetilde{\beta}_m, -\widetilde{\alpha}_m) = (-mp^{n-j}\beta + \beta_j, -mp^{n-j}\alpha + \alpha_j).$$

Then the coefficient of $\pi^{\tilde{\alpha}}t^{\tilde{\beta}}$ in B is equal to

$$\sum_{\tau=1}^{s} \varepsilon_{j_\tau}\mu^{-p^{n-j_\tau}}, \text{ where } \varepsilon_j = \begin{cases} \theta_n & \text{if } j = n, \\ -m\theta_n^{p^{n-j}}\theta_j & \text{if } j < n. \end{cases}$$

Note that $\varepsilon_{j_\tau} \in k^*$, $\tau = 1, \dots, s$, whence

$$\sum_{\tau=1}^{s} \varepsilon_{j_\tau}\mu^{-p^{n-j_\tau}} \ne 0,$$

because μ is transcendent over k. Therefore, $v_{\widetilde{T}}(B) = -\widetilde{\alpha}_m$, and

$$d_K(L\widetilde{T}/\widetilde{T}) = \frac{p-1}{p} e_{\widetilde{T}/K}^{-1} v_T(B) = \frac{p-1}{p} p^{-n}\alpha^{-1} \cdot \widetilde{\alpha}_m.$$

Now consider the general case where L/F is defined by

$$x^p - x = \theta^{(m)}\pi_0^{-m} + \theta^{(m-1)}\pi_0^{-m+1} + \dots,$$

where $m > 0$, $\theta^{(i)} \in k$ $(i = m, \dots, 1)$, $\theta^{(i)} = 0$, $(i, p) \ne 1$, $\theta^{(m)} \ne 0$. Then, in view of the above computation, $L\widetilde{T}/\widetilde{T}$ can be defined by the equation

$$x^p - x = B^{(m)} + \dots + B^{(1)},$$

where $B^{(i)} = 0$ for $\theta^{(i)} = 0$, and $v_{\widetilde{T}}(B^{(i)}) = -\widetilde{\alpha}_i$ otherwise. Since $\widetilde{\alpha}_i$ strictly increases with i, it follows that in the general case we have

$$d_K(L\widetilde{T}/\widetilde{T}) = \frac{p-1}{p} p^{-n} \alpha^{-1} \cdot \widetilde{\alpha}_m,$$

i.e., $d_K(L\widetilde{T}/\widetilde{T})$ depends only on m and has other properties from the Proposition. □

Corollary 2.2.1. *Let L_1/K and L_2/K be constant Galois extensions of degree p with equal ramification number, i.e., $d_K(L_1/K) = d_K(L_2/K)$. Then $D(L_1/K, L_2/K) = 0$.*

Proposition 2.3. *Let $F = k((\pi_0))$, and let $E/k((\pi_0))$, $E'/k((\pi_0))$ be totally ramified Galois extensions of the same degree p^n such that*

$$D(EK/K, E'K/K) = 0.$$

Then the Hasse–Herbrand functions $\psi_{E/F}$ and $\psi_{E'/F}$ coincide.

Proof. Let $h_1 \leq \cdots \leq h_n$ (respectively, $h'_1 \leq \cdots \leq h'_n$) be all ramification breaks of E/F (respectively, of E'/F) in upper numbering. For a positive integer c prime to p, denote by T_c/K_λ an arbitrary constant Galois extension of degree p with ramification number c, i.e., an extension defined by the Artin–Schreier equation with the right-hand side $(\lambda\pi)^{-c}$. By Proposition 1.3, we have

$$h_i(ET_c/T_c) = \begin{cases} h_i & \text{if} \quad h_i \leq c, \\ c + p(h_i - c) & \text{if} \quad h_i > c. \end{cases}$$

Formula (3) implies

$$d_{T_c}(ET_c/T_c) = \frac{h_1}{p^n} + \cdots + \frac{h_m}{p^{n-m+1}} + \frac{c + p(h_{m+1} - c)}{p^{n-m}} + \cdots + \frac{c + p(h_n - c)}{p},$$

where m is defined as the maximal integer such that $c \geq h_m$.

First, we assume that $h_1, \ldots, h_n$ and $h'_1, \ldots, h'_n$ are integral and prime to p.

Suppose that $\psi_{E/F} \neq \psi_{E'/F}$, i.e., $h_j \neq h'_j$ for some j; take the maximal j with this property. Without loss of generality we may assume that $h_j < h'_j$.

Since $D(EK/K, E'K/K) = 0$, we have $d_{T_c}(ET_c/T_c) = d_{T_c}(E'T_c/T_c)$ for all c. In particular, for $c = h'_j$ this yields

$$\frac{h_1}{p^n} + \cdots + \frac{h_j}{p^{n-j+1}} = \frac{h'_1}{p^n} + \cdots + \frac{h'_j}{p^{n-j+1}}, \tag{5}$$

and for $c = h_j$ we obtain

$$\frac{h_1}{p^n} + \cdots + \frac{h_j}{p^{n-j+1}} = \frac{h'_1}{p^n} + \cdots + \frac{h'_m}{p^{n-m+1}} + \frac{c + p(h'_{m+1} - c)}{p^{n-m}} + \cdots + \frac{c + p(h'_j - c)}{p^{n-j}}, \tag{6}$$

where m is maximal with the property $h'_m \leq h_j$. The last $j - m > 0$ terms on the right-hand side of (6) are larger than the corresponding terms on the right-hand side of (5), whereas the left-hand side in these relations are the same, a contradiction.

Finally, the general case can be reduced to that already considered if we replace F by its constant extension of degree p with ramification number $c = \min(h_1, h'_1)$ several times. Doing this, we first make all h_i and h'_i integral and at the last step we make them congruent to c modulo p and, therefore, prime to p. □

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