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ON A NEW TYPE OF ℓ-ADIC REGULATOR FOR ALGEBRAIC NUMBER FIELDS. II

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Dedicated to S. V. Vostokov on the occasion of his 70th anniversary.

Abstract. In a preceding paper of the author, a new type of an ℓ-adic regulator $\mathfrak{R}_\ell(K)$ was introduced for an algebraic number field K such that the prime ℓ splits completely in K. Nevertheless, the element $\mathfrak{R}_\ell(K) \in \mathbb{Z}_\ell$ is defined only up to an arbitrary factor in $(\mathbb{Z}_\ell^\times)^2$. In the present paper, under the assumption of the validity of the Shanuel conjecture (both Archimedean and ℓ-adic), the definition of $\mathfrak{R}_\ell(K)$ as a certain number in $\mathbb{Z}_\ell$ is given. For a real quadratic field K, such a defition can be obtained without using any additional conjectures.

§1. Introduction

Let K be an algebraic number field and ℓ a fixed prime number. Let $U(K)$ be the group of units of K. For the field K, there is a classical (Archimedean) regulator, which will be denoted by $R_\infty(K)$ in this paper (see [1, Chapter 2]). This regulator is defined in terms of embedding of the group $U(K)$ into the logarithmic space. Similarly, one can define the Leopoldt ℓ-adic regulator $\mathbf{R}_\ell(K)$ (see [2]). Unfortunately, this regulator is well defined only for totally real K.

As is well known, $R_\infty(K) \neq 0$ for any algebraic number field K. It is conjectured that $\mathbf{R}_\ell(K)$ is also nonzero for any prime ℓ and any totally real K. Herein, the relation $\mathbf{R}_\ell(K) \neq 0$ is equivalent to the validity of the Leopoldt conjecture.

In [3], the author proposed a new definition of an ℓ-adic regulator, which makes sense for any algebraic number field. (We shall denote this regulator by $R_\ell(K)$.) This new regulator turned out to be useful and yielded some interesting consequences even in the case of totally real fields [5].

Note that $R_\infty(K)$ is a uniquely defined positive real number. The Leopoldt regulator $\mathbf{R}_\ell(K)$ is an element of the algebraic closure $\bar{\mathbb{Q}}_\ell$ of the rational ℓ-adic number field $\mathbb{Q}_\ell$ and is defined uniquely up to a sign. The regulator $R_\ell(K)$ is a uniquely defined element of the field $\mathbb{Q}_\ell$. (This fact combined with [5, Proposition 2.1] shows that $\mathbf{R}_\ell(K)$ belongs to some quadratic extension of $\mathbb{Q}_\ell$ depending on K.) Note that the regulators $R_\infty(K), \mathbf{R}_\ell(K)$, and $R_\ell(K)$ are defined in terms of some fundamental system of units $\varepsilon_1, \dots, \varepsilon_r$ of K, where r is the rank of $U(K)$. If $\varepsilon_1, \dots, \varepsilon_r$ and $\varepsilon_1', \dots, \varepsilon_r'$ are two such systems linked by some transition matrix $C \in \mathrm{GL}(r, \mathbb{Z})$, then $\det C = \pm 1$, and the regulators $R_\infty(K)$ defined by these two systems differ by multiplication by $|\det C| = 1$, the regulators $\mathbf{R}_\ell(K)$ differ by multiplication by $\det(C) = \pm 1$, and, finally, the regulators $R_\ell(K)$ differ by multiplication by $\det(C)^2 = 1$.

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In [4], the author introduced a new ℓ-adic regulator $\mathfrak{R}_\ell(K)$, which is suitable for applications related to the Iwasawa theory. Let K_∞/K be the cyclotomic $\mathbb{Z}_\ell$-extension of K, $K_\infty = \cup_n K_n$, where $[K_n : K] = \ell^n$, and $U_S(K_n)[\ell]$ the pro-ℓ-completion of the group $U_S(K_n)$ of S-units of K_n, where S is the set of all places over ℓ in the field under consideration. Let $U_{S,1}(K)$ be the subgroup of global universal norms in $U_S(K)[\ell]/\mu_\ell(K)$, where $\mu_\ell(K)$ is the group of all roots of unity of ℓ-power degree in K, that is,

$$U_{S,1}(K) = \bigcap_{n=1}^{\infty} \big(N_{K_n/K}(U_S(K_n)[\ell]/\mu_\ell(K_n))\big).$$

The regulator $\mathfrak{R}_\ell(K)$, which will be defined in more detail in the next section, characterizes the $\mathbb{Z}_\ell$-module $U_{S,1}(K)$ and is defined in terms of a $\mathbb{Z}_\ell$-basis of that module. Thus, the element $\mathfrak{R}_\ell(K) \in \mathbb{Z}_\ell$ is defined up to multiplication by some element c^2, where $c \in \mathbb{Z}_\ell^\times$. The following question[1] arises: Can the definition of this regulator be refined so as to obtain $\mathfrak{R}_\ell(K)$ as a uniquely defined number, as in the case of the regulator $R_\ell(K)$?

Our goal here is to answer this question in the positive under the assumption that the Shanuel conjecture holds true (both ℓ-adic and Archimedean; see Conjectures 1 and 2 in §2). First, we assume K to be normal over $\mathbb{Q}$. In the definition of $\mathfrak{R}_\ell(K)$ in [4] we used an element $\alpha \in K^\times$ such that $(\alpha) = \mathfrak{l}_1^{\mathbf{h}}$, where $\mathfrak{l}_1$ is a fixed prime divisor of ℓ in K and $\mathbf{h}$ is the order of $\mathfrak{l}_1$ in the class group. Such an element is defined up to multiplication by an arbitrary unit of $U(K)$. So, the first problem we meet is the choice of a canonical representative $\tilde{\alpha}$ in the set $\alpha U(K)$. We prove (see Theorem 4) that this can be done, assuming that the Arcihmedean Shanuel conjecture is valid.

In §3, we define a new group $U^0_{S,3}(K)$ intimately related to $U_{S,1}(K)$ and, using the element $\tilde{\alpha}$ and some fundamental system of units in K, construct a "good" basis in $U^0_{S,3}(K)$. Then we prove that one can pass from one "good" basis to another via a transition matrix $C \in \mathrm{GL}(r, \mathbb{Z})$. Hence, any "good" basis determines the same regulator $\mathfrak{R}^0_{\ell,3}(K)$ of the group $U^0_{S,3}(K)$. Then it is easy to pass to the group $U_{S,1}(K)$ and its regulator $\mathfrak{R}_\ell(K)$. The precise definitions are given in (9) and (10). All these definitions can easily be carried over to the case of a nonnormal extension of $\mathbb{Q}$ (see (12) and (13)).

For a real quadratic K we prove (see Theorem 6) that the existence of the element $\tilde{\alpha}$ can be established without using any additional conjectures.

§2. Preliminary results

As in [4], we assume that the following particular case of the ℓ-adic Shanuel conjecture holds true.

Conjecture 1. *Suppose that the numbers $x_1, \dots, x_n \in \bar{\mathbb{Q}}_\ell$ are algebraic over $\mathbb{Q}$, and the numbers $\ell, x_1, \dots, x_n$ are multiplicatively independent. Then their ℓ-adic logarithms $\log x_1, \dots, \log x_n$ are algebraically independent over $\mathbb{Q}$.*

Also, we shall use a similar particular case of the Archimedean Shanuel conjecture.

Conjecture 2. *Let $x_1, \dots, x_n \in \mathbb{C}$ be complex numbers algebraic over $\mathbb{Q}$ and multiplicatively independent. Then their logarithms $\ln x_1, \dots, \ln x_n \in \mathbb{C}$ are algebraically independent over $\mathbb{Q}$.*

Now we recall the definition of the regulator $\mathfrak{R}_\ell(K)$. As in [4], let $D(K)$ be a free $\mathbb{Z}_\ell$-module generated by all prime divisors over ℓ. If $S = \{\mathfrak{l}_1, \dots, \mathfrak{l}_m\}$, where $m = [K : \mathbb{Q}]$,

[1] The author is gratelul to Don Zagier who has pointed out this problem to the author.

is the set of all such divisors, then any element of $D(K)$ may be represented uniquely in the form

$$\sum_{i=1}^{m} a_i \mathfrak{l}_i, \quad a_i \in \mathbb{Z}_\ell. \tag{1}$$

If $u \in U_S(K)$, then the element u gives rise to its principal divisor (u) of the form (1). In this case all the corresponding coefficients a_i are in $\mathbb{Z}$. This correspondence extends by continuity to a homomorphism $\operatorname{div}: U_S(K)[\ell] \to D(K)$, where $[\ell]$ means pro-ℓ-completion. As was proved in [4, Proposition 3.1], if the Leopoldt conjecture is valid for K and ℓ, then the homomorphism div induces an injection $\operatorname{div}: U_{S,1}(K) \hookrightarrow D(K)$.

One can define a standard scalar product on the $\mathbb{Z}_\ell$-module $D(K)$:

$$\langle\ ,\ \rangle: D(K) \times D(K) \longrightarrow \mathbb{Z}_\ell, \quad \langle x, y\rangle = \sum_{i=1}^{m} a_i b_i, \tag{2}$$

where $x, y \in D(K)$, $\quad x = \sum_{i=1}^{m} a_i \mathfrak{l}_i, \quad y = \sum_{i=1}^{m} b_i \mathfrak{l}_i, \quad a_i, b_i \in \mathbb{Z}_\ell$.

The regulator $\mathfrak{R}_\ell(K)$ is defined as the determinant of the matrix of the bilinear form $\langle\ ,\ \rangle$ defined on the free $\mathbb{Z}_\ell$-module $U_{S,1}(K)$. Thus, if $e_1, \dots, e_n$ is a $\mathbb{Z}_\ell$-basis of $U_{S,1}(K)$, then

$$\mathfrak{R}_\ell(K) = \det\bigl(\langle \operatorname{div}(e_i), \operatorname{div}(e_j)\rangle\bigr), \quad 1 \le i, j \le n.$$

This definition was discussed in more detail in [4, §3].

So, we want to define some collection of $\mathbb{Z}_\ell$-bases in $U_{S,1}(K)$ such that the passage from any basis of this collection to another basis can be done by using some transition matrix from $\mathrm{GL(n, \mathbb{Z})}$. Then we would be able to define $\mathfrak{R}_\ell(K)$ by formula (2), taking for the basis $e_1, \dots, e_n$ one of the bases of our collection.

First, we consider the case where K is normal over $\mathbb{Q}$. In this case the field K is either totally real or pure imaginary. We shall treat explicitly the latter case only, because the former can be treated in the same way. As in [4, §3], in our constructions we shall use an element $\alpha \in U_S(K)$ such that $(\alpha) = \mathfrak{l}_1^{\mathbf{h}}$, where $\mathfrak{l}_1$ is a prime divisor of ℓ in K, and $\mathbf{h}$ is the order of $\mathfrak{l}_1$ in the class group of K.

Obviously, the condition $(\alpha) = \mathfrak{l}_1^{\mathbf{h}}$ determines α up to multiplication by any element of the group $U(K)$. Thus, our first goal is to determine the element α canonically. For this, we shall use the homomorphism of the group $K^\times$ into the logarithmic space $\mathbb{R}_n$ that was described in detail in [1, Chapter. 2]. Presently, we assume that K is pure imaginary, that is, $m = [K : \mathbb{Q}] = 2n$. Let $\sigma_1, \bar\sigma_1, \sigma_2, \bar\sigma_2, \dots, \sigma_n, \bar\sigma_n$ be all injections of K into $\mathbb{C}$, where $\sigma_i, \bar\sigma_i$ are complex conjugate injections for $i = 1, 2, \dots, n$. Then with any element $x \in K^\times$ we can associate a vector $l(x) = (l_1(x), \dots, l_n(x))$, where $l_i(x) = \ln|\sigma_i(x)|^2 = \ln(\sigma_i(x)\bar\sigma_i(x))$. The map $l: K^* \to \mathbb{R}^n$ is a homomorphism, and $\ker l \cap U(K)$ coincides with the group $\mu(K)$ of all roots of unity in K. The homomorphism l maps the group of units $U(K)$ onto some lattice of rank $n-1$ contained in the subspace $V \subset \mathbb{R}^n$ determined by the condition

$$V = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n \ \mid\ \sum_{i=1}^{n} x_i = 0 \right\}.$$

Obviously, the vector $l(\alpha)$ does not belong to V, because $\sum_{i=1}^{n} l_i(\alpha) = \ln(N(\alpha)) = \ln(\ell^{\mathbf{h}})$.

In the space $\mathbb{R}^n$ we have the standard Eucledian norm such that

$$|x| = \sqrt{x_1^2 + \dots + x_n^2} \ \text{ for } \ x = (x_1, \dots, x_n) \in \mathbb{R}^n. \tag{3}$$

Let τ be an automorphism of K. Then

$$|l(x)| = |l(\tau(x))| \tag{4}$$

for any $x \in K$. Indeed, if σ_i and $\bar{\sigma}_i$ are complex conjugate, then so are $\sigma_i\tau$ and $\bar{\sigma}_i\tau$, that is, the vectors x and $\tau(x)$ have the same coordinates up to permutation.

For a set $T \subset \mathbb{C}$, finite or infinite, we denote by $\ln T$ the set $\{\ln t \mid t \in T\}$. Respectively, for a field $k \subset \mathbb{C}$ by $k(\ln T)$ we denote the field generated over k by the elements $\ln t$ for all $t \in T$.

Proposition 3. *Assume that Conjecture 2 holds true. Let K be a pure imaginary field normal over $\mathbb{Q}$ of degree $m = 2n$. Then the field $L := \mathbb{Q}(\ln U(K))$ is of trancendence degree $r = n-1$ over $\mathbb{Q}$ and $M := \mathbb{Q}(\ln U_S(K))$ is of trancendence degree $r+m$ over $\mathbb{Q}$. Let α be as above, and let $\sigma_1, \dots, \sigma_m$ be all automorphisms of K. Then the elements $\sigma_1(\alpha), \dots, \sigma_m(\alpha)$ are algebraically independent over L.*

Proof. Let $\varepsilon_1, \dots, \varepsilon_r$ be a fundamental system of units in $U(K)$. Then by Conjecture 2 the elements $\ln\varepsilon_1, \dots, \ln\varepsilon_r$ are algebraically independent over $\mathbb{Q}$, that is, $\deg \operatorname{tr}(L) \geq r$. Note that any unit $u \in U(K)$ can be represented (up to multiplication by some root of unity) in the form $u = \prod_{i=1}^{r} \varepsilon_i^{s_i}$ for some $s_i \in \mathbb{Z}$. Therefore, $\ln u = \sum_{i=1}^{r} s_i \ln\varepsilon_i$ and L is a pure trancendental extension of $\mathbb{Q}$ generated by the elements $\ln\varepsilon_1, \dots, \ln\varepsilon_r$. The elements $\varepsilon_1, \dots, \varepsilon_r, \sigma_1(\alpha), \dots, \sigma_m(\alpha)$ are multiplicatively independent and generate a subgroup of finite index in $U_S(K)$. Hence, their logarithms are algebraically independent and generate the field M over $\mathbb{Q}$. This proves the proposition. □

Theorem 4. *Assume that Conjecture 2 holds true. Let K be a normal extension of $\mathbb{Q}$, and suppose that the prime ℓ splits completely in K. Let $\mathfrak{l}_1$ be a fixed prime divisor of ℓ in K, and let $\alpha \in U_S(K)$ be an element such that $(\alpha) = \mathfrak{l}_1^{\mathbf{h}}$, where $\mathbf{h}$ is the order of $\mathfrak{l}_1$ in the class group. Put*

$$X(\alpha) = \{\alpha u \in K^\times \mid u \in U(K)\}.$$

Then in the set $X(\alpha)$ there is a unique vector $\widetilde{\alpha}$ of minimal length. Let $\mathfrak{l}_i$ be another prime divisor of ℓ in K and σ_i an automorphism of K such that $\sigma_i(\mathfrak{l}_1) = \mathfrak{l}_i$. Let $\widetilde{\alpha}_i \in K$ be an element constructed in the same manner as $\widetilde{\alpha}$ but relative to the prime divisor $\mathfrak{l}_i$. Then $\sigma_i(\widetilde{\alpha}) = \widetilde{\alpha}_i$.

Proof. We have $|l(\alpha u)| = |l(\alpha) + l(u)| \geq |l(u)| - |l(\alpha)|$. Hence, $|l(\alpha u)| > |l(\alpha)|$ if $|l(u)| > 2|l(\alpha)|$. Therefore, it suffices to search the minimal value of $|l(\alpha u)|$ on some finite subset of $l(U(K))$, which proves the existence of this minimal value. Let $|l(\alpha u_1)|$ be this minimal value. We need to check that for any $u_2 \in U(K)$ the condition $|l(\alpha u_1)| = |l(\alpha u_2)|$ implies $u_1 = u_2$. Indeed, if $|l(\alpha u_1)| = |l(\alpha u_2)|$, then

$$\begin{aligned} |l(\alpha u_1)|^2 &= \sum_{i=1}^{n} (l_i(\alpha u_1))^2 = \sum_{i=1}^{n} (l_i(\alpha) + l_i(u_1)))^2 \\ &= \sum_{i=1}^{n} \big(l_i(\alpha)^2 + 2l_i(\alpha)l_i(u_1) + l_i(u_1)^2\big) \\ &= |l(\alpha u_2)|^2 = \sum_{i=1}^{n} \big(l_i(\alpha)^2 + 2l_i(\alpha)l_i(u_2) + l_i(u_2)^2\big). \end{aligned} \tag{5}$$

Thus, (5) yields the identity

$$\sum_{i=1}^{n} 2(l_i(u_1) - l_i(u_2))l_i(\alpha) + \sum_{i=1}^{n} \big(l_i(u_1^2) - l_i(u_2^2)\big) = 0.$$

This gives an algebraic relation over L among the $l_i(\alpha)$. On the other hand, we have $l_i(\alpha) = \ln(\sigma_i(\alpha)\bar{\sigma}_i(\alpha))$, where σ_i and $\bar{\sigma}_i$ are complex conjugate injections. So, the algebraic independence of the elements $\ln\sigma_1(\alpha), \dots, \ln\sigma_m(\alpha)$ over L stated in Proposition 3 implies the algebraic independence over L of the elements $l_i(\alpha)$, $i = 1, \dots, n$. Thus, we obtain $l_i(u_1) = l_i(u_2)$ for $i = 1, 2, \dots, n$, that is, u_1 and u_2 coincide up to multiplication by a root of unity. This proves the uniqueness of $\widetilde{\alpha}$.

Now suppose that, acting as above, we have constructed elements $\widetilde{\alpha} = \widetilde{\alpha}_1$ and $\widetilde{\alpha}_i$, where the former corresponds to the prime divisor $\mathfrak{l}_1$ and the latter to $\mathfrak{l}_i$. Suppose that the automorphism σ_i sends $\mathfrak{l}_1$ to $\mathfrak{l}_i$. Then the element $\beta := \sigma_i(\widetilde{\alpha}_1)$ belongs to $\widetilde{\alpha}_i U(K)$, and (4) shows that $|l(\beta)|$ is minimal on this set. Therefore, $\beta = \widetilde{\alpha}_1$ up to multiplication by a root of unity. This completes the proof of the theorem. $\square$

§3. Definition and examples

Let K be normal over $\mathbb{Q}$. Suppose that ℓ splits completely in K, and that $\sigma_1, \dots, \sigma_m$ are all injections of K into $\mathbb{Q}_\ell$. Then any $x \in K^\times$ gives rise to a vector $\sigma_1(x), \dots, \sigma_m(x) \in K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong \mathbb{Q}_\ell^m$ and, applying the ℓ-adic logarithm map, we obtain a homomorphism

$$\log_\ell \colon U_S(K)[\ell] \longrightarrow \mathbb{Q}_\ell^m, \quad \log_\ell(u) = (\log \sigma_1(u), \dots, \log \sigma_m(u))$$

for any $u \in U_S(K)[\ell]$. This homomorphism was considered already in [4]. As in that paper, we assume that the following conjecture holds true.

Conjecture 5. *Let $\widetilde{\alpha}$ be the same as in §2. Then any $m - 1$ among the elements*

$$\log_\ell \sigma_1(\widetilde{\alpha}), \ \dots, \ \log_\ell \sigma_m(\widetilde{\alpha})$$

are linear independent over $\mathbb{Q}_\ell$.

Note that $N_{K/\mathbb{Q}}(\widetilde{\alpha}) = \ell^{\mathbf{h}}$, hence,

$$0 = \log_\ell N_{K/\mathbb{Q}}(\widetilde{\alpha}) = \sum_{i=1}^{m} \log \sigma_i(\widetilde{\alpha}). \tag{6}$$

This conjecture (in a more general form) was formulated in [4] as Conjecture 3.1. In that paper (see [4, Proposition 3.4]), it was proved that this conjecture is a consequence of our Conjecture 1, which appeared in [4] as Conjecture 2.1.

As in [4], let $U_{S,2}(K)$ be the subgroup of local universal norms in $U_S(K)[\ell]$. Then $U_{S,1}(K) \subseteq U_{S,2}(K)$ and, as it was proved in [4, (3.7)], the group $U_{S,2}(K)$ consists of all $x \in U_S(K)[\ell]$ such that $\log_\ell(x) = 0$. In [4, Proposition 3.5], it was shown that the group $U_{S,1}(K)$ is of finite index in $U_{S,2}(K)$ under the assumption that the Leopoldt conjecture and Conjecture 5 hold true. Thus, the last condition is satisfied if Conjecture 1 of the present paper is valid. In particular, using [4, Theorem 2.1], we see that $U_{S,2}(K)$ is a free $\mathbb{Z}_\ell$-module of rank $r + 1$, where $r = m/2 - 1$ is the rank of the group of units of K.

Put

$$U^0_{S,1}(K) = \{x \in U_{S,1}(K) \mid N_{K/\mathbb{Q}}(x) = 1\},$$
$$U^0_{S,2}(K) = \{x \in U_{S,2}(K) \mid N_{K/\mathbb{Q}}(x) = 1\}.$$

Then, assuming Conjecture 1 to be true, we conclude that $U^0_{S,1}(K)$ and $U^0_{S,2}(K)$ are free $\mathbb{Z}_\ell$-modules of rank r, and the index $(U^0_{S,2}(K) : U^0_{S,1}(K))$ is finite.

Now, let $\varepsilon_1, \dots, \varepsilon_r$ be a fundamental system of units of K. Then, as in the proof of Proposition 3.5 in [4], we see that for any $i = 1, \dots, r$ there is a uniquely defined collection of coefficients $a_{i,j} \in \mathbb{Q}_\ell$ such that

$$\sum_{j=1}^{m} a_{i,j} \log_\ell \sigma_j(\widetilde{\alpha}) = \log_\ell \varepsilon_i, \quad \sum_{j=1}^{m} a_{i,j} = 0, \tag{7}$$

where $\widetilde{\alpha}$ is the canonical element constucted in Theorem 1.

Put

$$\eta_i = \varepsilon_i^{-1} \prod_{j=1}^{m} \sigma_j(\widetilde{\alpha})^{a_{i,j}}. \tag{8}$$

Let $U^0_{S,3}$ be a $\mathbb{Z}_\ell$-module generated by the elements $\eta_1, \dots, \eta_r$. Then $\eta_j^{\ell^d} \in U_{S,2}(K)$ if d is so large that $\ell^d a_{i,j} \in \mathbb{Z}_\ell$ for all i, j. Therefore, the groups $U^0_{S,3}(K)$ and $U^0_{S,2}(K)$ are commensurable, that is, the index $(U^0_{S,3}(K) : U^0_{S,2}(K))$ is a well-defined integral power of ℓ.

Note that in the group $U^0_{S,3}(K)$ there is a special basis $\eta_1, \dots, \eta_r$, which depends on the fundamental system of units $\varepsilon_1, \dots, \varepsilon_r$. If one chooses another fundamental system of units $\varepsilon'_1, \dots, \varepsilon'_r$ related to the system $\varepsilon_1, \dots, \varepsilon_r$ via a transition matrix $C \in \mathrm{GL}(r, \mathbb{Z})$, and for the system $\varepsilon'_1, \dots, \varepsilon'_r$ one constructs the elements $\eta'_1, \dots, \eta'_r$ as above, then the elements $\eta'_1, \dots, \eta'_r$ will be related to the elements $\eta_1, \dots, \eta_r$ via the same transition matrix C. In particular, this means that the group $U^0_{S,3}(K)$ is well defined, that is, it does not depend on the particular choice of the fundamental system of units. Now we put

$$\mathfrak{R}^0_{\ell,3}(K) = \mathfrak{R}^0_\ell(U^0_{S,3}(K)) = \det(\langle \mathrm{div}(\eta_i), \mathrm{div}(\eta_j)\rangle), \quad 1 \le i, j \le r. \tag{9}$$

So, after the passage from $\eta_1, \dots, \eta_r$ to $\eta'_1, \dots, \eta'_r$, the number $\mathfrak{R}^0_{\ell,3}(K)$ is multiplied by $(\det C)^2 = 1$, that is, this number is well defined.

Now we put $U_{S,3}(K) = U^0_{S,3} \times \ell^{\mathbb{Z}_\ell}$. Thus, $U_{S,3}(K)$ is a multiplicative free $\mathbb{Z}_\ell$-module generated by the elements $\eta_1, \dots, \eta_r, \ell$, and from (7) it follows that $\langle \mathrm{div}(\eta_i), (\ell)\rangle = 0$ if $i = 1, \dots, r$. Taking the relation $\langle (\ell), (\ell)\rangle = [K : \mathbb{Q}] = m$ into account, finally we put

$$\mathfrak{R}_{\ell,3}(K) = \mathfrak{R}^0_{\ell,3}(K) m, \quad \mathfrak{R}_\ell(K) = \mathfrak{R}_{\ell,3}(K)(U_{S,1}(K) : U_{S,3}(K))^2, \tag{10}$$

where $(U_{S,1}(K) : U_{S,3}(K))$ is some integral power of ℓ.

Now we assume that k is an arbitrary algebraic number field, and that the prime ℓ splits completely in k. Let K be the Galois closure of k over $\mathbb{Q}$. Then all results obtained above are valid for the field K. In particular, the group $U^0_{S,3}(K)$ is defined. We may assume that there is some fixed $\mathbb{Z}_\ell$-basis $\eta_1, \dots, \eta_r$ of this group, which corresponds to the fundamental system of units $\varepsilon_1, \dots, \varepsilon_r$ of K. Let $\epsilon_1, \dots, \epsilon_s$ be a fundamental system of units of k. Then there is a representation

$$\epsilon_i = \prod_{k=1}^{r} \varepsilon_k^{b_{i,k}}, \quad 1 \le i \le s, \quad b_{i,k} \in \mathbb{Z}, \tag{11}$$

whose coefficients $b_{i,k}$ are defined uniquely. We put $\xi_i = \prod_{k=1}^{r} \eta_k^{b_{i,k}}$, $i = 1, \dots, s$. Then from (8) and (11) it follows that $\xi_i = \epsilon_i^{-1} A_i$ for $i = 1, \dots, s$, where

$$A_i = \prod_{k=1}^{r} \left(\prod_{j=1}^{m} \sigma_j(\widetilde{\alpha})^{a_{k,j}} \right)^{b_{i,k}}.$$

Let H be the subgroup corresponding to k in the Galois group $G = G(K/\mathbb{Q})$. Then the element $\log_\ell A_i = -\log_\ell \epsilon_i$ is fixed under the action of H. The element A_i can be written uniquely in the form $A_i = \prod_{j=1}^{m} \sigma_j(\widetilde{\alpha})^{c_{i,j}}$ with $c_{i,j} \in \mathbb{Q}_\ell$, and (7) implies that $\sum_{j=1}^{m} c_{i,j} = 0$ for $i = 1, \dots, s$. Thus, the divisor $\mathfrak{A}_i = \mathrm{div}(\xi_i) = \mathrm{div}(A_i)$ is fixed under the action of H. Therefore, there is $d \in \mathbb{Z}$ such that (in additive notation) $\ell^d \mathfrak{A}_i \in D(k)$. Put $t = |H| = [K : k]$. Denoting the pairing $\langle\ ,\ \rangle$ in the fields K and k by $\langle\ ,\ \rangle_K$ and $\langle\ ,\ \rangle_k$, respectively, we have $\langle x, y\rangle_k = t\langle x, y\rangle_K$ for any $x, y \in D(k)$.

We define $U^0_{S,3}(k)$ to be the $\mathbb{Z}_\ell$-module generated by the elements ξ_i, $i = 1, \dots, s$, and put

$$(12)\qquad \mathfrak{R}^0_{\ell,3}(k) = \det(\langle \operatorname{div}(\xi_i) \operatorname{div}(\xi_j)\rangle_K) = t^{-1} \det(\langle \operatorname{div}(\xi_i) \operatorname{div}(\xi_j)\rangle_k), \quad 1 \le i, j \le s.$$

Then we put $U_{S,3}(k) = U^0_{S,3}(k) \times \ell^{\mathbb{Z}_\ell}$ and $\mathfrak{R}_{\ell,3}(k) = \mathfrak{R}^0_{\ell,3}(k)[k : \mathbb{Q}]$.

Finally, as in (10), we put

$$(13)\qquad \mathfrak{R}_\ell(k) = \mathfrak{R}_{\ell,3}(k)(U_{S,1}(k) : U_{S,3}(k))^2.$$

Note that in the case of an Abelian field k we can avoid using Conjecture 1. Indeed, in this case the Leopoldt conjecture is proved, as well as Conjecture 5 (see [4, Proposition 3.3]).

Concerning Conjecture 2, the author has found the only nontrivial case where the assertion of Theorem 1 can be verified without using this conjecture.

Theorem 6. *Let K be a real quadratic field and ℓ a prime number splitting in K, $(\ell) = \mathfrak{l}_1\mathfrak{l}_2$. Let $\alpha \in U_S(K)$ be an element such that $(\alpha) = \mathfrak{l}_1^{\mathbf{h}}$, where $\mathbf{h}$ is the order of $\mathfrak{l}_1$ in the class group of K. Then the condition $|l(\alpha u_1)| = |l(\alpha u_2)|$ for some units $u_1, u_2 \in U(K)$ implies $l(u_1) = l(u_2)$, that is, $u_1 = \pm u_2$.*

Proof. For any $x \in K^\times$ the point $l(x)$ belongs to $\mathbb{R}_2$ and has coordinates $x_1 = \ln|\sigma_1(x)|$ and $x_2 = \ln|\sigma_2(x)|$, where σ_1, σ_2 are two different injections of K into $\mathbb{R}$. In particular, $l(U(K))$ is a one-dimensional lattice on the line L given by the equation $x_1 + x_2 = 0$. If $|l(\alpha u_1)| = |l(\alpha u_2)|$, then, multiplying by $l(u_1^{-1}u_2^{-1})$, we obtain $|l(\alpha u_1^{-1})| = |l(\alpha u_2^{-1})|$, i.e., $|l(\alpha) - l(u_1)| = |l(\alpha) - l(u_2)|$. This means that the triangle Δ with vertices $l(\alpha), l(u_1)$, and $l(u_2)$ is isosceles. Let $O = (l(u_1) + l(u_2))/2$ be the midpoint of the base $[l(u_1), l(u_2)]$. Then the segment $[l(\alpha), O]$ is a height of the triangle Δ, that is, the vector $\overrightarrow{O\,l(\alpha)}$ is orthogonal to the vector $(+1, -1)$, which is a directing vector of the line L. In its turn, this means that the coordinates of the vector $\overrightarrow{O\,l(\alpha)}$ are equal, i.e.,

$$\ln|\sigma_1(\alpha)| - \frac{1}{2}\ln|\sigma_1(u_1)| - \frac{1}{2}\ln|\sigma_1(u_2)| = \ln|\sigma_2(\alpha)| - \frac{1}{2}\ln|\sigma_2(u_1)| - \frac{1}{2}\ln|\sigma_2(u_2)|.$$

Since $\ln|\sigma_2(u_1)| = -\ln|\sigma_1(u_1)|$ and $\ln|\sigma_2(u_2)| = -\ln|\sigma_1(u_2)|$, from the preceding we obtain the equation

$$\ln|\sigma_1(\alpha)| - \ln|\sigma_2(\alpha)| - \ln|\sigma_1(u_1)| - \ln|\sigma_1(u_2)| = 0.$$

After exponentiation we get

$$\beta := \sigma_1(\alpha)\sigma_2(\alpha)^{-1}\sigma_1(u_1)^{-1}\sigma_1(u_2)^{-1} = \pm 1,$$

but this is impossible because β is an algebraic number with a divisor $\mathfrak{l}_1^{\mathbf{h}}\mathfrak{l}_2^{-\mathbf{h}}$. This proves the theorem. ☐

References

[1] Z. I. Borevič and I. R. Šafarevič, *The theory of numbers*, Nauka, Moscow, 1964; English transl., Newcomb Greenleaf Pure Appl. Math., vol. 20, Acad. Press, New York, 1966. MR0195803

[2] S. Lang, *Cyclotomic fields.* I, II, Comb. 2nd ed., Grad. Texts in Math., vol. 121, Springer-Verlag, New York, 1990. MR1029028

[3] L. V. Kuz′min, *Some remarks on an ℓ-adic Dirichlet theorem and ℓ-adic regulator*, Izv. Akad. Nauk SSSR Ser. Mat. **45** (1981), no. 6, 1203–1240; English transl., Math. USSR-Izv. **19** (1982), no. 3, 445–478. MR641800

[4] ———, *On a new type of ℓ-adic regulator for algebraic number fields* (*the ℓ-adic regulator without logarithms*), Izv. Ross. Akad. Nauk Ser. Mat. **79** (2015), no. 1, 115–152; English transl., Izv. Math. **79** (2015), no. 1, 109–144. MR3352584

[5] ———, *Some remarks on the ℓ-adic regulator.* V. *The growth on the ℓ-adic regulator in the cyclotomic $\mathbb{Z}_\ell$-extension of an algebraic number field*, Izv. Ross. Akad. Nauk Ser. Mat. **73** (2009), no. 5, 105–170; English transl., Izv. Math. **73** (2009), no. 5, 959–1021. MR2584230

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