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RIEMANN'S ZETA FUNCTION AND FINITE DIRICHLET SERIES

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To the 70th anniversary of Sergey Vladimirovich Vostokov

ABSTRACT. The paper describes computer experiments for calculating zeros and values of Riemann's zeta function and of its first derivative inside the critical strip and to the left of it with the help of finite Dirichlet series the coefficients of which are defined via initial nontrivial zeros of the zeta function.

§1. INTRODUCTION

A main object of study in number theory is prime numbers, in particular, the behavior of the function $\pi(x)$ giving the number of primes that are not greater than x.

A principal tools for the study of the distribution of primes among the natural numbers is Riemann's *zeta function* $\zeta(s)$, which can be defined via a *Dirichlet series*:

$$\zeta(s) = 1^{-s} + 2^{-s} + \cdots + n^{-s} + \ldots \tag{1.1}$$

The series converges for $\mathrm{Re}(s) > 1$ only, but it admits analytic continuation to the complex plane with the exception of the point $s = 1$, which is the only pole of $\zeta(s)$.

Leonhard Euler studed (1.1) for real values of s, in particular, he established that

$$1^{-s} + 2^{-s} + \cdots + n^{-s} + \ldots = \prod_{p \text{ is prime}} \frac{1}{1 - p^{-s}}. \tag{1.2}$$

This *Euler identity* is basically an analytical form of the *Fundamental Theorem of Arithmetic*, asserting the existence and uniqueness of factorization for natural numbers.

Expressed by (1.2), equality of a sum taken over all natural numbers and a product taken over primes only explains why the zeta function is a powerful tool for the study of prime numbers. Euler used identity (1.2) for a new proof of the infinitude of the set of prime numbers; this proof can be expressed by a single phrase: *Should this set be finite, the divergent harmonic series, which is the left-hand side of* (1.2) *for* $s = 1$, *would have finite value equal to the right-hand side of* (1.2).

Bernhard Riemann began to consider (1.1) for complex s and established that

$$\pi(x) = \mathrm{Li}(x) - \mathrm{Li}(x^{1/2})/2 + \sum_{\zeta(\rho)=0} \mathrm{Li}(x^{\rho}) + \text{small terms}, \tag{1.3}$$

where Li, named the *offset logarithmic integral*, is defined as

$$\mathrm{Li}(x) = \int_2^x \frac{1}{\ln(t)}\, dt.$$

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Summation in (1.3) is over the zeros of the zeta function, and the question *how well* $\mathrm{Li}(x)$ *approximates* $\pi(x)$ turns out to be closely related to the question *how large can be the real part of a zero of this function.* Euler indicated that the zeta function vanishes at the negative even numbers; nowadays these zeros are said to be *trivial.* Riemann established that there are no other real zeros, and that all nonreal, *nontrivial*, zeros lie in the *critical strip* $0 \leq \mathrm{Re}(s) \leq 1$.

The celebrated *Riemann hypothesis* predicts that these zeros lie on the *critical line* $\mathrm{Re}(s) = 1/2$, and this assertion is equivalent to

$$\pi(x) = \mathrm{Li}(x) + O(x^{1/2}\ln(x)).$$

The author ([6], see also [1]) used finite Dirichlet series with coefficients defined via the initial nontrivial zeros of the zeta function. Computer calculations showed that these series give surprisingly good approximations to the values of the zeta function inside the critical strip and to the left of it, and also allow one to find approximate values of other nontrivial zeros not used for constructing the series themseves. Also it turned out that the coefficients of these series have number-theoretical meaning, establishing new relationship between the zeta function and prime numbers.

The results of computer calculations are presented here in the form of *numerical observations*; they have been made, naturally, when considering a limited amount of finite Dirichlet series. However, based on these observations, one can formulate plausible assumptions about the properties of all series of this kind, and then try to prove these properties. Until now, no phenomenon discovered by computer calculations got a theoretical explanation/proof.

A bit different finite Dirichlet series were introduced in [7] (see also [1, 3]); these series have properties similar to the properties of the series considered in this paper, except those presented below in Subsections 5.3, 6.3, and 6.4.

The reader can follow the ongoing investigation of both types of series via [5].

§2. Objects of study

Riemann formulated his famous hypothesis in [8] not as a statement about the zeros of the zeta function, but as the property for the zeros of the function $\Xi(t) = \xi(1/2 + \mathrm{i}t)$ to be real, where

$$\xi(s) = g(s)\zeta(s),$$
$$g(s) = \pi^{-s/2}(s-1)\Gamma\big(s/2+1\big).$$

Riemann proved that the function $\xi(s)$ satisfies the *functional equation*

$$\xi(s) = \xi(1-s). \tag{2.1}$$

Accordingly,

$$\xi(s) = \sum_{n=1}^{\infty} g(s)n^{-s} \tag{2.2}$$

$$= \sum_{n=1}^{\infty} g(1-s)n^{s-1} \tag{2.3}$$

and we can write

$$\xi(s) = \sum_{n=1}^{\infty} h_n(s), \tag{2.4}$$

where

$$h_n(s) = \big(g(s)n^{-s} + g(1-s)n^{s-1}\big)/2.$$

Clearly, the functions $h_n(s)$ satisfy the following counterpart of (2.1):

$$h_n(s) = h_n(1-s). \tag{2.5}$$

However, identity (2.4) is purely formal, because the series (2.2) converges for $\mathrm{Re}(s) > 1$ only, while the series (2.3) converges for $\mathrm{Re}(s) < 0$ only.

Assuming the Riemann hypothesis, as well as the simplicity of all zeros of the zeta function, we can enumerate the zeros that lie in the upper half-plane in the ascending order of the imaginary parts:

$$\rho_1 = 1/2 + \gamma_1 \mathrm{i}, \dots, \rho_k = 1/2 + \gamma_k \mathrm{i}, \dots, \quad 0 < \gamma_1 < \dots < \gamma_k < \dots$$

In accordance with (2.1) the conjugates $\rho_{-1} = 1/2 - \gamma_1\mathrm{i}, \dots, \rho_{-k} = 1/2 - \gamma_k\mathrm{i}, \dots$, and only them, will be zeros of the zeta function in the lower half-plane.

Consider the determinant

$$\widetilde{\Delta}_N(x) = \begin{vmatrix} h_1(\rho_1) & \dots & h_1(\rho_{N-1}) & h_1(s) \\ \vdots & \ddots & \vdots & \vdots \\ h_N(\rho_1) & \dots & h_N(\rho_{N-1}) & h_N(s) \end{vmatrix}. \tag{2.6}$$

It can be expanded along the last column:

$$\widetilde{\Delta}_N(x) = \sum_{n=1}^{N} \widetilde{\delta}_{N,n} h_n(s). \tag{2.7}$$

We introduce the normalization

$$\delta_{N,n} = \widetilde{\delta}_{N,n}/\widetilde{\delta}_{N,1} \tag{2.8}$$

and, besides (2.7), shall consider the sum

$$\Delta_N(s) = \sum_{n=1}^{N} \delta_{N,n} n^{-s}; \tag{2.9}$$

notice that this sum *is not* a normalization of the sum (2.7).

The finite Dirichlet series (2.9) and its coefficients will be our objects of study.

§3. Zeros of $\widetilde{\Delta}_N(s)$ and of $\Delta_N(s)$

It is easily seen that $\widetilde{\Delta}_N(s)$ vanishes for $s = \rho_{\pm 1}, \dots, \rho_{\pm(N-1)}$.

Numerical observation 3.1. *The function $\Delta_N(s)$ has zeros close to the nontrivial zeros $\rho_{\pm 1}, \dots, \rho_{\pm(N-1)}$.*

Numerical observation 3.2. *The functions $\widetilde{\Delta}_N(s)$ and $\Delta_N(s)$ have zeros near $\rho_{\pm N}, \dots, \rho_{\pm(N+L)}$ for sufficiently small L.*

Numerical observation 3.3. *The functions $\widetilde{\Delta}_N(s)$ and $\Delta_N(s)$ have zeros near the trivial zeroes $-2, -4, \dots, -2L$ for sufficiently small L.*

Numerical observation 3.4. *The functions $\widetilde{\Delta}_N(s)$ and $\Delta_N(s)$ have zeros near the numbers $1 \pm 2\pi\mathrm{i}/\ln(2), \dots, 1 \pm 2\pi\mathrm{i}L/\ln(2)$ for sufficiently small L.*

The number of nearby zeros indicated in Observations 3.2–3.4 increases with the growth of N.

Tables 3.1–3.4 demonstrate Observations 3.1–3.4. In order to understand the origin of the extra zeros from the last observation, look at Figure 3.1 showing the typical behavior of the numbers $\delta_{N,n}$ for sufficiently large N. We see that the initial coefficients of the Dirichlet series (2.9) for $N = 201$ are close to the corresponding coefficients of the series

$$1^{-s} - 2^{-s} + \dots + (-1)^{n+1} n^{-s} + \dots = (1 - 2 \cdot 2^{-s})\zeta(s); \tag{3.1}$$

TABLE 3.1. Zeros of $\Delta_N(s)$ near initial nontrivial zeros of the zeta function for $N = 503$.

$$0 = \Delta_N(\rho_1 - 2.889\ldots\cdot 10^{-46} - 6.420\ldots\cdot 10^{-47}\mathrm{i})$$
$$0 = \Delta_N(\rho_{51} + 9.585\ldots\cdot 10^{-47} + 7.702\ldots\cdot 10^{-46}\mathrm{i})$$
$$0 = \Delta_N(\rho_{101} - 2.136\ldots\cdot 10^{-46} + 3.634\ldots\cdot 10^{-46}\mathrm{i})$$
$$0 = \Delta_N(\rho_{151} - 2.813\ldots\cdot 10^{-46} + 7.467\ldots\cdot 10^{-47}\mathrm{i})$$
$$0 = \Delta_N(\rho_{201} + 1.218\ldots\cdot 10^{-46} - 7.894\ldots\cdot 10^{-46}\mathrm{i})$$
$$0 = \Delta_N(\rho_{251} - 1.076\ldots\cdot 10^{-46} + 5.582\ldots\cdot 10^{-46}\mathrm{i})$$
$$0 = \Delta_N(\rho_{301} + 7.911\ldots\cdot 10^{-46} + 9.647\ldots\cdot 10^{-46}\mathrm{i})$$
$$0 = \Delta_N(\rho_{351} - 2.578\ldots\cdot 10^{-46} - 2.247\ldots\cdot 10^{-46}\mathrm{i})$$
$$0 = \Delta_N(\rho_{401} - 2.090\ldots\cdot 10^{-46} + 3.744\ldots\cdot 10^{-46}\mathrm{i})$$
$$0 = \Delta_N(\rho_{451} - 2.840\ldots\cdot 10^{-46} + 1.904\ldots\cdot 10^{-47}\mathrm{i})$$
$$0 = \Delta_N(\rho_{501} - 2.663\ldots\cdot 10^{-46} - 1.856\ldots\cdot 10^{-46}\mathrm{i})$$

TABLE 3.2. Zeros of $\widetilde{\Delta}_N(s)$ and of $\Delta_N(s)$ near larger nontrivial zeros of the zeta function for $N = 503$.

$$0 = \widetilde{\Delta}_N(\rho_{504} + 8.032\ldots\cdot 10^{-46}\mathrm{i}) = \Delta_N(\rho_{504} + 1.417\ldots\cdot 10^{-46} + 8.032\ldots\cdot 10^{-46}\mathrm{i})$$
$$0 = \widetilde{\Delta}_N(\rho_{514} + 2.075\ldots\cdot 10^{-44}\mathrm{i}) = \Delta_N(\rho_{514} + 8.947\ldots\cdot 10^{-44} + 2.075\ldots\cdot 10^{-44}\mathrm{i})$$
$$0 = \widetilde{\Delta}_N(\rho_{524} + 3.700\ldots\cdot 10^{-39}\mathrm{i}) = \Delta_N(\rho_{524} - 3.645\ldots\cdot 10^{-39} + 3.700\ldots\cdot 10^{-39}\mathrm{i})$$
$$0 = \widetilde{\Delta}_N(\rho_{534} - 3.086\ldots\cdot 10^{-35}\mathrm{i}) = \Delta_N(\rho_{534} - 2.593\ldots\cdot 10^{-36} - 3.086\ldots\cdot 10^{-35}\mathrm{i})$$
$$0 = \widetilde{\Delta}_N(\rho_{544} + 3.016\ldots\cdot 10^{-32}\mathrm{i}) = \Delta_N(\rho_{544} + 2.510\ldots\cdot 10^{-32} + 3.016\ldots\cdot 10^{-32}\mathrm{i})$$
$$0 = \widetilde{\Delta}_N(\rho_{554} - 1.008\ldots\cdot 10^{-29}\mathrm{i}) = \Delta_N(\rho_{554} - 1.821\ldots\cdot 10^{-29} - 1.008\ldots\cdot 10^{-29}\mathrm{i})$$
$$0 = \widetilde{\Delta}_N(\rho_{564} + 2.537\ldots\cdot 10^{-27}\mathrm{i}) = \Delta_N(\rho_{564} + 5.988\ldots\cdot 10^{-27} + 2.537\ldots\cdot 10^{-27}\mathrm{i})$$
$$0 = \widetilde{\Delta}_N(\rho_{574} + 1.080\ldots\cdot 10^{-24}\mathrm{i}) = \Delta_N(\rho_{574} - 1.418\ldots\cdot 10^{-24} + 1.080\ldots\cdot 10^{-24}\mathrm{i})$$
$$0 = \widetilde{\Delta}_N(\rho_{584} - 1.608\ldots\cdot 10^{-22}\mathrm{i}) = \Delta_N(\rho_{584} + 1.559\ldots\cdot 10^{-22} - 1.608\ldots\cdot 10^{-22}\mathrm{i})$$
$$0 = \widetilde{\Delta}_N(\rho_{594} + 4.413\ldots\cdot 10^{-21}\mathrm{i}) = \Delta_N(\rho_{594} - 7.341\ldots\cdot 10^{-21} + 4.413\ldots\cdot 10^{-21}\mathrm{i})$$
$$0 = \widetilde{\Delta}_N(\rho_{604} - 2.462\ldots\cdot 10^{-19}\mathrm{i}) = \Delta_N(\rho_{604} + 2.157\ldots\cdot 10^{-19} - 2.462\ldots\cdot 10^{-19}\mathrm{i})$$

TABLE 3.3. Zeros of $\Delta_N(s)$ near trivial zeroes of the zeta function for $N = 503$.

$$0 = \Delta_N(-2 - 5.980\ldots\cdot 10^{-39})$$
$$0 = \Delta_N(-4 + 1.281\ldots\cdot 10^{-34})$$
$$0 = \Delta_N(-6 - 1.544\ldots\cdot 10^{-30})$$
$$0 = \Delta_N(-8 + 1.200\ldots\cdot 10^{-26})$$
$$0 = \Delta_N(-10 - 6.366\ldots\cdot 10^{-23})$$
$$0 = \Delta_N(-12 + 2.294\ldots\cdot 10^{-19})$$
$$0 = \Delta_N(-14 - 4.793\ldots\cdot 10^{-16})$$

TABLE 3.4. Zeros of $\Delta_N(s)$ near zeroes of the extra factor for $N = 503$.

$$0 = \Delta_N(1 + 20\pi i/\ln(2) - 2.997\ldots \cdot 10^{-21} + 5.866\ldots \cdot 10^{-21} i)$$
$$0 = \Delta_N(1 + 50\pi i/\ln(2) - 1.238\ldots \cdot 10^{-20} + 5.088\ldots \cdot 10^{-21} i)$$
$$0 = \Delta_N(1 + 80\pi i/\ln(2) - 1.305\ldots \cdot 10^{-20} - 4.309\ldots \cdot 10^{-21} i)$$
$$0 = \Delta_N(1 + 110\pi i/\ln(2) - 3.869\ldots \cdot 10^{-21} - 6.406\ldots \cdot 10^{-21} i)$$
$$0 = \Delta_N(1 + 140\pi i/\ln(2) - 3.921\ldots \cdot 10^{-22} + 2.350\ldots \cdot 10^{-21} i)$$
$$0 = \Delta_N(1 + 170\pi i/\ln(2) - 8.513\ldots \cdot 10^{-21} + 7.125\ldots \cdot 10^{-21} i)$$
$$0 = \Delta_N(1 + 200\pi i/\ln(2) - 1.447\ldots \cdot 10^{-20} - 1.705\ldots \cdot 10^{-22} i)$$
$$0 = \Delta_N(1 + 230\pi i/\ln(2) + 5.114\ldots \cdot 10^{-11} - 1.419\ldots \cdot 10^{-10} i)$$

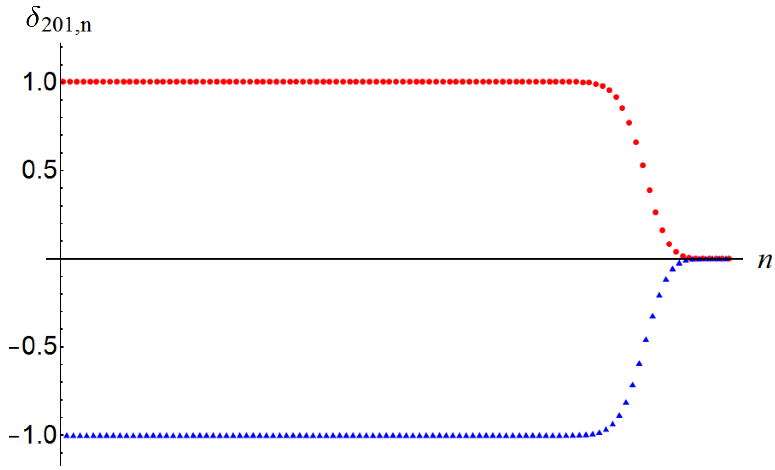

FIGURE 3.1. Numbers $\delta_{N,n}$, (disks for even n and triangles for odd n) for $N = 201$.

the latter series defines the *alternating zeta function* known also as the *Dirichlet eta function* $\eta(s)$. Such a very faint similarity relationship between two Dirichlet series will be symbolically recorded as

$$\Delta_N(s) \fallingdotseq \eta(s). \tag{3.2}$$

The zeros from Observation 3.4 are the zeros of the first factor on the right-hand side of (3.1). This factor vanishes also for $s = 1$, but this zero is canceled in (3.1) by the pole of the zeta function.

Informally, the observed phenomena can be described as follows: a few *initial nontrivial zeros of the zeta function "know about" a certain number of other nontrivial zeros, a certain number of initial trivial zeroes, and about the pole of the zeta function.*

Clearly, such phenomena do not occur for Taylor series – a polynomial the zeros of which are also zeroes of a certain meromorphic function does not deliver any information either about other zeros of this function or about its possible poles. Apparently, a significant role is played here by the fact that the zeta function is defined via a Dirichlet series. The strength of this property can be demonstrated, for example, by the *Hamburger theorem* stating that among all the functions that satisfy the functional equation (2.1) (and a few not very onerous additional conditions) the only function representable by a Dirichlet series is Riemann's zeta function.

§4. Almost linear relations

For a fixed value of N, one can observe a lot of almost linear relations among the numbers $\delta_{N,n}$ (these relations have either integral coefficients or rational coefficients with small denominators); two series of such relations are described below.

Consider the ratio

$$\nu_N(s) = \frac{\Delta_N(s)}{\zeta(s)} \tag{4.1}$$

$$= \frac{\sum_{n=1}^{N} \delta_{N,n} n^{-s}}{\sum_{n=1}^{\infty} n^{-s}} = \sum_{n=1}^{\infty} \mu_{N,n} n^{-s}. \tag{4.2}$$

It is easily seen that the numbers $\mu_{N,n}$ are linear combinations of the numbers $\delta_{N,n}$: extending the definition of $\delta_{N,n}$ by assuming that $\delta_{N,n} = 0$ for $n > N$, we have

$$\mu_{N,n} = \sum_{m|n} \mu\left(\tfrac{n}{m}\right) \delta_{N,m},$$

where $\mu(k)$ is the *Möbius function.*

In accordance with (3.2), one should expect that

$$\nu_N(s) \risingdotseq 1 - 2 \cdot 2^{-s},$$

and, indeed, there are the corresponding almost linear relations among the initial numbers $\delta_{N,n}$, see Table 4.1.

Now we consider the initial segments of the series (4.2):

$$\nu_{N,L}(s) = \sum_{n=1}^{L} \mu_{N,n} n^{-s}.$$

By (4.1) we have $\nu_N(1) = 0$, and, while the series (4.2), most likely, diverges, for relatively small L the values of $|\nu_{N,L}(1)|$ quickly decay with the growth of L, moreover, at the same rate as $|\mu_{N,L+1}/(L+1)|$.

Numerical observation 4.1. *For sufficiently small L, the ratio $\nu_{N,L}(1)/(\mu_{N,L+1}/(L+1))$ is close to -1.*

Table 4.2 demonstrates this observation which, obviously, produces yet another family of almost linear relations among the numbers $\delta_{N,n}$.

Table 4.1. Numbers $\mu_{N,n}$ for $N = 5001$.

$\delta_{N,3} - \delta_{N,1}$	$=$	$\mu_{N,3}$	$=$	$-1.047\ldots \cdot 10^{-208}$
$\delta_{N,4} - \delta_{N,2}$	$=$	$\mu_{N,4}$	$=$	$-3.507\ldots \cdot 10^{-479}$
$\delta_{N,5} - \delta_{N,1}$	$=$	$\mu_{N,5}$	$=$	$+1.808\ldots \cdot 10^{-699}$
$\delta_{N,6} - \delta_{N,3} - \delta_{N,2} + \delta_{N,1}$	$=$	$\mu_{N,6}$	$=$	$-4.322\ldots \cdot 10^{-872}$
$\delta_{N,7} - \delta_{N,1}$	$=$	$\mu_{N,7}$	$=$	$+1.554\ldots \cdot 10^{-1007}$
$\delta_{N,8} - \delta_{N,4}$	$=$	$\mu_{N,8}$	$=$	$-1.360\ldots \cdot 10^{-1114}$
$\delta_{N,9} - \delta_{N,3}$	$=$	$\mu_{N,9}$	$=$	$-8.367\ldots \cdot 10^{-1200}$
$\delta_{N,10} - \delta_{N,5} - \delta_{N,2} + \delta_{N,1}$	$=$	$\mu_{N,10}$	$=$	$+8.322\ldots \cdot 10^{-1268}$
$\delta_{N,11} - \delta_{N,1}$	$=$	$\mu_{N,11}$	$=$	$+6.055\ldots \cdot 10^{-1317}$
$\delta_{N,12} - \delta_{N,6} - \delta_{N,4} + \delta_{N,2}$	$=$	$\mu_{N,12}$	$=$	$-2.228\ldots \cdot 10^{-1361}$

TABLE 4.2. Ratios $\nu_{N,L}/(\mu_{N,L+1}/(L+1))$.

L	$\nu_{N,L}(1)/(\mu_{N,L+1}/(L+1))+1$		
	$N=3001$	$N=4001$	$N=5001$
1	$+1.335\ldots\cdot 10^{-125}$	$-3.944\ldots\cdot 10^{-168}$	$-3.492\ldots\cdot 10^{-209}$
2	$-7.059\ldots\cdot 10^{-161}$	$+6.245\ldots\cdot 10^{-216}$	$-2.510\ldots\cdot 10^{-271}$
3	$+1.803\ldots\cdot 10^{-127}$	$+1.197\ldots\cdot 10^{-174}$	$+4.125\ldots\cdot 10^{-221}$
4	$-5.663\ldots\cdot 10^{-99}$	$-2.907\ldots\cdot 10^{-136}$	$+1.991\ldots\cdot 10^{-173}$
5	$-3.823\ldots\cdot 10^{-76}$	$+9.485\ldots\cdot 10^{-106}$	$+3.082\ldots\cdot 10^{-136}$
6	$+4.778\ldots\cdot 10^{-58}$	$+1.963\ldots\cdot 10^{-83}$	$+7.657\ldots\cdot 10^{-108}$
7	$+1.065\ldots\cdot 10^{-38}$	$-1.528\ldots\cdot 10^{-63}$	$-5.467\ldots\cdot 10^{-86}$
8		$+2.608\ldots\cdot 10^{-48}$	$+8.951\ldots\cdot 10^{-69}$
9			$-1.836\ldots\cdot 10^{-48}$

§5. Arithmetical properties of the numbers $\delta_{N,n}$

The numbers $\delta_{N,n}$ (defined by (2.7) and (2.8)) are real. Nevertheles, they have definite number-theoretical meaning, in particular, they convey information about prime numbers, and this information can be extracted from them in diverse ways.

5.1. Sieve of Eratosthenes. Figure 5.1 shows $\log_{10}(|\delta_{5001,n}-1|)$. Visually, the points on this graph are situated at four *levels*:

- points from the top level correspond to the even values of n;
- points from the next level correspond to the odd values of n divisible by 3;
- the third level contains points corresponding to the values of n that are divisible by 5 but not divisible by 2 or by 3;
- the fourth level contains points corresponding to the values of n that are divisible by 7, but are relatively prime to $2\cdot 3\cdot 5$;
- points from the lowest level correspond to the values of n that are greater than 1 and are relatively prime to $2\cdot 3\cdot 5\cdot 7$.

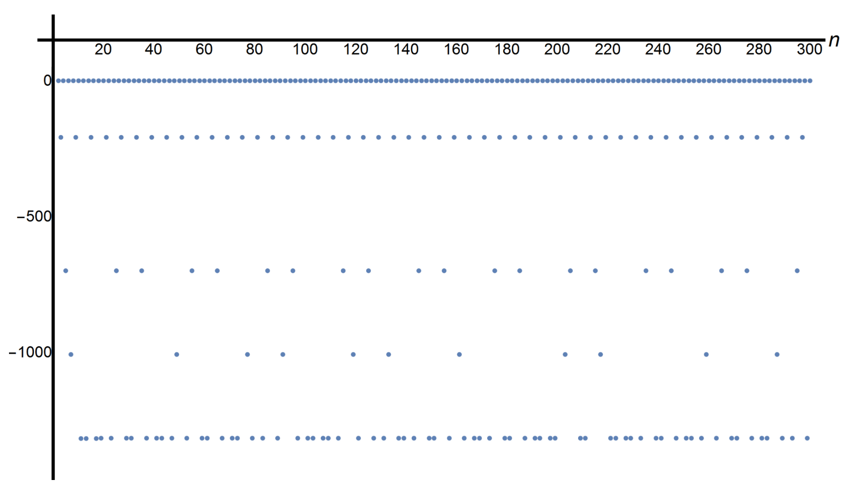

FIGURE 5.1. Numbers $\log_{10}(|\delta_{N,n}-1|)$ for $N=5001$.

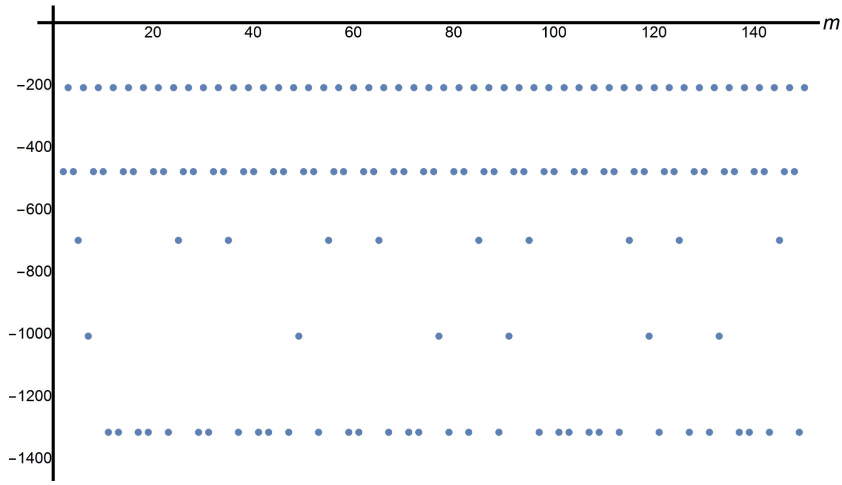

FIGURE 5.2. $\log_{10}(|\delta_{N,2m} - \delta_{N,2}|)$ for $N = 5001$.

In other words, the appearance of the point corresponding to a particular number at a certain level is governed by the smallest prime factor of this number, that is, there is a correspondence with steps in the construction of the sieve of Eratosthenes. For this reason, all levels but the lowest one will be called the *Eratosthenes levels*, and the lowest level will be called *post-Eratosthenes level.* The number of Eratosthenes levels increases when N grows up.

In Figure 5.1, it is impossible to notice that the ordinates of the points from the same Eratosthenes level are not equal. In fact, each such level splits in its turn into sublevels. Figure 5.2 shows values of $\log_{10}(|\delta_{5001,2m} - \delta_{5001,2}|)$. Again, we observe a picture similar to the sieve of Eratosthenes, but the rules for allocation of points to levels are a bit different. The changes affected the first two levels:

- points from the top level now correspond to multiples of 3;
- points from the next level correspond to even values of n not divisible by 3.

In other words, the sieve of Eratosthenes is now constructed in accordance with the following order of prime numbers: 3, 2, 5, 7, ...

Numerical observation 5.1. *Let n run over the arithmetical progression $d, 2d, \ldots, md, \ldots,$ where $d = 2^{k_2}3^{k_3}5^{k_5}\ldots$. For sufficiently large N, the corresponding Eratosthenes sublevel further splits into sublevels of the next rank in accordance with the divisibility of m by $q_1,\ q_2, \ldots,$ where these prime numbers are ordered in such a way that*

$$q_1^{k_{q_1}+1} < q_2^{k_{q_2}+1} < \cdots < q_j^{k_{q_j}+1} < \ldots$$

In other words, the behavior of the numbers $\delta_{N,n}$ has some fractal structure.

Figure 5.3 extends Figure 5.1 to larger values of n. We see that all levels are cut off suddenly when they reach some mysterious curve.

5.2. Inheritable divisors. Figure 5.4 shows, for $N = 2001$, numbers $\log_{10}|\alpha_{N,m}|$, where

$$\alpha_{N,m} = \sum_{n=1}^{m} \delta_{N,n}.$$

Once again, we observe points distributed on several levels, but now the rules are quite different:

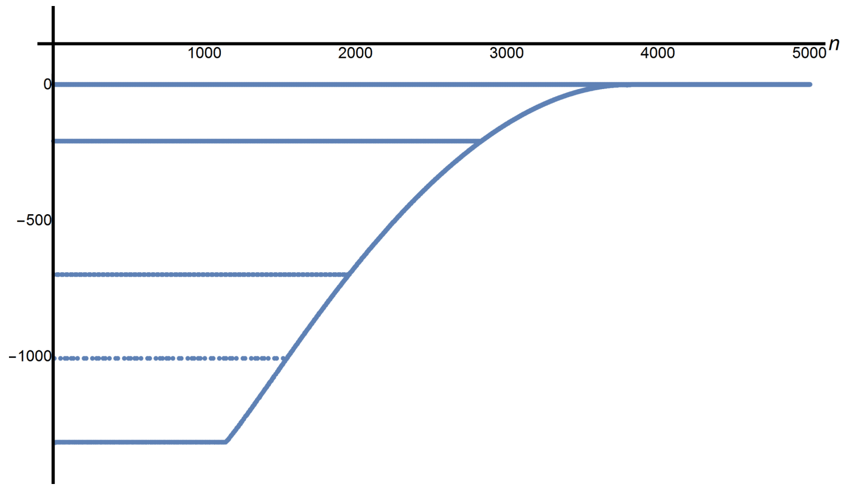

FIGURE 5.3. Numbers $\log_{10}(|\delta_{N,n} - 1|)$ for $N = 5001$.

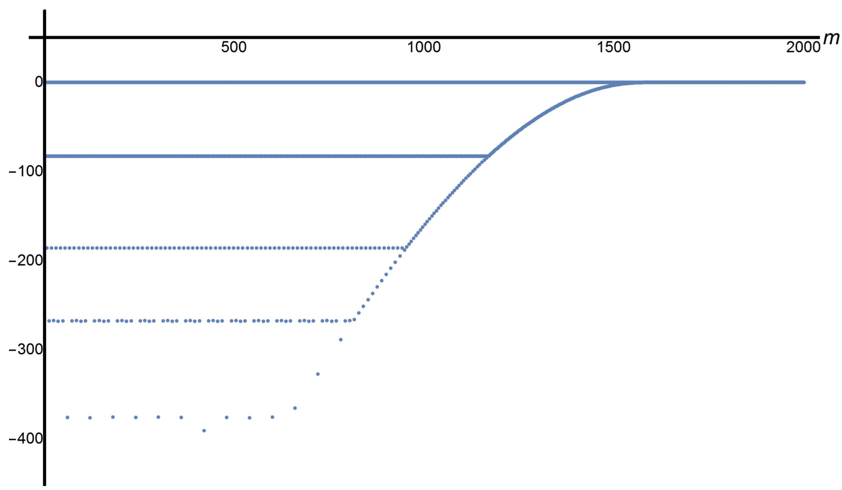

FIGURE 5.4. Numbers $\log_{10}|\alpha_{N,m}|$ for $N = 2001$.

- points from the top level correspond to the odd values of m;
- points from the next level corresponds to the even values of m not divisible by 3;
- the third level contains points corresponding to the values of m that are divisible by 6 but not divisible by 12;
- the forth level contains points corresponding to the values of m that are divisible by 12 but not divisible by 60;
- the fifth level contains points corresponding to the values of m that are divisible by 60 but divisible by 420;

In order to describe the levels formally, we introduce the notion of an *inheritable divisor* of a number m: this is any its divisor k such that all smaller natural numbers also divide m; symbolically this will be written as $k_{\leq}|m$.

Numerical observation 5.2. *The numbers $\alpha_{N,m}$ are distributed by levels in accordance with the maximal inheritable divisors of the index m.*

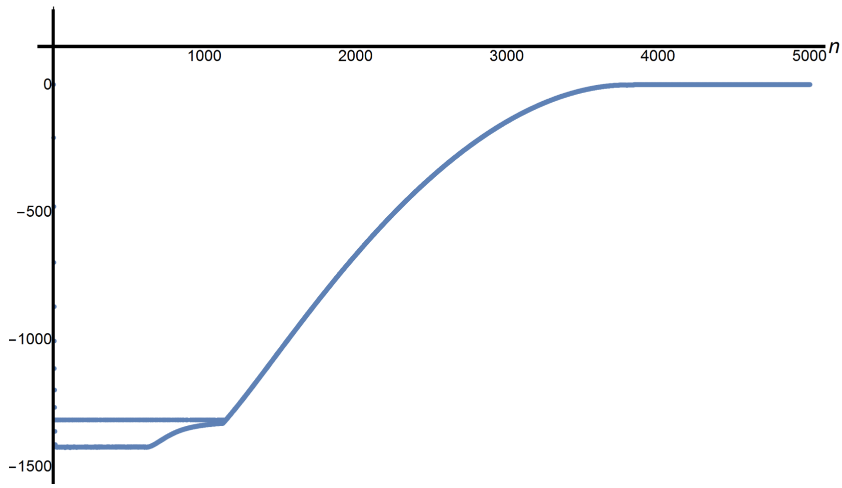

FIGURE 5.5. Numbers $\log_{10}(|\mu_{N,n}|)$ for $N = 5001$.

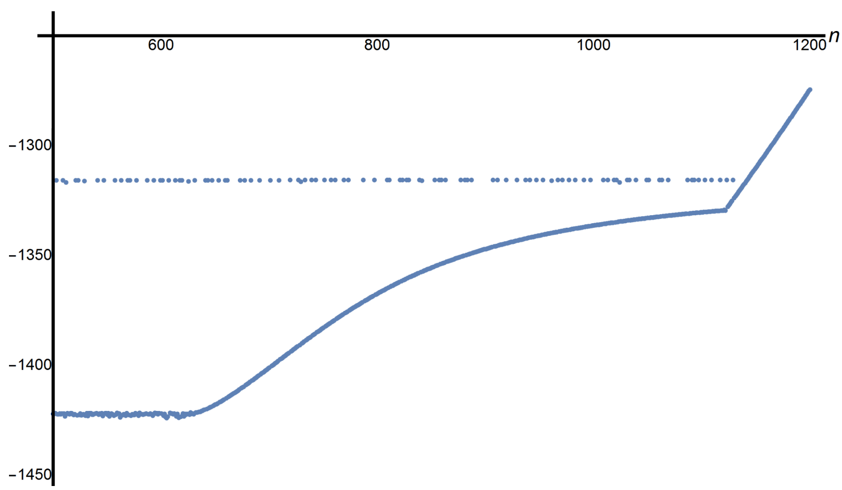

FIGURE 5.6. Numbers $\log_{10}(|\mu_{N,n}|)$ for $N = 5001$.

The levels of the numbers $\alpha_{N,n}$ have, like the numbers $\delta_{N,n}$, their own fractal structure.

5.3. Von Mangoldt function. Figures 5.5–5.7 show the values of $\log_{10}(|\mu_{N,n}|)$ for $N = 5001$ and diverse n. If n is a prime, then $\mu_{N,n} = \delta_{N,n} - 1$ and the corresponding points lie at the same places as on Figures 5.3 and 5.1. All other points move down and form the *postpost-Eratosthenes level* lying below the post-Eratosthenes level.

The range $1, \ldots, N$ of values of n can be split into 6 diapasons.

Diapason 1. In this diapason the values of $|\mu_{N,n}|$ quickly decay.

Diapason 2. In this diapason fibering begins: for certain n (Case (a)), the values $|\mu_{N,n}|$ still lie at the post-Eratosthenes level, for other n (Case (b)) they lie in between the post-Eratosthenes level and the postpost-Eratosthenes level.

Diapason 3. In this diapason there is a clear fibering for the post-Eratosthenes (Case (a)) and postpost-Eratosthenes (Case (b)) levels.

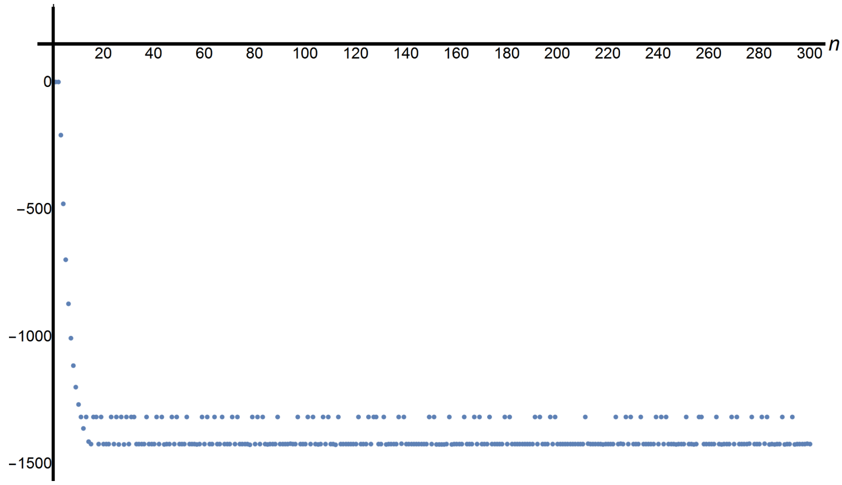

FIGURE 5.7. Numbers $\log_{10}(|\mu_{N,n}|)$ for $N = 5001$.

Diapason 4. In this diapason the fibering still exsits, for certain n (Case (a)), the values $|\mu_{N,n}|$ continue to lie at the post-Eratosthenes level, but for other n (Case (b)) values of $|\mu_{N,n}|$ start increasing until reaching the post-Eratosthenes level.

Diapason 5. The fibering is over, the increasing values of $|\mu_{N,n}|$ lie at the same misterious curve that was observerd earlier in Figure 5.3.

Diapason 6. Here the values of $\mu_{N,n}$ are close to ± 1.

It turns out that in diapasons 2–4 Cases (a) occur if and only if n is either a prime or a power of a prime; moreover, all numbers $\mu_{N,n}$ have the same sign.

Numerical observation 5.3. *For n in diapasons 2–4,*

$$\mu_{N,n} \approx \frac{\mu_{N,p}}{\ln(p)}\Lambda(n), \tag{5.1}$$

where $\Lambda(n)$ is the von Mangoldt function, and p is an arbitrary prime number from diapason 3.

In accordance with this observation, for all primes p the ratios $\omega_{N,p} = \mu_{N,p}/\ln(p)$ have almost the same value, and in approximate identities we shall write simply ω_N.

The approximate identity (5.1) indicates yet another way in which zeta zeros encode information about prime numbers. For $N = 5001$, the quality of this approximation in diapason 3 is demonstrated by the following inequalities: for $16 \le n \le 634$ we have

$$-10^{-1421} < \mu_{N,n} - \omega_N\Lambda(n) < 10^{-1421},$$

in particular, if n is a power of a prime, then

$$1 - 10^{-104} < \frac{\mu_{N,n}}{\omega_N\Lambda(n)} < 1 + 10^{-104},$$

where $\omega_{5001} \approx 2.5256.. \cdot 10^{-1317}$.

For $n > N$, the numbers $\mu_{N,n}$ show interesting but at present little understood patterns, see Figure 5.8.

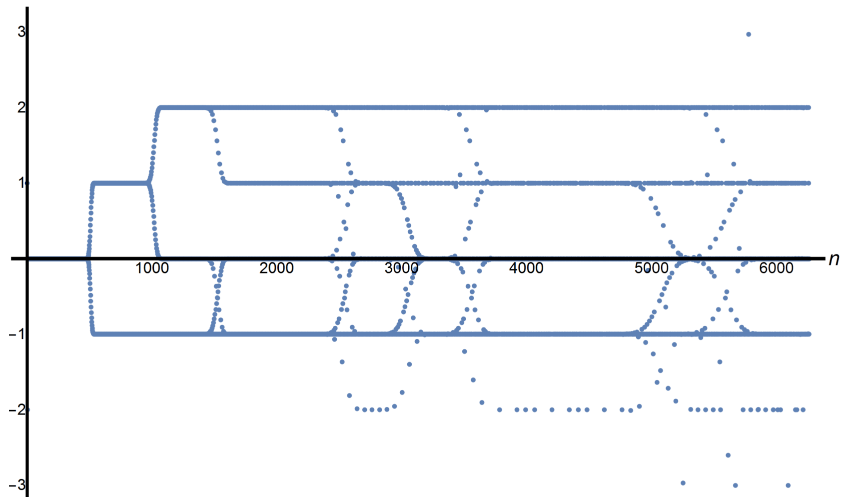

FIGURE 5.8. Numbers $\mu_{N,n}$ for $N = 626$.

§6. APPROXIMATIONS OF THE ZETA FUNCTION AND ITS FIRST DERIVATIVE

The left-hand and right-hand sides in (3.2) are connected via the relation $\leftrightharpoons$, which gives a very weak relationship of approximate termwise equality of two Dirichlet series. Nevertheless, in some cases this relation implies approximate numerical equality of the two sides.

6.1. Values of $\eta(s)$. The infinite alternating series for this function (the left-hand side in (3.1)) converges for $\mathrm{Re}(s) > 0$ only, but the finite sum on the left-hand side in (3.2) is defined for all s.

Numerical observation 6.1. *Values of $\eta(s)$ are well approximated by values of $\Delta_N(s)$ for sufficiently large N and not too large values of $|s|$, no matter whether s lies to the right of the critical strip, inside it, or to the left of it; the domain of good approximation and its accuracy increase with the growth of N.*

Table 6.1 demonstrates the accuracy of this approximation.

6.2. Values of $\zeta(s)$. Naturally, observation 6.1 can be reformulated as the statement that values of $\zeta(s)$ are well approximated by the ratio $\Delta_N(s)/(1 - 2 \cdot 2^{-s})$. Moreover, one can obtain even better approximation by replacing the denominator of this fraction by an initial segment of the series (4.2).

Numerical observation 6.2. *Values of $\zeta(s)$ are well approximated by the ratio of two finite series*

$$\frac{\Delta_N(s)}{\nu_{N,L}(s)} = \frac{\sum_{n=1}^{N} \delta_{N,n} n^{-s}}{\sum_{n=1}^{L} \mu_{N,n} n^{-s}} \tag{6.1}$$

for sufficiently large N, moderate L, and not too large value of $|s|$, no matter whether s lies to the right of the critical strip, inside it, or to the left of it; the domain of good approximation and its accuracy increase with the growth of N and with the growth of L up to certain limit increasing with the growth of N.

Table 6.2 demonstrates this observation. The accuracy of the approximation depends on L in a way similar to the accuracy of the initial segments of the asymptotic expansion

TABLE 6.1. Approximations of $\eta(s)$.

s	$\lvert\eta(s)/\Delta_N(s)-1\rvert$		
	$N = 3001$	$N = 4001$	$N = 5001$
1000	$2.492\ldots\cdot 10^{-426}$	$7.361\ldots\cdot 10^{-469}$	$6.519\ldots\cdot 10^{-510}$
-101	$1.221\ldots\cdot 10^{-107}$	$3.608\ldots\cdot 10^{-150}$	$3.195\ldots\cdot 10^{-191}$
-301	$2.019\ldots\cdot 10^{-72}$	$5.963\ldots\cdot 10^{-115}$	$5.281\ldots\cdot 10^{-156}$
$1/2+10\mathrm{i}$	$4.374\ldots\cdot 10^{-125}$	$1.291\ldots\cdot 10^{-167}$	$1.143\ldots\cdot 10^{-208}$
$1/2+100\mathrm{i}$	$8.710\ldots\cdot 10^{-125}$	$2.572\ldots\cdot 10^{-167}$	$2.278\ldots\cdot 10^{-208}$
$1/2+1000\mathrm{i}$	$2.048\ldots\cdot 10^{-125}$	$6.047\ldots\cdot 10^{-168}$	$5.356\ldots\cdot 10^{-209}$
$1/2+5000\mathrm{i}$	$2.157\ldots\cdot 10^{-109}$	$4.696\ldots\cdot 10^{-168}$	$4.159\ldots\cdot 10^{-209}$
$1/4+10\mathrm{i}$	$4.445\ldots\cdot 10^{-125}$	$1.312\ldots\cdot 10^{-167}$	$1.162\ldots\cdot 10^{-208}$
$1/4+100\mathrm{i}$	$7.179\ldots\cdot 10^{-125}$	$2.120\ldots\cdot 10^{-167}$	$1.877\ldots\cdot 10^{-208}$
$1/4+1000\mathrm{i}$	$2.304\ldots\cdot 10^{-125}$	$6.806\ldots\cdot 10^{-168}$	$6.027\ldots\cdot 10^{-209}$
$1/4+5000\mathrm{i}$	$3.313\ldots\cdot 10^{-109}$	$5.325\ldots\cdot 10^{-168}$	$4.716\ldots\cdot 10^{-209}$
$-10+10\mathrm{i}$	$1.163\ldots\cdot 10^{-123}$	$3.437\ldots\cdot 10^{-166}$	$3.044\ldots\cdot 10^{-207}$
$-10+100\mathrm{i}$	$1.168\ldots\cdot 10^{-123}$	$3.451\ldots\cdot 10^{-166}$	$3.056\ldots\cdot 10^{-207}$
$-10+1000\mathrm{i}$	$1.168\ldots\cdot 10^{-123}$	$3.449\ldots\cdot 10^{-166}$	$3.054\ldots\cdot 10^{-207}$
$-10+5000\mathrm{i}$	$4.339\ldots\cdot 10^{-108}$	$3.431\ldots\cdot 10^{-166}$	$3.039\ldots\cdot 10^{-207}$
$-100+10\mathrm{i}$	$8.145\ldots\cdot 10^{-108}$	$2.405\ldots\cdot 10^{-150}$	$2.130\ldots\cdot 10^{-191}$
$-100+100\mathrm{i}$	$8.145\ldots\cdot 10^{-108}$	$2.405\ldots\cdot 10^{-150}$	$2.130\ldots\cdot 10^{-191}$
$-100+1000\mathrm{i}$	$8.145\ldots\cdot 10^{-108}$	$2.405\ldots\cdot 10^{-150}$	$2.130\ldots\cdot 10^{-191}$
$-100+5000\mathrm{i}$	$1.674\ldots\cdot 10^{-97}$	$2.405\ldots\cdot 10^{-150}$	$2.130\ldots\cdot 10^{-191}$

TABLE 6.2. Approximations of $\zeta(s)$ for $s = 1/2 + 1000\mathrm{i}$.

L	$\lvert\zeta(s)/(\Delta_N(s)/\nu_{N,L}(s))-1\rvert$		
	$N = 3001$	$N = 4001$	$N = 5001$
2	$1.133\ldots\cdot 10^{-125}$	$3.345\ldots\cdot 10^{-168}$	$2.963\ldots\cdot 10^{-209}$
3	$9.236\ldots\cdot 10^{-286}$	$2.412\ldots\cdot 10^{-383}$	$8.589\ldots\cdot 10^{-480}$
4	$1.862\ldots\cdot 10^{-412}$	$3.231\ldots\cdot 10^{-557}$	$3.961\ldots\cdot 10^{-700}$
5	$1.155\ldots\cdot 10^{-510}$	$1.029\ldots\cdot 10^{-692}$	$8.643\ldots\cdot 10^{-873}$
6	$4.770\ldots\cdot 10^{-586}$	$1.054\ldots\cdot 10^{-797}$	$2.877\ldots\cdot 10^{-1008}$
7	$2.436\ldots\cdot 10^{-643}$	$2.213\ldots\cdot 10^{-880}$	$2.355\ldots\cdot 10^{-1115}$
8	$7.567\ldots\cdot 10^{-682}$	$3.588\ldots\cdot 10^{-943}$	$1.366\ldots\cdot 10^{-1200}$
9	$7.170\ldots\cdot 10^{-682}$	$2.332\ldots\cdot 10^{-991}$	$1.288\ldots\cdot 10^{-1268}$
10	$7.170\ldots\cdot 10^{-682}$	$2.397\ldots\cdot 10^{-991}$	$5.593\ldots\cdot 10^{-1317}$
11	$6.881\ldots\cdot 10^{-682}$	$2.300\ldots\cdot 10^{-991}$	$5.367\ldots\cdot 10^{-1317}$

of a function — there is an optimal amount of summands. To get a good approximation in (6.1) one can take for L any number from diapasons 1 or 2.

6.3. Values of $\zeta'(s)$. Since $\zeta(s)$ is well approximated by the ratio (6.1), we can differentiate it to get an approximation to the derivatives of the zeta function. This subsection

TABLE 6.3. Approximations of $\zeta'(\rho_k)$.

k	$\lvert\zeta'(\rho_k)/(\Delta_N(\rho_k)/\omega_N)+1\rvert$		
	$N = 3001$	$N = 4001$	$N = 5001$
1	$6.083\ldots\cdot 10^{-2}$	$3.247\ldots\cdot 10^{-2}$	$3.785\ldots\cdot 10^{-2}$
3	$3.022\ldots\cdot 10^{-4}$	$1.968\ldots\cdot 10^{-4}$	$8.142\ldots\cdot 10^{-5}$
10	$1.659\ldots\cdot 10^{-9}$	$9.476\ldots\cdot 10^{-10}$	$7.453\ldots\cdot 10^{-9}$
30	$1.565\ldots\cdot 10^{-17}$	$1.928\ldots\cdot 10^{-17}$	$7.436\ldots\cdot 10^{-17}$
100	$1.166\ldots\cdot 10^{-36}$	$7.915\ldots\cdot 10^{-38}$	$1.298\ldots\cdot 10^{-38}$
300	$1.482\ldots\cdot 10^{-69}$	$6.546\ldots\cdot 10^{-76}$	$1.982\ldots\cdot 10^{-78}$
1000	$5.494\ldots\cdot 10^{-74}$	$3.915\ldots\cdot 10^{-91}$	$1.605\ldots\cdot 10^{-106}$
1500	$2.050\ldots\cdot 10^{-74}$	$3.293\ldots\cdot 10^{-91}$	$2.153\ldots\cdot 10^{-106}$
2000	$3.008\ldots\cdot 10^{-74}$	$1.316\ldots\cdot 10^{-91}$	$8.729\ldots\cdot 10^{-107}$
2500	$2.291\ldots\cdot 10^{-63}$	$8.623\ldots\cdot 10^{-91}$	$1.064\ldots\cdot 10^{-106}$
3000		$1.599\ldots\cdot 10^{-91}$	$1.337\ldots\cdot 10^{-106}$
4000			$3.356\ldots\cdot 10^{-92}$

presents yet another, rather unexpected way of calculating the first derivative of this function via the numbers $\delta_{N,n}$.

The approximation (5.1) fails for n from diapason 1 and is not very accurate for n from diapason 2. "Adjusting" the series (4.2) for a few initial values of n, we can write

$$\nu_N(s) - \sum_{n=1}^{M} \big(\mu_{N,m} - \omega_N\Lambda(n)\big)n^{-s} \risingdotseq \sum_{n=1}^{\infty} \omega_N\Lambda(n)n^{-s} = -\omega_N\frac{\zeta'(s)}{\zeta(s)}, \tag{6.2}$$

where M is any number from diapason 3. The relationship between the two Dirichlet series in (6.2), expressed by the symbol $\risingdotseq$, is very weak — the approximate equality of coefficiens occurs for n from diapasons 1–4, but already in diapason 5 the values $\mu_{N,n}$ rapidly increase (see Figure 5.5 for a numerical example). However, surprisingly, this so weak relationship suggests very good numerical approximations.

Multiplying both sides in (6.2) by $-\zeta(s)/\omega_N$, we get

$$-\omega_N^{-1}\left(\Delta_N(s) - \zeta(s)\sum_{n=1}^{M}\big(\mu_{N,n} - \omega_N\Lambda(n)\big)n^{-s}\right) \risingdotseq \zeta'(s), \tag{6.3}$$

which immediately gives a method for calculating $\zeta'(s)$ when s is a zero of the zeta function.

Numerical observation 6.3. *The ratio $-\Delta_N(\rho_k)/\omega_N$ is a good approximation to $\zeta'(\rho_k)$, provided that k is not too far from $N/2$.*

Table 6.3 demonstrate the accuracy of this approximation.

In order to calculate $\zeta'(s)$ for generic s, we need to use the adjusting finite sum from the left-hand side of (6.2).

Numerical observation 6.4. *For large N, the left-hand side of* (6.3) *is a good approximation to $\zeta'(s)$ for s lying to the right, inside, or to the left of the critical strip and having not too big and not too small absolute value.*

Table 6.4 demonstrates the accuracy of this approximation. However, we have the following disadvantage: to calculate $\zeta'(s)$ we need to have very precise value of $\zeta(s)$,

TABLE 6.4. Approximations of $\zeta'(s)$ for $M = 19$.

s	$\left\|\omega_N^{-1}\left(\Delta_N(s) - \zeta(s)\sum_{n=1}^{M}(\mu_{N,n} - \omega_N\Lambda(n))n^{-s}\right)/\zeta'(s) + 1\right\|$		
	$N = 3001$	$N = 4001$	$N = 5001$
2	$6.769\ldots\cdot 10^{-4}$	$5.843\ldots\cdot 10^{-4}$	$4.656\ldots\cdot 10^{-4}$
1/2 + 100i	$1.206\ldots\cdot 10^{-16}$	$3.032\ldots\cdot 10^{-17}$	$1.785\ldots\cdot 10^{-16}$
−1/2 + 100i	$1.420\ldots\cdot 10^{-14}$	$5.150\ldots\cdot 10^{-15}$	$3.172\ldots\cdot 10^{-14}$
1/4 + 200i	$8.851\ldots\cdot 10^{-31}$	$1.275\ldots\cdot 10^{-32}$	$5.935\ldots\cdot 10^{-33}$
−3/4 + 300i	$7.955\ldots\cdot 10^{-44}$	$6.302\ldots\cdot 10^{-45}$	$6.774\ldots\cdot 10^{-46}$
1/2 + 1000i	$1.510\ldots\cdot 10^{-74}$	$7.345\ldots\cdot 10^{-92}$	$4.463\ldots\cdot 10^{-107}$
−1/4 + 1000i	$1.647\ldots\cdot 10^{-73}$	$2.041\ldots\cdot 10^{-90}$	$1.127\ldots\cdot 10^{-106}$
−3/2 + 2000i	$8.531\ldots\cdot 10^{-73}$	$1.504\ldots\cdot 10^{-89}$	$1.034\ldots\cdot 10^{-104}$
−3 + 3000i	$4.889\ldots\cdot 10^{-63}$	$3.156\ldots\cdot 10^{-88}$	$6.425\ldots\cdot 10^{-104}$
−20 + 3000i	$1.085\ldots\cdot 10^{-42}$	$5.065\ldots\cdot 10^{-67}$	$9.010\ldots\cdot 10^{-83}$

TABLE 6.5. The number of correct decimal digits of the solution of system (6.4)–(6.5) for $N_1 = 3001$, $N_2 = 5001$, $M = 19$.

s	$\zeta(s)$	$\zeta'(s)$
2	994	3
1/2 + 100i	1006	15
− 1/2 + 100i	1004	13
1/4 + 200i	1020	30
− 3/4 + 300i	1033	43
1/2 + 1000i	1064	73
− 1/4 + 1000i	1063	72
− 3/2 + 2000i	1063	72
− 3 + 3000i	1053	62
− 20 + 3000i	1038	41

TABLE 6.6. The number of correct decimal digits of the solution of system (6.6)–(6.9) for $N = 5001$, $M = 19$.

s	$\zeta(s)$	$\zeta(1-s)$	$\zeta'(s)$	$\zeta'(1-s)$
1/2 + 100i	1006	1006	16	16
− 1/2 + 100i	1005	1005	14	13
1/4 + 200i	1022	1022	32	31
− 3/4 + 300i	1034	1034	44	43
1/2 + 1000i	1081	1081	91	91
− 1/4 + 1000i	1080	1080	90	88
− 3/2 + 2000i	1079	1079	89	88
− 3 + 3000i	1078	1078	88	86
− 20 + 3000i	1062	1062	72	65

because (6.3) is the difference between two very close numbers. The accuracy produced by approximation (6.1) is not sufficient for this purpose. To work around this problem, we can calculate $\zeta(s)$ via (6.1) with a larger value of N. Another way is to use $\delta_{N,n}$ for two values $N = N_1$ and $N = N_2$, produce from (6.3) two approximate relations

$$-\omega_{N_1}^{-1}\left(\Delta_{N_1}(s) - \zeta(s)\sum_{n=1}^{M_1}\big(\mu_{N_1,m} - \omega_{N_1}\Lambda(n)\big)n^{-s}\right) \approx \zeta'(s), \tag{6.4}$$

$$-\omega_{N_2}^{-1}\left(\Delta_{N_1}(s) - \zeta(s)\sum_{n=1}^{M_2}\big(\mu_{N_2,m} - \omega_{N_2}\Lambda(n)\big)n^{-s}\right) \approx \zeta'(s), \tag{6.5}$$

and solve this system of two linear equations treating $\zeta(s)$ and $\zeta'(s)$ as uknowns. Table 6.5 demonstrates the accuracy of this approximation.

Instead of using two different values of N, we can do with a single N by using the functional equation. This allows us to write, instead of two of equations (6.4)–(6.5), four equations

$$-\,\omega_{N}^{-1}\left(\Delta_{N}(s) - \zeta(s)\sum_{n=1}^{M}\big(\mu_{N,m} - \omega_{N}\Lambda(n)\big)n^{-s}\right) \approx \zeta'(s), \tag{6.6}$$

$$-\,\omega_{N}^{-1}\left(\Delta_{N}(1-s) - \zeta(1-s)\sum_{n=1}^{M}\big(\mu_{N,m} - \omega_{N}\Lambda(n)\big)n^{s-1}\right) \approx \zeta'(1-s), \tag{6.7}$$

$$g(s)\zeta(s) = g(1-s)\zeta(1-s), \tag{6.8}$$

$$g'(s)\zeta(s) + g(s)\zeta'(s) = -g'(1-s)\zeta(1-s) - g(1-s)\zeta'(1-s), \tag{6.9}$$

binding the four "unknowns" $\zeta(s)$, $\zeta(1-s)$, $\zeta'(s)$, and $\zeta'(1-s)$. Table 6.6 demonstrates the accuracy of this approximation.

6.4. Values of $\nu_{N,N}(1)$. *The Lerch function*, defined by

$$\Psi(z,s,a) = \sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s},$$

for $z = a = 1$ coincides with the zeta function $\zeta(s)$, and for $z = -1$, $a = 1$ it is identical to the alternating zeta function $\eta(s)$. For the latter choice of the first two arguments and positive integral values of the third argument, there is yet another relationship between the Lerch function and zeta function.

Numerical observation 6.5. *For sufficiently large N, the ratio $\nu_{N,N}(1)/\Psi(-1,1,N+1)$ is very close to $(-1)^N$.*

Table 6.7 demonstrates this observation. We see that the values of $-\nu_{N,N}(1)$, which are originally defined via zeros of the "sophisticated" zeta function, admit approximate calculations via the "plain" sum

$$\sum_{n=N+1}^{\infty}\frac{(-1)^n}{n}. \tag{6.10}$$

By (4.1) we have $\nu_N(1) = 0$, hence the "expected" value of the sum

$$\sum_{n=N+1}^{\infty}\frac{\mu_{N,n}}{n} \tag{6.11}$$

is equal to $-\nu_{N,N}(1)$, i. e., is close to (6.10). Comparing (6.11) with (6.10), one might expect that for $n > N$ the numbers $\mu_{N,n}$ behave approximately as $(-1)^n$; however, Figure 5.8 show that this is not the case.

TABLE 6.7. Approximations of $\nu_{N,N}(1)$.

N	$\nu_{N,N}(1)/\Psi(-1,1,N+1)$
626	$+1-4.631\ldots\cdot 10^{-25}$
1251	$-1+4.144\ldots\cdot 10^{-54}$
1876	$+1+2.053\ldots\cdot 10^{-78}$
2501	$-1+4.424\ldots\cdot 10^{-105}$
3126	$+1-3.306\ldots\cdot 10^{-131}$
3751	$-1-3.680\ldots\cdot 10^{-157}$
4376	$+1-2.203\ldots\cdot 10^{-183}$
5001	$-1+1.047\ldots\cdot 10^{-208}$

§7. COMPUTATION ASPECTS

The matrices in (2.6) are almost singular, some of numbers $\mu_{N,n}$ and $\nu_{N,n}$ are, in absolute value, many orders less than $\delta_{N,n}$. This is the reason why for discovering the phenomena described above calculations should be performed with thousands of decimal digits. In particular, the first 40000 nontrivial zeta zeros were calculated with 40000 decimal digits, they are available at [2].

Moreover, one should work with matrices having thousands of rows and columns, because many effects have asymptotic character and can be seen only if the matrices that determine the numbers $\delta_{N,n}$ have sufficiently large size.

For computing determinants, there are many software packages, but none of them looked appropriate for our calculations. On the one hand, the great accuracy and the large size of matricies made it impossible to keep them entirely in RAM. On the other hand, the expansion (2.7) required the simultaneous computation of N minors of the matrix (2.6), containing obviously many common subexpressions. In view of the latter, an algorithm was developed for calculating simultaneously all the numbers $\delta_{N,n}$ for $n \leq N \leq M$; this algorithm has the complexity of the same order as the Gauss elimination for a system of M linear equations with M unknowns (see [1] for the details).

ACKNOWLEDGMENT

Initially, the calculation of the nontrivial zeros of the zeta function were performed by MATHEMATICA and SAGE. The accuracy of 40,000 decimal places was achieved thanks to the efficient algorithm that was implemented by Fredrik Johansson in his system ARB [4].

For calculation, computers of the following institutions were used: St.Petersburg Institute for Informatics and Automation of the Russian Academy of Sciences, Armenian National Grid Initiative Foundation, Isaac Newton Institute for Mathematical Sciences, Laboratoire d'Algorithmique, Complexité et Logique (Université Paris-Est Créteil), Laboratoire d'Informatique Algorithmique: Fondements et Applications (CNRS and Université Paris Diderot–Paris 7), Multi-modal Australian Sciences Imagine and Visualisation Environment, Wolfram Research.

The most time-consuming part of computing, calculation of the numbers $\Delta_{N,n}$, was performed by the author on the supercomputer "Chebyshev" of Moscow State University, and by Gleb Beliakov on Multi-modal Australian Sciences Imagine and Visualisation Environment.

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