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SUPERCHARACTER THEORY FOR GROUPS OF INVERTIBLE ELEMENTS OF REDUCED ALGEBRAS

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To Sergeĭ Vladimirovich Vostokov on the occasion of his 70th birthday

Abstract. A supercharacter theory is constructed for the group of invertible elements of a reduced algebra. For the case of the triangular group, the values of supercharacters on superclasses are calculated.

§1. Introduction

Traditionally the problem of classification of irreducible representations is viewed as the main problem in the representation theory of finite groups. However, for some groups like the unitriangular $\mathrm{UT}(n, \mathbb{F}_q)$, the problem of classification of irreducible representations turns out to be "wild". In 2008 in the paper [4], Diaconis and Isaacs proposed to replace the problem of classification of irreducible representations by the problem of construction of a supercharacter theory. In this theory, the supercharacters and superclasses play a role similar to that of the irreducible characters and classes of conjugate elements in the usual theory. In the same paper [4], the authors presented the supercharacter theory for the algebra groups, i.e., the groups of the type $1 + J$, where J is a nilpotent associative finite-dimensional algebra over a finite field. The main example is the theory of basic characters for the unitriangular group constructed by André [1, 2, 3] and Ning Yan [5].

A priori, there exist several supercharacter theories for a given group. The roughest one has only two supercharacters and two superclasses, and the sharpest one is the theory of irreducible representations. The main goal is to construct a supercharacter theory that gives the most accurate approximation of the theory of irreducible characters. The theory for the algebra groups mentioned above is optimal as far as nobody knows a better one. Since 2008, many papers have been devoted to the theory of supercharacters; see the bibliography in [6]. We only mention a few publications related to the topic of the present paper. In [7], the general supercharacter theory for semidirect products was presented. However, the construction suggested there produces a rather rough supercharacter theory, which can be improved in specific examples. In the case of a semidirect product with Abelian normal subgroup, there exist sharper supercharacter theories, see [8, 9].

In the present paper, we construct a supercharacter theory for the group of invertible elements of a reduced algebra (see Theorem 4.5). This group is a semidirect product of the Abelian group $(A/J)^*$ and the algebra group $1 + J$, where J is the radical of a given algebra A. The resulting theory is more accurate than in [7]. On the other hand, this supercharacter theory looks like an extension of the theory for the algebra groups mentioned above, see [4]. As an example, we obtain a supercharacter theory for the

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triangular group $\mathrm{T}(n, \mathbb{F}_q)$ (Theorem 5.2); the values of supercharacters on superclasses are calculated in Theorem 5.3.

Recall the main definitions of the paper [4]. Let G be a finite group, and let

$$\mathfrak{Ch} = \{\chi_\alpha \mid \alpha \in \mathfrak{A}\}$$

be a system of characters (representations) of G.

Definition 1.1. The system of characters $\mathfrak{Ch}$ determines a supercharacter theory of G if the characters in $\mathfrak{Ch}$ are pairwise disjoint and there exists a partition

$$\mathcal{K} = \{K_\beta \mid \beta \in \mathfrak{B}\}$$

of G with the following properties:

S1) $|\mathfrak{A}| = |\mathfrak{B}|$;

S2) each character χ_α is constant on each subset K_β;

S3) $\{1\} \in \mathcal{K}$ (hereinafter 1 is the unit element).

We refer to each character from $\mathfrak{Ch}$ as a *supercharacter* and to each subset in $\mathcal{K}$ as a *superclass.*

The easily verifiable condition S3 is very important; this can be seen from the following statement (see [4]). For the reader's convenience, we present it here with the proof.

Let $\mathcal{X}$ denote the system of subsets $\{X_\alpha : \alpha \in \mathfrak{A}\}$ in $\mathrm{Irr}(G)$, where each X_α is a subset of irreducible constituents of the character χ_α.

Proposition 1.2 ([4, Lemma 2.1])**.** *Suppose that a system of disjoint characters* $\mathfrak{Ch}$ *and a system of subsets* $\mathcal{K}$ *obey conditions* S1 *and* S2*. Then condition* S3 *is equivalent to the following condition* S4: *the system of subsets* $\mathcal{X}$ *determines a partition of* $\mathrm{Irr}(G)$*, and each character* χ_α *is equal to the character*

$$\sigma_\alpha = \sum_{\psi \in X_\alpha} \psi(1)\psi$$

up to a constant factor.

Proof. Suppose that S3 is fulfilled. Every system of disjoint characters is linearly independent. Since $|\mathfrak{A}| = |\mathfrak{B}|$, the system $\mathfrak{Ch}$ is a basis in the space of all complex-valued functions over G constant on the subsets in $\mathcal{K}$. The regular character $\rho(g)$ equals $|G|$ if $g = 1$, and equals 0 if $g \neq 1$. Condition S3 implies that $\rho(g)$ is constant on the subsets belonging to $\mathcal{K}$. We have

$$\rho = \sum a_\alpha \chi_\alpha, \quad \text{where} \quad a_\alpha \in \mathbb{C}.$$

On the other hand, each irreducible character is a constituent of ρ with multiplicity equal to its dimension. Therefore, every irreducible character is a constituent of exactly one χ_α, and $\chi_\alpha = \frac{1}{a_\alpha} \cdot \sigma_\alpha$. This proves S4.

Suppose that condition S4 holds true. The unit of the group belongs to one of the subsets from $\mathcal{K}$, say K_1. Condition S2 implies that $\chi_\alpha(g) = \chi_\alpha(e)$ for any $g \in K_1$ and $\alpha \in \mathfrak{A}$. By S4, we have $\rho(g) = \rho(1)$, whence $g = 1$ and $K_1 = \{1\}$. □

Corollary 1.3. *The system of supercharacters is uniquely determined up to constant factors by the partition* $\mathrm{Irr}(G) = \bigcup X_\alpha$.

We also recall some important properties of the systems $\mathfrak{Ch}$ and $\mathcal{K}$.

Proposition 1.4 ([4, Theorem 2.2.])**.** 1) *Each superclass is a union of classes of conjugate elements.*

2) *The partition* $\mathcal{K}$ *is uniquely determined by the partition* $\mathcal{X}$ *and vice versa.*

3) *The principal character is a supercharacter* (*up to a constant factor*).

§2. Group of invertible elements of a reduced algebra

Let A be a unital, associative, finite-dimensional algebra defined over a finite field $\mathbb{F}_q$ of q elements. Our main goal is to construct a supercharacter theory for the group $G = A^*$ of invertible elements of the algebra A.

It is known that, in the case of a finite field, there exists a subalgebra S complementary to the radical $J = J(A)$ of the algebra A, i.e., $A = S \oplus J$ (see [10, §11.6]). The subalgebra S is semisimple and is isomorphic to A/J. The algebra A is *reduced* if the algebra A/J is commutative (i.e., S is commutative). Then S is a direct sum of fields [10, §13.6]. Therefore,

$$S = k_1 e_1 \oplus \cdots \oplus k_n e_n,$$

where $\{e_1, \ldots, e_n\}$ is the system of primitive idempotents, and $k_1, \ldots, k_n$ are finite extensions of the field $\mathbb{F}_q$. Two elements of S are *associated* if they differ by a factor belonging to S^*. Any element of S is associated with some idempotent $f = e_{i_1} + \cdots + e_{i_m}$ in S.

The group G is a semidirect product of the subgroup $H = S^*$ of invertible elements of the subalgebra S and the normal subgroup $N = 1 + J$. We introduce the group $\widetilde{G}$ that consists of all triples $\tau = (t, a, b)$ with $t \in H$ and $a, b \in N$, equipped with the operation

$$((t_1, a_1, b_1) \cdot (t_2, a_2, b_2) = (t_1 t_2, t_2^{-1} a_1 t_2 a_2, t_2^{-1} b_1 t_2 b_2).$$

We define a representation of the group $\widetilde{G}$ in J by the formula

$$\rho(\tau)(x) = taxb^{-1}t^{-1}.$$

A $\widetilde{G}$-representation in the dual space J^* can be defined in a natural way:

$$\rho^*(\tau)\lambda(x) = \lambda(\rho(\tau^{-1})(x)).$$

Also, we can define left and right G-representations in J^* by the formulas $b\lambda(x) = \lambda(xb)$ and $\lambda a(x) = \lambda(ax)$. Then $\rho(\tau)(\lambda) = tb\lambda a^{-1} t^{-1}$.

For every idempotent $e \in A$, denote by A_e the subalgebra eAe. The subalgebra $J_e = eJe$ is the radical in A_e. Denote $e' = 1 - e$. We have Peirce decomposition:

$$J = eJe \oplus eJe' \oplus e'Je \oplus e'Je'.$$

We identify the dual space J_e^* with the subspace in J^* of all linear forms equal to zero on all components of the Peirce decomposition except for the first component.

Definition 2.1. 1) An element $x \in J$ is said to be *singular* if there exists an idempotent $e \in A$, $e \neq 1$, such that $x \in J_e$. Otherwise x is said to be *regular*.

2) An element $\lambda \in J^*$ is said to be *singular* if there exists an idempotent $e \in A$, $e \neq 1$, such that $\lambda \in J_e^*$. Otherwise, λ is said to be *regular*.

Lemma 2.2. *The following conditions are equivalent*:

1) *an element* $x \in J$ (*respectively,* $\lambda \in J^*$) *is singular*;

2) *there exists* $c \in A \setminus J$ *such that* $cx = xc = 0$ (*respectively,* $c\lambda = \lambda c = 0$).

Proof. We prove this for $x \in J$. The case of $\lambda \in J^*$ is treated similarly.

Note that the condition $x \in J_e$ is equivalent to $x \in J$ and $e'x = xe' = 0$. Therefore, an element x is singular if there exists an idempotent $f \neq 0$ such that $fx = xf = 0$.

Obviously, 1) implies 2). We prove that 2) implies 1). Let c belong to $A \setminus J$, and let $cx = xc = 0$. Suppose that $x \neq 0$ (the zero element is singular). The element c generates the commutative subalgebra $C = \mathbb{F}_q[c]$ that has the ideal $I = c\mathbb{F}_q[c]$. Since $x \neq 0$, the codimension of I in C is equal to 1.

Suppose that C is a local algebra. In this case, C has a unique ideal of codimension 1 and it consists of nilpotents; in particular, c is nilpotent. On the other hand, since c does not belong to J, it is not nilpotent. Therefore, C is not a local algebra.

The algebra C contains an idempotent f distinct from 0 or 1. If $f \in I$, then the claim is proved. Let $f \notin I$. There exists $\alpha \in \mathbb{F}_q$ such that $\alpha - f \in I$. Then

$$(\alpha - f)^2 = \alpha^2 - 2\alpha f + f^2 = \alpha^2 + (1 - 2\alpha) f \in I.$$

It is easily seen that

$$\begin{vmatrix} \alpha & -1 \\ \alpha^2 & 1 - 2\alpha \end{vmatrix} = \alpha(1 - \alpha). \tag{1}$$

If $\alpha \neq 0, 1$, then $1 \in I$ and $I = C$; this contradicts the fact that $\operatorname{codim}(I, C) = 1$.

If $\alpha = 0$, then $f \in I$; this contradicts the choice of f. Hence $\alpha = 1$. In this case, the idempotent $1 - f \neq 0$ belongs to I, and the claim follows. □

Proposition 2.3. *If an element $x \in J$ (respectively, $\lambda \in J^*$) is singular, then every element of its $\rho(\widetilde{G})$-orbit (respectively, $\rho^*(\widetilde{G})$-orbit) is also singular. A similar fact is true for the regular elements.*

Proof. Let $x \in J$. The case where $\lambda \in J^*$ is treated similarly. Let x be singular. There exists an idempotent $f \in A$, $f \neq 0$, such that $fx = xf = 0$. It suffices to prove that ax (respectively, xa) is singular for any $a \in N$. Then for $c = fa^{-1}$ we have $c \notin J$ and $c(ax) = (ax)c = 0$. Lemma 2.2 implies that the element ax is singular. □

Corollary 2.4. *For any singular $\widetilde{G}$-orbit $\mathcal{O} \subset J$ (respectively, $\mathcal{O}^* \subset J^*$), there exists an idempotent $e \in S$ such that $\mathcal{O} \cap J_e \neq \varnothing$ (respectively, $\mathcal{O}^* \cap J_e^* \neq \varnothing$).*

Proof. It is well known that every idempotent is conjugate to an idempotent in S (see, e.g., [11, Theorem 4.1]). □

We denote by $\widetilde{G_e}$ the group of invertible elements of A_e. Identifying $\tilde{g}_e \in \widetilde{G_e}$ with $\tilde{g}_e + e' \in \widetilde{G}$, where $e' = 1 - e$, we can treat $\widetilde{G_e}$ as a subgroup of $\widetilde{G}$.

Lemma 2.5. *Let e, f be two idempotents from S. Then*:

1) *if a $\widetilde{G}$-orbit $\mathcal{O}$ has nonempty intersection with both J_e and J_f, then $\mathcal{O}$ has nonempty intersection with J_{ef};*

2) *two elements x and y of J_e belong to a common $\widetilde{G}$-orbit if and only if x and y belong to a common $\widetilde{G_e}$-orbit.*

Proof. 1) Let $x \in \mathcal{O} \cap J_e$ and $y \in \mathcal{O} \cap J_f$ (i.e., $ex = xe = x$, $fy = yf = y$), and let $y = haxbh^{-1}$, where $h \in H$ and $a, b \in N$. Consider the element $z = eye$. Since $ef = fe$, we have $(ef)z = z(ef) = z$ and $z = eye = ehaxbh^{-1}e = heaxbeh^{-1}$. The element $a_1 = ea + e'$, where $e' = 1 - e$, obeys $a_1 x = eax + (1 - e)x = eax$. Since $a_1 = 1 \bmod J$, we obtain $a_1 \in N$. Similarly, $b_1 = be + e'$ belongs to N and $xb_1 = xbe + x(1 - e) = xbe$. Then $z = ha_1 x b_1 h^{-1}$, where $a_1, b_1 \in N$; thus, z lies in the common orbit $\mathcal{O}$.

2) Let $x, y \in J_e$ (i.e., $ex = xe = x$ and $ey = ye = y$). Obviously, if x and y belong to a common $\widetilde{G_e}$-orbit, then x and y belong to a common $\widetilde{G}$-orbit. We prove the converse statement.

Let x, y belong to a common $\widetilde{G}$-orbit, i.e., $y = haxbh^{-1}$, where $a, b \in N$ and $h \in H$. Then

$$y = eye = ehaxbh^{-1}e = (he)a(exe)beh^{-1} = (he)(eae)x(ebe)(h^{-1}e).$$

The element $he = eh$ is invertible in A_e and its inverse coincides with $h^{-1}e$. The elements eae and ebe belong to N_e. This proves that x and y lie in a common $\widetilde{G_e}$-orbit. □

Corollary 2.6. *For every $x \in J$ there exists a unique idempotent $e \in S$ such that $\mathcal{O}(x) \cap J_e \neq \varnothing$ and $\mathcal{O}(x) \cap J_e$ is a regular $\widetilde{G_e}$-orbit in J_e.*

Proof. This follows from statements 1) and 2) of the preceding lemma. □

The following properties of orbits in J^* are proved similarly.

Lemma 2.7. *Let e, f be idempotents in S. Then:*

1) *if an $\widetilde{G}$-orbit $\mathcal{O}^*$ has nonempty intersection with both J_e^* and J_f^*, then $\mathcal{O}^*$ has nonempty intersection with J_{ef}^*;*

2) *two elements λ and μ in J_e^* belong to a common $\widetilde{G}$-orbit if and only if λ and μ belong to a common $\widetilde{G_e}$-orbit.*

Corollary 2.8. *For any singular $\lambda \in J^*$ there exists a unique idempotent $e \in S$ such that $\mathcal{O}(\lambda) \cap J_e^* \neq \varnothing$ and $\mathcal{O}(\lambda) \cap J_e^*$ is a regular $\widetilde{G_e}$-orbit in J_e^*.*

Notation 2.9. $\mathcal{O}(J)$ is the set of all $\widetilde{G}$-orbits in J;
$n(J)$ is the number of all $\widetilde{G}$-orbits in J;
$\mathcal{O}(J_e)$ is the set of all $\widetilde{G}$-orbits of type $\mathcal{O}(x)$, where $x \in J_e$;
$n(J_e) = |\mathcal{O}(J_e)|$ (by Lemma 2.5 the number $n(J_e)$ coincides with the number of all $\widetilde{G_e}$-orbits in J_e);
$n_E(J)$ is the number of all regular $\widetilde{G}$-orbits in J;
$\mathcal{O}(J^*)$, $n(J^*)$, $\mathcal{O}(J_e^*)$, $n(J_e^*)$, $n_E(J^*)$ are defined similarly.

Corollary 2.10. 1) $\mathcal{O}(J_e) \cap \mathcal{O}(J_f) = \mathcal{O}(J_{ef})$,
2) $\mathcal{O}(J_e^*) \cap \mathcal{O}(J_f^*) = \mathcal{O}(J_{ef}^*)$.

Proposition 2.11. *The number of all regular $\widetilde{G}$-orbits in J coincides with the number of all regular $\widetilde{G}$-orbits in J^* (i.e., $n_E(J) = n_E(J^*)$).*

Proof. Let $\{e_1, \dots, e_n\}$ be a system of all primitive idempotents of the algebra S. Denote $e_i' = 1 - e_i$. For each idempotent $f \in S$, we denote by $l(f)$ the number of factors in the decomposition $f = e_{i_1}' \cdots e_{i_l}'$. For $f = 1$, we take $l(f) = 0$. Corollary 2.10 implies that $\mathcal{O}(J_f)$ coincides with the intersection $\bigcap \mathcal{O}(J_\phi)$, where ϕ runs through the set of idempotents $\{e_{i_1}', \dots, e_{i_l}'\}$.

The set of all singular orbits in J is the union of all $\mathcal{O}(J_{e_i'})$, where $1 \leq i \leq n$. Hence,

$$n_E(J) = \bigoplus (-1)^{l(f)} n(J_f),$$

where f runs through the set of all idempotents of S. A similar formula is true for the orbits in J^*:

$$n_E(J^*) = \bigoplus (-1)^{l(f)} n(J_f^*),$$

where f also runs through the set of all idempotents of S. For any finite subgroup of linear operators in a linear space V over a finite field, the number of orbits in V coincides with the number of orbits in the dual space V^* (see [4, Lemma 4.1]). For the orbits of the group $\widetilde{G}_f$ in J_f and J_f^*, we conclude that $n(J_f) = n(J_f^*)$. Therefore, $n_E(J) = n_E(J^*)$. □

§3. Superclasses

We define an action of the group $\widetilde{G}$ on G as follows:

(2) $$R_\tau(g) = 1 + ta(g-1)b^{-1}t^{-1}, \quad \text{where} \quad \tau = (t, a, b), \ t \in H, \ a, b \in N.$$

Notice that for $g = 1 + x \in N$ we obtain $R_\tau(1 + x) = 1 + \rho_\tau(x)$.

A *superclass* $\mathcal{K}(g)$ is the $\widetilde{G}$-orbit of the element $g \in G$. Every element $g \in G$ can be presented in the form $g = h + x$, where $h \in H$ and $x \in J$. It is easily seen that if $g = h + x$ and $g' = h' + x'$ lie in a common superclass, then $h = h'$.

Theorem 3.1. *Let $g = h + x$, where $h \in H$ and $x \in J$. Let $\mathcal{K} = \mathcal{K}(g)$. Let f be the idempotent associated with $h - 1$. Then:*

1) *there exists an idempotent $e \in S$, $e \perp f$, and a regular element $y \in J_e$ such that $h + y$ belongs to the superclass $\mathcal{K}$; let $\omega(y)$ denote the orbit of the element y in J_e with respect to the group $\widetilde{G_e}$;*

2) *the superclass $\mathcal{K}$ determines the quadruple $\beta = (e, f, h, \omega)$ uniquely.*

Proof. **Item 1**. We prove that there exists an element $h + y \in \mathcal{K}$ such that $y \in J_{f'}$, where $f' = 1 - f$.

Denote $s = h-1$. It suffices to show that there exist $a, b \in N$ such that $s+y = a(s+x)b$, where $yf = fy = 0$. Since s is associated with the idempotent f, it suffices to check that for any $x \in J$ there exist $a, b \in N$ and $y \in J$ such that $f + y = a(f + x)b$ and $yf = fy = 0$.

1) We show that there exists $u \in J$ such that $f + u = a(f + x)$ and $uf = 0$. Take $a = (1 + x)^{-1} \in N$. Then

$$uf = \big((1 + x)^{-1}(f + x) - f\big)f = ((1 + x)^{-1}(f + xf) - f = 0.$$

2) Let u be as in 1). We show that there exists $y \in J$ such that $f + y = (f + u)b$, $b \in N$, and $yf = fy = 0$. Take $b = (1 - fu) \in N$. Then

$$y = (f + u)(1 - fu) - f = (1 - f)u.$$

Hence, $fy = yf = 0$.

Item 2. Here, we prove that two elements $h + y$ and $h + y'$, where $y, y' \in J_{f'}$, belong to a common superclass if and only if y and y' belong to a common $\widetilde{G}_{f'}$-orbit in $J_{f'}$.

Obviously, if y and y' belong to a common $\widetilde{G}_{f'}$-orbit, then $h + y$ and $h + y'$ belong to a common superclass. We check the converse.

Let $h + y$ and $h + y'$ belong to a common supercalss. Then there exists $t \in H$ and $u, v \in J$ such that

$$h - 1 + y' = t(1 + u)(h - 1 + y)(1 + v)^{-1}t^{-1}. \tag{3}$$

Then

$$(h - 1 + y')t(1 + v) = t(1 + u)(h - 1 + y). \tag{4}$$

We multiply (4) by f' from the left and from the right. Since $f'(h - 1) = (h - 1)f' = 0$, $f'y = yf' = y$, $f'y' = y'f' = y'$, and $f't = tf'$, we obtain

$$y't(1 + f'vf') = t(1 + f'uf')y.$$

Thus, y and y' belong to a common $\widetilde{G}_{f'}$-orbit in $J_{f'}$.

Item 3. For any $\widetilde{G}_{f'}$-orbit ω in $J_{f'}$, there exists a unique idempotent $e < f'$ such that $\omega \cap J_e$ is a regular $\widetilde{G_e}$-orbit in J_e (see Corollary 2.6). We may assume that y is a regular element in J_e. This proves statement 1). Lemma 2.5 implies statement 2) of the theorem. □

We denote by $\mathfrak{B}$ the set of all quadruples $\beta = (e, f, h, \omega)$, where e, f is a pair of edempotents S, where $e \perp f$, $h \in H$, the element $h - 1$ is associated with f, and ω is a regular $\widetilde{G_e}$-orbit in J_e.

Notation 3.2. For any $\beta \in \mathfrak{B}$, we denote by $\mathcal{K}_\beta$ the superclass of the element $g = h+x$, where $x \in \omega$;

$$S_i = e_iS = Se_i; \quad H_i = \{1 - e_i + s_i, \text{ where } s_i \in S_i^*\};$$

$$H_f = \prod_{e_i \leq f} H_i; \quad m(f) = \sum_{e_i \leq f} (|H_i| - 1).$$

Corollary 3.3. *The superclasses* $\{\mathcal{K}_\beta \mid \beta \in \mathfrak{B}\}$ *form a partition of the group* G. *The number of superclasses is equal to*

$$|\mathfrak{B}| = \sum_{e\perp f}(n_E(J_e) + m(f)). \tag{5}$$

§4. Supercharacters

Let e, f be idempotents in S such that $e \perp f$. Let λ be a regular element in J_e^*. As before, $e' = 1 - e$ and

$$H_{e'} = \prod_{e_i \le e'} H_i.$$

Notation:

$$J_{\lambda,\mathrm{right}} = \{x \in J \mid \lambda x = 0\}, \quad N_{\lambda,\mathrm{right}} = \{a \in N \mid \lambda a = \lambda\}.$$

Observe that $N_{\lambda,\mathrm{right}} = 1 + J_{\lambda,\mathrm{right}}$. Similarly, we introduce $H_{\lambda,\mathrm{right}}$ and $H_{\lambda,\mathrm{left}}$.

It is easy to check that $H_{e'}$ is contained in $H_{\lambda,\mathrm{right}}$ and $H_{\lambda,\mathrm{left}}$. Since λ is a regular element in J_e^*, we see that

$$H_{e'} = H_{\lambda,\mathrm{right}} \cap H_{\lambda,\mathrm{left}}. \tag{6}$$

Consider the subgroup $G_\lambda = H_{e'} \cdot N_{\lambda,\mathrm{right}}$. The subgroup G_λ is a semidirect product of $H_{e'}$ and the normal subgroup $N_{\lambda,\mathrm{right}}$. Every element $g \in G_\lambda$ can be presented in the form $g = h + x$, where $h \in H_{e'}$ and $x \in J_{\lambda,\mathrm{right}}$.

We fix a nontrivial character $c \to \varepsilon^c$ of the additive group of the field $\mathbb{F}_q$ with values in the multiplicative group $\mathbb{C}^*$. Let θ be a linear character (one-dimensional representation) of the subgroup $H_{e'}$, and let a linear character of the subgroup G_λ be defined as follows:

$$\xi_{\theta,\lambda}(g) = \theta(h)\varepsilon^{\lambda(x)}, \tag{7}$$

where $g = h + x$, $h \in H_{e'}$, and $x \in J_{\lambda,\mathrm{right}}$. We show that $\xi = \xi_{\theta,\lambda}$ is indeed a linear character:

$$\begin{aligned}\xi(gg') &= \xi((h + x)(h' + x')) = \xi(hh' + h'x + x'h + xx') \\ &= \theta(hh')\varepsilon^{\lambda(h'x)}\varepsilon^{\lambda(x'h)}\varepsilon^{\lambda(xx')} = \theta(h)\theta(h')\varepsilon^{\lambda(x)}\varepsilon^{\lambda(x')} = \xi(g)\xi(g').\end{aligned}$$

The induced character

$$\chi_{\theta,\lambda} = \mathrm{ind}(\xi_{\theta,\lambda}, G_\lambda, G) \tag{8}$$

will be called a *supercharacter*.

Proposition 4.1. *The supercharacter* $\chi_{\theta,\lambda}$ *is constant on the superclasses.*

Proof. **Item 1**. Let $g \in G_\lambda$, $a \in N$. We show that $g' = 1 + (g - 1)a \in G_\lambda$ and $\chi_{\theta,\lambda}(g') = \chi_{\theta,\lambda}(g)$.

The element $g \in G_\lambda$ has the form $g = h + x$, where $h \in H_{e'}$ and $x \in J_{\lambda,\mathrm{right}}$. Let $a = 1 + u$ with $u \in J$. Then $g' = h + y$, where $h \in H_{e'}$ and $y = (h - 1)u + x + xu \in J$. Since

$$\lambda h = \lambda, \quad \lambda(h - 1)u = 0, \quad \lambda x = 0 \ \text{ and } \ \lambda xu = 0,$$

we have $\lambda y = \lambda(h - 1)u + \lambda x + \lambda xu = 0$. Hence, $y \in J_{\lambda,\mathrm{right}}$ and $g' \in G_\lambda$.

We obtain

$$\xi_{\theta,\lambda}(g') = \theta(h)\varepsilon^{\lambda((h-1)u+x+xu)} = \theta(h)\varepsilon^{\lambda(x)} = \xi_{\theta,\lambda}(g). \tag{9}$$

Denote $\Lambda(g) = \{s \in G | s^{-1}gs \in G_\lambda\}$.

We show that $\Lambda(g') = \Lambda(g)$. Indeed, $s^{-1}g's = 1 + (s^{-1}gs - 1)s^{-1}as$. As before, $s^{-1}gs \in G_\lambda$ implies $s^{-1}g's \in G_\lambda$. This proves that $\Lambda(g) \subset \Lambda(g')$. The converse inclusion is proved similarly.

Formula (9) shows that

$$\xi_{\theta,\lambda}(s^{-1}g's) = \xi_{\theta,\lambda}(s^{-1}gs) \tag{10}$$

for $s \in \Lambda(g)$. Therefore,

$$\chi_{\theta,\lambda}(g') = \frac{1}{|G_\lambda|}\sum_{s\in\Lambda(g')}\xi_{\theta,\lambda}(s^{-1}g's) = \frac{1}{|G_\lambda|}\sum_{s\in\Lambda(g)}\xi_{\theta,\lambda}(s^{-1}gs) = \chi_{\theta,\lambda}(g). \tag{11}$$

Item 2. The characters are constant on the classes of conjugate elements. Hence, $\chi_{\theta,\lambda}(g_0gg_0^{-1}) = \chi_{\theta,\lambda}(g)$ for every $g_0 \in G$. From formula (2) and Item 1, we conclude that $\chi_{\theta,\lambda}(R_\tau(g)) = \chi_{\theta,\lambda}(g)$ for every $\tau \in \widetilde{G}$ and $g \in G_\lambda$.

If a $\widetilde{G}$-orbit has empty intersection with G_λ, then $\chi_{\theta,\lambda}$ is zero on it. □

Proposition 4.2. *Let λ, λ' be regular elements of J_e^*. If λ, λ' belong to a common $\widetilde{G}_e$-orbit, then $\chi_{\theta,\lambda} = \chi_{\theta,\lambda'}$.*

Proof. This follows from the items 1 and 2 below.

Item 1. Consider the case where $\lambda' = a\lambda$ with $a = 1 + u \in N_e$. Since $J_{a\lambda,\mathrm{right}} = J_{\lambda,\mathrm{right}}$, we have $G_{a\lambda} = G_\lambda$. For any $g = h + x \in G_\lambda$, we obtain

$$\xi_{\theta,a\lambda}(g) = \theta(h)\varepsilon^{a\lambda(x)} = \theta(h)\varepsilon^{\lambda(x+xu)} = \theta(h)\varepsilon^{\lambda(x)} = \xi_{\theta,\lambda}(g),$$

where $g = h + x$, $h \in H_{e'}$, and $x \in J_{\lambda,\mathrm{right}}$. Hence, $\chi_{\theta,\lambda} = \chi_{\theta,a\lambda}$.

Item 2. Let $\lambda' = g_0\lambda g_0^{-1}$, where $g_0 \in G_e$. Then $G_{\lambda'} = g_0G_\lambda g_0^{-1}$ and

$$\xi_{\theta,\lambda'}(g) = \theta(h)\varepsilon^{g_0\lambda g_0^{-1}(x)} = \theta(h)\varepsilon^{\lambda(g_0^{-1}xg_0)} = \xi_{\theta,\lambda}(g_0^{-1}gg_0).$$

The corresponding induced representations are equivalent and $\chi_{\theta,\lambda} = \chi_{\theta,\lambda'}$. □

Proposition 4.3. *If $\theta \neq \theta'$, then the characters $\chi_{\theta,\lambda}$ and $\chi_{\theta',\lambda}$ are disjoint.*

Proof. Denote $\xi = \xi_{\theta,\lambda}$ and $\xi' = \xi_{\theta',\lambda}$. The Intertwining Number Theorem (see [12, Theorem 44.5]) says that the characters $\chi_{\theta,\lambda}$ and $\chi_{\theta',\lambda}$ are disjoint if and only if for every $s \in G$ there exists $g \in G_\lambda$ such that $sgs^{-1} \in G_\lambda$ and $\xi(sgs^{-1}) \neq \xi'(g)$.

It suffices to prove that for every $t \in H$ and $a \in N$ there exists $g \in G_\lambda$ such that aga^{-1} and tgt^{-1} belong to G_λ and

$$\xi(aga^{-1}) \neq \xi'(tgt^{-1}). \tag{12}$$

Item 1. Suppose that $a \in N$ and $t \in H$. We reduce the proof of (12) to the case where $a = 1 + ev$, with $v \in J$.

We show that there exists $w \in J$ such that $a(1 + e'w) = 1 + ev$, where $v \in J$ and $e' = 1 - e$. Indeed, let $a = 1 + u$, $u \in J$. Then for every $w \in J$ we have

$$a(1 + e'w) = (1 + eu + e'u)(1 + e'w) = 1 + eu(1 + e'w) + e'u + (1 + e'u)e'w.$$

It suffices to take $e'w = -(1 + e'u)^{-1}e'u$.

Since $\lambda e' = 0$, we see that $e'w \in J_{\lambda,\mathrm{right}}$ and $1 + e'w \in N_{\lambda,\mathrm{right}}$. Moreover,

$$t(1 + e'w)t^{-1} = 1 + e'twt^{-1} \in N_{\lambda,\mathrm{right}} \subset G_\lambda.$$

We conclude that it suffices to prove (12) for $a = 1 + ev$, where $v \in J$.

Item 2. Let $t \in H$, and let $a = 1 + ev = 1 + eve + eve'$, where $v \in J$. We prove (12) in the case where $eve' \in J_{\lambda,\mathrm{right}}$.

Observe that $(1 - eve')a = (1 - eve')(1 + eve + eve') = 1 + eve$. Since $1 - eve' \in N_{\lambda,\mathrm{right}} \subset G_\lambda$, it suffices to prove (12) for $a = 1 + eve$, where $v \in J$.

Let $g = h_0$ be an arbitrary element of the subgroup $H_{e'} \subset G_\lambda$. Then $h_0e = eh_0 = e$ and

$$ah_0a^{-1} = (1 + eve)h_0(1 + eve)^{-1} = h_0(1 + eve)(1 + eve)^{-1} = h_0 \in G_\lambda.$$

Since t and h_0 commute, $th_0t^{-1} = h_0 \in G_\lambda$.

By assumption, we have $\theta \neq \theta'$; there exists $h_0 \in H_{e'}$ such that $\theta(h_0) \neq \theta'(h_0)$. Taking $g = h_0$ in (12), we obtain

$$\xi(ah_0a^{-1}) = \xi(h_0) \neq \xi'(h_0) = \xi'(th_0t^{-1}).$$

Item 3. Let $a = 1 + x$, where $x = eve + eve' \in J$ and $eve' \notin J_{\lambda,\mathrm{right}}$. Then

$$\lambda(xe'J) = \lambda((eve + eve')e'J) = \lambda((eve')e'J) = \lambda(eve'(eJ + e'J)) = \lambda(eve'J) \neq 0.$$

Consider the chain of right ideals $e'J \supset e'J^2 \supset \cdots \supset e'J^k = \{0\}$. There exists a number i such that $\lambda(xe'J^i) \neq 0$ and $\lambda(xe'J^{i+1}) = 0$. Choose $y \in e'J^i$ such that $\lambda(xy) \neq 0$ and $\lambda(xyJ) = 0$. The last identity means that $xy \in J_{\lambda,\mathrm{right}}$. Since $\lambda e' = 0$, we conclude that $y \in J_{\lambda,\mathrm{right}}$.

Take $g = 1 + cy \in N_{\lambda,\mathrm{right}}$, where $c \in \mathbb{F}_q$. Then

$$aga^{-1} = (1 + x)(1 + cy)(1 + x)^{-1} = 1 + (1 + x)cy(1 + x)^{-1} \in 1 + cy + cxy + yJ + xyJ.$$

Therefore, $aga^{-1} \in N_{\lambda,\mathrm{right}} \subset G_\lambda$. Since $\lambda(e'J) = 0$, we have $\lambda(y) = 0$. Since $y, xy \in J_{\lambda,\mathrm{right}}$, we have $\lambda(yJ) = \lambda(xyJ) = 0$. We obtain $\xi(aga^{-1}) = \varepsilon^{c\lambda(xy)}$.

Since $y \in e'J$, we see that $tyt^{-1} \in e'J$ and

$$tgt^{-1} = 1 + tyt^{-1} \in N_{\lambda,\mathrm{right}} \subset G_\lambda.$$

We have $\xi'(tgt^{-1}) = \varepsilon^{c\lambda(tyt^{-1})} = 1$.

Since $\lambda(xy) \neq 0$, there exists $c \in \mathbb{F}_q$ such that $\varepsilon^{c\lambda(xy)} \neq 1$. This proves (12). □

Let $e \perp f$ be a pair of orthogonal idempotents. We say that a linear character θ of the subgroup $H_{e'}$ is *associated* with f if for any primitive idempotent $e_i \leq e'$, the following conditions are satisfied:

1) if $e_i \leq f$, then $\mathrm{Res}_{H_i}\,\theta \neq 1$;

2) if $e_i \not\leq f$, then $\mathrm{Res}_{H_i}\,\theta = 1$.

The number of characters associated with f is equal to $m(f)$ (see Notation 3.2).

We denote by $\mathfrak{A}$ the set of all quadruples $\alpha = (e, f, \theta, \omega^*)$, where e, f is a pair of idempotents from S, $e \perp f$, θ is a linear character of the subgroup $H_{e'}$ associated with f, and ω^* is a regular $\widetilde{G}_e$-orbit in J_e^*. For any $\alpha \in \mathfrak{A}$, we denote by χ_α the supercharacter equal to $\chi_{\theta,\lambda}$, where $\lambda \in \omega^*$ (see Proposition 4.2).

Proposition 4.4. *The characters $\{\chi_\alpha |\, \alpha \in \mathfrak{A}\}$ are pairwise disjoint.*

Proof. First, we recall the construction of supercharacters for the algebra group $N = 1+J$ (see [4]). For any $\mu \in J^*$, the supercharacter χ_μ of the group N is defined as the character induced by the character

$$\xi_\mu(1 + x) = \varepsilon^{\mu(x)}$$

of the subgroup $N_{\lambda,\mathrm{right}}$. Two supercharacters χ_μ and $\chi_{\mu'}$ are equal if and only if μ and μ' belong to a common $(N \times N)$-orbit. Otherwise, the supercharacters χ_μ and $\chi_{\mu'}$ are disjoint [4, Theorem 5.5].

Applying the decomposition formula for the restriction of the induced representation to a subgroup (see [12, Theorem 44.2], [13, Proposition 22]), we obtain

$$\mathrm{Res}_N(\chi_{\theta,\lambda}) = \sum_{t \in H_e} \chi_{Ad_t^*\lambda}. \tag{13}$$

We return to the proof of the proposition. Let $\alpha \neq \bar{\alpha}$, where $\alpha = (e, f, \theta, \omega^*)$ and $\bar{\alpha} = (\bar{e}, \bar{f}, \bar{\theta}, \bar{\omega}^*)$. If $e \neq \bar{e}$, then λ and $\bar{\lambda}$ belong to different $\widetilde{G}$-orbits (see Corollary 2.8). By formula (13), the restrictions of χ_α and $\chi_{\bar{\alpha}}$ to N are disjoint. It follows that χ_α and $\chi_{\bar{\alpha}}$ are disjoint.

The case where $e = \bar{e}$, $f = \bar{f}$, $\theta = \bar{\theta}$, and $\omega^* \neq \bar{\omega}^*$ is treated similarly (see Lemma 2.7).

It remains to consider the case where $e = \bar{e}$, $\omega^* = \bar{\omega}^*$, $f \neq \bar{f}$, or $\theta \neq \bar{\theta}$. The proof follows from Proposition 4.3. □

Theorem 4.5. *The systems of supercharacters* $\{\chi_\alpha \mid \alpha \in \mathfrak{A}\}$ *and superclasses* $\{\mathcal{K}_\beta \mid \beta \in \mathfrak{B}\}$ *give rise to a supercharacter theory on the group* G.

Proof. The supercharacters are disjoint by Proposition 4.4. The superclasses form a partition of G (see Corollary 3.3). The number of supercharacters is equal to

$$|\mathfrak{A}| = \sum_{e \perp f} (n_E(J_e^*) + m(f)). \tag{14}$$

Using formula (5) and Proposition 2.11, we conclude that $|\mathfrak{A}| = |\mathfrak{B}|$; this proves S1. The supercharacters are constant on the superclasses (see Proposition 4.1); this proves S2. Finally, $\{1\}$ is the superclass $K(g)$ for $g = 1$; this proves S3. □

§5. Supercharacter theory for the triangular group

Consider the algebra $A = \mathfrak{t}(n, \mathbb{F}_q)$ of all $(n \times n)$-matrices with entries in the field $\mathbb{F}_q$ and with zeros under the diagonal. The *triangular group* $G = \mathrm{T}(n, \mathbb{F}_q)$ is the group of invertible elements of the algebra A. The radical J coincides with the subalgebra $\mathfrak{n}(n, \mathbb{F}_q)$ of all triangular matrices with zeros on the diagonal. The complementary subalgebra S is the subalgebra of diagonal matrices; $H = S^*$ is the subgroup of diagonal matrices. In this section, we specialize the supercharacter theory constructed above for the triangular group.

Simplifying the language, we refer to a pair (i, j), where i, j are positive integers $1 \leq i < j \leq n$, as a *positive root*. Denote by Δ the set of all positive roots. The number $i = \mathrm{row}(\gamma)$ (respectively, $j = \mathrm{col}(\gamma)$) will be referred to as the number of the row (respectively, column) of the root $\gamma = (i, j)$. Following C. A. M. André, we introduce the definition of a basic subset. A *basic subset* is a subset $D \subset \Delta$ that has at most one positive root in any row and any column; see [1, 2].

Let $\{E_{ij} \mid 1 \leq i < j \leq n\}$ be a basis of matrix units of the radical J. With any basic subset D, we associate the element

$$x_D = \sum_{\gamma \in D} E_{ij}$$

in the radical J. Denote by $\mathcal{O}_D$ the $\widetilde{G}$-orbit of the element x_D. The orbits $\{\mathcal{O}_D\}$ form a partition of J (see [2]); therefore, the subsets $\{1 + \mathcal{O}_D\}$ form a partition of N.

Let $\{E_{ij}^*\}$ be the dual basis with respect to the basis $\{E_{ij}\}$. Next, D gives rise to the linear form

$$\lambda_D = \sum_{\gamma \in D} E_{ij}^*$$

and its $\widetilde{G}$-orbit $\mathcal{O}_D^*$. The orbits $\{\mathcal{O}_D^*\}$ form a partition of J^*, see [1].

Lemma 5.1. *The orbit* $\mathcal{O}_D$ *(respectively,* $\mathcal{O}_D^*$*) is regular if and only if* $\mathrm{row}(D) \cup \mathrm{col}(D) = [1, n]$.

Proof. We shall prove that the orbit $\mathcal{O}_D$ is singular if and only if $\mathrm{row}(D) \cup \mathrm{col}(D) \neq [1, n]$.

Suppose $\mathrm{row}(D) \cup \mathrm{col}(D) \neq [1, n]$, then the element x_D belongs to J_e, where $e = \sum E_{ii}$ and i runs through $\mathrm{row}(D) \cup \mathrm{col}(D)$. This proves that x_D (and the orbit $\mathcal{O}_D$) is singular.

On the other hand, suppose that $\mathcal{O}_D$ is a singular orbit; then there exists an idempotent $e \in S$, $e \neq 1$, such that $\mathcal{O}_D \cap J_e \neq \varnothing$ (see Corollary 2.4). The algebra J_e is isomorphic to $\mathfrak{n}(k, \mathbb{F}_q)$ for some $k < n$. The intersection $\mathcal{O}_D \cap J_e$ is a $\widetilde{G_e}$-orbit (see

Lemma 2.5); there exists $x_{D'} \in \mathcal{O}_D \cap J_e$ for some basic subset D'. Since x_D and $x_{D'}$ belong to a common $\widetilde{G}$-orbit, we conclude that $D = D'$. Therefore,

$$\mathrm{row}(D) \cup \mathrm{col}(D) = \mathrm{row}(D') \cup \mathrm{col}(D') \neq [1, n]. \qquad \square$$

Let $h = \mathrm{diag}(h_1, \dots, h_n)$ be an element of H such that $h_i = 1$ for each $i \in \mathrm{row}(D) \cup \mathrm{col}(D)$. Let $g_{h,D} = h + x_D \in G$, and let $K_{h,D}$ be a $\widetilde{G}$-orbit of the element $g_{h,D}$ in the group G.

Let θ be a linear character of H such that $\mathrm{Res}_{H_i}(\theta) = 1$ for each $i \in \mathrm{row}(D) \cup \mathrm{col}(D)$. Like in (8), for λ_D and θ we define the induced character

$$\chi_{\theta,D} = \chi_{\theta,\lambda_D}.$$

Theorem 5.2. *The systems of characters $\{\chi_{\theta,D}\}$ and subsets $\{K_{h,D}\}$ determine a supercharacter theory on the group $G = \mathrm{T}(n, \mathbb{F}_q)$.*

Proof. The proof follows from Lemma 5.1 and Theorem 4.5. $\square$

To calculate the values of supercharacters $\chi_{\theta,D}$ on superclases $K_{h,D'}$, we need new notation.

For each positive root $\gamma = (i, j)$, $1 \le i < j \le n$, we denote

$$\Delta'(\gamma) = \{(i, k) \,|\, i < k < j\}, \qquad \Delta''(\gamma) = \{(k, j) \,|\, i < k < j\}.$$

The numbers of elements of these two subsets coincide and are equal to $j - i - 1$.

Let $\delta'(D, D') = 0$ if there exists $\gamma \in D$ and $\gamma \in D'$ such that $\gamma' \in \Delta'(\gamma)$. Otherwise we put $\delta'(D, D') = 1$. The number $\delta''(D, D')$ is defined similarly.

Let $\delta_0(D, h) = 1$ if $h_i = 1$ for each $i \in \mathrm{row}(D) \cup \mathrm{col}(D)$; otherwise $\delta_0(D, h) = 0$. Denote

$$\delta(D, h, D') = \delta'(D, D')\delta''(D, D')\delta_0(D, h). \tag{15}$$

For each positive root $\gamma = (i, j)$, $1 \le i < j \le n$, we denote by $P(\gamma)$ the submatrix of the matrix $g-1$ with system of rows and columns $[i+1, j-1]$. Put $m(\gamma, h, D') = \mathrm{corank} P(\gamma)$. Since the matrix P_γ has at most one nonzero entry in any row and any column, $m(\gamma, h, D')$ is equal to the number of zero rows (columns). We put

$$m(D, h, D') = \sum_{\gamma \in D} m(\gamma, h, D'), \quad s(D, D') = |D| + |D \setminus D'|. \tag{16}$$

For $D = \varnothing$, we take $m(D, h, D') = 0$.

Theorem 5.3. *The value of a supercharacter on a superclass is equal to*

$$\chi_{\theta,D}(K_{h,D'}) = \delta(D, h, D')(-1)^{|D \cap D'|} q^{m(D,h,D')} (q-1)^{s(D,D')} \theta(h). \tag{17}$$

Proof. Denote by $\phi(\theta, D, h, D')$ the expression on the right-hand side of (17). For simplicity, put $\chi = \chi_{\theta,D}$, $\xi = \xi_{\theta,D}$, $g = g_{h,D'}$. We calculate the induced character $\chi(g) = \mathrm{ind}(\xi, G_\lambda, G)$ by using the well-known formula

$$\chi(g) = \frac{1}{|G_\lambda|} \sum \xi(s^{-1} g s), \tag{18}$$

where $s^{-1} g s \in G_\lambda$. If the set of s with $s^{-1} g s \in G_\lambda$ is empty, we put $\chi(g) = 0$.

Let $D = \{\gamma_1, \dots, \gamma_r\}$. For each $1 \le k \le r$, we denote

$$\lambda_k = \sum_{a=k}^{r} E^*_{\gamma_a}, \quad D_k = \{\gamma_1, \dots \gamma_k\}, \quad D_0 = \varnothing.$$

Here $\lambda_1 = \lambda_D$ and $D_r = D$. Set $\lambda_{r+1} = 0$. Observe that λ_k is the sum of the elements of the dual basis E^*_γ over $\gamma \in D \setminus D_{k-1}$.

The subgroups $G_k = G_{\lambda_k}$ form the chain

$$G_\lambda = G_1 \subset G_2 \subset \cdots \subset G_r \subset G_{r+1} = G.$$

We define a character χ_k of the group G_k by the formula

$$\chi_k = \mathrm{ind}(\xi, G_\lambda, G_k).$$

Then $\chi_1 = \xi$ is a character of the subgroup $G_\lambda = G_1$. For each $1 \le k \le r$ we have

$$\chi_{k+1} = \mathrm{ind}(\chi_k, G_k, G_{k+1}).$$

We prove the formula

$$\chi_{k+1}(g) = \phi(\theta, D_k, h, D')\varepsilon^{\lambda_{k+1}(x_{D'})}, \tag{19}$$

by using induction on k, $0 \le k \le r$. For $k = 0$, (19) is valid because $\chi_1(g) = \theta(h)\varepsilon^{\lambda(x_{D'})}$. Suppose that relation (19) is true for $k-1$; we prove it for k.

Let $\gamma = (i,j)$, $1 \le i < j \le n$, be an arbitrary positive root. Let $P'(\gamma)$ be the submatrix of the matrix $g-1$ with the rows $\{i\} \cup [i+1, j-1]$ and columns $[i+1, j-1]$. Respectively, $P''(\gamma)$ is the submatrix of the matrix $g-1$ with the rows $[i+1, j-1]$ and columns $[i+1, j-1] \cup \{j\}$.

The inductive assumption and formula (18) imply

$$\chi_{k+1}(g) = \phi(\theta, D_{k-1}, h, D')M(\gamma_k, h, D)\varepsilon^{\lambda_{k+1}(x_{D'})}, \tag{20}$$

where

$$M(\gamma_k, h, D) = \delta_0(\gamma_k, h) \sum_{t_i, t_j \in \mathbb{F}_q^*} \varepsilon^{t_i^{-1} t_j} \sum_{(t_i, \bar{x})P'(\gamma_k) = 0} \varepsilon^{(\bar{x}, \bar{p})}, \tag{21}$$

$\bar{x} = (x_1, \ldots, x_d)$, $\bar{p} = (p_1, \ldots, p_d)$ is the last column of the submatrix $P''(\gamma_k)$, $(\bar{x}, \bar{p}) = x_1 p_1 + \cdots + x_d p_d$, and $d = j - i - 1$.

If there exists $\gamma' \in D'$ lying in $\Delta(\gamma_k)$ (i.e., $\delta'(\{\gamma_k\}, h, D') = 0$), then

$$\mathrm{rank} P(\gamma_k) < \mathrm{rank} P'(\gamma_k).$$

Since the second sum in (21) is taken over the empty set, we have

$$M(\gamma_k, h, D) = 0.$$

If there is no $\gamma' \in D'$ lying in $\Delta(\gamma_k)$ (i.e., $\delta'(\{\gamma_k\}, h, D') = 1$), then the first row of the submatrix $P'(\gamma_k)$ is zero. Hence,

$$M(\gamma_k, h, D) = \delta_0(\gamma_k, h)\delta'(\{\gamma_k\}, h, D') \left(\sum_{t_i, t_j \in \mathbb{F}_q^*} \varepsilon^{t_i^{-1} t_j} \right) \left(\sum_{\bar{x} P(\gamma_k) = 0} \varepsilon^{(\bar{x}, \bar{p})} \right) \tag{22}$$

We calculate the third factor M_3 in (22):

$$M_3 = \sum_{t_i, t_j \in \mathbb{F}_q^*} \varepsilon^{t_i^{-1} t_j} = \begin{cases} (q-1)^2 & \text{if } \gamma_k \notin D', \\ (q-1)(-1) & \text{otherwise.} \end{cases}$$

Then

$$M_3 = (-1)^{|\{\gamma_k\} \cap D'|} (q-1)^{s(\{\gamma_k\}, D')}.$$

To calculate the forth factor in (22), we note that

$$\sum_{\bar{x} \in W} \varepsilon^{(\bar{x}, \bar{p})} = \begin{cases} |W| & \text{if } \bar{p} \in W^\perp, \\ 0 & \text{otherwise} \end{cases}$$

for any linear subspace W. We calculate the forth factor M_4 in (22):

$$M_4 = \sum_{\bar{x}P(\gamma_k)=0} \varepsilon^{(\bar{x},\bar{p})} = \begin{cases} q^{\operatorname{corank} P(\gamma_k)} & \text{if } \bar{p} \in \operatorname{Im}(P(\gamma_k)), \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$M_4 = \delta''(\{\gamma_k\}, h, D')q^{m(\{\gamma_k\},h,D')}.$$

Since

$$\delta_0(\gamma_k, h)\delta'(\{\gamma_k\}, h, D')\delta''(\{\gamma_k\}, h, D') = \delta(\{\gamma_k\}, h, D'),$$

substitution of M_3 and M_4 in (22) yields

$$M(\gamma_k, h, D) = \delta(\{\gamma_k\}, h, D')(-1)^{|\{\gamma_k\}\cap D'|}(q-1)^{s(\{\gamma_k\},D')}q^{m(\{\gamma_k\},h,D')}. \tag{23}$$

Substituting (23) in (20), we obtain

$$\chi_{k+1}(g) = \phi(\theta, D_k, h, D')\varepsilon^{\lambda_{k+1}(x_{D'})},$$

which proves (19) for k.

Taking $k = r$ in (19), we finally obtain $\chi(g) = \phi(\theta, D, h, D')$. □

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