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SPARSE RADEMACHER CHAOS IN SYMMETRIC SPACES

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Dedicated to
Evgenii Mikhailovich Semenov
on the occasion of his 75th birthday

Abstract. A dispersed Rademacher chaos whose combinatorial dimension equals its order d is treated. It is proved that its unconditionality in a symmetric space X guarantees the equivalence of this chaos in X to the canonical basis of ℓ_2. In its turn, the latter property occurs if and only if $X \supset G_{2/d}$, where $G_{2/d}$ is the separable part of the Orlicz space $\mathrm{ExpL}^{2/d}$ corresponding to the function $M(u) \sim \exp(u^{2/d})$. Furthermore, it is shown that a chaos of an arbitrary order constructed by an arbitrary system of stochastically independent symmetric random variables is a basic sequence in any ambient symmetric space.

Introduction

As usual, the Rademacher functions are defined as follows: for $0 \le t \le 1$, we put $r_n(t) := \operatorname{sign}(\sin(2^n \pi t))$, $n = 1, 2, \dots$. They are independent, symmetrically distributed, and form an incomplete orthonormal sequence on $[0,1]$. By the classical Khintchine inequality (see [1]), for every $p \ge 1$ and arbitrary $a_k \in \mathbb{R}$, $k = 1, 2, \dots$, we have

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{L_p[0,1]} \le \sqrt{p} \Big(\sum_{k=1}^{\infty} a_k^2 \Big)^{\frac{1}{2}}. \tag{1}$$

This relation was the origin of extensive study and many generalizations, and found numerous applications in various branches of analysis. In particular, by the well-known Rodin–Semenov theorem (see [2]), the sequence $\{r_n\}_{n=1}^{\infty}$ in a symmetric function space X is equivalent to the standard basis of ℓ_2 if and only if $X \supset G_2$, where G_2 is the separable part of the Orlicz space $\operatorname{Exp} L^2$ constructed by the function $M(u) \sim \exp(u^2)$.

The main object of study in the present paper is polylinear forms (the term "chaos" is also applied) constructed starting with the Rademacher system. See, e.g., [3, Chapter 6] concerning more general relevant definitions.

Definition 1. *The Rademacher chaos of order* $d \in \mathbb{N}$ is the set of all functions of the form $r_{i_1 i_2 \dots i_d}(t) := r_{i_1}(t) \cdot r_{i_2}(t) \cdot \dots \cdot r_{i_d}(t)$, where $i_1 > i_2 > \dots > i_d \ge 1$.

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Thus, the system $\{r_n\}_{n=1}^{\infty}$ itself is the chaos of order 1. Next, adjoining the function $r_0(t) \equiv 1$ to the union of the collections described in Definition 1 of arbitrary order, we obtain the classical Walsh system $\{w_n\}_{n=0}^{\infty}$. We recall its definition: $w_0(t) = r_0(t) \equiv 1$ and, if $n \in \mathbb{N}$ is represented in the form $n = \sum_{i=0}^{k} \varepsilon_i 2^i$, where $\varepsilon_k = 1$ and ε_i is 0 or 1 for $i = 0, 1, \dots, k-1$, then, in accordance with the Paley enumeration, $w_n(t) = \prod_{i=0}^{k} \big(r_{i+1}(t)\big)^{\varepsilon_i}$. It is well known (see, e.g., [4, Statements 1.1.5 and 2.6.3]) that the sequence $\{w_n\}_{n=0}^{\infty}$ is a complete orthonormal system on $[0,1]$.

It can easily be shown (see the beginning of the paper [5]) that, though Rademacher chaos of order greater than 1 is not a usual (say, Hadamard) lacunary system of Walsh functions, it possesses nevertheless certain properties relating it to such sequences. For example, in [5] it was proved that the Rademacher chaos of order d is a 2^{-d}-uniqueness system (i.e., the convergence to zero of the series $\sum_{i_1>i_2>\dots>i_d\ge 1} a_{i_1 i_2 \dots i_d} r_{i_1 i_2 \dots i_d}(t)$ on an arbitrary set $E \subset [0,1]$ of Lebesgue measure greater than $1-2^{-d}$ implies that $a_{i_1 i_2 \dots i_d} = 0$ for all $i_1 > \dots > i_d$).

In [6] and [7] (see also [8]), properties of the subspace of a symmetric space generated by a Rademacher chaos of arbitrary order were studied. In particular, it was shown that the sequence $\{r_{i_1 i_2 \dots i_d}\}_{r_1>\dots>r_d\ge 1}$ is unconditional in a symmetric space X if and only if it spans a Hilbert subspace in X. In the present paper, we treat similar questions for a dispersed (i.e., not full) chaos.

On the basis of the notion of a fractional Cartesian product introduced in [9], R. Blei came up with the following definition of the combinatorial dimension of a set (see [10] and also [11, Chapter XI]; the latter contains quite a few interesting examples of application of this notion). Throughout, the symbol $|A|$ will denote the cardinality of a finite set A; also, we put $\mathbb{N}^d := \mathbb{N} \times \mathbb{N} \times \dots \times \mathbb{N}$ (d factors), where $\mathbb{N}$ is the set of positive integers.

Definition 2. We say that a set $S \subset \mathbb{N}^d$ *has combinatorial dimension* α if

1) for every $\beta > \alpha$ there exists $C_\beta > 0$ such that for every collection $A_1, A_2, \dots, A_d \subset \mathbb{N}$ of sets with $|A_1| = |A_2| = \dots = |A_d| = m$, we have

$$|S \cap (A_1 \times A_2 \times \dots \times A_d)| < C_\beta m^\beta;$$

2) for every $\gamma < \alpha$ and $k \in \mathbb{N}$ there exist sets $A_1, A_2, \dots, A_d \subset \mathbb{N}$, $|A_1| = |A_2| = \dots = |A_d| = m > k$, with

$$|S \cap (A_1 \times A_2 \times \dots \times A_d)| > m^\gamma.$$

It is known that for every real $\alpha \in [1, d]$ there exists a set of dimension α, see [11, Chapter XIII]. At the same time, in §3 it will be shown that in the case of the maximal combinatorial dimension (equal to d) this definition is equivalent to the following much simpler property: for every $n \in \mathbb{N}$ there exists a collection of sets $B_1, B_2, \dots, B_d \subset \mathbb{N}$, $|B_1| = |B_2| = \dots = |B_d| = n$, with $B_1 \times B_2 \times \dots \times B_d \subset S$. We shall give two proofs of this important fact. The first is based on the deep theorem by Erdős on hypergraphs, see [12]; the second proof does not involve the theory of hypergraphs and makes the exposition self-contained. With this characterization at hand, in §4 we shall prove the main result of the paper: if a set $S \subset \mathbb{N}^d$ has combinatorial dimension d, then the unconditionality of the sequence $\{r_{i_1 i_2 \dots i_d}\}_{(i_1, i_2, \dots, i_d) \in S}$ in a symmetric space X guarantees the equivalence of this sequence in X to the standard basis of ℓ_2. Moreover, the latter property is equivalent to the inclusion $X \supset G_{2/d}$, where $G_{2/d}$ is the separable part of the Orlicz space $\operatorname{Exp} L^{2/d}$ constructed by the function $M(u) \sim \exp(u^{2/d})$.

In the beginning of the paper, in §2, after introducing the necessary definitions and after proving some auxiliary statements, we shall consider the properties of a chaos constructed on the basis of an arbitrary system of stochastically independent symmetrically distributed functions. We shall show that this is a basic sequence in an arbitrary symmetric space containing this chaos, thus refining a result established in [7], where the same was proved for the Rademacher chaos of the second order and symmetric interpolation spaces relative to the Banach couple (L_1, L_∞).

§1. Preliminaries and auxiliary results

The symbol $\mathcal{S}$ stands for the set of all functions on $[0,1]$ that are measurable and finite almost everywhere (a.e.) with respect to Lebesgue measure. As usual, functions coinciding a.e. are identified. We write $x \le y$, where $x, y \in \mathcal{S}$, if $x(t) \le y(t)$ for a.e. $t \in [0,1]$. We recall the definitions of certain Banach spaces of measurable functions.

Definition 3. A Banach space X of functions in $\mathcal{S}$ is called an *ideal space* if the conditions $x \in X$, $y \in \mathcal{S}$, and $|y| \le |x|$ imply $y \in X$ and $\|y\|_X \le \|x\|_X$.

Two functions x and y in $\mathcal{S}$ are said to be *equimeasurable* if

$$\mu\{t \in [0,1] \,:\, |x(t)| > \tau\} = \mu\{t \in [0,1] \,:\, |y(t)| > \tau\} \text{ for all } \tau > 0.$$

Here μ denotes the usual Lebesgue measure on $[0,1]$. For every $x \in \mathcal{S}$ there exists its rearrangement x^*, i.e., a (unique) nonnegative monotone nonincreasing function equimeasurable with x and continuous from the left (see, e.g., [13, §II.2]).

Definition 4. A Banach ideal space X on $[0,1]$ is said to be *symmetric* if the conditions $x \in X$, $y \in \mathcal{S}$, and $y^* = x^*$ imply $y \in X$ and $\|y\|_X = \|x\|_X$.

This definition shows that, along with a function x, every symmetric space contains all functions equimeasurable with x. We denote by $\chi_A = \chi_A(t)$ the characteristic function (the indicator) of a measurable set $A \subset [0,1]$. Important information about a symmetric space X is given by its fundamental function

$$\phi_X(t) := \|\chi_{(0,t)}\|_X, \quad t \in [0,1].$$

Examples of symmetric spaces are provided by L_p, Lorentz, Marcinkiewicz, and Orlicz spaces.

As usual $L_p = L_p[0,1]$, $1 \le p < \infty$ is the set of all functions $x \in \mathcal{S}$ with $\|x\|_p := \big(\int_0^1 |x(t)|^p\,dt\big)^{1/p} < \infty$. The limit case is the space L_∞ with the norm

$$\|x\|_\infty := \inf\big\{C \,:\, \mu\{t \in [0,1] \,:\, |x(t)| > C\} = 0\big\}.$$

For every symmetric space on $[0,1]$ we have the following continuous embeddings: $L_\infty \subset X \subset L_1$, see [13, Theorem II.4.1].

We denote by Φ the class of all continuous, monotone increasing, and concave functions on $[0,1]$ that vanish at zero. If $\varphi \in \Phi$, then the *Lorentz space* $\Lambda(\varphi)$ consists of all functions $x \in \mathcal{S}$ with

$$\|x\|_{\Lambda(\varphi)} := \int_0^1 x^*(t)\,d\varphi(t) < \infty.$$

The Lorentz space $\Lambda(\varphi)$ possesses the following extremal property in the class of symmetric spaces: if $\phi_X(t) \le C\varphi(t)$ for some $C > 0$ and all $t \in [0,1]$, then $\Lambda(\varphi) \subset X$ (see [13, Theorem II.5.5]). This will be used repeatedly in the present paper.

The *Marcinkiewicz space* $\mathcal{M}(\varphi)$ consists of all functions $x \in \mathcal{S}$ whose norm

$$\|x\|_{\mathcal{M}(\varphi)} := \sup_{t \in (0,1]} \frac{\varphi(t)}{t} \int_0^t x^*(s)\,ds$$

is finite.

Lemma 1. *Let X be a symmetric space, and let $1/\varphi \in X$. Then $\mathcal{M}(\varphi) \subset X$.*

Proof. Since

$$\|x\|_{\mathcal{M}(\varphi)} = \sup_{t\in(0,1]} \frac{\varphi(t)}{t} \int_0^t x^*(s)\,ds \geq \sup_{t\in(0,1]} \varphi(t)x^*(t),$$

we see that every $x \in \mathcal{M}(\varphi)$ obeys the inequality

$$x^*(t) \leq \frac{\|x\|_{\mathcal{M}(\varphi)}}{\varphi(t)}, \quad t \in (0,1],$$

whence

$$\|x\|_X \leq \|x\|_{\mathcal{M}(\varphi)} \cdot \left\| \frac{1}{\varphi} \right\|_X. \qquad \square$$

If a function φ does not belong to Φ but coincides with a function $\varphi_1 \in \Phi$ on some interval $(0, t_0)$, then by $\Lambda(\varphi)$ and $\mathcal{M}(\varphi)$ we understand the spaces $\Lambda(\varphi_1)$ and $\mathcal{M}(\varphi_1)$, respectively. The choice of a specific function φ_1 does not influence this space as a set. In particular, this remark is applicable to the spaces mentioned in the following statement.

Corollary 1. *Let $\alpha > \beta > 0$. Then*

$$\mathcal{M}(\ln^{-\beta}(e/t)) \subset \Lambda(\ln^{-\alpha}(e/t)).$$

Proof. Indeed, since $\beta/\alpha < 1$, we have

$$\left\| \ln^{\beta}(e/t) \right\|_{\Lambda(\ln^{-\alpha}(e/t))} = \int_0^1 \ln^{\beta}(e/t)\, d\ln^{-\alpha}(e/t) = \int_0^1 x^{-\beta/\alpha}\, dx < \infty.$$

It remains to use Lemma 1. $\square$

The closure of the set of bounded measurable functions in a symmetric space X will be called its *separable part* and will be denoted by X^0. Then X^0 is a symmetric space, which is separable provided $X \neq L_\infty$. We put $\mathcal{M}^0(\varphi) := (\mathcal{M}(\varphi))^0$.

Lemma 2. *Suppose X is a symmetric space, $\varphi \in \Phi$, and all functions of the form*

$$f_m(t) = \min\left\{ \frac{1}{\varphi(t)}, m \right\}, \quad m \in \mathbb{N},$$

have uniformly bounded norms in X, i.e., for some $B > 0$ we have

$$\sup_{m\in\mathbb{N}} \|f_m\|_X \leq B.$$

Then $\mathcal{M}^0(\varphi) \subset X$ with an embedding constant not exceeding B.

Proof. By [13, Chapter II, formula (5.16)], for every $x \in \mathcal{M}^0(\varphi)$ we have

$$\lim_{t\to 0+} x^*(t)\varphi(t) = 0.$$

Thus, for every $\varepsilon > 0$ we can choose a sequence $t_n \downarrow 0$ such that

$$x^*(t) \leq \frac{\varepsilon}{2^n \varphi(t)} \tag{2}$$

for all $t \in (0, t_n)$, $n = 1, 2, \ldots$. Next, since

$$\|x\|_{\mathcal{M}^0(\varphi)} = \|x\|_{\mathcal{M}(\varphi)} \geq \sup_{t\in(0,1]} x^*(t)\varphi(t),$$

we see that the following estimate holds true for all $t \in (0,1]$:

$$x^*(t) \leq \frac{\|x\|_{\mathcal{M}^0(\varphi)}}{\varphi(t)}.$$

Together with (2), this implies

$$x^*(t) \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} f_{m_n}(t) + \|x\|_{\mathcal{M}^0(\varphi)} f_{m_0}(t),$$

where m_n is chosen so as to have $m_n \geq 1/\varphi(t_{n+1})$ for every $n = 0, 1, \ldots.$ Then $x \in X$ and

$$\|x\|_X \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} \|f_{m_n}\|_X + \|x\|_{\mathcal{M}^0(\varphi)} \cdot \|f_{m_0}\|_X \leq \big(\varepsilon + \|x\|_{\mathcal{M}^0(\varphi)}\big) B.$$

Since $\varepsilon > 0$ is arbitrary, the claim follows. □

Let $M = M(u)$ be an *Orlicz function*, i.e., a continuous, concave, monotone increasing function on $[0, +\infty)$ with $M(0) = 0$. The *Orlicz space* L_M consists of all functions in $\mathcal{S}$ for which the following norm is finite:

$$\|x\|_{\mathrm{L}_M} := \inf \left\{ \lambda > 0 \,:\, \int_0^1 M\left(\frac{|x(t)|}{\lambda}\right) dt \leq 1 \right\}.$$

In particular, if $M(u) = u^p$, $1 \leq p < \infty$, then $L_M = L_p$. For an Orlicz function M and an arbitrary function F, we write $M(u) \sim F(u)$ if for some constants $c, C > 0$ the inequality $F(cu) \leq M(u) \leq F(Cu)$ is fulfilled for sufficiently large u. Note that if M_1 and M_2 are two Orlicz functions with $M_1(u) \sim F(u)$ and $M_2(u) \sim F(u)$, then $L_{M_1} = L_{M_2}$, with equivalence of the norms. In §4, we shall need the following statement about Orlicz spaces.

Lemma 3. *Let $z = z(u,t)$ be a measurable function on the square $[0,1] \times [0,1]$, and let M be an Orlicz function. Suppose that the function $z_t(u) := z(u,t)$ belongs to L_M for a.e. $t \in [0,1]$, and*

$$\operatorname*{ess\,sup}_{t \in [0,1]} \|z(\,\cdot\,, t)\|_{\mathrm{L}_M(\,\cdot\,)} < \infty.$$

Then $z(u, \,\cdot\,) \in \mathrm{L}_M$ for a.e. $u \in [0,1]$ (as a function of t), and

$$\int_0^1 \|z(u, \,\cdot\,)\|_{\mathrm{L}_M(\,\cdot\,)}\, du \leq 2 \operatorname*{ess\,sup}_{t \in [0,1]} \|z(\,\cdot\,, t)\|_{\mathrm{L}_M(\,\cdot\,)}.$$

Proof. Denote

$$a := \operatorname*{ess\,sup}_{t \in [0,1]} \|z(\,\cdot\,, t)\|_{\mathrm{L}_M(\,\cdot\,)}, \quad l(u) := 1 + \int_0^1 M\left(\frac{|z(u,t)|}{a}\right) dt.$$

Then

$$\int_0^1 M\left(\frac{|z(u,t)|}{a}\right) du \leq 1$$

for a.e. $t \in [0,1]$ and, consequently,

$$\int_0^1 l(u)\, du = 1 + \int_0^1 \int_0^1 M\left(\frac{|z(u,t)|}{a}\right) dt\, du = 1 + \int_0^1 \int_0^1 M\left(\frac{|z(u,t)|}{a}\right) du\, dt \leq 2.$$

On the other hand, since $l(u) \geq 1$, the function M is convex, and

$$\int_0^1 M\left(\frac{|z(u,t)|}{a}\right) dt \leq l(u),$$

it follows that

$$\int_0^1 M\left(\frac{|z(u,t)|}{a l(u)}\right) dt \leq \ 1 \text{ and } \ \|z(u, \,\cdot\,)\|_{\mathrm{L}_M(\,\cdot\,)} \leq a l(u).$$

As a result,

$$\int_0^1 \|z(u,\,\cdot\,)\|_{\mathrm{L}_M(\,\cdot\,)}\,du \le a\int_0^1 l(u)\,du \le 2a. \qquad \square$$

For the fundamental function of an Orlicz space, we have the formula

$$\phi_{\mathrm{L}_M}(t) = \frac{1}{M^{-1}(1/t)}.$$

If, moreover, $1/\phi_{\mathrm{L}_M} \in \mathrm{L}_M$, then the Orlicz space L_M coincides with the Marcinkiewicz space $\mathcal{M}(\phi_{\mathrm{L}_M})$. This follows from Lemma 1 and the embedding $X \subset \mathcal{M}(\varphi)$, which is valid for every symmetric space X provided $\varphi(t) \le C\phi_X(t)$, see [13, Theorem II.5.7]. In particular, if $M(u) \sim \exp(u^\alpha)$, $\alpha > 0$, then the corresponding Orlicz space L_M, to be denoted by $\operatorname{Exp} L^\alpha$, coincides with the Marcinkiewicz space $\mathcal{M}(\ln^{-1/\alpha}(e/t))$. Surely, the same is true for the separable parts of these spaces.

More information about symmetric spaces can be found in the monographs [13, 14] and [15].

As usual, the interval $[0,1]$ will be treated sometimes as a probability space with Lebesgue measure in the role of probability. Then the independence of functions is understood in the stochastic sense, i.e., as the independence of random variables. A random variable ξ is said to be *symmetrically distributed* if its distribution function coincides with that for $-\xi$. The Rademacher functions $r_n(t) := \operatorname{sign}\sin(2^n\pi t)$, $n \in \mathbb{N}$, provide an important example of independent and symmetrically distributed random variables.

Recall that a sequence $\{x_k\}_{k=1}^\infty$ of elements of a Banach space X is said to be *basic* if it is a basis in the closure of its linear span. If for every bijection $\pi : \mathbb{N} \to \mathbb{N}$ the sequence $\{x_{\pi(k)}\}_{k=1}^\infty$ is also basic, we call $\{x_k\}_{k=1}^\infty$ an *unconditional basic sequence.* The following criterion is well known, see [16, Chapter 1, Theorem 10].

Theorem 1. *A basic sequence $\{x_k\}_{k=1}^\infty$ in a Banach space X is unconditional if and only if there exists a number D such that for every $n \in \mathbb{N}$, an arbitrary collection of signs $\{\theta_k\}_{k=1}^n$, $\theta_k = \pm 1$, and all $a_k \in \mathbb{R}$ we have*

$$\Big\|\sum_{k=1}^n \theta_k a_k x_k\Big\|_X \le D\Big\|\sum_{k=1}^n a_k x_k\Big\|_X.$$

A sequence $\{x_k\}_{k=1}^\infty$ satisfying the assumptions of the last theorem with D is said to be *D-unconditional.* In particular, a sequence of independent and symmetrically distributed random variables is a 1-unconditional basic sequence in every symmetric space that contains it, see [17, Proposition 1.14].

We shall denote by C a constant independent of the main parameters. Its particular value may change even within one chain of inequalities.

§2. Chaos of independent functions is a basic sequence in symmetric spaces

In the next theorem we use the lexicographic order: an index $\{i_1, i_2, \dots, i_k\}$ precedes another index $\{j_1, j_2, \dots j_l\}$ if for the first m with i_m different from j_m we have $j_m > i_m$. If there is no such m, the shorter index precedes the longer. Note that this order agrees with the well-known Paley enumeration for the Walsh system (see the Introduction or [16, §4.5]).

Theorem 2. *Let $d \in \mathbb{N}$, and let $\{x_i\}_{i=1}^\infty$ be a sequence of independent symmetrically distributed functions* (*random variables*) *on $[0,1]$, $x_i \ne 0$, $i = 1, 2, \dots,$ such that all functions of the form*

$$x_{i_1 i_2 \dots i_d} = x_{i_1 i_2 \dots i_d}(t) := x_{i_1}(t)x_{i_2}(t)\dots x_{i_d}(t), \quad t \in [0,1],$$

belong to a symmetric space X.

Then the sequence $\{x_{i_1 i_2 \dots i_k}\}_{i_1 > i_2 > \dots > i_k,\, k \le d}$ is basic in X.

We preface the proof by the following lemma.

Lemma 4. *Let $d \in \mathbb{N}$. Suppose that a random variable z and symmetrically distributed random variables $x_1, x_2, \dots$ are mutually independent, and, moreover, z and all products of the form $z x_{i_1} x_{i_2} \dots x_{i_l}$, $l \le d$, belong to a symmetric space X. Then for any real numbers $a_0, a_1, a_2, \dots, a_k$ and any $0 \le m < k$ we have*

$$\Big\| z \sum_{i=0}^{m} a_i y_i \Big\| \le d \Big\| z \sum_{i=0}^{k} a_i y_i \Big\|, \tag{3}$$

where $y_0 \equiv 1$, $y_1, y_2, \dots$ is the sequence of the products $\{x_{i_1} x_{i_2} \dots x_{i_l}\}$ with $l \le d$ and $i_1 > i_2 > \dots > i_l$ (in the lexicographic order).

Proof. We proceed by induction on d. Let $d = 1$. Put $a'_i = a_i$ for $i = 1, 2, \dots, m$ and $a'_i = -a_i$ for $i > m$. Then ($x_0 := y_0 \equiv 1$)

$$\begin{aligned}\Big\| z \sum_{i=0}^{m} a_i x_i \Big\| &= \frac{1}{2} \Big\| z \sum_{i=0}^{k} a_i x_i + z \sum_{i=0}^{k} a'_i x_i \Big\| \\ &\le \frac{1}{2} \Big(\Big\| z \sum_{i=0}^{k} a_i x_i \Big\| + \Big\| z \sum_{i=0}^{k} a'_i x_i \Big\| \Big) = \Big\| z \sum_{i=0}^{k} a_i x_i \Big\|,\end{aligned}$$

because the random vectors

$$(z, x_1, \dots, x_m, x_{m+1}, \dots, x_k) \quad \text{and} \quad (z, x_1, \dots, x_m, -x_{m+1}, \dots, -x_k)$$

are identically distributed by assumption and, thus, the random variables $z \sum_{i=0}^{k} a_i x_i$ and $z \sum_{i=0}^{k} a'_i x_i$ are also identically distributed; therefore, these random variables have the same norms in X.

To complete induction, we must show how to pass from $d - 1$ to d. First, for every $N \in \mathbb{N}$ we verify the relation

$$\Big\| z \sum_{\substack{i_1 \le N, \\ k \le d}} a_{i_1 \dots i_k} x_{i_1} \dots x_{i_k} \Big\| \le \Big\| z \sum_{\substack{i_1 \le N+1, \\ k \le d}} a_{i_1 \dots i_k} x_{i_1} \dots x_{i_k} \Big\|. \tag{4}$$

Put $a'_{(N+1) i_2 \dots i_k} = -a_{(N+1) i_2 \dots i_k}$ and $a'_{i_1 i_2 \dots i_k} = a_{i_1 i_2 \dots i_k}$ if $i_1 \le N$. Then, again because the random variables $(z, x_1, \dots, x_N, x_{N+1})$ and $(z, x_1, \dots, x_N, -x_{N+1})$ are equidistributed, we have

$$\begin{aligned}\Big\| z \sum_{\substack{i_1 \le N \\ k \le d}} a_{i_1 \dots i_k} x_{i_1} \dots x_{i_k} \Big\| &= \frac{1}{2} \Big\| z \sum_{\substack{i_1 \le N+1 \\ k \le d}} a_{i_1 \dots i_k} x_{i_1} \dots x_{i_k} + z \sum_{\substack{i_1 \le N+1 \\ k \le d}} a'_{i_1 \dots i_k} x_{i_1} \dots x_{i_k} \Big\| \\ &\le \frac{1}{2} \Big(\Big\| z \sum_{\substack{i_1 \le N+1 \\ k \le d}} a_{i_1 \dots i_k} x_{i_1} \dots x_{i_k} \Big\| + \Big\| z \sum_{\substack{i_1 \le N+1 \\ k \le d}} a'_{i_1 \dots i_k} x_{i_1} \dots x_{i_k} \Big\| \Big) \\ &= \Big\| z \sum_{\substack{i_1 \le N+1 \\ k \le d}} a_{i_1 \dots i_k} x_{i_1} \dots x_{i_k} \Big\|,\end{aligned}$$

and (4) is proved.

Now, let A be an arbitrary initial fragment of the list $\{i_2 \dots i_k\}$, $i_2 \le N$, taken in the lexicographic order. Using (4) and the inductive hypothesis, we arrive at

$$\begin{aligned}
&\Big\| z \sum_{\substack{i_1 \le N\\ k\le d}} a_{i_1\dots i_k} x_{i_1}\dots x_{i_k} + z \sum_{\substack{i_2\dots i_k\in A\\ k\le d}} a_{(N+1)i_2\dots i_k} x_{N+1} x_{i_2}\dots x_{i_k}\Big\| \\
&\qquad \le \Big\| z \sum_{\substack{i_1 \le N\\ k\le d}} a_{i_1\dots i_k} x_{i_1}\dots x_{i_k}\Big\| + \Big\| z x_{N+1} \sum_{\substack{i_2\dots i_k\in A\\ k\le d}} a_{(N+1)i_2\dots i_k} x_{i_2}\dots x_{i_k}\Big\| \qquad (5)\\
&\qquad \le \Big\| z \sum_{\substack{i_1 \le N+1\\ k\le d}} a_{i_1\dots i_k} x_{i_1}\dots x_{i_k}\Big\| + (d-1)\Big\| z x_{N+1} \sum_{\substack{i_2\le N\\ k\le d}} a_{(N+1)i_2\dots i_k} x_{i_2}\dots x_{i_k}\Big\|.
\end{aligned}$$

Putting $a'_{i_1\dots i_k} = a_{i_1\dots i_k}$ if $i_1 \le N$ and $a'_{(N+1)i_2\dots i_k} = -a_{(N+1)i_2\dots i_k}$, we estimate as before:

$$\begin{aligned}
&\Big\| z x_{N+1} \sum_{\substack{i_2\le N\\ k\le d}} a_{(N+1)i_2\dots i_k} x_{i_2}\dots x_{i_k}\Big\| \\
&\qquad = \frac{1}{2}\Big\| z \sum_{\substack{i_1 \le N+1\\ k\le d}} a_{i_1\dots i_k} x_{i_1}\dots x_{i_k} - z \sum_{\substack{i_1 \le N+1\\ k\le d}} a'_{i_1\dots i_k} x_{i_1}\dots x_{i_k}\Big\| \\
&\qquad\qquad \le \Big\| z \sum_{\substack{i_1 \le N+1\\ k\le d}} a_{i_1\dots i_k} x_{i_1}\dots x_{i_k}\Big\|.
\end{aligned}$$

Together with (5), this implies

$$\Big\| z \sum_{\substack{i_1 \le N\\ k\le d}} a_{i_1\dots i_k} x_{i_1}\dots x_{i_k} + z \sum_{\substack{i_2\dots i_k\in A\\ k\le d}} a_{(N+1)i_2\dots i_k} x_{N+1} x_{i_2}\dots x_{i_k}\Big\| \le d \Big\| z \sum_{\substack{i_1 \le N+1\\ k\le d}} a_{i_1\dots i_k} x_{i_1}\dots x_{i_k}\Big\|.$$

Since the order is lexicographic, from the last inequality and (4) we finally obtain (3) for arbitrary m and k with $0 \le m < k$. ☐

Proof of the theorem. Since $x_{i_1 i_2\dots i_d} \in X$ and the random variables x_i are nondegenerate and independent, it easily follows that $x_{i_1 i_2\dots i_l} \in X$ for every $l \le d$ and arbitrary $i_1, i_2, \dots, i_l$. Putting $z \equiv 1$ and $a_0 = 0$ in Lemma 2, we see that the sequence

$$\{y_i\}_{i=1}^\infty := \{x_{i_1 i_2\dots i_k}\}_{i_1 > i_2 > \dots > i_k,\, k\le d}$$

(taken in the lexicographic order) satisfies the "basis property inequality"

$$\Big\| \sum_{i=1}^m a_i y_i \Big\| \le d \Big\| \sum_{i=1}^k a_i y_i \Big\| \quad \text{for all} \quad 1 \le m < k \quad \text{and} \quad a_i \in \mathbb{R}.$$

The minimality of the system $\{y_i\}_{i=1}^\infty$ (i.e, the fact that, for every $m \in \mathbb{N}$, the function y_m does not belong to the closed linear span of the set $\{y_i\}_{i\ne m}$) follows from the above inequality and the condition $x_i \ne 0$, $i = 1, 2, \dots$. Applying the Banach theorem about bases (see [16, Theorem 1.6]) to the subspace generated by the sequence $\{y_i\}_{i=1}^\infty$, we complete the proof. ☐

Corollary 2. *For every $d \in \mathbb{N}$, the Rademacher chaos $\{r_{i_1} r_{i_2}\dots r_{i_d}\}_{i_1>i_2>\dots i_d}$, viewed as a subsystem of the Walsh system in the Paley enumeration, is a basic sequence in any symmetric space X.*

In [7] the last statement was proved under the additional assumption that X is an interpolation space for the Banach couple (L_1, L_∞).

§3. On the combinatorial dimension of subsets of the Cartesian product of countable sets

In what follows, we shall consider subsystems $\{r_{i_1 i_2 \dots i_d}\}_{(i_1,i_2,\dots,i_d)\in S}$ of a Rademacher chaos for which the set S of indices has combinatorial dimension (see the Introduction for the definition) equal to the order d of the chaos. The following description of such sets is crucial.

Theorem 3. *A set $S \subset \mathbb{N}^d$ has combinatorial dimension $\alpha = d$ if and only if S possesses the following property: for every natural number n there exists a collection of sets $B_1, B_2, \dots, B_d \subset \mathbb{N}$, $|B_1| = |B_2| = \dots = |B_d| = n$, such that*

$$B_1 \times B_2 \times \dots \times B_d \subset S.$$

We shall prove Theorem 3 by two methods. It can be shown that this statement is a consequence of the deep theorem by Erdős about hypergraphs, see [12] (in the case where $d = 2$, see also the earlier paper [18]). But before that we give a direct proof of Theorem 3, not involving the graph theory and more preferable for making the exposition self-contained. We start with an auxiliary statement.

Lemma 5. *Let $m, a, b \in \mathbb{N}$, let X and Y be finite sets with $|X| = m^a$, $|Y| = m^b$, and let $Z \subset X \times Y$. If $|Z| > m^{a+b-\varepsilon}$, where $\varepsilon \in (0,1)$, then*

$$\left|\left\{x \in X \,:\, |\{y \in Y \,:\, (x,y) \in Z\}| > \frac{m^{b-\varepsilon}}{2}\right\}\right| > \frac{m^{a-\varepsilon}}{2}.$$

Proof. We denote

$$X_1 := \left\{x \in X \,:\, |\{y \in Y \,:\, (x,y) \in Z\}| > \frac{m^{b-\varepsilon}}{2}\right\}.$$

Then

$$\begin{aligned} m^{a+b-\varepsilon} < |Z| &= \sum_{x \in X_1} |\{y \in Y \,:\, (x,y) \in Z\}| + \sum_{x \in X \setminus X_1} |\{y \in Y \,:\, (x,y) \in Z\}| \\ &\le |X_1| \cdot m^b + (m^a - |X_1|) \cdot \frac{m^{b-\varepsilon}}{2} \le |X_1| \cdot m^b + \frac{m^{a+b-\varepsilon}}{2}, \end{aligned}$$

whence we obtain the required inequality: $|X_1| > m^{a-\varepsilon}/2$. □

Since in the limiting case where $\alpha = d$ condition 1) of definition 2 is fulfilled for every subset $S \subset \mathbb{N}^d$, we see that, to prove Theorem 3, it suffices to verify the following statement.

Theorem 4. *Let $d \in \mathbb{N}$. For every $n \in \mathbb{N}$ there exist $\varepsilon = \varepsilon(n,d) > 0$ and $l = l(n,d) \in \mathbb{N}$ such that the inclusion*

$$S \subset A_1 \times A_2 \times \dots \times A_d,$$

where $|A_1| = |A_2| = \dots = |A_d| = m > l$, $|S| > m^{d-\varepsilon}$, implies that S includes a set of the form

$$B_1 \times B_2 \times \dots \times B_d,$$

where $B_i \subset A_i$ and $|B_i| = n$, $i = 1, 2, \dots, d$.

Proof. We argue by induction. For $d = 1$ the claim is obvious. Suppose that the claim is true for $d = k$; we verify it for $d = k+1$. Putting $\varepsilon(n, k+1) := \varepsilon(n,k)/(n+2)$, we denote for short $\delta := \varepsilon(n,k)$, $\eta := \varepsilon(n, k+1)$.

Let $X := A_1 \times A_2 \times \dots \times A_d$, $Y := A_{d+1}$, and let

$$S \subset (A_1 \times A_2 \times \dots \times A_d) \times A_{d+1} = X \times Y$$

satisfy $|S| > m^{d+1-\eta}$. It suffices to prove that for m sufficiently large there exist sets $X_1 \subset X$ and $B_{d+1} \subset Y$ with $X_1 \times B_{d+1} \subset S$, $|X_1| > m^{d-\delta}$, and $|B_{d+1}| = n$, because in this case, by the inductive hypothesis, X_1 must contain a subset of the form $B_1 \times B_2 \times \cdots \times B_d$, where $|B_i| = n$, $i = 1, 2, \ldots, n$. Suppose that, to the contrary, the sets X_1 and B_{d+1} of the above form do not exist. This means that for every $E \subset Y$ with $|E| = n$, we have

$$|\{x \in X \,:\, x \times E \subset S\}| \le m^{d-\delta}. \tag{6}$$

Putting $S_x := \{y \in Y \,:\, (x, y) \in S\}$, for every pair (F, G) of sets we introduce the function

$$\chi(F, G) = \begin{cases} 1 & \text{if } F \subset G, \\ 0 & \text{if } F \not\subset G. \end{cases}$$

Then inequality (6) can be rewritten in the form

$$\sum_{x \in X} \chi(E, S_x) \le m^{d-\delta}.$$

Since $E \subset Y$, $|E| = n$ is arbitrary, it follows that

$$\sum_{\substack{E \subset Y \\ |E|=n}} \sum_{x \in X} \chi(E, S_x) \le C_m^n \cdot m^{d-\delta}, \tag{7}$$

where $C_m^n = m!/(n!(m-n)!)$. On the other hand, for arbitrary $G \subset Y$ we have

$$\sum_{\substack{E \subset Y \\ |E|=n}} \chi(E, G) = C_{|G|}^n,$$

whence it follows that

$$\sum_{\substack{E \subset Y \\ |E|=n}} \sum_{x \in X} \chi(E, S_x) = \sum_{x \in X} \sum_{\substack{E \subset Y \\ |E|=n}} \chi(E, S_x) = \sum_{x \in X} C_{|S_x|}^n. \tag{8}$$

By Lemma 5, there exists a set $X_2 \subset X$ such that

$$|X_2| > \frac{m^{d-\eta}}{2} \quad \text{and} \qquad |S_x| > \frac{m^{1-\eta}}{2} \quad \text{for every} \quad x \in X_2.$$

Consequently,

$$\begin{aligned} \sum_{x \in X} C_{|S_x|}^n \ge \sum_{x \in X_2} C_{|S_x|}^n > \frac{m^{d-\eta}}{2} \cdot C_{\lfloor 0{,}5 m^{1-\eta} \rfloor}^n \\ > \frac{m^{d-\eta}(0.5m^{1-\eta} - 1)(0.5m^{1-\eta} - 2) \ldots (0.5m^{1-\eta} - n)}{2n!}. \end{aligned} \tag{9}$$

Comparing (7), (8), and (9), we arrive at the inequality

$$C_m^n \cdot m^{d-\delta} > \frac{m^{d-\eta}(0.5m^{1-\eta} - 1)(0.5m^{1-\eta} - 2) \ldots (0.5m^{1-\eta} - n)}{2n!}$$

or, after elementary transformations,

$$2m(m-1) \ldots (m-n+1) > m^{\delta-\eta} \cdot (0.5m^{1-\eta} - 1)(0.5m^{1-\eta} - 2) \ldots (0.5m^{1-\eta} - n).$$

For fixed n and m sufficiently large, this contradicts the choice of $\eta < \delta/(n+1)$. Thus, if m is sufficiently large, the sets X_1 and B_{d+1} mentioned before inequality (6) exist, and the theorem is proved. □

To formulate the theorem of Erdős mentioned above, we need some notions of the graph theory.

Definition 5. A *hypergraph* is an ordered pair $G = (V, E)$, where V is an arbitrary set and E is an arbitrary subset of 2^V. The elements of V are called *vertices* and the elements of E are called *edges*.

Definition 6. A hypergraph is called an *r-graph* if all its edges have the same cardinality r.

Thus, a 2-graph is a usual graph without loops and multiple edges.

Definition 7. We denote by K_n^r the *complete r-partite hypergraph with parts of size n*, i.e., the r-graph whose set V of vertices consists of r mutually nonintersecting sets $V_1, V_2, \dots, V_r$ having n vertices each, and

$$E = \big\{(v_1, v_2, \dots, v_r) \,:\, v_1 \in V_1, v_2 \in V_2, \dots, v_r \in V_r\big\}.$$

Consequently, K_n^r has precisely nr vertices and n^r edges.

Theorem 5 (Erdős; see [12, Theorem 1])**.** *For arbitrary positive integers n, r and arbitrary $k > k_0(n, r)$, every r-graph $G = (V, E)$ with $|V| = k$ and*

$$|E| \ge k^{r - \frac{1}{n^{r-1}}}$$

contains K_n^r as a subgraph.

Proof of Theorem 3. The set $S \subset \mathbb{N}^d$ can be viewed as a d-graph $S = (V_S, E_S)$ with infinitely many vertices:

$$V_S = \mathbb{N}_1 \cup \mathbb{N}_2 \cup \dots \cup \mathbb{N}_d, \quad E_S = \{(n_1, n_2, \dots, n_d) \,:\, (n_1, n_2, \dots, n_d) \in S\}.$$

If S has dimension d in the sense of Definition 2, then, by item 2) in that definition, for every $\gamma < d$ the set S contains a finite subgraph $G = (V, E)$ with $|V| = dm$ and $|E| > m^\gamma$. Moreover, m can be chosen arbitrarily large. Now, for an arbitrary $n \in \mathbb{N}$, we choose $\gamma > d - 1/n^{d-1}$ and a number m so large that the corresponding finite subgraph $G = (V, E)$ c $|V| = dm$ obeys the following inequalities (the notation k_0 has the same meaning as in the Erdős theorem):

$$dm > k_0(n, d), \quad |E| > m^\gamma > (dm)^{d - \frac{1}{n^{d-1}}}.$$

Applying the Erdős theorem to $G = (V, E) \subset S = (V_S, E_S)$, we arrive at the conclusion that G contains a subgraph $K_n^d \subset G \subset S$. Reversing the terminology, we obtain the sets $B_1, B_2, \dots, B_d$ (they are the parts of the hypergraph K_n^d) mentioned in Theorem 3. □

Remark 1. In [19], a slightly different, more rigorous definition of the combinatorial dimension was used. Specifically, a set $S \subset \mathbb{N}^d$ is of combinatorial dimension α if there exist N_0, $C > 0$, and $c > 0$ such that

1′) the conditions $A_i \subset \mathbb{N}$, $|A_i| = m$, $i = 1, 2, \dots, d$, imply

$$|S \cap (A_1 \times A_2 \times \dots \times A_d)| \le C m^\alpha;$$

2′) for all $N \ge N_0$, we have

$$|S \cap \{1, 2, \dots, N\}^d| \ge c N^\alpha.$$

In the context of Definition 2, it is natural to modify condition 2′), replacing it by the following one:

2″) for every $k \in \mathbb{N}$, there exist sets $A_i \subset \mathbb{N}$, $|A_i| = m > k$, $i = 1, 2, \dots, d$, with

$$|S \cap (A_1 \times A_2 \times \dots \times A_d)| > c m^\alpha.$$

If α is the combinatorial dimension of the set $S \subset \mathbb{N}^d$ in this sense (i.e, properties 1′) and 2″) are fulfilled), then it is easily seen that S has the same combinatorial dimension also in the sense of Definition 2. At the same time, for every $\alpha \in [1, d)$ there exists a set of dimension α in the sense of Definition 2 that does not possess property $1')$ (see [11, p. 486, formula (6.22)] or [20, Theorem 2.3]). In [21] (see also [22, Subsection 6.1]) sets of dimension $\alpha \in (1, d]$ in the sense of Definition 2 that do not have property $2'')$ were considered (yet without any proof of their existence). However, for $\alpha = d$ these definitions are equivalent by Theorem 3. It should also be noted that in [20, Theorem 5.4] and [21, Theorem 4.4] a result close to the equivalence 2) $\Leftrightarrow$ 3) in Theorem 6 of the present paper was established for S of arbitrary dimension α, but only in the case where X belongs to a certain class of Orlicz spaces.

In the next section we shall consider subsystems of the Rademacher chaos of order d indexed by a set S of dimension d. The dimension is understood in the sense of Definition 2. However, a key point in our arguments is the claim of Theorem 3, i.e., the possibility for each n to choose a collection of sets $B_1, B_2, \dots, B_d$ with $|B_1| = |B_2| = \dots = |B_d| = n$ and $B_1 \times B_2 \times \dots \times B_d \subset S$.

§4. Criterion of unconditionality for a sparse Rademacher chaos

Recall that $r_{i_1 i_2 \dots i_d}(t) := r_{i_1}(t) r_{i_2}(t) \dots r_{i_d}(t)$, $0 \le t \le 1$, where the $r_i(t)$ are the Rademacher functions. By $\triangle^d$, $d \in \mathbb{N}$, we shall denote the "lower-triangular" part of $\mathbb{N}^d$, i.e.,

$$\triangle^d := \{(i_1, i_2, \dots, i_d) \in \mathbb{N}^d : i_1 > i_2 > \dots > i_d\}.$$

The Rademacher functions form a 1-unconditional sequence in every symmetric space X. As to the equivalence of the sequence $\{r_i\}_{i=1}^\infty$ in X to the canonical basis in ℓ_2, it was mentioned in the Introduction that this happens if and only if we have the embedding $X \supset G_2$, where G_2 is the separable part of the Orlicz space $\operatorname{Exp} L^2$ constructed by the function $M(u) \sim \exp(u^2)$. The following unconditionality criterion for a Rademacher chaos whose dimension is equal to its order $d \ge 2$ is the principal result of the paper. On the one hand, it shows that, in the chaos case, the relationship between the properties in question is quite different: a Rademacher chaos is unconditional in a symmetric space X if and only if it is equivalent to the canonical basis of ℓ_2. On the other hand, again, a necessary and sufficient condition for that is expressed in terms of embeddings related to spaces of the family G_α (which also involves G_2). For this reason, the equivalence 2) $\Leftrightarrow$ 3) in the next theorem can be regarded as an extension of the Rodin–Semenov theorem (see the Introduction, and also [2]) to the case of a sparse Rademacher chaos.

Theorem 6. *Let X be a symmetric space, let $d \in \mathbb{N}$, $d \ge 2$, and let a set $S \subset \triangle^d$ have combinatorial dimension d. The following conditions are equivalent*:

1) *$\{r_{i_1 i_2 \dots i_d}\}_{(i_1, i_2, \dots, i_d) \in S}$ is an unconditional basic sequence in X*;

2) *the sequence $\{r_{i_1 i_2 \dots i_d}\}_{(i_1, i_2, \dots, i_d) \in S}$ is equivalent in X to the standard basis of ℓ_2, i.e, for some constant C_X we have*

$$\begin{aligned} C_X^{-1} \big\| \{a_{i_1 i_2 \dots i_d}\}_{(i_1, i_2, \dots, i_d) \in S} \big\|_{\ell_2} &\le \Big\| \sum_{(i_1, i_2, \dots, i_d) \in S} a_{i_1 i_2 \dots i_d} r_{i_1 i_2 \dots i_d} \Big\|_X \\ &\le C_X \big\| \{a_{i_1 i_2 \dots i_d}\}_{(i_1, i_2, \dots, i_d) \in S} \big\|_{\ell_2}; \end{aligned}$$

3) *$X \supset G_{2/d}$, where $G_{2/d}$ is the separable part of the Orlicz space $\operatorname{Exp} L^{2/d}$ constructed by the Orlicz function $M(u) \sim \exp(u^{2/d})$.*

The proof of this theorem requires several auxiliary statements. The first of them is well known (see [23, 24] or [11, Chapter VII, Theorem 32]). However, we present a short proof for the reader's convenience.

Lemma 6. *Let U be a finite subset of $\triangle^d$, $d \in \mathbb{N}$. There is a constant C_d depending only on d and such that for every $p \geq 1$ and every sequence $\{a_{i_1 i_2 \dots i_d}\}_{(i_1,i_2,\dots,i_d)\in U}$ of reals we have*

$$\Big\| \sum_{(i_1,i_2,\dots,i_d)\in U} a_{i_1 i_2 \dots i_d} r_{i_1 i_2 \dots i_d} \Big\|_p \leq C_d p^{\frac{d}{2}} \Big(\sum_{(i_1,i_2,\dots,i_d)\in U} a^2_{i_1 i_2 \dots i_d} \Big)^{\frac{1}{2}}$$

and

$$\Big(\sum_{(i_1,i_2,\dots,i_d)\in U} a^2_{i_1 i_2 \dots i_d} \Big)^{\frac{1}{2}} \leq 4^d C_d^2 \cdot \Big\| \sum_{(i_1,i_2,\dots,i_d)\in U} a_{i_1 i_2 \dots i_d} r_{i_1 i_2 \dots i_d} \Big\|_1 .$$

Proof. By [3, Theorem 6.5.1], if a sequence of independent symmetrically distributed random variables $\{\xi_i\}_{i=1}^n$ satisfies the inequality

$$\Big\| a_0 + \sum_{i=1}^n a_i \xi_i \Big\|_p \leq \kappa \Big\| a_0 + \sum_{i=1}^n a_i \xi_i \Big\|_q$$

for some $p \geq q \geq 1$, where the constant κ does not depend on $a_i \in \mathbb{R}$, $i = 0, 1, \dots$, then there exists a constant $C_d \geq 1$ depending only on d such that for every sequence $\{a_{i_1 i_2 \dots i_d}\}_{(i_1,i_2,\dots,i_d)\in U}$ of reals we have

$$\Big\| \sum_{n \geq i_1 > i_2 > \dots > i_d \geq 1} a_{i_1 i_2 \dots i_d} \xi_{i_1} \xi_{i_2} \dots \xi_{i_d} \Big\|_p \leq C_d \kappa^d \Big\| \sum_{n \geq i_1 > i_2 > \dots > i_d \geq 1} a_{i_1 i_2 \dots i_d} \xi_{i_1} \xi_{i_2} \dots \xi_{i_d} \Big\|_q .$$

Taking $\xi_i = r_i$ and $q = 2$, we refer to the Khintchine inequality (1) with $p \geq 2$ and the fact that the system $\{r_{i_1 i_2 \dots i_d}\}$ is orthonormal to deduce that

$$\Big\| \sum_{(i_1,i_2,\dots,i_d)\in U} a_{i_1 i_2 \dots i_d} r_{i_1 i_2 \dots i_d} \Big\|_p \leq C_d p^{\frac{d}{2}} \Big(\sum_{(i_1,i_2,\dots,i_d)\in U} a^2_{i_1 i_2 \dots i_d} \Big)^{\frac{1}{2}}$$

for every finite set U. If $p \in [1, 2)$, this inequality is immediate from the embedding $L_2 \subset L_p$. Next, the Hölder inequality shows that

$$\|x\|_2^3 \leq \|x\|_1 \cdot \|x\|_4^2$$

for every bounded function x. In particular,

$$\begin{aligned}
&\Big\| \sum_{(i_1,i_2,\dots,i_d)\in U} a_{i_1 i_2 \dots i_d} r_{i_1 i_2 \dots i_d} \Big\|_2^3 \\
&\quad \leq \Big\| \sum_{(i_1,i_2,\dots,i_d)\in U} a_{i_1 i_2 \dots i_d} r_{i_1 i_2 \dots i_d} \Big\|_1 \cdot \Big\| \sum_{(i_1,i_2,\dots,i_d)\in U} a_{i_1 i_2 \dots i_d} r_{i_1 i_2 \dots i_d} \Big\|_4^2 \\
&\quad \leq \Big\| \sum_{(i_1,i_2,\dots,i_d)\in U} a_{i_1 i_2 \dots i_d} r_{i_1 i_2 \dots i_d} \Big\|_1 \cdot C_d^2 \cdot 4^d \cdot \Big\| \sum_{(i_1,i_2,\dots,i_d)\in U} a_{i_1 i_2 \dots i_d} r_{i_1 i_2 \dots i_d} \Big\|_2^2 ,
\end{aligned}$$

$$\begin{aligned}
\Big(\sum_{(i_1,i_2,\dots,i_d)\in U} a^2_{i_1 i_2 \dots i_d} \Big)^{\frac{1}{2}} &= \Big\| \sum_{(i_1,i_2,\dots,i_d)\in U} a_{i_1 i_2 \dots i_d} r_{i_1 i_2 \dots i_d} \Big\|_2 \\
&\leq C_d^2 \cdot 4^d \cdot \Big\| \sum_{(i_1,i_2,\dots,i_d)\in U} a_{i_1 i_2 \dots i_d} r_{i_1 i_2 \dots i_d} \Big\|_1 . \qquad \square
\end{aligned}$$

Lemma 7. *Let $n \in \mathbb{N}$, and let sets $B_1, B_2, \dots, B_d \subset \mathbb{N}$ have cardinality n each. Then there exists a collection of signs*

$$\theta_{i_1 i_2 \dots i_d} = \pm 1, \quad (i_1, i_2, \dots, i_d) \in B_1 \times B_2 \times \dots \times B_d,$$

with

$$\Big\| \sum_{i_1\in B_1, i_2\in B_2,\dots,i_d\in B_d} \theta_{i_1 i_2\dots i_d} r_{i_1 i_2 \dots i_d} \Big\|_{L_\infty} \le \sqrt{2d} n^{\frac{d+1}{2}}.$$

Proof. Fixing $t \in [0,1]'$, where $[0,1]'$ is the set of dyadic-irrational points of $[0,1]$, we consider the random variable

$$F_t(\omega) := \sum_{i_1\in B_1, i_2\in B_2,\dots,i_d\in B_d} \Theta_{i_1 i_2\dots i_d}(\omega) r_{i_1 i_2 \dots i_d}(t),$$

where the $\Theta_{i_1 i_2\dots i_d}$ are mutually independent symmetric Bernoulli random variables on some probability space Ω that take the values ± 1. By the well-known inequality for the probability of large deviations in a Bernoulli scheme (see, e.g., [25, Chapter I, §6, formula (42)]), for every $\lambda > 0$ and $t \in [0,1]'$ we have

$$\mathbb{P}\{\omega : |F_t(\omega)| > \lambda\} < 2e^{-\lambda^2/2n^d}.$$

In particular, putting $\lambda = \sqrt{2d} n^{(d+1)/2}$, we deduce that

$$\mathbb{P}\Big\{\omega : |F_t(\omega)| > \sqrt{2d} n^{\frac{d+1}{2}}\Big\} < 2e^{-dn}.$$

Now we denote by A_t the set of $\omega \in \Omega$ for which $|F_t(\omega)| \le \sqrt{2d} n^{(d+1)/2}$ and put $\overline{A_t} := \Omega \setminus A_t$. The last inequality shows that $\mathbb{P}\{\overline{A_t}\} < 2e^{-dn}$. When t runs through $[0,1]'$, we obtain precisely 2^n different variants of values for the sequence of signs $\{r_{i_k}(t)\}_{i_k \in B_k}$ for every $k = 1, 2, \dots, d$ and, consequently, at most 2^{dn} different collections of signs for the sequence $\{r_{i_1 i_2\dots i_d}(t)\}_{i_k\in B_k}$. Therefore, among the sets A_t there are at most 2^{dn} different; we denote them by $A_1, A_2, \dots, A_l$, $l \le 2^{dn}$. Consequently,

$$\mathbb{P}\Big\{\bigcap_{t\in[0,1]'} A_t\Big\} = 1 - \mathbb{P}\Big\{\bigcup_{t\in[0,1]'} \overline{A_t}\Big\} \ge 1 - \sum_{k=1}^{l} \mathbb{P}\{\overline{A_k}\} \ge 1 - 2^{dn}\cdot 2e^{-dn} > 0$$

for $dn \ge 3$. Thus, there exists $\omega_0 \in \Omega$ such that

$$\Big| \sum_{i_1\in B_1, i_2\in B_2,\dots,i_d\in B_d} \Theta_{i_1 i_2\dots i_d}(\omega_0) r_{i_1 i_2 \dots i_d}(t) \Big| \le \sqrt{2d} n^{\frac{d+1}{2}}$$

for a.e. $t \in [0,1]$. As a result, the claim of the lemma is fulfilled for $\theta_{i_1 i_2\dots i_d} = \Theta_{i_1 i_2\dots i_d}(\omega_0)$ (if $dn < 3$, the claim is obvious). ☐

Lemma 8. *For every $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that*

$$\Big(\frac{1}{\sqrt{n}} \sum_{k=1}^{n} r_k\Big)^*(t) \ge \frac{1}{2}\min\Big\{\ln^{\frac{1}{2}}(e/t), m\Big\}$$

for all $t \in \big(0, \frac{1}{10}\big)$.

Proof. By the de Moivre–Laplace integral limit theorem, we have

$$\mu\Big\{t\in[0,1] : \Big|\frac{1}{\sqrt{n}}\sum_{k=1}^{n} r_k(t)\Big| \ge x\Big\} \stackrel{n\to\infty}{\longrightarrow} \frac{2}{\sqrt{2\pi}}\int_x^\infty e^{-u^2/2}\,du$$

uniformly in all $x \ge 0$. Since

$$\int_x^\infty e^{-u^2/2}\,du \ge \int_x^{2x} e^{-u^2/2}\,du \ge xe^{-2x^2},$$

for $x \ge 1$ we have

$$\mu\Big\{t\in[0,1] : \Big|\frac{1}{\sqrt{n}}\sum_{k=1}^{n} r_k(t)\Big| \ge x\Big\} \ge \frac{2}{\sqrt{2\pi}} e^{-2x^2} - \varepsilon_n,$$

where ε_n tends to 0 as $n \to \infty$ and does not depend on x. Thus, for $t = \frac{2}{\sqrt{2\pi}} e^{-2x^2} - \varepsilon_n \le \frac{2}{\sqrt{2\pi}} e^{-2} - \varepsilon_n$ we have

$$\left(\frac{1}{\sqrt{n}} \sum_{k=1}^{n} r_k \right)^*(t) \ge x = \frac{1}{\sqrt{2}} \ln^{\frac{1}{2}} \left(\frac{1}{\sqrt{2\pi}(t+\varepsilon_n)} \right).$$

Since $\varepsilon_n \to 0$, for large n (say, with $\varepsilon_n < \frac{1}{1000}$) this inequality holds true for all $t \in \left(0, \frac{1}{10}\right) \subset \left(0, \frac{2}{\sqrt{2\pi}} e^{-2} - \frac{1}{1000}\right)$. Moreover, on every interval $\left(t_0, \frac{1}{10}\right)$, $t_0 > 0$, for n sufficiently large we have

$$\frac{1}{\sqrt{2}} \ln^{\frac{1}{2}} \left(\frac{2}{\sqrt{2\pi}(t+\varepsilon_n)} \right) \ge \sqrt{\frac{3}{7}} \ln^{\frac{1}{2}} \left(\frac{2}{\sqrt{2\pi} t} \right) \ge \sqrt{\frac{3}{7}} \left(-\frac{1}{4} + \ln \frac{1}{t} \right)^{\frac{1}{2}} \ge \frac{1}{2} \ln^{\frac{1}{2}} \left(\frac{e}{t} \right),$$

because $-1/4 + \ln(1/t) \ge 7/12 \ln(e/t)$ for $t \in (0, e^{-2})$. Consequently, for every $t_0 \in \left(0, \frac{1}{10}\right)$ and sufficiently large n we obtain

$$\left(\frac{1}{\sqrt{n}} \sum_{k=1}^{n} r_k \right)^*(t) \ge \frac{1}{2} \min \left\{ \ln^{\frac{1}{2}}(e/t), \ln^{\frac{1}{2}}(e/t_0) \right\},$$

and the claim follows. □

Lemma 9. *Let $x_1, x_2, \ldots, x_m$ be mutually stochastically independent functions on* $[0,1]$. *Then for every $t_1, t_2, \ldots, t_m \in (0,1]$ and $t = t_1 t_2 \ldots t_m$ we have*

$$\left(\prod_{k=1}^{m} x_k \right)^*(t) \ge \prod_{k=1}^{m} x_k^*(t_k).$$

Proof. Since the functions $x_1, x_2, \ldots, x_m$ are mutually independent, we see that the functions $y(s) := \left| \prod_{k=1}^{m} x_k(s) \right|$ on $[0,1]$ and

$$\widetilde{y}(s_1, s_2, \ldots, s_m) := \prod_{k=1}^{m} x_k^*(s_k) \quad \text{on} \quad [0,1]^m.$$

are equidistributed. On the set $[0,t_1] \times [0,t_2] \times \cdots \times [0,t_m]$, we have

$$\widetilde{y}(s_1, s_2, \ldots, s_m) = \prod_{k=1}^{m} x_k^*(s_k) \ge \prod_{k=1}^{m} x_k^*(t_k),$$

and, consequently,

$$\left(\prod_{k=1}^{m} x_k \right)^*(t) = y^*(t) = \widetilde{y}^*(t) \ge \prod_{k=1}^{m} x_k^*(t_k)$$

for $t = t_1 t_2 \ldots t_m$. □

We formulate a consequence of Lemmas 8 and 9.

Corollary 3. *Let $d, m \in \mathbb{N}$. If $n \in \mathbb{N}$ is sufficiently large, then for every collection of mutually nonintersecting sets $B_1, B_2, \ldots, B_d \subset \mathbb{N}$ with $|B_1| = |B_2| = \cdots = |B_d| = n$ and all $t \in (0, 1/10^d)$ we have*

$$\left(\prod_{k=1}^{d} \left(\frac{1}{\sqrt{n}} \sum_{i_k \in B_k} r_{i_k} \right) \right)^*(t) \ge \frac{1}{(2\sqrt{d})^d} \min \left\{ \ln^{d/2}(e/t), m \right\}.$$

Proof. Since the sets $B_1, B_2, \dots, B_d \subset \mathbb{N}$ are mutually disjoint, the functions

$$x_k(t) := \frac{1}{\sqrt{n}} \sum_{i_k \in B_k} r_{i_k}(t), \quad k = 1, 2, \dots,$$

are mutually independent. For every $k = 1, 2, \dots, d$, the function x_k is equimeasurable with

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} r_i(t),$$

and Lemma 8 shows that

$$x_k^*(t^{1/d}) \geq \frac{1}{2} \min\left\{ \ln^{\frac{1}{2}} \left(\frac{e}{t^{1/d}} \right), m^{1/d} \right\} \geq \frac{1}{2\sqrt{d}} \min\left\{ \ln^{\frac{1}{2}}(e/t), m^{1/d} \right\}$$

for all $t \in (0, 1/10^d)$ and sufficiently large n. Lemma 9 implies

$$\left(\prod_{k=1}^{d} x_k \right)^*(t) \geq \prod_{k=1}^{d} x_k^*(t^{1/d}),$$

whence

$$\left(\prod_{k=1}^{d} x_k \right)^*(t) \geq \left(\frac{1}{2\sqrt{d}} \min\left\{ \ln^{\frac{1}{2}}(e/t), m^{1/d} \right\} \right)^d = \frac{1}{(2\sqrt{d})^d} \min\left\{ \ln^{d/2}(e/t), m \right\}. \qquad \square$$

Now we pass to the proof of the main theorem.

Proof of Theorem 6. First, we prove that conditions 2) and 3) are equivalent, and then we verify the implication 1) $\Rightarrow$ 2). The reverse implication 2) $\Rightarrow$ 1) is obvious.

Assume that condition 2) of the theorem is true. By Theorem 3, for arbitrary $n \in \mathbb{N}$ we find a collection of sets $B_1, B_2, \dots, B_d$, of cardinality n each, with $B_1 \times B_2 \times \dots \times B_d \subset S$. Then, on the one hand, by condition 2) of the theorem,

$$\left\| \prod_{k=1}^{d} \left(\frac{1}{\sqrt{n}} \sum_{i_k \in B_k} r_{i_k} \right) \right\|_X \leq C_X \Bigg(\sum_{(i_1, \dots, i_d), i_k \in B_k} \left(\frac{1}{n^{d/2}} \right)^2 \Bigg)^{\frac{1}{2}} = C_X.$$

On the other hand, since $S \subset \triangle^d$, the sets B_k are disjoint and Corollary 3 shows that

$$\left(\prod_{k=1}^{d} \left(\frac{1}{\sqrt{n}} \sum_{i_k \in B_k} r_{i_k} \right) \right)^*(t) \geq \frac{1}{(2\sqrt{d})^d} \min\left\{ \ln^{d/2}(e/t), m \right\} \chi_{(0, 1/10^d)}(t)$$

for an arbitrary $m \in \mathbb{N}$ for if $n = n(m) \in \mathbb{N}$ sufficiently large. Since X is symmetric, by the previous estimates we obtain

$$\begin{aligned} &\left\| \min\left\{ \ln^{d/2} \left(\frac{e}{t} \right), m \right\} \right\|_X \\ &\qquad \leq (20\sqrt{d})^d \left\| \frac{1}{(2\sqrt{d})^d} \min\left\{ \ln^{d/2} \left(\frac{e}{t} \right), m \right\} \chi_{(0, 1/10^d)}(t) \right\|_X \leq (20\sqrt{d})^d C_X. \end{aligned}$$

for all $m \in \mathbb{N}$. Applying Lemma 2 to the function $\varphi(t) = \ln^{-d/2}(e/t)$, we conclude that $\mathcal{M}^0(\ln^{-d/2}(e/t)) \subset X$. Since $\mathcal{M}^0(\ln^{-d/2}(e/t)) = G_{2/d}$ (see §1), we have proved the implication 2) $\Rightarrow$ 3).

Conversely assume that $G_{2/d} \subset X$. By Lemma 6, for every finite set $U \subset S$ we have

$$\sup_{p \geq 1} p^{-\frac{d}{2}} \left\| \sum_{(i_1, i_2, \dots, i_d) \in U} a_{i_1 i_2 \dots i_d} r_{i_1 i_2 \dots i_d} \right\|_p \leq C_d \Bigg(\sum_{(i_1, i_2, \dots, i_d) \in U} a_{i_1 i_2 \dots i_d}^2 \Bigg)^{\frac{1}{2}}.$$

It is well known (see, e.g., [26, p. 22, 23], [27, p. 275, Corollary 1], or a more general statement in [28, Theorem 4.7]) that for some C depending only on $\alpha > 0$ we have

$$C^{-1} \sup_{p\geq 1} p^{-1/\alpha} \|x\|_p \leq \|x\|_{\mathrm{ExpL}^\alpha} \leq C \sup_{p\geq 1} p^{-1/\alpha} \|x\|_p.$$

We put here $\alpha = 2/d$; by the preceding inequality and the inclusion $G_{2/d} \subset X$, we obtain

$$\begin{aligned} &\Big\| \sum_{(i_1,i_2,\dots,i_d)\in U} a_{i_1 i_2 \dots i_d} r_{i_1 i_2 \dots i_d} \Big\|_X \\ &\leq C \Big\| \sum_{(i_1,i_2,\dots,i_d)\in U} a_{i_1 i_2 \dots i_d} r_{i_1 i_2 \dots i_d} \Big\|_{G_{2/d}} \leq C \Big(\sum_{(i_1,i_2,\dots,i_d)\in U} a^2_{i_1 i_2 \dots i_d} \Big)^{\frac{1}{2}}, \end{aligned} \tag{10}$$

where the constant C depends only on X and d. Next, by a standard argument involving the completeness of X, we obtain the right-hand inequality in condition 2) of the theorem. The left-hand inequality is a consequence of the second inequality in Lemma 6, the embedding $X \subset L_1$, and the limit passage along a system $U_1 \subset U_2 \subset \dots \subset U_n \subset \dots$ of finite subsets that exhaust S. This proves the equivalence 2) $\Leftrightarrow$ 3).

Now, we prove the implication 1) $\Rightarrow$ 2). First, we show that if the sequence

$$\{r_{i_1 i_2 \dots i_d}\}_{(i_1,i_2,\dots,i_d)\in S}$$

is basic and unconditional in X, then $X \supset \mathcal{M}(\ln^{-\frac{1}{2}}(e/t))$.

By Theorem 1, for every finite set $U \subset S$ and every $\theta_{i_1 i_2 \dots i_d} = \pm 1$, $(i_1, i_2, \dots, i_d) \in U$, we have

$$\Big\| \sum_{(i_1,i_2,\dots,i_d)\in U} r_{i_1 i_2 \dots i_d} \Big\|_X \leq C \Big\| \sum_{(i_1,i_2,\dots,i_d)\in U} \theta_{i_1 i_2 \dots i_d} r_{i_1 i_2 \dots i_d} \Big\|_X. \tag{11}$$

Since the set $S \subset \triangle^d$ has dimension d, Theorem 3 shows that for every $n \in \mathbb{N}$ there exists a set $U_n \subset S$ of the form $U_n = B_1 \times B_2 \times \dots \times B_d$, $|B_1| = |B_2| = \dots = |B_d| = n$. Choosing signs as in Lemma 7 and using the preceding inequality and the embedding $L_\infty \subset X$, we obtain

$$\Big\| \sum_{(i_1,i_2,\dots,i_d)\in U_n} r_{i_1 i_2 \dots i_d} \Big\|_X \leq C \Big\| \sum_{(i_1,i_2,\dots,i_d)\in U_n} \theta_{i_1 i_2 \dots i_d} r_{i_1 i_2 \dots i_d} \Big\|_{L_\infty} \leq C n^{\frac{d+1}{2}}, \tag{12}$$

where the constant C depends on X and d, but not on n. On the other hand, there exists a set $A \subset [0, 1]$ of measure 2^{-dn} on which all Rademacher functions involved in the sum over U_n have value $+1$. On this set, we have

$$\sum_{(i_1,i_2,\dots,i_d)\in U_n} r_{i_1 i_2 \dots i_d} \geq |U_n| = n^d,$$

and since the sum is symmetrically distributed, it follows that

$$\Big(\sum_{(i_1,i_2,\dots,i_d)\in U_n} r_{i_1 i_2 \dots i_d} \Big)^*(t) \geq n^d \chi_{(0,2^{1-dn})}(t). \tag{13}$$

Combined with (12), this yields the following estimate for the fundamental function of X:

$$\phi_X\big(2^{1-dn}\big) \leq C n^{\frac{d+1}{2} - d} = C n^{\frac{1-d}{2}} = C d^{\frac{d-1}{2}} \log_2^{\frac{1-d}{2}} \Big(\frac{2}{2^{1-nd}} \Big).$$

Since the points of the form 2^{1-dn} constitute a geometric progression and the functions involved in the preceding inequality are concave for sufficiently small values of the argument, we see that there exists a constant $C > 0$ such that the inequality

$$\phi_X(t) \leq C \ln^{\frac{1-d}{2}}(e/t)$$

is fulfilled for all $t \in (0,1]$. By the extremal property of Lorentz spaces mentioned in §1, it follows that

$$\Lambda\big(\ln^{\frac{1-d}{2}}(e/t)\big) \subset X. \tag{14}$$

If $d > 2$, then $(d-1)/2 > \frac{1}{2}$, and the required embedding $\mathcal{M}(\ln^{-1/2}(e/t)) \subset X$ immediately follows from Corollary 1. In the case of $d = 2$ some additional arguments will be in order.

First, by Lemma 7 and inequality (10), for some $\theta_{i_1 i_2} = \pm 1$, $(i_1, i_2) \in U_n$, we have

$$\Big\| \sum_{(i_1,i_2)\in U_n} \theta_{i_1 i_2} r_{i_1 i_2} \Big\|_{L_\infty} \le 2n^{\frac{3}{2}}$$

and

$$\Big\| \sum_{(i_1,i_2)\in U_n} \theta_{i_1 i_2} r_{i_1 i_2} \Big\|_{\mathcal{M}(\ln^{-1}(e/t))} \le C\Big(\sum_{(i_1,i_2)\in U_n} \theta^2_{i_1 i_2} \Big)^{\frac{1}{2}} = Cn,$$

respectively. These relations and the definition of the norm in a Marcinkiewicz space show that

$$\Big(\sum_{(i_1,i_2)\in U_n} \theta_{i_1 i_2} r_{i_1 i_2} \Big)^*(t) \le C \min\Big\{ n^{\frac{3}{2}}, n \ln\Big(\frac{e}{t}\Big) \Big\}, \quad 0 < t \le 1,$$

whence

$$\Big\| \sum_{(i_1,i_2)\in U_n} \theta_{i_1 i_2} r_{i_1 i_2} \Big\|_{\Lambda(\ln^{-1/2}(e/t))} \le C \int_0^1 \min\Big\{ n^{\frac{3}{2}}, n \ln\Big(\frac{e}{t}\Big) \Big\}\, d\ln^{-1/2}\Big(\frac{e}{t}\Big)$$

$$= C\Big(\int_0^{e^{1-\sqrt{n}}} n^{\frac{3}{2}}\, d\ln^{-1/2}\Big(\frac{e}{t}\Big) + \int_{e^{1-\sqrt{n}}}^1 n \ln\Big(\frac{e}{t}\Big)\, d\ln^{-1/2}\Big(\frac{e}{t}\Big) \Big) < Cn^{\frac{5}{4}}.$$

Since the system $\{r_{i_1 i_2}\}_{(i_1,i_2)\in S}$ is unconditional, by inequality (13) and the embedding (14) for $d = 2$, we obtain

$$\phi_X\left(2^{1-2n}\right) \le \frac{1}{n^2} \Big\| \sum_{(i_1,i_2)\in U_n} r_{i_1 i_2} \Big\|_X$$

$$\le \frac{C}{n^2} \Big\| \sum_{(i_1,i_2)\in U_n} \theta_{i_1 i_2} r_{i_1 i_2} \Big\|_X \le Cn^{\frac{5}{4}-2} = C \log_2^{-\frac{3}{4}}\Big(\frac{2}{2^{1-2n}}\Big).$$

Since the Lorentz space is extremal (see §1), by Corollary 1 we obtain

$$\mathcal{M}(\ln^{-1/2}(e/t)) \subset \Lambda(\ln^{-3/4}(e/t)) \subset X.$$

Next, let $r'_{i_1 i_2 \dots i_d}(u)$ be the sequence of Rademacher functions enumerated arbitrarily. It is well known (see, e.g., [29, Theorem V.8.7]) that for every finite set $U \subset S$ and every sequence $\{a_{i_1 i_2 \dots i_d}\}_{(i_1,i_2,\dots,i_d)\in U}$ we have

$$\Big\| \sum_{(i_1,i_2,\dots,i_d)\in U} r'_{i_1 i_2 \dots i_d} a_{i_1 i_2 \dots i_d} \Big\|_{\mathrm{ExpL}^2} \le C\Big(\sum_{(i_1,i_2,\dots,i_d)\in U} a^2_{i_1 i_2 \dots i_d} \Big)^{\frac{1}{2}}.$$

(This can also be deduced from (10) if we put $d=1$ there). Since the space $\mathcal{M}(\ln^{-1/2}(e/t))$ coincides with the Orlicz space $\operatorname{Exp} L^2$, we can use inequality (11), the embedding $X \supset$

$\mathcal{M}(\ln^{-1/2}(e/t))$, and Lemma 3 to deduce that

$$\begin{aligned}\Big\| \sum_{(i_1,i_2,\dots,i_d)\in U} a_{i_1 i_2\dots i_d} r_{i_1 i_2\dots i_d}\Big\|_X &\\ \le C\int_0^1 \Big\| \sum_{(i_1,i_2,\dots,i_d)\in U} r'_{i_1 i_2\dots i_d}(u) a_{i_1 i_2\dots i_d} r_{i_1 i_2\dots i_d}\Big\|_X du &\\ \le C\int_0^1 \Big\| \sum_{(i_1,i_2,\dots,i_d)\in U} r'_{i_1 i_2\dots i_d}(u) a_{i_1 i_2\dots i_d} r_{i_1 i_2\dots i_d}\Big\|_{\mathrm{ExpL}^2} du &\\ \le C \operatorname*{ess\,sup}_{t\in[0,1]} \Big\| \sum_{(i_1,i_2,\dots,i_d)\in U} r'_{i_1 i_2\dots i_d}(\,\cdot\,) a_{i_1 i_2\dots i_d} r_{i_1 i_2\dots i_d}(t)\Big\|_{\mathrm{ExpL}^2(\,\cdot\,)} &\\ \le C\Big(\sum_{(i_1,i_2,\dots,i_d)\in U} a^2_{i_1 i_2\dots i_d}\Big)^{\frac12}&\end{aligned}$$

with some C depending only on X and d, where $U \subset S$ is an arbitrary finite set. By standard arguments, it follows that

$$\Big\| \sum_{(i_1,i_2,\dots,i_d)\in S} a_{i_1 i_2\dots i_d} r_{i_1 i_2\dots i_d}\Big\|_X \le C\big\|\{a_{i_1 i_2\dots i_d}\}_{(i_1,i_2,\dots,i_d)\in S}\big\|_{\ell_2}.$$

The reverse inequality is an immediate consequence of the inequality in Lemma 6 and the embedding $X \subset L_1$. This proves the theorem. ☐

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