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HOMOGENIZATION OF HIGH ORDER ELLIPTIC OPERATORS WITH PERIODIC COEFFICIENTS

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ABSTRACT. A selfadjoint strongly elliptic operator A_ε of order $2p$ given by the expression $b(\mathbf{D})^* g(\mathbf{x}/\varepsilon) b(\mathbf{D})$, $\varepsilon > 0$, is studied in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. Here $g(\mathbf{x})$ is a bounded and positive definite $(m \times m)$-matrix-valued function on $\mathbb{R}^d$; it is assumed that $g(\mathbf{x})$ is periodic with respect to some lattice. Next, $b(\mathbf{D}) = \sum_{|\alpha|=p} b_\alpha \mathbf{D}^\alpha$ is a differential operator of order p with constant coefficients; the b_α are constant $(m \times n)$-matrices. It is assumed that $m \geq n$ and that the symbol $b(\boldsymbol{\xi})$ has maximal rank. For the resolvent $(A_\varepsilon - \zeta I)^{-1}$ with $\zeta \in \mathbb{C} \setminus [0,\infty)$, approximations are obtained in the norm of operators in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ and in the norm of operators acting from $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to the Sobolev space $H^p(\mathbb{R}^d;\mathbb{C}^n)$, with error estimates depending on ε and ζ.

INTRODUCTION

0.1. Operator-theoretic approach to homogenization theory. The paper concerns homogenization theory for periodic differential operators (DO's) in the small period limit. This is a wide area of theoretical and applied science. A broad literature is devoted to homogenization problems; first of all, we mention the books [BeLP, BaPa, ZhKO].

In a series of papers by M. Sh. Birman and T. A. Suslina [BSu1, BSu2, BSu3, BSu4], an operator-theoretic approach to homogenization problems was suggested, making it possible to study a wide class of selfadjoint matrix second order operators $\mathcal{A}$ acting in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. It was assumed that $\mathcal{A}$ admits a factorization

$$\mathcal{A} = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}). \tag{0.1}$$

Here $g(\mathbf{x})$ is a bounded and uniformly positive definite $(m \times m)$-matrix-valued function periodic with respect to some lattice $\Gamma \subset \mathbb{R}^d$. Let Ω denote the elementary cell of the lattice Γ. Next, $b(\mathbf{D})$ is an $(m \times n)$-matrix homogeneous first order DO. It is assumed that $m \geq n$ and that the symbol $b(\boldsymbol{\xi})$ has rank n for any $0 \neq \boldsymbol{\xi} \in \mathbb{R}^d$. Under the above assumptions, $\mathcal{A}$ is strongly elliptic. The simplest example of the operator (0.1) is the scalar elliptic operator $-\operatorname{div} g(\mathbf{x})\nabla$ (the acoustics operator); the operator of elasticity theory also can be represented in the form (0.1). These and other examples were considered in [BSu1, BSu3, BSu4] in detail.

Let $\varepsilon > 0$ be a small parameter. We denote $F^\varepsilon(\mathbf{x}) := F(\varepsilon^{-1}\mathbf{x})$. Consider the operator $\mathcal{A}_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})$ whose coefficients oscillate rapidly as $\varepsilon \to 0$.

In [BSu1], it was shown that, as $\varepsilon \to 0$, the resolvent $\left(\mathcal{A}_\varepsilon + I\right)^{-1}$ converges in the operator norm in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to the resolvent of the *effective operator* $\mathcal{A}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$.

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Here g^0 is a constant *effective matrix*. It was proved that

$$\big\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}^0 + I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le C\varepsilon. \tag{0.2}$$

In [BSu2, BSu3], a more accurate approximation of the resolvent $(\mathcal{A}_\varepsilon + I)^{-1}$ in the operator norm in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ with error $O(\varepsilon^2)$ was obtained. In [BSu4], an approximation of the resolvent $(\mathcal{A}_\varepsilon + I)^{-1}$ was found in the norm of operators acting from $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to the Sobolev space $H^1(\mathbb{R}^d;\mathbb{C}^n)$:

$$\big\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}^0 + I)^{-1} - \varepsilon\mathcal{K}(\varepsilon)\big\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \le C\varepsilon. \tag{0.3}$$

Here $\mathcal{K}(\varepsilon)$ is the so-called *corrector*. The operator $\mathcal{K}(\varepsilon)$ involves rapidly oscillating factors, and so it depends on ε; herewith, $\|\mathcal{K}(\varepsilon)\|_{L_2\to H^1} = O(\varepsilon^{-1})$.

Estimates (0.2) and (0.3) are order-sharp; the constants are controlled explicitly in terms of the problem data. Such results are called *operator error estimates* in homogenization theory. The method of [BSu1, BSu2, BSu3, BSu4] is based on the scaling transformation, the direct integral expansion for the periodic operator $\mathcal{A}$ (based on the Floquet–Bloch theory), and the analytic perturbation theory. It turned out that the resolvent of the operator $\mathcal{A}_\varepsilon$ can be approximated in terms of the threshold characteristics of the operator $\mathcal{A}$ at the bottom of the spectrum. In this sense, the homogenization procedure is a *spectral threshold effect*.

We also mention the recent paper [Su], where analogs of estimates (0.2), (0.3) for the resolvent $(\mathcal{A}_\varepsilon - \zeta I)^{-1}$ at an arbitrary point $\zeta \in \mathbb{C}\setminus\mathbb{R}_+$ were obtained; the corresponding error estimates are two-parametric: they depend on ε and ζ.

A different approach to operator error estimates (the modified first approximation method) was suggested by V. V. Zhikov; by this method, in [Zh] and [ZhPas] analogs of estimates (0.2) and (0.3) were obtained for the acoustics operator and the elasticity operator.

Homogenization problems for periodic elliptic DO's of high even order are of special interest. In the paper [V] by N. A. Veniaminov, the method suggested in [BSu1] was developed for such operators. The homogenization problem was studied for the operator

$$\mathcal{B}_\varepsilon = (\mathbf{D}^p)^* g^\varepsilon(\mathbf{x})\mathbf{D}^p. \tag{0.4}$$

Here $g(\mathbf{x})$ is a symmetric, uniformly positive definite, and bounded tensor of order $2p$, periodic with respect to the lattice Γ. The operator (0.4) with $p = 2$ arises in the theory of elastic plates (see [ZhKO]).

The effective operator for $\mathcal{B}_\varepsilon$ is given by $\mathcal{B}^0 = (\mathbf{D}^p)^* g^0\mathbf{D}^p$, where g^0 is an *effective tensor*. In [V], the following analog of estimate (0.2) was proved:

$$\big\|(\mathcal{B}_\varepsilon + I)^{-1} - (\mathcal{B}^0 + I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le C\varepsilon. \tag{0.5}$$

0.2. Main results. We study a class of high order elliptic periodic DO's that is more general than (0.4). Consider the operator

$$A = A(g) = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}), \tag{0.6}$$

where $g(\mathbf{x})$ is a uniformly positive definite and bounded $(m\times m)$-matrix-valued function, periodic with respect to the lattice Γ, and $b(\mathbf{D})$ is an $(m\times n)$-matrix homogeneous DO of order p. The precise definition of the operator (0.6) is given in Subsection 4.1. We study the homogenization problem for the operator $A_\varepsilon = A(g^\varepsilon)$.

The main results of the paper are approximations of the resolvent $(A_\varepsilon - \zeta I)^{-1}$, where $\zeta \in \mathbb{C}\setminus\mathbb{R}_+$, in various operator norms with two-parametric error estimates (depending on ε and ζ). Theorem 8.1 gives approximation of the resolvent in the operator norm in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ (an analog of estimate (0.5)):

$$\big\|(A_\varepsilon - \zeta I)^{-1} - (A^0 - \zeta I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le C_1(\zeta)\varepsilon. \tag{0.7}$$

In Theorem 8.2, approximation of the resolvent is obtained in the "energy" norm (i.e., in the norm of operators acting from $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to the Sobolev space $H^p(\mathbb{R}^d;\mathbb{C}^n)$):

$$\big\|(A_\varepsilon-\zeta I)^{-1}-(A^0-\zeta I)^{-1}-\varepsilon^p K(\zeta;\varepsilon)\big\|_{L_2(\mathbb{R}^d)\to H^p(\mathbb{R}^d)}\le C_2(\zeta)\varepsilon. \tag{0.8}$$

It is shown that the effective operator A^0 has the same structure as the initial operator: $A^0=b(\mathbf{D})^*g^0b(\mathbf{D})$. The corrector $K(\zeta;\varepsilon)$ involves rapidly oscillating factors, and $\|K(\zeta;\varepsilon)\|_{L_2\to H^p}=O(\varepsilon^{-p})$. The dependence of $C_1(\zeta)$ and $C_2(\zeta)$ on the spectral parameter ζ is traced.

Besides estimate (0.8), we obtain an approximation of the operator $g^\varepsilon b(\mathbf{D})(A_\varepsilon-\zeta I)^{-1}$ (corresponding to the "flux") in the norm of operators from $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to $L_2(\mathbb{R}^d;\mathbb{C}^m)$.

In the general case, the corrector $K(\zeta;\varepsilon)$ involves an auxiliary smoothing operator. We distinguish a condition under which the standard corrector, without the smoothing operator, can be used (see Theorem 8.6).

0.3. The method. The method is a further development of the operator-theoretic approach.

By the scaling transformation, the dependence on the parameter ε is carried from the coefficients of the operator over to the resolvent point. Namely, we have the following unitary equivalence:

$$(A_\varepsilon-\zeta I)^{-1}\sim\varepsilon^{2p}(A-\zeta\varepsilon^{2p}I)^{-1},$$
$$(A^0-\zeta I)^{-1}\sim\varepsilon^{2p}(A^0-\zeta\varepsilon^{2p}I)^{-1}.$$

Then estimate (0.7) reduces to the inequality

$$\big\|(A-\zeta\varepsilon^{2p}I)^{-1}-(A^0-\zeta\varepsilon^{2p}I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\le C_1(\zeta)\varepsilon^{1-2p}. \tag{0.9}$$

To prove (0.8), we combine (0.7) and the auxiliary inequality

$$\big\|A_\varepsilon^{1/2}\big((A_\varepsilon-\zeta I)^{-1}-(A^0-\zeta I)^{-1}-\varepsilon^pK(\zeta;\varepsilon)\big)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\le C_3(\zeta)\varepsilon.$$

By the scaling transformation, the last inequality is equivalent to

$$\big\|A^{1/2}\big((A-\zeta\varepsilon^{2p}I)^{-1}-(A^0-\zeta\varepsilon^{2p}I)^{-1}-\widetilde{K}(\zeta;\varepsilon)\big)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\le C_3(\zeta)\varepsilon^{1-p}. \tag{0.10}$$

Estimates (0.9) and (0.10) with arbitrary $\zeta\in\mathbb{C}\setminus\mathbb{R}_+$ are deduced from the same inequalities with $\zeta=-1$ by suitable identities for resolvents; this trick is borrowed from [Su]. Therefore, the main considerations concern the case where $\zeta=-1$.

The operator A is expanded in the direct integral of operators $A(\mathbf{k})$ acting in $L_2(\Omega;\mathbb{C}^n)$ and depending on the parameter $\mathbf{k}$ (the *quasimomentum*). The operator $A(\mathbf{k})$ is given by the expression $b(\mathbf{D}+\mathbf{k})^*g(\mathbf{x})b(\mathbf{D}+\mathbf{k})$ with periodic boundary conditions. As in [BSu1], we distinguish the one-dimensional parameter $t=|\mathbf{k}|$, with respect to which the family $A(\mathbf{k})$ is a *polynomial operator pencil* of order $2p$. In [V], an abstract approach was developed for such operator pencils. Using it, we prove estimate (0.9) (with $\zeta=-1$). To check (0.10), we develop the abstract scheme for the polynomial pencils by analogy with [BSu4].

0.4. Organization of the paper. The paper consists of 8 sections. §§1–3 are devoted to the abstract approach. In §1, we describe a factorized operator family $A(t)=X(t)^*X(t)$ and introduce the spectral germ. In §2, the results obtained in [V] (the threshold approximations and the principal order approximation for $(A(t)+\varepsilon^{2p}I)^{-1}$) are described. In §3, by further development of the abstract method, we obtain approximation of the resolvent $(A(t)+\varepsilon^{2p}I)^{-1}$ with the corrector taken into account. In §4, the class of periodic differential operators A acting in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ is introduced; the operator A is expanded in the direct integral of the operators $A(\mathbf{k})$ acting in $L_2(\Omega;\mathbb{C}^n)$. In §5, the operator family $A(\mathbf{k})$ is studied with the help of the abstract results, the effective operator is introduced, and the properties of the effective matrix are described. In §6, we obtain approximation for the resolvent $(A(\mathbf{k})+\varepsilon^{2p}I)^{-1}$, by applying theorems of the abstract approach. In §7, using the results of §6 and the direct integral expansion for A, we deduce theorems about approximation of the resolvent $(A+\varepsilon^{2p}I)^{-1}$; next, with the help of appropriate identities for the resolvents, these theorems are carried over to the case of the resolvent $(A-\zeta\varepsilon^{2p}I)^{-1}$ at an arbitrary point $\zeta\in\mathbb{C}\setminus\mathbb{R}_+$. We distinguish the condition under which the smoothing operator in the corrector can be eliminated. In §8, we use the scaling transformation to deduce the main results (theorems about approximations of the resolvent $(A_\varepsilon-\zeta I)^{-1}$ in various operator norms) from the estimates of §7.

0.5. Notation. Let $\mathfrak{H}$ and $\mathfrak{G}$ be separable Hilbert spaces. The symbols $\|\cdot\|_{\mathfrak{H}}$ and $(\cdot,\cdot)_{\mathfrak{H}}$ stand for the norm and the inner product in $\mathfrak{H}$, respectively; the symbol $\|\cdot\|_{\mathfrak{H}\to\mathfrak{G}}$ denotes the norm of a continuous linear operator acting from $\mathfrak{H}$ to $\mathfrak{G}$. Sometimes we omit the indices. If G is a linear operator acting from $\mathfrak{H}$ to $\mathfrak{G}$, then $\operatorname{Dom} G$ and $\operatorname{Ker} G$ denote its domain and its kernel, respectively. If $\mathfrak{N}$ is a subspace of $\mathfrak{H}$, then $\mathfrak{N}^\perp$ denotes its orthogonal complement.

The symbols $\langle\cdot,\cdot\rangle$ and $|\cdot|$ stand for the inner product and the norm in $\mathbb{C}^n$, respectively; $\mathbf{1}_n$ is the identity $(n\times n)$-matrix. If a is an $(m\times n)$-matrix, then $|a|$ denotes the norm of a viewed as an operator from $\mathbb{C}^n$ to $\mathbb{C}^m$.

The L_q-classes of $\mathbb{C}^n$-valued functions in a domain $\mathcal{O}\subset\mathbb{R}^d$ are denoted by $L_q(\mathcal{O};\mathbb{C}^n)$, $1\le q\le\infty$. The Sobolev classes of $\mathbb{C}^n$-valued functions (in a domain $\mathcal{O}\subseteq\mathbb{R}^d$) of order s and integrability degree q are denoted by $W_q^s(\mathcal{O};\mathbb{C}^n)$. If $q=2$, we use the notation $H^s(\mathcal{O};\mathbb{C}^n)$, $s\in\mathbb{R}$. If $n=1$, we write $L_q(\mathcal{O})$, $W_q^s(\mathcal{O})$, $H^s(\mathcal{O})$, but sometimes we use this abbreviated notation also for spaces of vector-valued or matrix-valued functions.

The bold font is used for vectors. We use the notation

$$\mathbf{x}=(x_1,\dots,x_d)\in\mathbb{R}^d,$$
$$iD_j=\partial_j=\partial/\partial x_j,\quad j=1,\dots,d,$$
$$\mathbf{D}=-i\nabla=(D_1,\dots,D_d).$$

Next, if $\alpha=(\alpha_1,\dots,\alpha_d)\in\mathbb{Z}_+^d$ is a multiindex and $\mathbf{k}\in\mathbb{R}^d$, then $|\alpha|=\sum_{j=1}^d\alpha_j$, $\mathbf{k}^\alpha=k_1^{\alpha_1}\dots k_d^{\alpha_d}$, and $\mathbf{D}^\alpha=D_1^{\alpha_1}\dots D_d^{\alpha_d}$. If α and β are two multiindices, we write $\beta\le\alpha$ if $\beta_j\le\alpha_j$, $j=1,\dots,d$; the binomial coefficients are denoted by $C_\alpha^\beta=C_{\alpha_1}^{\beta_1}\cdot\ldots\cdot C_{\alpha_d}^{\beta_d}$.

We denote $\mathbb{R}_+=[0,\infty)$. By C, B, c, $\mathcal{C}$, and $\mathfrak{C}$ (possibly, with indices and marks) we denote various constants in estimates.

0.6. Added in proof. The authors have learned that, in parallel with the present work, similar homogenization results for high order elliptic equations were obtained by S. E. Pastukhova in [Pas1, Pas2] by a different approach mentioned above (the modified method of first approximation).

§1. Abstract approach. The spectral germ

1.1. Polynomial pencils of the form $X(t)^*X(t)$. Let $\mathfrak{H}$ and $\mathfrak{H}_*$ be separable complex Hilbert spaces. Consider a family (polynomial pencil) of operators

$$X(t)=\sum_{j=0}^{p}X_jt^j,\quad t\in\mathbb{R},\quad p\in\mathbb{N},\quad p\ge2.$$

(The case where $p=1$ was studied in [BSu1, BSu2, BSu4] in detail.) The operators $X(t)$ and X_j act from $\mathfrak{H}$ to $\mathfrak{H}_*$:

$$X(t),\quad X_j\colon\mathfrak{H}\to\mathfrak{H}_*.$$

It is assumed that the operator X_0 is *densely defined and closed*, and that X_p is *bounded*. Moreover, we impose the following condition on the domains of these operators.

Condition 1.1.

$$\operatorname{Dom} X(t) = \operatorname{Dom} X_0 \subset \operatorname{Dom} X_j \subset \operatorname{Dom} X_p = \mathfrak{H}, \quad j = 1, \dots, p-1, \quad t \in \mathbb{R}.$$

We also assume that the intermediate operators X_j with $j = 1, \dots, p-1$ are subordinate to X_0.

Condition 1.2. *For $j = 0, \dots, p-1$ and any $u \in \operatorname{Dom} X_0$, we have*

$$\|X_j u\|_{\mathfrak{H}_*} \le \widetilde{C} \, \|X_0 u\|_{\mathfrak{H}_*} , \tag{1.1}$$

where $\widetilde{C}$ is some constant (obviously, $\widetilde{C} \ge 1$).

Note that for $j = 0$ estimate (1.1) is trivial. Under the above assumptions, the operator $X(t)$ is *closed* on the domain $\operatorname{Dom} X(t) = \operatorname{Dom} X_0$.

From Condition 1.2 it follows that

$$\operatorname{Ker} X_0 \subset \operatorname{Ker} X_j, \quad j = 1, \dots, p-1. \tag{1.2}$$

Our main object is the family of selfadjoint nonnegative operators given by

$$A(t) = X(t)^* X(t), \quad t \in \mathbb{R}, \tag{1.3}$$

in $\mathfrak{H}$. The operator (1.3) is generated by the closed quadratic form

$$a(t)[u,u] = \|X(t)u\|^2_{\mathfrak{H}_*}, \quad u \in \operatorname{Dom} X_0.$$

We denote $A(0) = X_0^* X_0 =: A_0$ and put

$$\mathfrak{N} := \operatorname{Ker} A_0 = \operatorname{Ker} X_0, \qquad \mathfrak{N}_* := \operatorname{Ker} A_0^* = \operatorname{Ker} X_0^*.$$

Let P and P_* be the orthogonal projections of $\mathfrak{H}$ onto $\mathfrak{N}$ and of $\mathfrak{H}_*$ onto $\mathfrak{N}_*$, respectively.

Condition 1.3. *Suppose that the point $\lambda_0 = 0$ is an isolated point in the spectrum of A_0, and*

$$n := \dim \mathfrak{N} < \infty, \quad n \le n_* := \dim \mathfrak{N}_* \le \infty.$$

The distance from $\lambda_0 = 0$ to the rest of the spectrum of A_0 is denoted by d^0. Let $F(t,s)$ denote the spectral projection of $A(t)$ for the interval $[0,s]$, and let $\mathfrak{F}(t,s) := F(t,s)\mathfrak{H}$. Next, fixing a positive number δ such that $\delta \le \min\{d^0/36, 1/4\}$, we put

$$t^0 = \delta^{1/2} (\widehat{C})^{-1}, \tag{1.4}$$

where

$$\widehat{C} = \max\bigl\{(p-1)\widetilde{C}, \|X_p\|\bigr\}. \tag{1.5}$$

Here $\widetilde{C}$ is the constant from (1.1). Note that $t^0 \le 1$.

In [V, Lemma 3.9], it was shown that Condition 1.2 implies the inequality

$$\|X_0 f\|_{\mathfrak{H}_*} \le 2\bigl(\|X(t) f\|_{\mathfrak{H}_*} + \sqrt{\delta}\|f\|_{\mathfrak{H}}\bigr), \quad f \in \operatorname{Dom} X_0, \quad |t| \le t^0. \tag{1.6}$$

It turns out (see [V, Proposition 3.10]) that for $|t| \le t^0$ we have

$$F(t,\delta) = F(t, 3\delta), \quad \operatorname{rank} F(t,\delta) = n. \tag{1.7}$$

This means that for $|t| \le t^0$ the operator $A(t)$ has exactly n eigenvalues (counted with multiplicities) on the interval $[0,\delta]$, while the interval $(\delta, 3\delta)$ is free of the spectrum. For convenience, we denote

$$F(t) := F(t,\delta), \quad \mathfrak{F}(t) := \mathfrak{F}(t,\delta).$$

1.2. The operators Z, R, and the spectral germ S. We put $\mathcal{D} = \operatorname{Dom} X_0 \cap \mathfrak{N}^\perp$. Since the point $\lambda_0 = 0$ is isolated in the spectrum of A_0, we can interpret $\mathcal{D}$ as a Hilbert space with the inner product $(X_0\varphi, X_0\eta)_{\mathfrak{H}_*}$, $\varphi, \eta \in \mathcal{D}$. Let $u \in \mathfrak{H}_*$. Consider the equation $X_0^*(X_0\psi - u) = 0$ for $\psi \in \mathcal{D}$, understood in the weak sense:

$$(X_0\psi, X_0\zeta)_{\mathfrak{H}_*} = (u, X_0\zeta)_{\mathfrak{H}_*}, \quad \zeta \in \mathcal{D}. \tag{1.8}$$

The right-hand side of (1.8) is an antilinear continuous functional of $\zeta \in \mathcal{D}$. Hence, there exists a unique solution ψ such that $\|X_0\psi\|_{\mathfrak{H}_*} \le \|u\|_{\mathfrak{H}_*}$. Now, let

$$\omega \in \mathfrak{N}, \quad u = -X_p\omega; \tag{1.9}$$

in this case we denote the solution of equation (1.8) by $\psi(\omega)$. Let $Z\colon \mathfrak{H} \to \mathfrak{H}$ be the bounded operator defined by

$$Z\omega = \psi(\omega),\ \omega \in \mathfrak{N}; \quad Z\varphi = 0,\ \varphi \in \mathfrak{N}^\perp. \tag{1.10}$$

In order to estimate the norm of Z, we write (1.8) with $u = -X_p\omega$ and $\zeta = \psi(\omega)$:

$$\|X_0\psi(\omega)\|^2_{\mathfrak{H}_*} = -\big(X_p\omega, X_0\psi(\omega)\big)_{\mathfrak{H}_*} \le \|X_0\psi(\omega)\|_{\mathfrak{H}_*}\|X_p\omega\|_{\mathfrak{H}_*},$$

whence

$$\big(A_0\psi(\omega), \psi(\omega)\big)_{\mathfrak{H}} \le \|X_p\|^2\,\|\omega\|^2_{\mathfrak{H}}.$$

Recalling that $d^0 \ge 36\delta$ and $\psi(\omega) \in \mathfrak{N}^\perp$, we obtain

$$36\delta\|\psi(\omega)\|^2_{\mathfrak{H}} \le \big(A_0\psi(\omega), \psi(\omega)\big)_{\mathfrak{H}} \le \|X_p\|^2\|\omega\|^2_{\mathfrak{H}}.$$

Hence,

$$\|Z\| \le (1/6)\delta^{-1/2}\|X_p\|. \tag{1.11}$$

Now, we put

$$\omega_* := X_0\psi(\omega) + X_p\omega \in \mathfrak{N}_* \tag{1.12}$$

and define an operator R by the relations

$$R\colon \mathfrak{N} \to \mathfrak{N}_*, \quad R\omega = \omega_*. \tag{1.13}$$

The operator R can be represented as

$$R = P_*X_p|_{\mathfrak{N}}. \tag{1.14}$$

By definition, the *spectral germ* of the operator family $A(t)$ at the point $t = 0$ is the selfadjoint operator

$$S = R^*R\colon \mathfrak{N} \to \mathfrak{N}. \tag{1.15}$$

From (1.14) and (1.15) it follows that

$$S = PX_p^*P_*X_p\big|_{\mathfrak{N}}.$$

The germ S is said to be *nondegenerate* if $\operatorname{Ker} S = \{0\}$, or, equivalently, $\operatorname{rank} R = n$.

1.3. Analytic branches of eigenvalues and eigenvectors of $A(t)$. With the help of analytic perturbation theory, important properties of the first n eigenvalues and the corresponding eigenvectors of $A(t)$ with sufficiently small t were found in [V, Subsection 3.3]. Namely, for $|t| \le t^0$ there exist real-analytic functions $\lambda_j(t)$ (branches of eigenvalues) and real-analytic $\mathfrak{H}$-valued functions $\varphi_j(t)$ (branches of eigenvectors) such that

$$A(t)\varphi_j(t) = \lambda_j(t)\varphi_j(t), \quad j = 1, \dots, n, \quad |t| \le t^0, \tag{1.16}$$

and the set $\{\varphi_j(t)\}_{j=1}^n$ forms an orthonormal basis in $\mathfrak{F}(t)$. If $t_* \le t^0$ is sufficiently small, then the following power series expansions are convergent for $|t| \le t_*$ (see [V, Theorem 3.15]):

$$\lambda_j(t) = \gamma_j t^{2p} + \dots, \quad \gamma_j \ge 0, \quad j = 1, \dots, n, \quad |t| \le t_*, \tag{1.17}$$

$$\varphi_j(t) = \omega_j + t\varphi_j^{(1)} + t^2\varphi_j^{(2)} + \dots, \quad j = 1, \dots, n, \quad |t| \le t_*. \tag{1.18}$$

Here, the set $\{\omega_j\}_{j=1}^n$ is an orthonormal basis in $\mathfrak{N}$. The numbers γ_j and the vectors ω_j are eigenvalues and eigenvectors of the spectral germ S, i.e.,

$$S\omega_j = \gamma_j \omega_j, \quad j = 1, \dots, n.$$

We have

$$P = \sum_{j=1}^{n} (\,\cdot\,, \omega_j)_{\mathfrak{H}} \omega_j, \tag{1.19}$$

$$SP = \sum_{j=1}^{n} \gamma_j (\,\cdot\,, \omega_j)_{\mathfrak{H}} \omega_j. \tag{1.20}$$

As was shown in [V, Subsection 3.3], the elements $\varphi_j^{(i)}$ occurring in (1.18) satisfy

$$\varphi_j^{(i)} \in \mathfrak{N}, \quad j = 1, \dots, n, \quad i = 1, \dots, p-1; \tag{1.21}$$

$$\varphi_j^{(p)} - \psi(\omega_j) \in \mathfrak{N}, \quad j = 1, \dots, n. \tag{1.22}$$

From (1.7) and (1.16) it follows that

$$F(t) = \sum_{j=1}^{n} \left(\,\cdot\,, \varphi_j(t)\right)_{\mathfrak{H}} \varphi_j(t), \quad |t| \le t^0, \tag{1.23}$$

$$A(t)F(t) = \sum_{j=1}^{n} \lambda_j(t) \left(\,\cdot\,, \varphi_j(t)\right)_{\mathfrak{H}} \varphi_j(t), \quad |t| \le t^0. \tag{1.24}$$

Substituting expansions (1.17), (1.18) in (1.23), (1.24) and taking (1.19) and (1.20) into account, we obtain the power series expansions

$$F(t) = P + tF_1 + \dots, \quad |t| \le t_*,$$
$$A(t)F(t) = t^{2p} SP + \dots, \quad |t| \le t_*,$$

convergent for $|t| \le t_*$ (where $t_* \le t^0$ is sufficiently small). However, what we need is not these expansions, but approximations for $F(t)$ and $A(t)F(t)$ with one or several first terms (*threshold approximations*) and with error estimates on the entire interval $|t| \le t^0$.

§2. Abstract approach: Threshold approximations

This section contains the main results of the abstract method, as obtained in [V].

2.1. Auxiliary material. We need a version of the resolvent identity in the case where the domains of two operators may be different, but the domains of the corresponding quadratic forms do coincide. The relevant version of the resolvent identity was found in [BSu1, Chapter 1, §2].

Let a and b be two closed nonnegative quadratic forms in $\mathfrak{H}$ defined on a common domain

$$\mathfrak{d} := \operatorname{Dom} a = \operatorname{Dom} b, \tag{2.1}$$

which is dense in $\mathfrak{H}$. The operators corresponding to the forms a and b are denoted by A and B, respectively. Consider the sesquilinear form

$$a_\gamma[u, v] = a[u, v] + \gamma(u, v)_{\mathfrak{H}}, \quad \gamma > 0. \tag{2.2}$$

The corresponding quadratic form is positive definite. The form b_γ is introduced in a similar way. The linear space $\mathfrak{d}$ is a Hilbert space $\mathfrak{d}(a_\gamma)$ with respect to the inner product (2.2). The norm in $\mathfrak{d}(a_\gamma)$ is denoted by $\| \cdot \|_{\mathfrak{d}}$:

$$\|u\|_{\mathfrak{d}} = a_\gamma[u, u]^{1/2}, \quad u \in \mathfrak{d}. \tag{2.3}$$

By (2.1), the form b_γ is continuous on $\mathfrak{d}(a_\gamma)$ and determines an equivalent norm on this space. Let $\alpha > 0$ be a constant defined by

$$\alpha^2 = \sup_{0 \neq u \in \mathfrak{d}} \frac{a_\gamma[u, u]}{b_\gamma[u, u]}.$$

Consider the form $\mathfrak{t} = b - a$. Obviously, it is a_γ-continuous and generates a selfadjoint operator T_γ in $\mathfrak{d}(a_\gamma)$:

$$\mathfrak{t}[u, v] = a_\gamma[T_\gamma u, v], \quad u, \, v \in \mathfrak{d}.$$

We denote

$$\Omega_z(A) := I + (z + \gamma) R_z(A) = (A + \gamma I) R_z(A),$$

where $R_z(A) = (A - zI)^{-1}$ is the resolvent of A at the point $z \in \rho(A)$. (Here $\rho(A)$ is the resolvent set of A.) Similar notation is introduced for the operator B. Then we have

$$R_z(B) - R_z(A) = -\Omega_z(A) T_\gamma R_z(B), \quad z \in \rho(A) \cap \rho(B), \tag{2.4}$$

see [BSu1, Chapter 1, §2]. From the definition of the norm (2.3) in $\mathfrak{d}$, it follows directly that

$$\|u\|_{\mathfrak{H}} \leq \gamma^{-1/2} \|u\|_{\mathfrak{d}}, \tag{2.5}$$

$$\|R_z(A)\|_{\mathfrak{H} \to \mathfrak{d}} \leq \gamma^{-1/2} \, \|\Omega_z(A)\|_{\mathfrak{H} \to \mathfrak{H}} , \tag{2.6}$$

$$\|R_z(A)\|_{\mathfrak{d} \to \mathfrak{d}} \leq \gamma^{-1} \, \|\Omega_z(A)\|_{\mathfrak{H} \to \mathfrak{H}} ,$$

$$\|\Omega_z(A)\|_{\mathfrak{d} \to \mathfrak{d}} \leq 1 + |z + \gamma| \gamma^{-1} \|\Omega_z(A)\|_{\mathfrak{H} \to \mathfrak{H}}. \tag{2.7}$$

In their turn, in [V] some other estimates were deduced from here.

2.2. Estimates for the difference of the resolvents. Threshold approximations. In this subsection, we formulate some estimates obtained in [V, Subsection 4.2].

Let $\Gamma_\delta \subset \mathbb{C}$ be a contour that envelopes the real interval $[0, \delta]$ equidistantly at the distance δ. Recall that δ was chosen in Subsection 1.1. Let $z \in \Gamma_\delta$, and let $|t| \leq t^0$. The roles of the operators A and B are played by $A_0 = A(0)$ and $A(t)$. For brevity, we write $R_z(t)$ instead of $R_z(A(t))$ and $\Omega_z(t)$ instead of $\Omega_z(A(t))$.

We apply the approach of Subsection 2.1, putting

$$\gamma = \delta, \quad \mathfrak{d} = \operatorname{Dom} X_0, \quad a[u, u] = \|X_0 u\|^2_{\mathfrak{H}_*}, \quad b[u, u] = \|X(t) u\|^2_{\mathfrak{H}_*}.$$

It is easily seen that $\alpha \le 3$. Obviously, $|z| \le 2\delta$ for $z \in \Gamma_\delta$. By (1.7), $\|R_z(t)\|_{\mathfrak{H}\to\mathfrak{H}} \le \delta^{-1}$ for $z \in \Gamma_\delta$ and $|t| \le t^0$. Hence,

$$\|\Omega_z(t)\|_{\mathfrak{H}\to\mathfrak{H}} \le 4, \quad z \in \Gamma_\delta, \quad |t| \le t^0. \tag{2.8}$$

Together with (2.6) this yields

$$\|R_z(0)\|_{\mathfrak{H}\to\mathfrak{d}} \le 4\delta^{-1/2}, \quad z \in \Gamma_\delta. \tag{2.9}$$

Similarly, (2.7) and (2.8) imply that

$$\|\Omega_z(0)\|_{\mathfrak{d}\to\mathfrak{d}} \le 13. \tag{2.10}$$

Now, consider the form

$$\begin{aligned}\mathfrak{t}[u,u] &= \|X(t)u\|^2_{\mathfrak{H}_*} - \|X_0 u\|^2_{\mathfrak{H}_*}\\ &= 2\,\mathrm{Re}\left((tX_1 + \dots + t^p X_p)u, X_0 u\right)_{\mathfrak{H}_*} + \|(tX_1 + \dots + t^p X_p)u\|^2_{\mathfrak{H}_*}.\end{aligned}$$

In the space $\mathfrak{d}$ with the metric determined by a_δ, this quadratic form generates an operator $T_\delta = T_\delta(t)$, which can be represented as

$$T_\delta(t) = \sum_{j=1}^{2p} t^j T_\delta^{(j)}, \tag{2.11}$$

where the operators $T_\delta^{(j)}$ do not depend on t. The norms of $T_\delta(t)$ and $T_\delta^{(j)}$ were estimated in [V, Propositions 4.3, 4.4].

Proposition 2.1. *Let t^0 be given by* (1.4). *Then*

$$\|T_\delta(t)\|_{\mathfrak{d}\to\mathfrak{d}} \le C_\circ |t|, \quad |t| \le t^0, \tag{2.12}$$

$$\|T_\delta^{(j)}\|_{\mathfrak{d}\to\mathfrak{d}} \le \widetilde{B}, \quad j = 1, \dots, 2p, \tag{2.13}$$

where the constants $C_\circ$ and $\widetilde{B}$ are given by

$$C_\circ = 5\widehat{C}\delta^{-1/2} = 5(t^0)^{-1}, \tag{2.14}$$

$$\widetilde{B} = p\widetilde{C}^2 + \|X_p\|^2\delta^{-1}, \tag{2.15}$$

and $\widetilde{C}$ is the constant in (1.1) (*obviously, $\widetilde{B} \ge 1$*).

In what follows, we shall use an inequality equivalent to (2.12):

$$\left| \|X(t)u\|^2_{\mathfrak{H}_*} - \|X_0 u\|^2_{\mathfrak{H}_*} \right| \le \left(\|X_0 u\|^2_{\mathfrak{H}_*} + \delta \|u\|^2_{\mathfrak{H}} \right) C_\circ |t|, \quad u \in \mathfrak{d}, \quad |t| \le t^0. \tag{2.16}$$

The following inequality was deduced from (2.4) in [V, (4.16)]:

$$\|R_z(t) - R_z(0)\|_{\mathfrak{H}\to\mathfrak{H}} \le 48 C_\circ \delta^{-1} |t|, \quad |t| \le t^0, \quad z \in \Gamma_\delta. \tag{2.17}$$

Estimates for the difference of resolvents allow us to obtain the *threshold approximations* for the operators $F(t)$ and $A(t)F(t)$. The spectral projection $F(t)$ admits the following representation (see [K]):

$$F(t) = -\frac{1}{2\pi i} \oint_{\Gamma_\delta} R_z(t)\, dz,$$

where the integration over Γ_δ is in the positive direction. Then

$$F(t) - P = -\frac{1}{2\pi i} \oint_{\Gamma_\delta} \left(R_z(t) - R_z(0) \right) dz. \tag{2.18}$$

Since the length of the contour Γ_δ is equal to $2\delta + 2\pi\delta$, from (2.17) and (2.18) we obtain

$$\|F(t) - P\|_{\mathfrak{H}\to\mathfrak{H}} \le C_1 |t|, \quad |t| \le t^0, \quad C_1 = 48(1 + \pi^{-1}) C_\circ. \tag{2.19}$$

Also, in [V, (4.25), (4.27)] the following approximation for the operator $A(t)F(t)$ was found:

$$\|A(t)F(t) - t^{2p}SP\|_{\mathfrak{H}\to\mathfrak{H}} \le C_2|t|^{2p+1}, \quad |t| \le t^0, \tag{2.20}$$

with

$$C_2 = c(p)(\widetilde{B}^{2p} + C_\circ^{2p+1}), \tag{2.21}$$

where $C_\circ$, $\widetilde{B}$ are defined by (2.14), (2.15), and $c(p)$ depends only on p.

2.3. Approximation of the resolvent $(A(t)+\varepsilon^{2p}I)^{-1}$. In this subsection, we prove a theorem about approximation of the resolvent $(A(t)+\varepsilon^{2p}I)^{-1}$. For this, yet another condition is needed.

Condition 2.2. *The eigenvalues $\lambda_j(t)$ of $A(t)$ satisfy*

$$\lambda_j(t) \ge c_* t^{2p}, \quad j = 1, \dots, n, \quad c_* > 0, \quad |t| \le t^0.$$

From Condition 2.2 and relations (1.17), (1.20) it follows that the spectral germ satisfies

$$S \ge c_* I_{\mathfrak{N}}, \tag{2.22}$$

so that S is nondegenerate.

The following statement was obtained in [V, Proposition 4.9]; for completeness, we provide the proof.

Proposition 2.3. *For $\varepsilon > 0$ and $|t| \le t^0$ we have*

$$\varepsilon^{2p-1}\left\|(A(t)+\varepsilon^{2p}I)^{-1}F(t) - (t^{2p}SP+\varepsilon^{2p}I)^{-1}P\right\|_{\mathfrak{H}\to\mathfrak{H}} \le C_3. \tag{2.23}$$

The constant $C_3 = c_*^{-\frac{1}{2p}}(2C_1 + c_*^{-1}C_2)$ depends only on p, δ, the constant $\widetilde{C}$ in (1.1), *the norm $\|X_p\|$, and c_*.*

Proof. We use the identity

$$\begin{aligned} G(t,\varepsilon) &:= (A(t)+\varepsilon^{2p}I)^{-1}F(t) - (t^{2p}SP+\varepsilon^{2p}I)^{-1}P \\ &= (A(t)+\varepsilon^{2p}I)^{-1}F(t)(F(t)-P) + (F(t)-P)(t^{2p}SP+\varepsilon^{2p}I)^{-1}P \\ &\quad - F(t)(A(t)+\varepsilon^{2p}I)^{-1}(A(t)F(t)-t^{2p}SP)(t^{2p}SP+\varepsilon^{2p}I)^{-1}P. \end{aligned} \tag{2.24}$$

By Condition 2.2,

$$\left\|(A(t)+\varepsilon^{2p}I)^{-1}\right\|_{\mathfrak{H}\to\mathfrak{H}} \le (c_* t^{2p}+\varepsilon^{2p})^{-1}, \quad |t| \le t^0, \tag{2.25}$$

$$\left\|(t^{2p}SP+\varepsilon^{2p}I)^{-1}P\right\|_{\mathfrak{H}\to\mathfrak{H}} \le (c_* t^{2p}+\varepsilon^{2p})^{-1}. \tag{2.26}$$

From (2.19), (2.20), and (2.24)–(2.26) it follows that

$$\|G(t,\varepsilon)\|_{\mathfrak{H}\to\mathfrak{H}} \le 2C_1|t|\big(c_* t^{2p}+\varepsilon^{2p}\big)^{-1} + C_2|t|^{2p+1}\big(c_* t^{2p}+\varepsilon^{2p}\big)^{-2}, \quad |t| \le t^0,$$

which implies (2.23). □

By (1.7), for $\varepsilon > 0$ and $|t| \le t^0$ we have

$$\begin{aligned} &\varepsilon^{2p-1}\big\|\big(A(t)+\varepsilon^{2p}I\big)^{-1}F(t)^\perp\big\|_{\mathfrak{H}\to\mathfrak{H}} \\ &\qquad \le \varepsilon^{2p-1}\big\|\big(A(t)+\varepsilon^{2p}I\big)^{-1+1/2p}\big\|\,\big\|\big(A(t)+\varepsilon^{2p}I\big)^{-1/2p}F(t)^\perp\big\| \le (3\delta)^{-1/2p}. \end{aligned}$$

Combining this with Proposition 2.3, we arrive at the following result.

Theorem 2.4. *Let $A(t)$ be the operator family* (1.3) *satisfying the assumptions of Subsection* 1.1 *and Condition* 2.2. *Let P be the orthogonal projection of $\mathfrak{H}$ onto the subspace $\mathfrak{N}$, and let S be the spectral germ of the family $A(t)$ at $t=0$. Then*

$$\varepsilon^{2p-1}\left\|(A(t)+\varepsilon^{2p}I)^{-1}-(t^{2p}SP+\varepsilon^{2p}I)^{-1}P\right\|_{\mathfrak{H}\to\mathfrak{H}}\le C_A,\quad \varepsilon>0,\quad |t|\le t^0. \tag{2.27}$$

Here δ is the quantity chosen in Subsection 1.1, *and t^0 is defined by* (1.4). *The constant C_A is given by*

$$C_A=C_3+(3\delta)^{-1/2p}=c_*^{-\frac{1}{2p}}\left(2C_1+c_*^{-1}C_2\right)+(3\delta)^{-1/2p} \tag{2.28}$$

and depends only on p, δ, the constant $\widetilde{C}$ in (1.1), *the norm $\|X_p\|$, and c_*.*

Remark 2.5. It is possible to write down a cumbersome explicit expression for the constant C_A by using relations (1.5), (2.14), (2.15), (2.19), (2.21), and (2.28). For further application to differential operators, it is important to know how this constant depends on the problem data. After possible overstating, C_A can be viewed as a polynomial in the variables $\widetilde{C}$, $\|X_p\|$, $\delta^{-1/2p}$, and $c_*^{-1/2p}$ with positive coefficients depending only on p.

§3. The abstract approach: Approximation of the resolvent $(A(t)+\varepsilon^{2p}I)^{-1}$ with the corrector taken into account

In the present section, we obtain approximation of the resolvent $(A(t)+\varepsilon^{2p}I)^{-1}$ with the corrector taken into account. Our goal is to prove the following theorem.

Theorem 3.1. *Let $A(t)$ be the operator family* (1.3) *satisfying the assumptions of Subsection* 1.1 *and Condition* 2.2. *Let P be the orthogonal projection of $\mathfrak{H}$ onto the subspace $\mathfrak{N}$, let Z be the operator* (1.10), *and let S be the spectral germ of $A(t)$ at the point $t=0$. Then for $\varepsilon>0$ and $|t|\le t^0$ we have*

$$\varepsilon^{p-1}\left\|A(t)^{1/2}\left((A(t)+\varepsilon^{2p}I)^{-1}-(I+t^pZ)\,(t^{2p}SP+\varepsilon^{2p}I)^{-1}P\right)\right\|_{\mathfrak{H}\to\mathfrak{H}}\le\check{C}_A. \tag{3.1}$$

Here t^0 is defined by (1.4). *The constant $\check{C}_A$ depends only on p, δ, the constant $\widetilde{C}$ in* (1.1), *the norm $\|X_p\|$, and c_*.*

3.1. Proof of Theorem 3.1: Step 1.
Denote

$$\mathfrak{A}_\varepsilon(t)=A(t)^{1/2}(A(t)+\varepsilon^{2p}I)^{-1}, \tag{3.2}$$

$$\Xi(t,\varepsilon)=(t^{2p}SP+\varepsilon^{2p}I)^{-1}P. \tag{3.3}$$

We need to estimate the operator

$$\Upsilon(t,\varepsilon):=\mathfrak{A}_\varepsilon(t)-A(t)^{1/2}(I+t^pZ)\Xi(t,\varepsilon). \tag{3.4}$$

Note that the operator (3.3) satisfies (2.26).

The operator (3.4) can be represented as the sum of four terms

$$\Upsilon(t,\varepsilon)=J_1(t,\varepsilon)+J_2(t,\varepsilon)+J_3(t,\varepsilon)+J_4(t,\varepsilon), \tag{3.5}$$

where

$$J_1(t,\varepsilon):=\mathfrak{A}_\varepsilon(t)F(t)^\perp, \tag{3.6}$$

$$J_2(t,\varepsilon):=\mathfrak{A}_\varepsilon(t)F(t)(F(t)-P), \tag{3.7}$$

$$J_3(t,\varepsilon):=F(t)\mathfrak{A}_\varepsilon(t)P-F(t)A(t)^{1/2}\Xi(t,\varepsilon), \tag{3.8}$$

$$J_4(t,\varepsilon):=A(t)^{1/2}(F(t)-P)\Xi(t,\varepsilon)-t^pA(t)^{1/2}Z\Xi(t,\varepsilon). \tag{3.9}$$

To estimate the operator (3.6), we apply the Young inequality in the form

$$\left(\lambda+\varepsilon^{2p}\right)^{-1}\le\lambda^{-1/2-1/2p}\varepsilon^{1-p},\quad \lambda>0,\quad \varepsilon>0. \tag{3.10}$$

By (1.7), (3.2), (3.6), and (3.10),

$$\|J_1(t,\varepsilon)\|_{\mathfrak{H}\to\mathfrak{H}} \le \sup_{\lambda\ge 3\delta} \lambda^{1/2}(\lambda+\varepsilon^{2p})^{-1} \le (3\delta)^{-1/2p}\varepsilon^{1-p}, \quad |t|\le t^0. \tag{3.11}$$

Condition 2.2 and (3.10) imply that

$$\begin{aligned} \|\mathfrak{A}_\varepsilon(t)F(t)\|_{\mathfrak{H}\to\mathfrak{H}} &= \sup_{1\le l\le n} \sqrt{\lambda_l(t)}(\lambda_l(t)+\varepsilon^{2p})^{-1} \\ &\le \varepsilon^{1-p} \sup_{1\le l\le n} \lambda_l(t)^{-1/2p} \le c_*^{-1/2p}|t|^{-1}\varepsilon^{1-p}, \quad 0<|t|\le t^0. \end{aligned} \tag{3.12}$$

Together with (2.19) this yields the following estimate for the operator (3.7):

$$\|J_2(t,\varepsilon)\|_{\mathfrak{H}\to\mathfrak{H}} \le C_1 c_*^{-1/2p}\varepsilon^{1-p}, \quad |t|\le t^0. \tag{3.13}$$

Next, to estimate the operator (3.8), we apply the following analog of the resolvent identity:

$$F(t)(A(t)+\varepsilon^{2p}I)^{-1}P - F(t)\Xi(t,\varepsilon) = -F(t)(A(t)+\varepsilon^{2p}I)^{-1}(A(t)F(t)-t^{2p}SP)\Xi(t,\varepsilon)$$

and multiply the two sides by $A(t)^{1/2}$ from the left. Then the operator (3.8) takes the form

$$J_3(t,\varepsilon) = -F(t)\mathfrak{A}_\varepsilon(t)\left(A(t)F(t)-t^{2p}SP\right)\Xi(t,\varepsilon).$$

Combining this with (2.20), (2.26), and (3.12), we obtain

$$\begin{aligned} \|J_3(t,\varepsilon)\|_{\mathfrak{H}\to\mathfrak{H}} &\le C_2 c_*^{-1/2p}(c_* t^{2p}+\varepsilon^{2p})^{-1}|t|^{2p+1}|t|^{-1}\varepsilon^{1-p} \\ &\le C_2 c_*^{-1/2p-1}\varepsilon^{1-p}, \quad |t|\le t^0. \end{aligned} \tag{3.14}$$

As a result, relations (3.5), (3.11), (3.13), and (3.14) imply that

$$\|\Upsilon(t,\varepsilon)\|_{\mathfrak{H}\to\mathfrak{H}} \le C_4\varepsilon^{1-p} + \|J_4(t,\varepsilon)\|_{\mathfrak{H}\to\mathfrak{H}}, \quad |t|\le t^0, \tag{3.15}$$

where

$$C_4 = (3\delta)^{-1/2p} + C_1 c_*^{-1/2p} + C_2 c_*^{-1/2p-1}. \tag{3.16}$$

Thus, the proof of estimate (3.1) reduces to estimation of the operator (3.9).

3.2. The iteration procedure. Now, we rewrite the resolvent identity (2.4) as

$$R_z(t) = R_z(0) - \Omega_z(0)T_\delta(t)R_z(t). \tag{3.17}$$

We shall iterate this using representation (2.11) for $T_\delta(t)$. After p iterations we get

$$R_z(t) - R_z(0) = t\Psi_1(z) + \dots + t^p\Psi_p(z) + \Psi_*(t,z). \tag{3.18}$$

Together with (2.18), this implies

$$F(t) - P = tF_1 + \dots + t^pF_p + F_*(t), \tag{3.19}$$

where

$$F_i = -\frac{1}{2\pi i}\oint_{\Gamma_\delta} \Psi_i(z)\,dz, \quad i=1,\dots,p, \tag{3.20}$$

$$F_*(t) = -\frac{1}{2\pi i}\oint_{\Gamma_\delta} \Psi_*(t,z)\,dz. \tag{3.21}$$

By (3.19), the operator (3.9) can be represented in the form

$$J_4(t,\varepsilon) = J_4^{(1)}(t,\varepsilon) + J_4^{(2)}(t,\varepsilon) + J_4^{(3)}(t,\varepsilon), \tag{3.22}$$

where

$$J_4^{(1)}(t,\varepsilon) := \sum_{i=1}^{p-1} t^i A(t)^{1/2} F_i \Xi(t,\varepsilon), \tag{3.23}$$

$$J_4^{(2)}(t,\varepsilon) := t^p A(t)^{1/2} (F_p - Z)\Xi(t,\varepsilon), \tag{3.24}$$

$$J_4^{(3)}(t,\varepsilon) := A(t)^{1/2} F_*(t)\Xi(t,\varepsilon). \tag{3.25}$$

3.3. Estimates for the operators F_i. Now we find expressions for the operators F_i in terms of the coefficients of the expansions (1.18) for the eigenvectors $\varphi_j(t)$. By (1.23),

$$\begin{aligned}
F(t) &= \sum_{j=1}^n \big(\,\cdot\,,\varphi_j(t)\big)_{\mathfrak{H}}\varphi_j(t)\\
&= \sum_{j=1}^n \big(\,\cdot\,,\omega_j + t\varphi_j^{(1)} + \dots + t^p\varphi_j^{(p)}\big)_{\mathfrak{H}}\big(\omega_j + t\varphi_j^{(1)} + \dots + t^p\varphi_j^{(p)}\big) + O(t^{p+1})\\
&= \sum_{j=1}^n \big(\,\cdot\,,\omega_j\big)_{\mathfrak{H}}\omega_j + t\sum_{j=1}^n \big\{\big(\,\cdot\,,\omega_j\big)_{\mathfrak{H}}\varphi_j^{(1)} + \big(\,\cdot\,,\varphi_j^{(1)}\big)_{\mathfrak{H}}\omega_j\big\} + \dots\\
&\quad + t^p\sum_{j=1}^n \big\{\big(\,\cdot\,,\omega_j\big)_{\mathfrak{H}}\varphi_j^{(p)} + \big(\,\cdot\,,\varphi_j^{(1)}\big)_{\mathfrak{H}}\varphi_j^{(p-1)} + \dots + \big(\,\cdot\,,\varphi_j^{(p)}\big)_{\mathfrak{H}}\omega_j\big\} + F_*(t).
\end{aligned}$$

In accordance with (3.19),

$$F_i = \sum_{j=1}^n\sum_{k=0}^i \big(\,\cdot\,,\varphi_j^{(k)}\big)\varphi_j^{(i-k)}, \quad i = 0,\dots,p, \tag{3.26}$$

where, for convenience, it is assumed that $\varphi_j^{(0)} = \omega_j$, $j = 1,\dots,n$, and $F_0 = P$.

By (1.21), we have $\varphi_j^{(l)} \in \mathfrak{N}$, $l = 0,\dots,p-1$, $j = 1,\dots,n$. Hence, the operators F_i with $i = 1,\dots,p-1$ take $\mathfrak{N}$ to $\mathfrak{N}$ and take $\mathfrak{N}^\perp$ to $\{0\}$. This fact will help us to estimate the norm of the operator (3.23) by $O(\varepsilon^{-p+1})$.

To estimate the operators F_i, we use the invariant representations (3.20) and estimate the integrands uniformly in z. For this, first we find invariant representations for the operators $\Psi_i(z)$. We iterate identity (3.17) by using (2.11). The symbol "$\sim$" will be used instead of "$=$" if terms of order t^k with $k > p$ are dropped. We have:

$$\begin{aligned}
R_z(t) &= R_z(0) - \Omega_z(0)T_\delta(t)\big(R_z(0) - \Omega_z(0)T_\delta(t)R_z(t)\big)\\
&\sim R_z(0) - \Omega_z(0)\sum_{i_1=1}^p t^{i_1}T_\delta^{(i_1)}R_z(0) + (\Omega_z(0)T_\delta(t))^2R_z(t)\\
&\sim R_z(0) - \Omega_z(0)\sum_{i_1=1}^p t^{i_1}T_\delta^{(i_1)}R_z(0)\\
&\quad + \Omega_z(0)\sum_{i_1=1}^{p-1} t^{i_1}T_\delta^{(i_1)}\Omega_z(0)\sum_{i_2=1}^{p-1} t^{i_2}T_\delta^{(i_2)}R_z(0) - (\Omega_z(0)T_\delta(t))^3R_z(t)\\
&\sim R_z(0) - \Omega_z(0)\sum_{i_1=1}^p t^{i_1}T_\delta^{(i_1)}R_z(0) + \Omega_z(0)\sum_{i_1=1}^{p-1} t^{i_1}T_\delta^{(i_1)}\Omega_z(0)\sum_{i_2=1}^{p-1} t^{i_2}T_\delta^{(i_2)}R_z(0)\\
&\quad - \Omega_z(0)\sum_{i_1=1}^{p-2} t^{i_1}T_\delta^{(i_1)}\Omega_z(0)\sum_{i_2=1}^{p-2} t^{i_2}T_\delta^{(i_2)}\Omega_z(0)\sum_{i_3=1}^{p-2} t^{i_3}T_\delta^{(i_3)}R_z(0)\\
&\quad + (\Omega_z(0)T_\delta(t))^4R_z(t).
\end{aligned}$$

We continue this iteration procedure till the last term is $(\Omega_z(0)T_\delta(t))^{p+1}R_z(t)$. The final expression takes the form

$$\begin{aligned} R_z(t) \sim R_z(0) &- \Omega_z(0)\sum_{i_1=1}^{p} t^{i_1}T_\delta^{(i_1)}R_z(0) \\ &+ \Omega_z(0)\sum_{i_1=1}^{p-1} t^{i_1}T_\delta^{(i_1)}\Omega_z(0)\sum_{i_2=1}^{p-1} t^{i_2}T_\delta^{(i_2)}R_z(0) + \dots \\ &+ (-1)^k\Omega_z(0)\sum_{i_1=1}^{p+1-k} t^{i_1}T_\delta^{(i_1)}\cdot\ \dots\ \cdot\Omega_z(0)\sum_{i_k=1}^{p+1-k} t^{i_k}T_\delta^{(i_k)}R_z(0) + \dots \\ &+ (-1)^p t^p(\Omega_z(0)T_\delta^{(1)})^p R_z(0). \end{aligned}$$

Let us extract $\Psi_i(z)$. For simplicity, we use the following notation. Let $\gamma^k = \left(\gamma_1^k, \dots, \gamma_k^k\right)$ be a multiindex of length k such that $\gamma_i^k \geq 1$, $i = 1, \dots, k$. Denote

$$\left(\Omega_z(0)T_\delta^{(\,\cdot\,)}\right)^{\gamma^k} = \Omega_z(0)T_\delta^{(\gamma_1^k)}\cdot\ \dots\ \cdot\Omega_z(0)T_\delta^{(\gamma_k^k)}.$$

Then $\Psi_i(z)$ is given by

$$\Psi_i(z) = \sum_{k=1}^{i}(-1)^k\sum_{|\gamma^k|=i}\left(\Omega_z(0)T_\delta^{(\,\cdot\,)}\right)^{\gamma^k}R_z(0), \quad i = 1, \dots, p. \tag{3.27}$$

Now, relations (2.5), (2.9), (2.10), and (2.13) imply the following estimates for the operators (3.27):

$$\begin{aligned} \|\Psi_i(z)\|_{\mathfrak{H}\to\mathfrak{H}} &\leq \delta^{-1/2}\|\Psi_i(z)\|_{\mathfrak{H}\to\mathfrak{d}} \\ &\leq \delta^{-1/2}\sum_{k=1}^{i}\sum_{|\gamma^k|=i}\left\|\left(\Omega_z(0)T_\delta^{(\,\cdot\,)}\right)^{\gamma^k}R_z(0)\right\|_{\mathfrak{H}\to\mathfrak{d}} \\ &\leq 4\delta^{-1}\sum_{k=1}^{i}\sum_{|\gamma^k|=i}\left(13\widetilde{B}\right)^k, \quad i = 1, \dots, p. \end{aligned}$$

Since $\widetilde{B} \geq 1$, the quantity $(13\widetilde{B})^k$ does not exceed $(13\widetilde{B})^i$ for $k = 1, \dots, i$, whence

$$\|\Psi_i(z)\|_{\mathfrak{H}\to\mathfrak{H}} \leq 4\delta^{-1}\left(13\widetilde{B}\right)^i\left(\sum_{k=1}^{i}\sum_{|\gamma^k|=i}1\right), \quad i = 1, \dots, p.$$

Using (3.20) and observing that the length of the contour Γ_δ is equal to $2\pi\delta + 2\delta$, we obtain

$$\|F_i\|_{\mathfrak{H}\to\mathfrak{H}} \leq 4\left(13\widetilde{B}\right)^i\left(\sum_{k=1}^{i}\sum_{|\gamma^k|=i}1\right)\left(\pi^{-1}+1\right) := C^{(i)}, \quad i = 1, \dots, p. \tag{3.28}$$

3.4. Estimation of the operator $J_4^{(1)}(t,\varepsilon)$. Let $u \in \mathfrak{H}$, and let $v = \Xi(t,\varepsilon)u \in \mathfrak{N}$. We estimate the norms

$$\left\|A(t)^{1/2}F_i\Xi(t,\varepsilon)u\right\|_{\mathfrak{H}} = \|X(t)F_iv\|_{\mathfrak{H}_*}, \quad i = 1, \dots, p-1. \tag{3.29}$$

As has already been mentioned, the operators F_i, $i = 1, \dots, p-1$, take $\mathfrak{N}$ to $\mathfrak{N}$ and $\mathfrak{N}^\perp$ to $\{0\}$. Combining this with (1.2), we simplify the expression under the norm sign on the right-hand side of (3.29):

$$X(t)F_i v = \big(X_0 + tX_1 + \dots + t^{p-1}X_{p-1} + t^p X_p\big)\, F_i v = t^p X_p F_i v, \quad v \in \mathfrak{N}, \quad i = 1, \dots, p-1.$$

Consequently, by (2.26) and (3.28),

$$\|X(t)F_i v\|_{\mathfrak{H}_*} \le C^{(i)} |t|^p \|X_p\| \|v\|_{\mathfrak{H}} \le C^{(i)} |t|^p \|X_p\| \big(c_* t^{2p} + \varepsilon^{2p}\big)^{-1} \|u\|_{\mathfrak{H}}. \tag{3.30}$$

By the Young inequality (3.10), we have

$$\big(c_* t^{2p} + \varepsilon^{2p}\big)^{-1} \le c_*^{-1/2-1/2p} |t|^{-1-p} \varepsilon^{1-p}. \tag{3.31}$$

Combining this with (3.29), (3.30) and observing that $t^0 \le 1$, we obtain the following estimate for the operator (3.23):

$$\|J_4^{(1)}(t,\varepsilon)\|_{\mathfrak{H} \to \mathfrak{H}} \le C_5 \varepsilon^{1-p}, \quad |t| \le t^0, \tag{3.32}$$

where

$$C_5 = c_*^{-1/2-1/2p} \|X_p\| \Big(\sum_{i=1}^{p-1} C^{(i)} \Big). \tag{3.33}$$

3.5. Estimation of the operator $J_4^{(2)}(t,\varepsilon)$. Now, we consider the operator F_p. By (3.26),

$$F_p = \sum_{j=1}^{n} \sum_{i=0}^{p} (\,\cdot\,, \varphi_j^{(i)}) \varphi_j^{(p-i)}.$$

Denote $\widetilde{\omega}_j := \varphi_j^{(p)} - Z\omega_j$. Relations (1.10) and (1.22) show that $\widetilde{\omega}_j \in \mathfrak{N}$. The operator F_p can be represented as

$$F_p = \check{F}_p + \widetilde{F}_p, \tag{3.34}$$

where

$$\check{F}_p = \sum_{j=1}^{n} \big((\,\cdot\,, \omega_j) Z\omega_j + (\,\cdot\,, Z\omega_j)\omega_j \big), \tag{3.35}$$

$$\widetilde{F}_p = \sum_{j=1}^{n} \sum_{i=1}^{p-1} (\,\cdot\,, \varphi_j^{(i)}) \varphi_j^{(p-i)} + \sum_{j=1}^{n} \big((\,\cdot\,, \omega_j) \widetilde{\omega}_j + (\,\cdot\,, \widetilde{\omega}_j)\omega_j \big). \tag{3.36}$$

Since $\widetilde{\omega}_j \in \mathfrak{N}$, we can use (1.21) to check that the operator (3.36) takes $\mathfrak{N}$ to $\mathfrak{N}$ and takes $\mathfrak{N}^\perp$ to $\{0\}$.

Using (3.35) and (1.19), we see that $\check{F}_p = ZP + PZ^*$. From the definition (1.10) of Z it follows that $PZ = 0$, whence $Z^*P = 0$. Consequently, $(\check{F}_p - Z)P = 0$. Together with (3.34), this implies

$$(F_p - Z)P = \widetilde{F}_p P. \tag{3.37}$$

Thus, the operator (3.24) can be written as

$$J_4^{(2)}(t,\varepsilon) = t^p A(t)^{1/2} \widetilde{F}_p \Xi(t,\varepsilon). \tag{3.38}$$

By (1.11), (3.28), and (3.37),

$$\|\widetilde{F}_p P\|_{\mathfrak{H} \to \mathfrak{H}} \le (1/6)\delta^{-1/2} \|X_p\| + C^{(p)}. \tag{3.39}$$

Again, let $u \in \mathfrak{H}$, and let $v = \Xi(t,\varepsilon)u \in \mathfrak{N}$. The fact that $\widetilde{F}_p$ takes $\mathfrak{N}$ to $\mathfrak{N}$ together with (1.2) allows us to simplify the expression for $X(t)\widetilde{F}_p v$:

$$X(t)\widetilde{F}_p v = \big(X_0 + tX_1 + \dots + t^{p-1}X_{p-1} + t^p X_p\big)\, \widetilde{F}_p v = t^p X_p \widetilde{F}_p v.$$

Hence, using (2.26), (3.38), and (3.39), we obtain

$$\begin{aligned}\big\|J_4^{(2)}(t,\varepsilon)u\big\|_{\mathfrak{H}} &= |t|^p\big\|X(t)\widetilde{F}_p v\big\|_{\mathfrak{H}_*} = t^{2p}\big\|X_p\widetilde{F}_p v\big\|_{\mathfrak{H}_*}\\ &\le t^{2p}\|X_p\|\big(C^{(p)}+(1/6)\delta^{-1/2}\|X_p\|\big)\big(c_* t^{2p}+\varepsilon^{2p}\big)^{-1}\|u\|_{\mathfrak{H}}.\end{aligned}$$

Combining this with (3.31) and recalling that $t^0 \le 1$, we arrive at

$$\big\|J_4^{(2)}(t,\varepsilon)u\big\|_{\mathfrak{H}} \le C_6\varepsilon^{1-p}, \quad |t|\le t^0, \tag{3.40}$$

where

$$C_6 = \|X_p\|\big(C^{(p)}+(1/6)\delta^{-1/2}\|X_p\|\big)c_*^{-1/2-1/2p}. \tag{3.41}$$

3.6. Estimation of the operator $J_4^{(3)}(t,\varepsilon)$. To estimate the operator (3.25), we use the representation (3.21). First, we estimate the operator $A(t)^{1/2}\Psi_*(t,z)$ uniformly with respect to $z\in\Gamma_\delta$. For this, we need some auxiliary statements.

Lemma 3.2. *We have*

$$\|A(t)^{1/2}\|_{\mathfrak{d}\to\mathfrak{H}} \le \sqrt{6}, \quad |t|\le t^0.$$

Proof. Let $u\in\mathfrak{d}$. The definition of the operator $A(t)^{1/2}$ implies $\|A(t)^{1/2}u\|_{\mathfrak{H}} = \|X(t)u\|_{\mathfrak{H}_*}$. By (2.16),

$$\|X(t)u\|^2_{\mathfrak{H}_*} \le \|X_0u\|^2_{\mathfrak{H}_*} + \big(\|X_0u\|^2_{\mathfrak{H}_*}+\delta\|u\|^2_{\mathfrak{H}}\big)C_\circ|t|, \quad |t|\le t^0.$$

Recalling that $\|u\|^2_{\mathfrak{d}} = \|X_0u\|^2_{\mathfrak{H}_*}+\delta\|u\|^2_{\mathfrak{H}}$ and taking (2.14) into account, we arrive at

$$\|A(t)^{1/2}u\|^2_{\mathfrak{H}} \le (1+C_\circ t^0)\|u\|^2_{\mathfrak{d}} = 6\|u\|^2_{\mathfrak{d}}, \quad u\in\mathfrak{d}, \quad |t|\le t^0. \qquad \square$$

Lemma 3.3. *We have*

$$\|R_z(t)\|_{\mathfrak{H}\to\mathfrak{d}} \le (2\sqrt{3}+3)\delta^{-1/2}, \quad z\in\Gamma_\delta, \quad |t|\le t^0. \tag{3.42}$$

Proof. Let $u\in\mathfrak{H}$. By the definition of the norm in $\mathfrak{d}$, we have

$$\|R_z(t)u\|_{\mathfrak{d}} \le \delta^{1/2}\,\|R_z(t)u\|_{\mathfrak{H}} + \|X_0R_z(t)u\|_{\mathfrak{H}_*}.$$

Together with (1.6) this implies

$$\begin{aligned}\|R_z(t)u\|_{\mathfrak{d}} &\le 2\,\|X(t)R_z(t)u\|_{\mathfrak{H}_*} + 3\delta^{1/2}\,\|R_z(t)u\|_{\mathfrak{H}}\\ &\le 2\,\|X(t)R_z(t)u\|_{\mathfrak{H}_*} + 3\delta^{-1/2}\,\|u\|_{\mathfrak{H}}, \quad |t|\le t^0, \quad z\in\Gamma_\delta.\end{aligned} \tag{3.43}$$

Since $|z|\le 2\delta$ for $z\in\Gamma_\delta$, we get

$$\begin{aligned}\|X(t)R_z(t)u\|^2_{\mathfrak{H}_*} &= (A(t)R_z(t)u, R_z(t)u)_{\mathfrak{H}}\\ &= (u, R_z(t)u)_{\mathfrak{H}} + z\|R_z(t)u\|^2_{\mathfrak{H}}\\ &\le \|R_z(t)\|_{\mathfrak{H}\to\mathfrak{H}}\|u\|^2_{\mathfrak{H}} + 2\delta\|R_z(t)\|^2_{\mathfrak{H}\to\mathfrak{H}}\|u\|^2_{\mathfrak{H}} \le 3\delta^{-1}\|u\|^2_{\mathfrak{H}}.\end{aligned} \tag{3.44}$$

Combining (3.43) and (3.44), we arrive at the required inequality (3.42). $\square$

Now, we estimate the operator $\Psi_*(t,z)$ in the $(\mathfrak{H}\to\mathfrak{d})$-norm. For this, we apply the iteration procedure once again in order to extract the term $\Psi_*(t,z)$ from the expansion (3.18). As before, we iterate (3.17) using (2.11). Our goal is to estimate the norms of all operators multiplied by t^k with $k>p$. Therefore, we shall use the sign $\sim$ if some terms of order t^k with $k\le p$ are dropped. The first iteration yields

$$\begin{aligned}R_z(t) &= R_z(0) - \Omega_z(0)T_\delta(t)\big(R_z(0) - \Omega_z(0)T_\delta(t)R_z(t)\big)\\ &= R_z(0) - \Omega_z(0)\big(tT_\delta^{(1)}+\cdots+t^pT_\delta^{(p)}\big)R_z(0)\\ &\quad - \Omega_z(0)\big(t^{p+1}T_\delta^{(p+1)}+\cdots+t^{2p}T_\delta^{(2p)}\big)R_z(0) + (\Omega_z(0)T_\delta(t))^2R_z(t).\end{aligned}$$

Hence,

$$R_z(t) \sim -\Omega_z(0)\big(t^{p+1}T_\delta^{(p+1)} + \dots + t^{2p}T_\delta^{(2p)}\big)R_z(0) + (\Omega_z(0)T_\delta(t))^2 R_z(t). \tag{3.45}$$

We denote the first term on the right-hand side of (3.45) by $\mathcal{I}_1(t,z)$ and estimate it with the help of (2.9), (2.10), (2.13), and the inequality $t^0 \le 1$:

$$\|\mathcal{I}_1(t,z)\|_{\mathfrak{H}\to\mathfrak{d}} \le C_{(1)}|t|^{p+1}, \quad |t| \le t^0, \quad C_{(1)} = 52\delta^{-1/2}p\widetilde{B}.$$

Now we write out the unrecorded terms for the second iteration retaining only the terms with t^k, $k > p$:

$$\begin{aligned}
(\Omega_z(0)T_\delta(t))^2 R_z(t) &= (\Omega_z(0)T_\delta(t))^2 (R_z(0) - \Omega_z(0)T_\delta(t)R_z(t)) \\
&= t^2\Omega_z(0)\big(T_\delta^{(1)}\Omega_z(0)T_\delta^{(1)}\big)R_z(0) \\
&\quad + t^3\Omega_z(0)\big(T_\delta^{(1)}\Omega_z(0)T_\delta^{(2)} + T_\delta^{(2)}\Omega_z(0)T_\delta^{(1)}\big)R_z(0) + \dots \\
&\quad + t^p\Omega_z(0)\big(T_\delta^{(1)}\Omega_z(0)T_\delta^{(p-1)} + \dots + T_\delta^{(p-1)}\Omega_z(0)T_\delta^{(1)}\big)R_z(0) \\
&\quad + t^{p+1}\Omega_z(0)\big(T_\delta^{(1)}\Omega_z(0)T_\delta^{(p)} + \dots + T_\delta^{(p)}\Omega_z(0)T_\delta^{(1)}\big)R_z(0) + \dots \\
&\quad + t^{4p}\Omega_z(0)\big(T_\delta^{(2p)}\Omega_z(0)T_\delta^{(2p)}\big)R_z(0) - (\Omega_z(0)T_\delta(t))^3 R_z(t) \\
&\sim t^{p+1}\Omega_z(0)\big(T_\delta^{(1)}\Omega_z(0)T_\delta^{(p)} + \dots + T_\delta^{(p)}\Omega_z(0)T_\delta^{(1)}\big)R_z(0) + \dots \\
&\quad + t^{4p}\Omega_z(0)\big(T_\delta^{(2p)}\Omega_z(0)T_\delta^{(2p)}\big)R_z(0) - (\Omega_z(0)T_\delta(t))^3 R_z(t) \\
&=: \mathcal{I}_2(t,z) - (\Omega_z(0)T_\delta(t))^3 R_z(t).
\end{aligned}$$

We estimate the extracted term $\mathcal{I}_2(t,z)$ with the help of (2.9), (2.10), (2.13), and the inequality $t^0 \le 1$:

$$\|\mathcal{I}_2(t,z)\|_{\mathfrak{H}\to\mathfrak{d}} \le C_{(2)}|t|^{p+1}, \quad |t| \le t^0,$$

where

$$C_{(2)} = 4 \cdot 13^2\delta^{-1/2}c_p^{(2)}\widetilde{B}^2, \quad c_p^{(2)} = p + (p+1) + \dots + 2p + (2p-1) + \dots + 1.$$

Now the unrecorded term is $-\big(\Omega_z(0)T_\delta(t)\big)^3 R_z(t)$.

We continue this iteration procedure till the unrecorded term is

$$(-1)^{p+1}\big(\Omega_z(0)T_\delta(t)\big)^{p+1} R_z(t) =: \mathcal{I}^0(t,z).$$

All the terms $\mathcal{I}_j(t,z)$, $j = 1,\dots,p$, extracted in the consequent iterations are estimated as follows:

$$\|\mathcal{I}_j(t,z)\|_{\mathfrak{H}\to\mathfrak{d}} \le C_{(j)}|t|^{p+1}, \quad |t| \le t^0, \quad j = 1,\dots,p, \tag{3.46}$$

where

$$C_{(j)} = 4 \cdot 13^j\delta^{-1/2}c_p^{(j)}\widetilde{B}^j, \tag{3.47}$$

and $c_p^{(j)}$ depends only on p and j. Finally, the term $\mathcal{I}^0(t,z)$ is estimated by using (2.10), (2.12), and Lemma 3.3:

$$\|\mathcal{I}^0(t,z)\|_{\mathfrak{H}\to\mathfrak{d}} \le C_{(p+1)}|t|^{p+1}, \quad |t| \le t^0, \tag{3.48}$$

where

$$C_{(p+1)} = (2\sqrt{3}+3) \cdot 13^{p+1}\delta^{-1/2}C_\circ^{p+1}. \tag{3.49}$$

Clearly, $\Psi_*(t,z) = \mathcal{I}_1(t,z) + \dots + \mathcal{I}_p(t,z) + \mathcal{I}^0(t,z)$. As a result, relations (3.46) and (3.48) imply that

$$\|\Psi_*(t,z)\|_{\mathfrak{H}\to\mathfrak{d}} \le C_7|t|^{p+1}, \quad z \in \Gamma_\delta, \quad |t| \le t^0, \tag{3.50}$$

where

$$C_7 = \sum_{j=1}^{p+1} C_{(j)}. \tag{3.51}$$

Now, Lemma 3.2 and estimate (3.50) yield

$$\|A(t)^{1/2}\Psi_*(t,z)\|_{\mathfrak{H}\to\mathfrak{H}} \le \sqrt{6}C_7|t|^{p+1}, \quad z\in\Gamma_\delta, \quad |t|\le t^0.$$

Combining this with (3.21) and estimating the length of the contour Γ_δ by $2\pi\delta + 2\delta \le 2\pi + 2$, we obtain

$$\|A(t)^{1/2}F_*(t)\|_{\mathfrak{H}\to\mathfrak{H}} \le (1+\pi^{-1})\sqrt{6}C_7|t|^{p+1}, \quad |t|\le t^0. \tag{3.52}$$

As a result, relations (2.26) and (3.52) imply the following estimate for the operator (3.25):

$$\|J_4^{(3)}(t,\varepsilon)\|_{\mathfrak{H}\to\mathfrak{H}} \le (1+\pi^{-1})\sqrt{6}C_7|t|^{p+1}(c_* t^{2p}+\varepsilon^{2p})^{-1}, \quad |t|\le t^0.$$

Using (3.31), we deduce the inequality

$$\|J_4^{(3)}(t,\varepsilon)\|_{\mathfrak{H}\to\mathfrak{H}} \le C_8\varepsilon^{1-p}, \quad |t|\le t^0, \tag{3.53}$$

where

$$C_8 = (1+\pi^{-1})\sqrt{6}C_7 c_*^{-1/2-1/2p}. \tag{3.54}$$

3.7. Completion of the proof of Theorem 3.1. Relations (3.4), (3.15), (3.22), (3.32), (3.40), and (3.53) imply the required estimate (3.1) with the constant

$$\check{C}_A = C_4 + C_5 + C_6 + C_8. \tag{3.55}$$

Remark 3.4. It is possible to write out a cumbersome explicit expression for the constant $\check{C}_A$, using relations (1.5), (2.14), (2.15), (2.19), (2.21), (3.16), (3.28), (3.33), (3.41), (3.47), (3.49), (3.51), (3.54), and (3.55). For further application to differential operators, it is important to know how this constant depends on the problem data. After possible overstating, $\check{C}_A$ can be regarded as a polynomial in the variables $\widetilde{C}$, $\|X_p\|$, $\delta^{-1/2p}$, and $c_*^{-1/2p}$ with positive coefficients depending only on p.

§4. Periodic differential operators in $\mathbb{R}^d$. Direct integral expansion

4.1. Factorized operators of order $2p$ in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, we consider differential operators given formally by the expression

$$A = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}). \tag{4.1}$$

Here $g(\mathbf{x})$ is a uniformly positive definite and bounded $(m\times m)$-matrix-valued function (in general, $g(\mathbf{x})$ is a Hermitian matrix with complex entries):

$$\begin{gathered} g,\ g^{-1}\in L_\infty(\mathbb{R}^d), \\ g(\mathbf{x}) \ge c\mathbf{1}_m, \quad c>0, \quad \mathbf{x}\in\mathbb{R}^d. \end{gathered} \tag{4.2}$$

The operator $b(\mathbf{D})$ is given by

$$b(\mathbf{D}) = \sum_{|\alpha|=p} b_\alpha \mathbf{D}^\alpha, \tag{4.3}$$

where the b_α are constant $(m\times n)$-matrices, in general, with complex entries. It is assumed that $m\ge n$ and that the symbol $b(\boldsymbol{\xi}) = \sum_{|\alpha|=p} b_\alpha \boldsymbol{\xi}^\alpha$ is such that

$$\operatorname{rank} b(\boldsymbol{\xi}) = n, \quad 0\ne\boldsymbol{\xi}\in\mathbb{R}^d.$$

This condition is equivalent to the following estimates:

$$\alpha_0 \mathbf{1}_n \le b(\boldsymbol{\theta})^* b(\boldsymbol{\theta}) \le \alpha_1 \mathbf{1}_n, \quad \boldsymbol{\theta} \in \mathbb{S}^{d-1}, \quad 0 < \alpha_0 \le \alpha_1 < \infty. \tag{4.4}$$

Without loss of generality, we assume that

$$|b_\alpha| \le \alpha_1^{1/2}, \quad |\alpha| = p. \tag{4.5}$$

The precise definition of the operator A is given in terms of quadratic forms. By (4.2), the matrix g can be written as

$$g(\mathbf{x}) = h(\mathbf{x})^* h(\mathbf{x}),$$

where h, $h^{-1} \in L_\infty$. For instance, one can put $h = g^{1/2}$.

Consider the operator X acting from $L_2(\mathbb{R}^d; \mathbb{C}^n)$ to $L_2(\mathbb{R}^d; \mathbb{C}^m)$ and defined by

$$(X\mathbf{u})(\mathbf{x}) = h(\mathbf{x}) b(\mathbf{D}) \mathbf{u}(\mathbf{x}), \quad \mathbf{u} \in \operatorname{Dom} X = H^p(\mathbb{R}^d; \mathbb{C}^n),$$

and the quadratic form

$$a[\mathbf{u}, \mathbf{u}] = \|X\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \langle g(\mathbf{x}) b(\mathbf{D}) \mathbf{u}(\mathbf{x}), b(\mathbf{D})\mathbf{u}(\mathbf{x}) \rangle \, d\mathbf{x}, \quad \mathbf{u} \in H^p(\mathbb{R}^d; \mathbb{C}^n). \tag{4.6}$$

Using the Fourier transformation and relations (4.2), (4.4), it is easy to check that

$$c_0 \int_{\mathbb{R}^d} |\mathbf{D}^p \mathbf{u}|^2 \, d\mathbf{x} \le a[\mathbf{u}, \mathbf{u}] \le c_1 \int_{\mathbb{R}^d} |\mathbf{D}^p \mathbf{u}|^2 \, d\mathbf{x}, \quad \mathbf{u} \in H^p(\mathbb{R}^d; \mathbb{C}^n), \tag{4.7}$$

where the notation $|\mathbf{D}^p \mathbf{u}|^2 := \sum_{|\alpha|=p} |\mathbf{D}^\alpha \mathbf{u}|^2$ is used. Indeed, by the Parseval identity, we have

$$\|g^{-1}\|^{-1}_{L_\infty} \int_{\mathbb{R}^d} |b(\boldsymbol{\xi}) \widehat{\mathbf{u}}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \le a[\mathbf{u}, \mathbf{u}] \le \|g\|_{L_\infty} \int_{\mathbb{R}^d} |b(\boldsymbol{\xi}) \widehat{\mathbf{u}}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi},$$

where $\widehat{\mathbf{u}}(\boldsymbol{\xi})$ is the Fourier image of the function $\mathbf{u}(\mathbf{x})$. Together with (4.4) this yields

$$\alpha_0 \|g^{-1}\|^{-1}_{L_\infty} \int_{\mathbb{R}^d} |\boldsymbol{\xi}|^{2p} |\widehat{\mathbf{u}}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \le a[\mathbf{u}, \mathbf{u}] \le \alpha_1 \|g\|_{L_\infty} \int_{\mathbb{R}^d} |\boldsymbol{\xi}|^{2p} |\widehat{\mathbf{u}}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi}. \tag{4.8}$$

Using the elementary inequalities

$$c'_p \sum_{|\alpha|=p} |\boldsymbol{\xi}^\alpha|^2 \le |\boldsymbol{\xi}|^{2p} \le c''_p \sum_{|\alpha|=p} |\boldsymbol{\xi}^\alpha|^2, \tag{4.9}$$

with constants c'_p, c''_p depending only on d and p, we arrive at the required inequalities (4.7) with

$$c_0 = c'_p \alpha_0 \|g^{-1}\|^{-1}_{L_\infty}, \quad c_1 = c''_p \alpha_1 \|g\|_{L_\infty}. \tag{4.10}$$

Hence, the form (4.6) is closed and nonnegative. The selfadjoint operator in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ corresponding to this form is denoted by A.

4.2. Lattices in $\mathbb{R}^d$. In what follows, it is assumed that the matrix-valued functions g and h are *periodic* with respect to some lattice $\Gamma \subset \mathbb{R}^d$:

$$g(\mathbf{x} + \mathbf{n}) = g(\mathbf{x}), \quad h(\mathbf{x} + \mathbf{n}) = h(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d, \quad \mathbf{n} \in \Gamma.$$

Let $\mathbf{n}_1, \dots, \mathbf{n}_d$ be a basis in $\mathbb{R}^d$ generating the lattice Γ:

$$\Gamma = \Big\{ \mathbf{n} \in \mathbb{R}^d \, : \, \mathbf{n} = \sum_{i=1}^d l_i \mathbf{n}_i, \, l_i \in \mathbb{Z} \Big\},$$

and let Ω be the elementary cell of the lattice Γ:

$$\Omega = \left\{ \mathbf{x} \in \mathbb{R}^d \,:\, \mathbf{x} = \sum_{i=1}^{d} t_i \mathbf{n}_i,\, 0 < t_i < 1 \right\}.$$

The basis $\mathbf{s}^1, \dots, \mathbf{s}^d$ in $\mathbb{R}^d$ dual to $\mathbf{n}_1, \dots, \mathbf{n}_d$ is defined by the relations $\langle \mathbf{s}^i, \mathbf{n}_j \rangle_{\mathbb{R}^d} = 2\pi\delta^i_j$. This basis generates the *lattice* $\widetilde{\Gamma}$ *dual to* Γ:

$$\widetilde{\Gamma} = \left\{ \mathbf{s} \in \mathbb{R}^d \,:\, \mathbf{s} = \sum_{i=1}^{d} q_i \mathbf{s}^i,\, q_i \in \mathbb{Z} \right\}.$$

Instead of the cell of the dual lattice, it is convenient to consider the *central Brillouin zone*

$$\widetilde{\Omega} = \left\{ \mathbf{k} \in \mathbb{R}^d \,:\, |\mathbf{k}| < |\mathbf{k} - \mathbf{s}|,\, 0 \neq \mathbf{s} \in \widetilde{\Gamma} \right\},$$

which is a fundamental domain of the lattice $\widetilde{\Gamma}$. We denote $|\Omega| = \operatorname{meas}\Omega$, $|\widetilde{\Omega}| = \operatorname{meas}\widetilde{\Omega}$ and recall that $|\Omega|\,|\widetilde{\Omega}| = (2\pi)^d$. Let r_0 be the maximal radius of the ball contained in $\operatorname{clos}\widetilde{\Omega}$. Then

$$2r_0 = \min|\mathbf{s}|, \quad 0 \neq \mathbf{s} \in \widetilde{\Gamma}. \tag{4.11}$$

Denote

$$\mathcal{B}(r) = \left\{ \mathbf{k} \in \mathbb{R}^d \,:\, |\mathbf{k}| \leq r \right\}, \quad r > 0.$$

With the lattice Γ, a discrete Fourier transformation $\{\widehat{v}_{\mathbf{s}}\}_{\mathbf{s}\in\widetilde{\Gamma}} \mapsto v$ is associated:

$$v(\mathbf{x}) = |\Omega|^{-1/2} \sum_{\mathbf{s}\in\widetilde{\Gamma}} \widehat{v}_{\mathbf{s}} \exp\left(i\langle \mathbf{s}, \mathbf{x}\rangle\right), \quad \mathbf{x} \in \Omega,$$

which is a unitary mapping of $l_2(\widetilde{\Gamma})$ onto $L_2(\Omega)$:

$$\int_\Omega |v(\mathbf{x})|^2 d\mathbf{x} = \sum_{\mathbf{s}\in\widetilde{\Gamma}} |\widehat{v}_{\mathbf{s}}|^2.$$

By $\widetilde{W}^s_q(\Omega;\mathbb{C}^n)$ we denote the subspace of all functions in $W^s_q(\Omega;\mathbb{C}^n)$ whose Γ-periodic extension to $\mathbb{R}^d$ belongs to $W^s_{q,\mathrm{loc}}(\mathbb{R}^d;\mathbb{C}^n)$. If $q = 2$, we use the notation $\widetilde{H}^s(\Omega;\mathbb{C}^n)$.

4.3. The operators $X(\mathbf{k})$ and $A(\mathbf{k})$ in $L_2(\Omega;\mathbb{C}^n)$. Let $\mathbf{k} \in \mathbb{R}^d$. We consider the operator $X(\mathbf{k})\colon L_2(\Omega;\mathbb{C}^n) \to L_2(\Omega;\mathbb{C}^m)$ defined on the domain

$$\operatorname{Dom} X(\mathbf{k}) = \widetilde{H}^p(\Omega;\mathbb{C}^n)$$

by the relation

$$(X(\mathbf{k})\mathbf{u})(\mathbf{x}) = h(\mathbf{x})b(\mathbf{D}+\mathbf{k})\mathbf{u}(\mathbf{x}). \tag{4.12}$$

Consider the quadratic form $a(\mathbf{k})$ given by

$$a(\mathbf{k})[\mathbf{u},\mathbf{u}] = \|X(\mathbf{k})\mathbf{u}\|^2_{L_2(\Omega)} = \int_\Omega \langle g(\mathbf{x})b(\mathbf{D}+\mathbf{k})\mathbf{u}, b(\mathbf{D}+\mathbf{k})\mathbf{u}\rangle\, d\mathbf{x}, \quad \mathbf{u} \in \widetilde{H}^p(\Omega;\mathbb{C}^n). \tag{4.13}$$

Using the discrete Fourier transformation and relations (4.2), (4.4), it is easy to check that

$$\alpha_0\|g^{-1}\|^{-1}_{L_\infty} a_*(\mathbf{k})[\mathbf{u},\mathbf{u}] \leq a(\mathbf{k})[\mathbf{u},\mathbf{u}] \leq \alpha_1\|g\|_{L_\infty} a_*(\mathbf{k})[\mathbf{u},\mathbf{u}], \quad \mathbf{u} \in \widetilde{H}^p(\Omega;\mathbb{C}^n), \tag{4.14}$$

for any $\mathbf{k} \in \mathbb{R}^d$, where

$$a_*(\mathbf{k})\,[\mathbf{u},\mathbf{u}] := \sum_{\mathbf{s}\in\widetilde{\Gamma}} |\mathbf{s}+\mathbf{k}|^{2p}|\widehat{\mathbf{u}}_{\mathbf{s}}|^2, \quad \mathbf{u} \in \widetilde{H}^p(\Omega;\mathbb{C}^n). \tag{4.15}$$

Together with (4.9) this yields

$$c_0 \int_\Omega |(\mathbf{D}+\mathbf{k})^p \mathbf{u}|^2\, d\mathbf{x} \le a(\mathbf{k})\,[\mathbf{u},\mathbf{u}] \le c_1 \int_\Omega |(\mathbf{D}+\mathbf{k})^p \mathbf{u}|^2\, d\mathbf{x}, \quad \mathbf{u} \in \widetilde{H}^p(\Omega;\mathbb{C}^n),$$

where the constants c_0, c_1 are as in (4.10). Hence, the operator $X(\mathbf{k})$ is closed, and the form (4.13) is closed and nonnegative. The selfadjoint operator in $L_2(\Omega;\mathbb{C}^n)$ corresponding to the form $a(\mathbf{k})$ is denoted by $A(\mathbf{k})$. Formally, we have

$$A(\mathbf{k}) = b(\mathbf{D}+\mathbf{k})^* g(\mathbf{x}) b(\mathbf{D}+\mathbf{k}).$$

4.4. Direct integral expansion for the operator A. Initially, the Gelfand transformation $\mathcal{U}$ is defined on the Schwartz class $\mathcal{S}(\mathbb{R}^d;\mathbb{C}^n)$ by the relation

$$\begin{aligned}&\tilde{\mathbf{v}}(\mathbf{k},\mathbf{x}) = (\mathcal{U}\mathbf{v})\,(\mathbf{k},\mathbf{x}) = |\widetilde{\Omega}|^{-1/2} \sum_{\mathbf{n}\in\Gamma} \exp(-i\langle \mathbf{k}, \mathbf{x}+\mathbf{n}\rangle)\mathbf{v}(\mathbf{x}+\mathbf{n}),\\ &\mathbf{v} \in \mathcal{S}(\mathbb{R}^d;\mathbb{C}^n), \quad \mathbf{x}\in\mathbb{R}^d, \quad \mathbf{k}\in\mathbb{R}^d.\end{aligned}$$

Then

$$\int_{\widetilde{\Omega}} \int_\Omega |\tilde{\mathbf{v}}\,(\mathbf{k},\mathbf{x})|^2\, d\mathbf{x}\, d\mathbf{k} = \int_{\mathbb{R}^d} |\mathbf{v}(\mathbf{x})|^2\, d\mathbf{x}, \quad \tilde{\mathbf{v}} = \mathcal{U}\mathbf{v},$$

and $\mathcal{U}$ extends by continuity up to unitary mapping

$$\mathcal{U}\colon L_2(\mathbb{R}^d;\mathbb{C}^n) \to \int_{\widetilde{\Omega}} \oplus L_2(\Omega;\mathbb{C}^n) d\mathbf{k} =: \mathcal{K}. \tag{4.16}$$

The relation $\mathbf{v} \in H^p(\mathbb{R}^d;\mathbb{C}^n)$ is equivalent to the fact that $\tilde{\mathbf{v}}(\mathbf{k},\,\cdot\,) \in \widetilde{H}^p(\Omega;\mathbb{C}^n)$ for a. e. $\mathbf{k} \in \widetilde{\Omega}$ and

$$\int_{\widetilde{\Omega}} \int_\Omega \left(|(\mathbf{D}+\mathbf{k})^p \tilde{\mathbf{v}}(\mathbf{k},\mathbf{x})|^2 + |\tilde{\mathbf{v}}(\mathbf{k},\mathbf{x})|^2 \right) d\mathbf{x}\, d\mathbf{k} < \infty.$$

Under the Gelfand transformation $\mathcal{U}$, the operator of multiplication by a bounded Γ-periodic function in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ turns into the operator of multiplication by the same function on the fibers of the direct integral $\mathcal{K}$ in (4.16). On these fibers, the action of the operator $b(\mathbf{D})$ on $\mathbf{v} \in H^p(\mathbb{R}^d;\mathbb{C}^n)$ turns into the action of the operator $b(\mathbf{D}+\mathbf{k})$ on $\tilde{\mathbf{v}}(\mathbf{k},\,\cdot\,) \in \widetilde{H}^p(\Omega;\mathbb{C}^n)$.

It follows that the form (4.6) can be written as

$$a\,[\mathbf{v},\mathbf{v}] = \int_{\widetilde{\Omega}} a(\mathbf{k})[\tilde{\mathbf{v}}(\mathbf{k},\,\cdot\,), \tilde{\mathbf{v}}(\mathbf{k},\,\cdot\,)]\, d\mathbf{k}, \quad \mathbf{v} \in H^p(\mathbb{R}^d;\mathbb{C}^n).$$

Thus, the operator A is unitarily equivalent (with the affinity $\mathcal{U}$) to the direct integral of the operators $A(\mathbf{k})$:

$$\mathcal{U}A\mathcal{U}^{-1} = \int_{\widetilde{\Omega}} \oplus A(\mathbf{k}) d\mathbf{k}. \tag{4.17}$$

§5. Differential operators $A(\mathbf{k})$ on the cell Ω. Application of the abstract method

5.1. Investigation of the operators $X(\mathbf{k})$ and $A(\mathbf{k})$. Our goal is to check that the abstract approach can be applied to the operators $A(\mathbf{k})$. As in [BSu1], we put $t := |\mathbf{k}|$ and $\boldsymbol{\theta} := \mathbf{k}/t$. The operators $X(\mathbf{k}) =: X(t,\boldsymbol{\theta})$ and $A(\mathbf{k}) =: A(t,\boldsymbol{\theta})$ depend on the one-dimensional parameter t and the additional parameter $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$ (the latter was absent in the abstract outline). We shall try to make our constructions and estimates uniform with respect to $\boldsymbol{\theta}$.

By (4.3) and (4.12),

$$X(\mathbf{k}) = h \sum_{|\alpha|=p} b_\alpha (\mathbf{D}+\mathbf{k})^\alpha = h \sum_{|\alpha|=p} b_\alpha \sum_{\beta \le \alpha} C_\alpha^\beta \mathbf{k}^{\alpha-\beta} \mathbf{D}^\beta$$
$$= h \sum_{|\alpha|=p} b_\alpha \sum_{\beta \le \alpha} C_\alpha^\beta t^{|\alpha-\beta|} \boldsymbol{\theta}^{\alpha-\beta} \mathbf{D}^\beta .$$

Hence, the operator $X(\mathbf{k})$ can be written as

$$X(\mathbf{k}) = X(t,\boldsymbol{\theta}) = X_0 + \sum_{j=1}^{p} t^j X_j(\boldsymbol{\theta}). \tag{5.1}$$

Here, the operator

$$X_0 = h \sum_{|\alpha|=p} b_\alpha \mathbf{D}^\alpha = hb(\mathbf{D}) \tag{5.2}$$

is closed on the domain

$$\operatorname{Dom} X_0 = \widetilde{H}^p(\Omega;\mathbb{C}^n), \tag{5.3}$$

the "intermediate" operators $X_j(\boldsymbol{\theta})$, $j = 1,\dots,p-1$, are given by

$$X_j(\boldsymbol{\theta}) = h \sum_{|\alpha|=p} b_\alpha \sum_{\substack{\beta\le\alpha,\\ |\beta|=p-j}} C_\alpha^\beta \boldsymbol{\theta}^{\alpha-\beta} \mathbf{D}^\beta \tag{5.4}$$

on the domains

$$\operatorname{Dom} X_j(\boldsymbol{\theta}) = \widetilde{H}^{p-j}(\Omega;\mathbb{C}^n), \tag{5.5}$$

and the operator

$$X_p(\boldsymbol{\theta}) = h \sum_{|\alpha|=p} b_\alpha \boldsymbol{\theta}^\alpha = hb(\boldsymbol{\theta})$$

is bounded from $L_2(\Omega;\mathbb{C}^n)$ to $L_2(\Omega;\mathbb{C}^m)$.

From (5.3) and (5.5) it follows that Condition 1.1 is satisfied:

$$\operatorname{Dom} X_0 \subset \operatorname{Dom} X_j(\boldsymbol{\theta}) \subset \operatorname{Dom} X_p(\boldsymbol{\theta}) = L_2(\Omega;\mathbb{C}^n), \quad j=1,\dots,p-1.$$

By (4.4), we have

$$\|X_p(\boldsymbol{\theta})\| \le \alpha_1^{1/2} \|g\|_{L_\infty}^{1/2} . \tag{5.6}$$

Now we consider the kernels of the operators $X_j(\boldsymbol{\theta})$.

Proposition 5.1. *The kernel of X_0 consists of constant vector-valued functions*:

$$\mathfrak{N} := \operatorname{Ker} X_0 = \{\mathbf{u} \in L_2(\Omega;\mathbb{C}^n) : \mathbf{u}(\mathbf{x}) = \mathbf{c} \in \mathbb{C}^n\}. \tag{5.7}$$

For $j = 1,\dots,p-1$ we have

$$\mathfrak{N} \subset \operatorname{Ker} X_j(\boldsymbol{\theta}), \quad \boldsymbol{\theta} \in \mathbb{S}^{d-1}. \tag{5.8}$$

Proof. Let $\mathbf{u} \in \mathfrak{N}$. This means that $\mathbf{u} \in \widetilde{H}^p(\Omega;\mathbb{C}^n)$ and $b(\mathbf{D})\mathbf{u} = 0$. Using the Parseval identity for the discrete Fourier transformation, we write this condition as

$$0 = \int_\Omega |b(\mathbf{D})\mathbf{u}(\mathbf{x})|^2 \, d\mathbf{x} = \sum_{\mathbf{s}\in\widetilde{\Gamma}} |b(\mathbf{s})\hat{\mathbf{u}}_\mathbf{s}|^2 = \sum_{\mathbf{s}\in\widetilde{\Gamma}} \langle b(\mathbf{s})^* b(\mathbf{s})\hat{\mathbf{u}}_\mathbf{s}, \hat{\mathbf{u}}_\mathbf{s}\rangle_{\mathbb{C}^n} . \tag{5.9}$$

By (4.4), relation (5.9) is equivalent to

$$|\mathbf{s}|^{2p} |\hat{\mathbf{u}}_\mathbf{s}|^2 = 0, \quad \mathbf{s} \in \widetilde{\Gamma}.$$

In its turn, this is equivalent to the fact that all the Fourier coefficients $\hat{\mathbf{u}}_{\mathbf{s}}$ except for $\hat{\mathbf{u}}_{\mathbf{0}}$ are equal to zero, i.e., $\mathbf{u}(\mathbf{x}) = \mathbf{c} \in \mathbb{C}^n$.

By (5.4), relations (5.8) are obvious. □

The orthogonal projection of $L_2(\Omega; \mathbb{C}^n)$ onto $\mathfrak{N}$ acts as averaging over the cell:

$$P\mathbf{u} = |\Omega|^{-1} \int_\Omega \mathbf{u}(\mathbf{x})\, d\mathbf{x}, \quad \mathbf{u} \in L_2(\Omega; \mathbb{C}^n).$$

Let $n_* = \operatorname{Ker} X_0^*$. The relation $m \geq n$ ensures that $n_* \geq n$. Moreover, since

$$\mathfrak{N}_* = \operatorname{Ker} X_0^* = \{\mathbf{q} \in L_2(\Omega; \mathbb{C}^m) \,:\, h^*\mathbf{q} \in \widetilde{H}^p(\Omega; \mathbb{C}^m) \,:\, b(\mathbf{D})^* h^* \mathbf{q} = 0\},$$

the following alternative occurs: either $n_* = \infty$ (if $m > n$), or $n_* = n$ (if $m = n$).

Now we check that Condition 1.2 is satisfied.

Proposition 5.2. *For $j = 1, \dots, p-1$ we have*

$$\|X_j(\boldsymbol{\theta})\mathbf{u}\|_{L_2(\Omega)} \leq \widetilde{C}_j \|X_0 \mathbf{u}\|_{L_2(\Omega)}, \quad \mathbf{u} \in \widetilde{H}^p(\Omega; \mathbb{C}^n). \tag{5.10}$$

Here the constants $\widetilde{C}_j$ do not depend on the parameter $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$ and depend only on d, p, j, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, α_0, α_1, and r_0.

Proof. By (4.5) and (5.4),

$$\|X_j(\boldsymbol{\theta})\mathbf{u}\|_{L_2(\Omega)} \leq \alpha_1^{1/2} \|g\|_{L_\infty}^{1/2} \sum_{|\alpha|=p} \sum_{\substack{\beta \leq \alpha, \\ |\beta| = p-j}} C_\alpha^\beta \|\mathbf{D}^\beta \mathbf{u}\|_{L_2(\Omega)}. \tag{5.11}$$

We expand $\mathbf{u} \in \widetilde{H}^p(\Omega; \mathbb{C}^n)$ in the Fourier series

$$\mathbf{u}(\mathbf{x}) = |\Omega|^{-1/2} \sum_{\mathbf{s} \in \widetilde{\Gamma}} \hat{\mathbf{u}}_{\mathbf{s}} e^{i\langle \mathbf{x}, \mathbf{s}\rangle}. \tag{5.12}$$

By (4.11), for $j = 1, \dots, p-1$ we have

$$|\mathbf{s}^\beta|^2 \leq |\mathbf{s}|^{2|\beta|} \leq (2r_0)^{-2j} |\mathbf{s}|^{2p}, \quad \mathbf{0} \neq \mathbf{s} \in \widetilde{\Gamma}, \quad |\beta| = p-j. \tag{5.13}$$

By the Parseval identity for the Fourier series, from (5.13) we deduce that

$$\|\mathbf{D}^\beta \mathbf{u}\|_{L_2(\Omega)}^2 = \sum_{\mathbf{0} \neq \mathbf{s} \in \widetilde{\Gamma}} |\mathbf{s}^\beta \hat{\mathbf{u}}_{\mathbf{s}}|^2 \leq (2r_0)^{-2j} \sum_{\mathbf{0} \neq \mathbf{s} \in \widetilde{\Gamma}} |\mathbf{s}|^{2p} |\hat{\mathbf{u}}_{\mathbf{s}}|^2, \quad |\beta| = p-j. \tag{5.14}$$

Next, from the definition (5.2), (5.3) of X_0, the expansion (5.12), and the lower estimate (4.4) it follows that

$$\|X_0 \mathbf{u}\|_{L_2(\Omega)}^2 \geq \left\|g^{-1}\right\|_{L_\infty}^{-1} \alpha_0 \sum_{\mathbf{s} \in \widetilde{\Gamma}} |\mathbf{s}|^{2p} |\hat{\mathbf{u}}_{\mathbf{s}}|^2, \quad \mathbf{u} \in \widetilde{H}^p(\Omega; \mathbb{C}^n). \tag{5.15}$$

As a result, combining (5.11), (5.14), and (5.15), we arrive at the required inequality (5.10) with the constant

$$\widetilde{C}_j = \alpha_1^{1/2} \alpha_0^{-1/2} \|g\|_{L_\infty}^{1/2} \|g^{-1}\|_{L_\infty}^{1/2} (2r_0)^{-j} \left(\sum_{|\alpha|=p} \sum_{\substack{\beta \leq \alpha \\ |\beta| = p-j}} C_\alpha^\beta \right). \tag{5.16}$$

□

From the compactness of the embedding of $\operatorname{Dom} a(0) = \widetilde{H}^p(\Omega; \mathbb{C}^n)$ into $L_2(\Omega; \mathbb{C}^n)$, it follows that the spectrum of the operator $A(0)$ is discrete. The point $\lambda_0 = 0$ is an isolated eigenvalue of the operator $A(0) = X_0^* X_0$ of multiplicity n; the corresponding eigenspace $\mathfrak{N}$ consists of constant vector-valued functions (see (5.7)). We estimate the distance d^0 from the point $\lambda_0 = 0$ to the rest of the spectrum of $A(0)$. From (4.11) and (5.15) it follows that

$$a(0)[\mathbf{u},\mathbf{u}] \geq \alpha_0 \|g^{-1}\|^{-1}_{L_\infty} (2r_0)^{2p} \|\mathbf{u}\|^2_{L_2(\Omega)}, \quad \mathbf{u} \in \widetilde{H}^p(\Omega;\mathbb{C}^n), \quad \int_\Omega \mathbf{u}(\mathbf{x})\, d\mathbf{x} = 0.$$

Thus,

$$d^0 \geq \alpha_0 \|g^{-1}\|^{-1}_{L_\infty} (2r_0)^{2p}. \tag{5.17}$$

In accordance with the abstract outline, we should fix a positive number δ such that $\delta \leq \min\{d^0/36,\, 1/4\}$. Taking (5.17) into account, we put

$$\delta = \min\{\alpha_0 \|g^{-1}\|^{-1}_{L_\infty} (2r_0)^{2p}/36,\, 1/4\}. \tag{5.18}$$

Inequalities (5.10) allow us to choose $\widetilde{C}$ (see (1.1)) as follows:

$$\widetilde{C} = \max\{1,\, \widetilde{C}_1,\, \dots,\, \widetilde{C}_{p-1}\}, \tag{5.19}$$

where the constants $\widetilde{C}_j$ are defined by (5.16).

Now the constant $\widehat{C} = \max\{(p-1)\widetilde{C},\, \|X_p(\boldsymbol{\theta})\|\}$ (see (1.5)) depends on $\boldsymbol{\theta}$. Using (5.6), we take the following overstated value

$$\widehat{C} = \max\{(p-1)\widetilde{C},\, \alpha_1^{1/2}\|g\|^{1/2}_{L_\infty}\},$$

which does not depend on $\boldsymbol{\theta}$. As in (1.4), we put

$$t^0 = \frac{\delta^{1/2}}{\widehat{C}} = \frac{\delta^{1/2}}{\max\{(p-1)\widetilde{C},\, \alpha_1^{1/2}\|g\|^{1/2}_{L_\infty}\}}. \tag{5.20}$$

5.2. Incorporation of the operators $A(t,\boldsymbol{\theta})$ in the framework of the abstract method. We shall apply the abstract method, putting

$$\mathfrak{H} = L_2(\Omega;\mathbb{C}^n), \quad \mathfrak{H}_* = L_2(\Omega;\mathbb{C}^m).$$

The role of the polynomial pencil $X(t)$ is played by $X(t,\boldsymbol{\theta}) := X(\mathbf{k}) = X(t\boldsymbol{\theta})$ (see (5.1)); this pencil depends also on $\boldsymbol{\theta}$. In Subsection 5.1 it was checked that Conditions 1.1 and 1.2 are satisfied. The role of $A(t)$ is played by the operator $A(t,\boldsymbol{\theta}) := A(\mathbf{k}) = A(t\boldsymbol{\theta})$. By the definition of $A(\mathbf{k})$ (see Subsection 4.3), we have

$$A(t,\boldsymbol{\theta}) = X(t,\boldsymbol{\theta})^* X(t,\boldsymbol{\theta}).$$

In Subsection 5.1 it was checked that Condition 1.3 is also satisfied. The kernel $\mathfrak{N} = \operatorname{Ker} A(0) = \operatorname{Ker} X_0$ is described in (5.7).

It remains to check that Condition 2.2 is satisfied. Denote by $E_j(\mathbf{k})$, $j \in \mathbb{N}$, $\mathbf{k} \in \widetilde{\Omega}$, the consecutive eigenvalues of the operator $A(\mathbf{k})$ counted with multiplicities.

We use the two-sided estimates (4.14) for the form $a(\mathbf{k})$ in terms of the auxiliary form (4.15). The selfadjoint operator in $\mathfrak{H}$ corresponding to the form (4.15) is denoted by $A_*(\mathbf{k})$, and its consecutive eigenvalues are denoted by $E_j^0(\mathbf{k})$, $j \in \mathbb{N}$. For a different way of enumeration, the eigenvalues of $A_*(\mathbf{k})$ can be found from (4.15): they coincide with the numbers $|\mathbf{s}+\mathbf{k}|^{2p}$, $\mathbf{s} \in \widetilde{\Gamma}$, and every such eigenvalue is of multiplicity n (they are enumerated by the index $\mathbf{s} \in \widetilde{\Gamma}$). Then it easily follows that

$$E_l^0(\mathbf{k}) = |\mathbf{k}|^{2p}, \quad l = 1,\dots,n, \quad \mathbf{k} \in \operatorname{clos}\widetilde{\Omega}, \tag{5.21}$$

$$E_1^0(\mathbf{k}) \geq r^{2p}, \quad \mathbf{k} \in \operatorname{clos}\widetilde{\Omega} \setminus \mathcal{B}(r), \quad 0 < r \leq r_0. \tag{5.22}$$

Combining the lower estimate (4.14) with (5.21) and (5.22), we obtain

$$E_l(\mathbf{k}) \geq \alpha_0 \left\|g^{-1}\right\|_{L_\infty}^{-1} |\mathbf{k}|^{2p}, \quad l = 1, \dots, n, \quad \mathbf{k} \in \operatorname{clos} \widetilde{\Omega},$$

$$E_1(\mathbf{k}) \geq \alpha_0 \left\|g^{-1}\right\|_{L_\infty}^{-1} r^{2p}, \quad \mathbf{k} \in \operatorname{clos} \widetilde{\Omega} \setminus \mathcal{B}(r), \quad 0 < r \leq r_0, \tag{5.23}$$

which implies that Condition 2.2 is satisfied with the constant

$$c_* = \alpha_0 \left\|g^{-1}\right\|_{L_\infty}^{-1}. \tag{5.24}$$

5.3. Construction of the spectral germ. The operators Z, R, and S introduced in Subsection 1.2 now depend on $\boldsymbol{\theta}$. To construct them, we introduce an auxiliary operator Λ. Let

$$\mathfrak{M} = \{\mathbf{w} \in L_2(\Omega; \mathbb{C}^m) : \mathbf{w}(\mathbf{x}) = \mathbf{C} \in \mathbb{C}^m\}$$

be the subspace of constant vector-valued functions in $\mathfrak{H}_* = L_2(\Omega; \mathbb{C}^m)$.

The operator $\Lambda\colon \mathfrak{M} \to \mathfrak{H}$ takes a vector $\mathbf{C} \in \mathfrak{M}$ to the *weak* Γ*-periodic solution* $\mathbf{v}_{\mathbf{C}} \in \widetilde{H}^p(\Omega; \mathbb{C}^n)$ of the problem

$$b(\mathbf{D})^* g(\mathbf{x}) \left(b(\mathbf{D})\mathbf{v}_{\mathbf{C}}(\mathbf{x}) + \mathbf{C}\right) = 0, \quad \int_\Omega \mathbf{v}_{\mathbf{C}}(\mathbf{x})\, d\mathbf{x} = 0. \tag{5.25}$$

Problem (5.25) with $\mathbf{C} = b(\boldsymbol{\theta})\mathbf{c}$, $\mathbf{c} \in \mathbb{C}^n$, is a realization of problem (1.8), (1.9) (now $\omega = \mathbf{c} \in \mathfrak{N}$). Now we describe how the operator Λ acts. Let $\mathbf{e}_1, \dots, \mathbf{e}_m$ be the standard orthonormal basis in $\mathbb{C}^m$, and let $\mathbf{v}_j = \mathbf{v}_{\mathbf{e}_j}$. In the standard basis in $\mathbb{C}^n$, the vector-valued functions $\mathbf{v}_j(\mathbf{x})$ can be written as columns of length n. Let $\Lambda(\mathbf{x})$ be the $(n \times m)$-matrix with the columns $\mathbf{v}_1(\mathbf{x}), \dots, \mathbf{v}_m(\mathbf{x})$. Then Λ is the operator of multiplication by the matrix-valued function $\Lambda(\mathbf{x})$. Note that $\Lambda(\mathbf{x})$ is a Γ-periodic solution of the problem

$$b(\mathbf{D})^* g(\mathbf{x}) \left(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m\right) = 0, \quad \int_\Omega \Lambda(\mathbf{x})\, d\mathbf{x} = 0. \tag{5.26}$$

In accordance with (1.10), we obtain

$$(Z(\boldsymbol{\theta})\mathbf{c})(\mathbf{x}) = \Lambda(\mathbf{x}) b(\boldsymbol{\theta})\mathbf{c}, \quad \mathbf{c} \in \mathbb{C}^n = \mathfrak{N},$$

$$Z(\boldsymbol{\theta})\mathbf{u} = 0, \quad \mathbf{u} \in \mathfrak{N}^\perp.$$

Thus,

$$Z(\boldsymbol{\theta}) = \Lambda b(\boldsymbol{\theta}) P. \tag{5.27}$$

The operator $R(\boldsymbol{\theta})$ defined as in (1.12) and (1.13) takes the form

$$(R(\boldsymbol{\theta})\mathbf{c})(\mathbf{x}) = h(\mathbf{x}) \left(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m\right) b(\boldsymbol{\theta})\mathbf{c}, \quad \mathbf{c} \in \mathbb{C}^n = \mathfrak{N}.$$

Then the spectral germ $S(\boldsymbol{\theta}) = R(\boldsymbol{\theta})^* R(\boldsymbol{\theta})\colon \mathfrak{N} \to \mathfrak{N}$ is given by the formula

$$S(\boldsymbol{\theta}) = P b(\boldsymbol{\theta})^* (b(\mathbf{D})\Lambda + \mathbf{1}_m)^* g (b(\mathbf{D})\Lambda + \mathbf{1}_m) b(\boldsymbol{\theta})\big|_{\mathfrak{N}}$$

and acts as multiplication by the matrix $b(\boldsymbol{\theta})^* g^0 b(\boldsymbol{\theta})$, where

$$g^0 = |\Omega|^{-1} \int_\Omega \big(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m\big)^* g(\mathbf{x}) \big(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m\big)\, d\mathbf{x}. \tag{5.28}$$

Taking (5.26) into account, we rewrite g^0 as

$$g^0 = |\Omega|^{-1} \int_\Omega \widetilde{g}(\mathbf{x})\, d\mathbf{x}, \tag{5.29}$$

where

$$\widetilde{g}(\mathbf{x}) = g(\mathbf{x}) \left(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m\right). \tag{5.30}$$

The constant matrix (5.29) is called the *effective matrix.* Automatically, g^0 is positive definite, which can easily be deduced from (5.28). Thus, we have proved that the spectral germ of the operator family $A(t,\boldsymbol{\theta})$ has the form

$$S(\boldsymbol{\theta}) = b(\boldsymbol{\theta})^* g^0 b(\boldsymbol{\theta}).$$

As was mentioned in Subsection 2.3, Condition 2.2 implies that

$$S(\boldsymbol{\theta}) \geq c_* I_{\mathfrak{N}}, \tag{5.31}$$

where the constant c_* is defined by (5.24) and does not depend on $\boldsymbol{\theta}$. So, the spectral germ $S(\boldsymbol{\theta})$ is nondegenerate.

5.4. The effective operator. The properties of the effective matrix. We put

$$S(\mathbf{k}) = t^{2p} S(\boldsymbol{\theta}) = b(\mathbf{k})^* g^0 b(\mathbf{k}), \quad \mathbf{k} = t\boldsymbol{\theta} \in \mathbb{R}^d; \tag{5.32}$$

this is the symbol of the differential operator

$$A^0 = b(\mathbf{D})^* g^0 b(\mathbf{D}) \tag{5.33}$$

with constant coefficients, called the *effective operator* for A. Relations (5.31) and (5.32) imply the following estimate for the symbol of the effective operator:

$$b(\mathbf{k})^* g^0 b(\mathbf{k}) \geq c_* |\mathbf{k}|^{2p} \mathbf{1}_n, \quad \mathbf{k} \in \mathbb{R}^d. \tag{5.34}$$

Now we discuss some properties of the effective matrix g^0.

Proposition 5.3. *Denote*

$$\overline{g} := |\Omega|^{-1} \int_\Omega g(\mathbf{x}) d\mathbf{x}, \quad \underline{g} := \left(|\Omega|^{-1} \int_\Omega g(\mathbf{x})^{-1} d\mathbf{x} \right)^{-1}.$$

Then the effective matrix g^0 satisfies the inequalities

$$\underline{g} \leq g^0 \leq \overline{g}. \tag{5.35}$$

If $m = n$, then $g^0 = \underline{g}$.

Proof. The proof is similar to that of Theorem 1.5 in [BSu1, Chapter 3], where the second order DO's were studied. Let $\mathbf{C} \in \mathbb{C}^m$, and let $\mathbf{v}_\mathbf{C}$ be the periodic solution of problem (5.25). Obviously,

$$h\mathbf{C} = h(b(\mathbf{D})\mathbf{v}_\mathbf{C} + \mathbf{C}) - hb(\mathbf{D})\mathbf{v}_\mathbf{C}. \tag{5.36}$$

The summands on the right-hand side of (5.36) are orthogonal to each other in $L_2(\Omega; \mathbb{C}^m)$, because the first of them belongs to $\operatorname{Ker} X_0^*$, while the second belongs to $\operatorname{Ran} X_0$. Hence,

$$\|h(b(\mathbf{D})\mathbf{v}_\mathbf{C} + \mathbf{C})\|^2_{L_2(\Omega)} \leq \|h\mathbf{C}\|^2_{L_2(\Omega)}, \quad \mathbf{C} \in \mathbb{C}^m.$$

By (5.28), this implies

$$\langle g^0 \mathbf{C}, \mathbf{C} \rangle \leq \langle \overline{g} \mathbf{C}, \mathbf{C} \rangle, \quad \mathbf{C} \in \mathbb{C}^m,$$

which is equivalent to the upper estimate (5.35).

To prove the lower estimate, note that $\mathfrak{P} := (h^*)^{-1}\mathfrak{M} \subset \mathfrak{N}_* = \operatorname{Ker} X_0^*$. We put

$$Q\mathbf{w} = |\Omega|^{-1} (h^*)^{-1} \underline{g} \int_\Omega h^{-1} \mathbf{w} \, d\mathbf{x}, \quad \mathbf{w} \in \mathfrak{H}_* = L_2(\Omega; \mathbb{C}^m).$$

It is easily checked that $Q\mathbf{w} \in \mathfrak{P}$ for $\mathbf{w} \in \mathfrak{H}_*$, $Q\mathbf{w} = \mathbf{w}$ for $\mathbf{w} \in \mathfrak{P}$, and

$$(Q\mathbf{w}, \mathbf{w})_{\mathfrak{H}_*} = |\Omega|^{-1} \langle \underline{g} \mathbf{C}_\mathbf{w}, \mathbf{C}_\mathbf{w} \rangle, \quad \mathbf{C}_\mathbf{w} = \int_\Omega h^{-1} \mathbf{w} \, d\mathbf{x}, \quad \mathbf{w} \in \mathfrak{H}_*. \tag{5.37}$$

It follows that Q is the orthogonal projection of $\mathfrak{H}_*$ onto the subspace $\mathfrak{P}$. We apply Q to the two sides of (5.36). Since $hb(\mathbf{D})\mathbf{v}_\mathbf{C} \in \operatorname{Ran} X_0 = \mathfrak{N}_*^\perp$, we have the relation $Qh\mathbf{C} = Qh(b(\mathbf{D})\mathbf{v}_\mathbf{C} + \mathbf{C})$. Hence, by (5.28),

$$\begin{aligned} \|Qh\mathbf{C}\|^2_{\mathfrak{H}_*} &= \|Qh(b(\mathbf{D})\mathbf{v}_\mathbf{C} + \mathbf{C})\|^2_{\mathfrak{H}_*} \\ &\le \|h(b(\mathbf{D})\mathbf{v}_\mathbf{C} + \mathbf{C})\|^2_{\mathfrak{H}_*} = |\Omega|\langle g^0\mathbf{C}, \mathbf{C}\rangle, \quad \mathbf{C} \in \mathbb{C}^m. \end{aligned} \tag{5.38}$$

From (5.37) with $\mathbf{w} = h\mathbf{C}$ it follows that

$$\|Qh\mathbf{C}\|^2_{\mathfrak{H}_*} = (Qh\mathbf{C}, h\mathbf{C})_{\mathfrak{H}_*} = |\Omega|\langle \underline{g}\mathbf{C}, \mathbf{C}\rangle. \tag{5.39}$$

Together with (5.38), this implies the lower estimate (5.35).

If $m = n$, we have $n_* = m = n$. Then $\mathfrak{P} \subset \mathfrak{N}_*$, $\dim \mathfrak{P} = m = n$, and $\dim \mathfrak{N}_* = n_* = n$. Hence, $\mathfrak{P} = \mathfrak{N}_*$. Since $h(b(\mathbf{D})\mathbf{v}_\mathbf{C} + \mathbf{C}) \in \mathfrak{N}_*$, we have equality in (5.38). This means that $g^0 = \underline{g}$. □

Estimates of the form (5.35) are known in homogenization theory for specific DO's as the Voigt–Reuss bracketing. The following estimates for the norms of the effective matrix and its inverse are implied by (5.35):

$$|g^0| \le \|g\|_{L_\infty}, \quad |(g^0)^{-1}| \le \|g^{-1}\|_{L_\infty}. \tag{5.40}$$

Now we distinguish the cases where one of the inequalities in (5.35) becomes an identity. The following two statements are similar to Propositions 1.6 and 1.7 in [BSu1, Chapter 3].

Proposition 5.4. *Let $\mathbf{g}_k(\mathbf{x})$, $k = 1, \dots, m$, be the columns of the matrix $g(\mathbf{x})$. Then the identity $g^0 = \overline{g}$ is equivalent to the relations*

$$b(\mathbf{D})^*\mathbf{g}_k(\mathbf{x}) = 0, \quad k = 1, \dots, m. \tag{5.41}$$

Proof. By (5.28), the identity $g^0 = \overline{g}$ is equivalent to the relation

$$\|h(b(\mathbf{D})\mathbf{v}_\mathbf{C} + \mathbf{C})\|^2_{\mathfrak{H}_*} = \|h\mathbf{C}\|^2_{\mathfrak{H}_*}, \quad \mathbf{C} \in \mathbb{C}^m. \tag{5.42}$$

As has already been mentioned, the terms on the right-hand side of (5.36) are orthogonal to each other. Therefore, (5.42) is equivalent to the fact that $hb(\mathbf{D})\mathbf{v}_\mathbf{C} = 0$ for any $\mathbf{C} \in \mathbb{C}^m$. By (5.25), this condition is satisfied if and only if $b(\mathbf{D})^*g(\mathbf{x})\mathbf{C} = 0$ for any $\mathbf{C} \in \mathbb{C}^m$. This is equivalent to (5.41). □

Proposition 5.5. *Let $\mathbf{l}_k(\mathbf{x})$, $k = 1, \dots, m$, be the columns of the matrix $g(\mathbf{x})^{-1}$. The identity $g^0 = \underline{g}$ is equivalent to the representations*

$$\mathbf{l}_k(\mathbf{x}) = \mathbf{l}_k^0 + b(\mathbf{D})\mathbf{v}_k(\mathbf{x}), \quad \mathbf{l}_k^0 \in \mathbb{C}^m, \quad \mathbf{v}_k \in \widetilde{H}^p(\Omega; \mathbb{C}^n); \quad k = 1, \dots, m. \tag{5.43}$$

Proof. By (5.38) and (5.39), $g^0 = \underline{g}$ if and only if $h(b(\mathbf{D})\mathbf{v}_\mathbf{C} + \mathbf{C}) \in \mathfrak{P}$ for any $\mathbf{C} \in \mathbb{C}^m$. In other words, for any $\mathbf{C} \in \mathbb{C}^m$ there exists a vector $\mathbf{C}_* \in \mathbb{C}^m$ such that $h(b(\mathbf{D})\mathbf{v}_\mathbf{C} + \mathbf{C}) = (h^*)^{-1}\mathbf{C}_*$. This means that

$$g(\mathbf{x})^{-1}\mathbf{C}_* = b(\mathbf{D})\mathbf{v}_\mathbf{C}(\mathbf{x}) + \mathbf{C}, \quad \mathbf{C} \in \mathbb{C}^m. \tag{5.44}$$

Integrating over Ω, we obtain $\underline{g}^{-1}\mathbf{C}_* = \mathbf{C}$.

Obviously, relation (5.44) is true for any $\mathbf{C} \in \mathbb{C}^m$ if and only if so it is for $\mathbf{C} = \underline{g}^{-1}\mathbf{e}_k$ (i.e., $\mathbf{C}_* = \mathbf{e}_k$), $k = 1, \dots, m$. The last condition is equivalent to the representations (5.43) for the columns $\mathbf{l}_k(\mathbf{x})$, $k = 1, \dots, m$. □

Remark 5.6. From the proof of Proposition 5.5 it can be seen that if $g^0 = \underline{g}$, then the matrix (5.30) is constant: $\widetilde{g}(\mathbf{x}) = g^0 = \underline{g}$.

5.5. Estimates for the matrix-valued function Λ. In what follows, we need estimates for the norms of Λ.

Lemma 5.7. *Let $\mathbf{v}_j(\mathbf{x})$, $j = 1, \dots, m$, be the columns of the matrix-valued function $\Lambda(\mathbf{x})$ that is a Γ-periodic solution of problem* (5.26). *Then*

$$\|b(\mathbf{D})\mathbf{v}_j\|_{L_2(\Omega)} \le |\Omega|^{1/2}\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty}^{1/2}, \quad j = 1, \dots, m, \tag{5.45}$$

$$\|\mathbf{v}_j\|_{L_2(\Omega)} \le \alpha_0^{-1/2}(2r_0)^{-p}|\Omega|^{1/2}\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty}^{1/2}, \quad j = 1, \dots, m. \tag{5.46}$$

Proof. Recall that the function $\mathbf{v}_j \in \widetilde{H}^p(\Omega;\mathbb{C}^n)$ satisfies the identity

$$(g(b(\mathbf{D})\mathbf{v}_j + \mathbf{e}_j), b(\mathbf{D})\mathbf{w})_{L_2(\Omega)} = 0, \quad \mathbf{w} \in \widetilde{H}^p(\Omega;\mathbb{C}^n), \tag{5.47}$$

and also the condition $\int_\Omega \mathbf{v}_j\, d\mathbf{x} = 0$. From (5.47) it follows that

$$\|hb(\mathbf{D})\mathbf{v}_j\|_{L_2(\Omega)} \le \|h\mathbf{e}_j\|_{L_2(\Omega)} \le |\Omega|^{1/2}\|g\|_{L_\infty}^{1/2}, \quad j = 1, \dots, m,$$

which implies (5.45).

In order to estimate $\|\mathbf{v}_j\|_{L_2(\Omega)}$, we use Fourier series, (4.4), (4.11), and the condition $\int_\Omega \mathbf{v}_j\, d\mathbf{x} = 0$, obtaining

$$\|b(\mathbf{D})\mathbf{v}_j\|^2_{L_2(\Omega)} \ge \alpha_0 \sum_{\mathbf{0}\ne\mathbf{s}\in\widetilde{\Gamma}} |\mathbf{s}|^{2p}|\hat{\mathbf{v}}_{j,\mathbf{s}}|^2 \ge \alpha_0(2r_0)^{2p}\|\mathbf{v}_j\|^2_{L_2(\Omega)}, \quad j = 1, \dots, m, \tag{5.48}$$

where the $\hat{\mathbf{v}}_{j,\mathbf{s}}$, $\mathbf{s} \in \widetilde{\Gamma}$, are the Fourier coefficients of the function $\mathbf{v}_j$. Relations (5.45) and (5.48) imply (5.46). □

Corollary 5.8. *Suppose that the matrix-valued function $\Lambda(\mathbf{x})$ is the Γ-periodic solution of problem* (5.26). *Then*

$$\|\Lambda\|_{L_2(\Omega)} \le |\Omega|^{1/2}C_\Lambda^{(1)}, \tag{5.49}$$

$$\|b(\mathbf{D})\Lambda\|_{L_2(\Omega)} \le |\Omega|^{1/2}C_\Lambda^{(2)}, \tag{5.50}$$

$$\|\Lambda\|_{H^p(\Omega)} \le |\Omega|^{1/2}C_\Lambda, \tag{5.51}$$

where

$$C_\Lambda^{(1)} := m^{1/2}\alpha_0^{-1/2}(2r_0)^{-p}\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty}^{1/2},$$
$$C_\Lambda^{(2)} := m^{1/2}\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty}^{1/2},$$
$$C_\Lambda := C_\Lambda^{(2)}\alpha_0^{-1/2}\Big(\sum_{|\beta|\le p}(2r_0)^{-2(p-|\beta|)}\Big)^{1/2}.$$

Proof. Inequalities (5.49) and (5.50) follow directly from (5.46) and (5.45), respectively.

To check (5.51), we apply the Fourier series expansion. As in (5.14), taking (4.4) into account, we have

$$\begin{aligned}\|\mathbf{D}^\beta\Lambda\|^2_{L_2(\Omega)} &\le (2r_0)^{-2(p-|\beta|)}\sum_{\mathbf{0}\ne\mathbf{s}\in\widetilde{\Gamma}}|\mathbf{s}|^{2p}\big|\widehat{\Lambda}_\mathbf{s}\big|^2 \\ &\le (2r_0)^{-2(p-|\beta|)}\alpha_0^{-1}\sum_{\mathbf{0}\ne\mathbf{s}\in\widetilde{\Gamma}}\big|b(\mathbf{s})\widehat{\Lambda}_\mathbf{s}\big|^2 \\ &= (2r_0)^{-2(p-|\beta|)}\alpha_0^{-1}\|b(\mathbf{D})\Lambda\|^2_{L_2(\Omega)}, \quad |\beta| \le p.\end{aligned}$$

Hence,

$$\|\Lambda\|^2_{H^p(\Omega)} \le \alpha_0^{-1}\|b(\mathbf{D})\Lambda\|^2_{L_2(\Omega)}\Big(\sum_{|\beta|\le p}(2r_0)^{-2(p-|\beta|)}\Big).$$

Combined with (5.50), this implies (5.51). □

§6. Approximation of the resolvent $(A(\mathbf{k}) + \varepsilon^{2p}I)^{-1}$

6.1. Approximation of the resolvent $(A(\mathbf{k}) + \varepsilon^{2p}I)^{-1}$ in the operator norm in $L_2(\Omega;\mathbb{C}^n)$. We apply Theorem 2.4 to the operator family $A(t,\boldsymbol{\theta})$. The number t^0 is defined by (5.20) and does not depend on $\boldsymbol{\theta}$. Due to the presence of the projection P onto the subspace (5.7) of constant vector-valued functions, (5.32) implies that

$$t^{2p}S(\boldsymbol{\theta})P = S(\mathbf{k})P = b(\mathbf{k})^*g^0b(\mathbf{k})P = b(\mathbf{D}+\mathbf{k})^*g^0b(\mathbf{D}+\mathbf{k})P = A^0(\mathbf{k})P. \tag{6.1}$$

Hence, the operator occurring under the norm sign in (2.27) takes the form

$$(A(\mathbf{k}) + \varepsilon^{2p}I)^{-1} - (A^0(\mathbf{k}) + \varepsilon^{2p}I)^{-1}P.$$

Now the constant C_A depends on $\boldsymbol{\theta}$. In accordance with Remark 2.5, this constant is a polynomial in the variables $\widetilde{C}$, $\|X_p(\boldsymbol{\theta})\|$, $\delta^{-1/2p}$, and $c_*^{-1/2p}$ with positive coefficients depending only on p. Relations (5.16), (5.18), (5.19), and (5.24) show that the constants δ, $\widetilde{C}$, and c_* do not depend on $\boldsymbol{\theta}$; by (5.6), the norm $\|X_p(\boldsymbol{\theta})\|$ can be replaced by $\alpha_1^{1/2}\|g\|_{L_\infty}^{1/2}$. Thus, after possible overstating, we may assume that the constant C_A does not depend on $\boldsymbol{\theta}$; it depends only on d, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and r_0.

Applying Theorem 2.4, we arrive at the inequality

$$\begin{aligned}\varepsilon^{2p-1}\big\|(A(\mathbf{k}) + \varepsilon^{2p}I)^{-1} - (A^0(\mathbf{k}) + \varepsilon^{2p}I)^{-1}P\big\|_{L_2(\Omega)\to L_2(\Omega)} \le C_A,\\ \varepsilon>0, \quad |\mathbf{k}|\le t^0.\end{aligned} \tag{6.2}$$

We show that the projection P under the norm sign in (6.2) can be replaced by the identity operator; this will only lead to changing the constant on the right. Using the discrete Fourier transformation, we see that

$$\begin{aligned}&\varepsilon^{2p-1}\big\|(A^0(\mathbf{k}) + \varepsilon^{2p}I)^{-1}P^\perp\big\|_{L_2(\Omega)\to L_2(\Omega)}\\ &\quad= \varepsilon^{2p-1}\sup_{\mathbf{0}\ne\mathbf{s}\in\widetilde{\Gamma}}\big|\big(b(\mathbf{s}+\mathbf{k})^*g^0b(\mathbf{s}+\mathbf{k}) + \varepsilon^{2p}\mathbf{1}_n\big)^{-1}\big|\\ &\quad\le \varepsilon^{2p-1}\sup_{\mathbf{0}\ne\mathbf{s}\in\widetilde{\Gamma}}(c_*|\mathbf{s}+\mathbf{k}|^{2p}+\varepsilon^{2p})^{-1} \le c_*^{-1/2p}r_0^{-1}, \quad \varepsilon>0, \quad |\mathbf{k}|\le t^0.\end{aligned}$$

Here we have used (5.34) and (4.11). Combining this with (6.2), we obtain

$$\begin{aligned}\varepsilon^{2p-1}\big\|(A(\mathbf{k}) + \varepsilon^{2p}I)^{-1} - (A^0(\mathbf{k}) + \varepsilon^{2p}I)^{-1}\big\|_{L_2(\Omega)\to L_2(\Omega)} \le C_A + c_*^{-1/2p}r_0^{-1},\\ \varepsilon>0, \quad |\mathbf{k}|\le t^0.\end{aligned} \tag{6.3}$$

If $\mathbf{k}\in\operatorname{clos}\widetilde{\Omega}\setminus\mathcal{B}(t^0)$, then estimates are trivial. Each summand under the norm sign in (6.3) is estimated separately by using (5.23) for the first eigenvalue of $A(\mathbf{k})$ and a similar estimate for the effective operator. We have

$$\begin{aligned}\varepsilon^{2p-1}\big\|(A(\mathbf{k}) + \varepsilon^{2p}I)^{-1}\big\|_{L_2(\Omega)\to L_2(\Omega)} &\le c_*^{-1/2p}(t^0)^{-1},\\ \varepsilon^{2p-1}\big\|(A^0(\mathbf{k}) + \varepsilon^{2p}I)^{-1}\big\|_{L_2(\Omega)\to L_2(\Omega)} &\le c_*^{-1/2p}(t^0)^{-1},\\ \varepsilon>0, \quad \mathbf{k}\in\operatorname{clos}\widetilde{\Omega}&\setminus\mathcal{B}(t^0).\end{aligned} \tag{6.4}$$

Combining (6.3) and (6.4), and denoting

$$\widetilde{C}_A = \max\big\{C_A + c_*^{-1/2p}r_0^{-1}, 2c_*^{-1/2p}(t^0)^{-1}\big\},$$

we arrive at the following result.

Theorem 6.1. *Let $A(\mathbf{k}) = b(\mathbf{D}+\mathbf{k})^* g b(\mathbf{D}+\mathbf{k})$ be the operator defined in Subsection* 4.3, *and let $A^0(\mathbf{k}) = b(\mathbf{D}+\mathbf{k})^* g^0 b(\mathbf{D}+\mathbf{k})$, where g^0 is the effective matrix defined in Subsection* 5.3. *Then*

$$\big\|(A(\mathbf{k}) + \varepsilon^{2p} I)^{-1} - (A^0(\mathbf{k}) + \varepsilon^{2p} I)^{-1}\big\|_{L_2(\Omega)\to L_2(\Omega)} \le \widetilde{C}_A \varepsilon^{1-2p}, \quad \varepsilon > 0, \quad \mathbf{k} \in \operatorname{clos} \widetilde{\Omega}.$$

The constant $\widetilde{C}_A$ depends only on d, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.

6.2. Approximation of the resolvent $\big(A(\mathbf{k}) + \varepsilon^{2p} I\big)^{-1}$ in the energy norm. Now we apply Theorem 3.1 to the operator family $A(t, \boldsymbol{\theta})$. As in (6.1), from (5.27) we deduce the identity

$$t^p Z(\boldsymbol{\theta}) = \Lambda b(\mathbf{k}) P = \Lambda b(\mathbf{D}+\mathbf{k}) P. \tag{6.5}$$

By (6.1) and (6.5), the operator under the norm sign in (3.1) takes the form

$$\mathcal{E}(\mathbf{k}, \varepsilon) := A(\mathbf{k})^{1/2} \left((A(\mathbf{k}) + \varepsilon^{2p} I)^{-1} - (I + \Lambda b(\mathbf{D}+\mathbf{k}))(A^0(\mathbf{k}) + \varepsilon^{2p} I)^{-1} P \right). \tag{6.6}$$

By Remark 3.4 and relations (5.6), (5.16), (5.18), (5.19), and (5.24), after possible overstating, we may assume that $\check{C}_A$ does not depend on $\boldsymbol{\theta}$; it depends only on d, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and r_0.

Applying Theorem 3.1, we arrive at the inequality

$$\varepsilon^{p-1} \|\mathcal{E}(\mathbf{k}, \varepsilon)\|_{L_2(\Omega)\to L_2(\Omega)} \le \check{C}_A, \quad \varepsilon > 0, \quad |\mathbf{k}| \le t^0. \tag{6.7}$$

Now we estimate the norm of the operator (6.6) for $\mathbf{k} \in \operatorname{clos} \widetilde{\Omega} \setminus \mathcal{B}(t^0)$. The operator (6.6) can be represented as the sum of three terms:

$$\begin{aligned} \mathcal{E}(\mathbf{k}, \varepsilon) = A(\mathbf{k})^{1/2} (A(\mathbf{k}) + \varepsilon^{2p} I)^{-1} &- A(\mathbf{k})^{1/2} (A^0(\mathbf{k}) + \varepsilon^{2p} I)^{-1} P \\ &- A(\mathbf{k})^{1/2} \Lambda P_m b(\mathbf{D}+\mathbf{k}) (A^0(\mathbf{k}) + \varepsilon^{2p} I)^{-1} P, \end{aligned} \tag{6.8}$$

where P_m is the orthogonal projection onto the subspace $\mathfrak{M}$ of constant vector-valued functions in $L_2(\Omega; \mathbb{C}^m)$.

The first term is estimated with the help of (5.23):

$$\varepsilon^{p-1} \|A(\mathbf{k})^{1/2} (A(\mathbf{k}) + \varepsilon^{2p} I)^{-1}\|_{L_2(\Omega)\to L_2(\Omega)} \le c_*^{-1/2p} (t^0)^{-1}, \quad \mathbf{k} \in \operatorname{clos} \widetilde{\Omega} \setminus \mathcal{B}(t^0). \tag{6.9}$$

Using the presence of the projection P and relations (4.4) and (5.34), we obtain the following estimate for the second term:

$$\begin{aligned} &\big\|A(\mathbf{k})^{1/2} (A^0(\mathbf{k}) + \varepsilon^{2p} I)^{-1} P\big\|_{L_2(\Omega)\to L_2(\Omega)} \\ &\quad = \big\|h b(\mathbf{D}+\mathbf{k}) (A^0(\mathbf{k}) + \varepsilon^{2p} I)^{-1} P\big\|_{L_2(\Omega)\to L_2(\Omega)} \\ &\quad \le \|g\|_{L_\infty}^{1/2} \big|b(\mathbf{k}) (b(\mathbf{k})^* g^0 b(\mathbf{k}) + \varepsilon^{2p} \mathbf{1}_n)^{-1}\big| \\ &\quad \le \alpha_1^{1/2} \|g\|_{L_\infty}^{1/2} |\mathbf{k}|^p (c_* |\mathbf{k}|^{2p} + \varepsilon^{2p})^{-1}. \end{aligned} \tag{6.10}$$

Hence,

$$\varepsilon^{p-1} \big\|A(\mathbf{k})^{1/2} (A^0(\mathbf{k}) + \varepsilon^{2p} I)^{-1} P\big\|_{L_2(\Omega)\to L_2(\Omega)} \le \alpha_1^{1/2} \|g\|_{L_\infty}^{1/2} c_*^{-1/2-1/2p} (t^0)^{-1}, \quad \mathbf{k} \in \operatorname{clos} \widetilde{\Omega} \setminus \mathcal{B}(t^0). \tag{6.11}$$

As in (6.10), for the third term on the right-hand side of (6.8) we have

$$\begin{aligned}&\big\|A(\mathbf{k})^{1/2}\Lambda P_m b(\mathbf{D}+\mathbf{k})(A^0(\mathbf{k})+\varepsilon^{2p}I)^{-1}P\big\|_{L_2(\Omega)\to L_2(\Omega)}\\ &\quad\le \big\|A(\mathbf{k})^{1/2}\Lambda P_m\big\|_{L_2(\Omega)\to L_2(\Omega)}\big\|b(\mathbf{D}+\mathbf{k})(A^0(\mathbf{k})+\varepsilon^{2p}I)^{-1}P\big\|_{L_2(\Omega)\to L_2(\Omega)}\\ &\quad\le \alpha_1^{1/2}|\mathbf{k}|^p(c_*|\mathbf{k}|^{2p}+\varepsilon^{2p})^{-1}\|A(\mathbf{k})^{1/2}\Lambda P_m\|_{L_2(\Omega)\to L_2(\Omega)}.\end{aligned}\tag{6.12}$$

Obviously,

$$\begin{aligned}\|A(\mathbf{k})^{1/2}\Lambda P_m\|_{L_2(\Omega)\to L_2(\Omega)} &= \|hb(\mathbf{D}+\mathbf{k})\Lambda P_m\|_{L_2(\Omega)\to L_2(\Omega)}\\ &\le |\Omega|^{-1/2}\|g\|^{1/2}_{L_\infty}\|b(\mathbf{D}+\mathbf{k})\Lambda\|_{L_2(\Omega)}.\end{aligned}\tag{6.13}$$

Recall that the norm $\|b(\mathbf{D})\Lambda\|_{L_2(\Omega)}$ satisfies (5.50). By (4.4) and (5.49),

$$\|b(\mathbf{k})\Lambda\|_{L_2(\Omega)} \le |\mathbf{k}|^p\alpha_1^{1/2}|\Omega|^{1/2}C_\Lambda^{(1)} \le r_1^p\alpha_1^{1/2}|\Omega|^{1/2}C_\Lambda^{(1)},\quad \mathbf{k}\in\operatorname{clos}\widetilde{\Omega},\tag{6.14}$$

where $2r_1=\operatorname{diam}\widetilde{\Omega}$. Relations (5.50), (6.13), and (6.14) imply that

$$\|A(\mathbf{k})^{1/2}\Lambda P_m\|_{L_2(\Omega)\to L_2(\Omega)} \le \|g\|^{1/2}_{L_\infty}\big(C_\Lambda^{(2)}+r_1^p\alpha_1^{1/2}C_\Lambda^{(1)}\big),\quad \mathbf{k}\in\operatorname{clos}\widetilde{\Omega}.\tag{6.15}$$

From (6.12) and (6.15) it follows that, for $|\mathbf{k}|>t^0$, the third term in (6.8) satisfies

$$\begin{aligned}&\varepsilon^{p-1}\big\|A(\mathbf{k})^{1/2}\Lambda P_m b(\mathbf{D}+\mathbf{k})(A^0(\mathbf{k})+\varepsilon^{2p}I)^{-1}P\big\|_{L_2(\Omega)\to L_2(\Omega)}\le C_9,\\ &\mathbf{k}\in\operatorname{clos}\widetilde{\Omega}\setminus\mathcal{B}(t^0),\quad C_9:=\alpha_1^{1/2}\|g\|^{1/2}_{L_\infty}\big(C_\Lambda^{(2)}+r_1^p\alpha_1^{1/2}C_\Lambda^{(1)}\big)c_*^{-1/2-1/2p}(t^0)^{-1}.\end{aligned}\tag{6.16}$$

As a result, relations (6.8), (6.9), (6.11), and (6.16) imply the following estimate for the operator (6.6) for $|\mathbf{k}|>t^0$:

$$\begin{aligned}&\varepsilon^{p-1}\|\mathcal{E}(\mathbf{k},\varepsilon)\|_{L_2(\Omega)\to L_2(\Omega)}\le \widehat{C}_A,\quad \varepsilon>0,\quad \mathbf{k}\in\operatorname{clos}\widetilde{\Omega}\setminus\mathcal{B}(t^0),\\ &\widehat{C}_A:=c_*^{-1/2p}(t^0)^{-1}\big(1+c_*^{-1/2}\alpha_1^{1/2}\|g\|^{1/2}_{L_\infty}\big)+C_9.\end{aligned}\tag{6.17}$$

Inequalities (6.7) and (6.17) lead to the following result.

Theorem 6.2. *Let $A(\mathbf{k})=b(\mathbf{D}+\mathbf{k})^*gb(\mathbf{D}+\mathbf{k})$ be the operator defined in Subsection* 4.3*, and let $A^0(\mathbf{k})=b(\mathbf{D}+\mathbf{k})^*g^0b(\mathbf{D}+\mathbf{k})$, where g^0 is the effective matrix defined in Subsection* 5.3*. Let P be the orthogonal projection of $L_2(\Omega;\mathbb{C}^n)$ onto the subspace* (5.7)*. Suppose that $\Lambda(\mathbf{x})$ is a Γ-periodic solution of problem* (5.26)*. Then*

$$\begin{aligned}&\big\|A(\mathbf{k})^{1/2}\big((A(\mathbf{k})+\varepsilon^{2p}I)^{-1}-(I+\Lambda b(\mathbf{D}+\mathbf{k}))(A^0(\mathbf{k})+\varepsilon^{2p}I)^{-1}P\big)\big\|_{L_2(\Omega)\to L_2(\Omega)}\\ &\qquad\le C_A^\circ\varepsilon^{1-p},\quad \varepsilon>0,\quad \mathbf{k}\in\operatorname{clos}\widetilde{\Omega}.\end{aligned}$$

The constant $C_A^\circ=\max\{\check{C}_A,\widehat{C}_A\}$ depends only on m, d, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.

§7. Approximation of the resolvent of the operator A

7.1. Approximation of the resolvent $(A+\varepsilon^{2p}I)^{-1}$ in the operator norm in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. We return to the operator A acting in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. By (4.17), we have

$$(A+\varepsilon^{2p}I)^{-1}=\mathcal{U}^{-1}\left(\int_{\widetilde{\Omega}}\oplus(A(\mathbf{k})+\varepsilon^{2p}I)^{-1}\,d\mathbf{k}\right)\mathcal{U}.\tag{7.1}$$

The operator $(A^0+\varepsilon^{2p}I)^{-1}$ admits a similar expansion. Hence,

$$\begin{aligned}&\big\|(A+\varepsilon^{2p}I)^{-1}-(A^0+\varepsilon^{2p}I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\\ &\quad=\operatorname*{ess\,sup}_{\mathbf{k}\in\widetilde{\Omega}}\big\|(A(\mathbf{k})+\varepsilon^{2p}I)^{-1}-(A^0(\mathbf{k})+\varepsilon^{2p}I)^{-1}\big\|_{L_2(\Omega)\to L_2(\Omega)}.\end{aligned}$$

Together with Theorem 6.1, this implies the following result.

Theorem 7.1. *Let $A = b(\mathbf{D})^* g b(\mathbf{D})$ be the operator defined in Subsection* 4.1, *and let $A^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$ be the effective operator. Then we have*

$$\big\|(A + \varepsilon^{2p} I)^{-1} - (A^0 + \varepsilon^{2p} I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \widetilde{C}_A \varepsilon^{1-2p}, \quad \varepsilon > 0. \tag{7.2}$$

The constant $\widetilde{C}_A$ depends only on d, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.

7.2. Approximation of the resolvent $(A + \varepsilon^{2p} I)^{-1}$ in the energy norm. Now we use Theorem 6.2 and expansion (7.1) to approximate the resolvent $\big(A + \varepsilon^{2p} I\big)^{-1}$ with the corrector taken into account. Recall that, under the Gelfand transformation, the operator $b(\mathbf{D})$ expands in the direct integral of operators $b(\mathbf{D} + \mathbf{k})$; and the operator of multiplication by the periodic matrix-valued function $\Lambda(\mathbf{x})$ turns into multiplication by the same function on the fibers of the direct integral $\mathcal{K}$ (see (4.16)). We shall also need the operator $\Pi := \mathcal{U}^{-1}[P]\mathcal{U}$ acting in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. Here $[P]$ is the operator in $\mathcal{K}$ that acts as P on the fibers. It is easily seen (see [BSu3, (6.8)]) that Π is the pseudodifferential operator with the symbol $\chi_{\widetilde{\Omega}}(\boldsymbol{\xi})$, where $\chi_{\widetilde{\Omega}}$ is the characteristic function of the set $\widetilde{\Omega}$, i.e.,

$$(\Pi \mathbf{u})(\mathbf{x}) = (2\pi)^{-d/2} \int_{\widetilde{\Omega}} e^{i\langle \mathbf{x}, \boldsymbol{\xi}\rangle} \widehat{\mathbf{u}}(\boldsymbol{\xi})\, d\boldsymbol{\xi}. \tag{7.3}$$

(Note that the operator Π is smoothing.)

It follows that the operator

$$\mathcal{E}(\varepsilon) := A^{1/2}\left((A + \varepsilon^{2p} I)^{-1} - (I + \Lambda b(\mathbf{D}))(A^0 + \varepsilon^{2p} I)^{-1}\Pi\right)$$

expands in the direct integral of operators $\mathcal{E}(\mathbf{k}, \varepsilon)$ (see (6.6)). Therefore, by Theorem 6.2,

$$\varepsilon^{p-1} \|\mathcal{E}(\varepsilon)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le C_A^\circ, \quad \varepsilon > 0. \tag{7.4}$$

From (7.4) we deduce the following result.

Theorem 7.2. *Let $A = b(\mathbf{D})^* g b(\mathbf{D})$ be the operator defined in Subsection* 4.1, *and let $A^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$ be the effective operator. Let $\Lambda(\mathbf{x})$ be the Γ-periodic solution of problem* (5.26), *and let $\widetilde{g}(\mathbf{x})$ be the matrix-valued function* (5.30). *Let Π be the operator* (7.3). *Then for $\varepsilon > 0$ we have*

$$\big\|(A + \varepsilon^{2p} I)^{-1} - (I + \Lambda b(\mathbf{D})\Pi)(A^0 + \varepsilon^{2p} I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le C_A^{(1)} \varepsilon^{1-2p}, \tag{7.5}$$

$$\big\|A^{1/2}\big((A + \varepsilon^{2p} I)^{-1} - (I + \Lambda b(\mathbf{D})\Pi)(A^0 + \varepsilon^{2p} I)^{-1}\big)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le C_A^{(2)} \varepsilon^{1-p}, \tag{7.6}$$

$$\big\|g b(\mathbf{D})(A + \varepsilon^{2p} I)^{-1} - \widetilde{g} b(\mathbf{D})(A^0 + \varepsilon^{2p} I)^{-1}\Pi\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le C_A^{(3)} \varepsilon^{1-p}. \tag{7.7}$$

The constants $C_A^{(1)}$, $C_A^{(2)}$, $C_A^{(3)}$ depend only on m, d, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.

Proof. To check (7.5), we use (7.2) and estimate the operator

$$\Lambda b(\mathbf{D})\Pi(A^0 + \varepsilon^{2p} I)^{-1} = \Lambda \Pi_m b(\mathbf{D})(A^0 + \varepsilon^{2p} I)^{-1}\Pi.$$

Here Π_m is the pseudodifferential operator with the symbol $\chi_{\widetilde{\Omega}}(\boldsymbol{\xi})$ in $L_2(\mathbb{R}^d;\mathbb{C}^m)$. Since the operator $[\Lambda]\Pi_m$ is unitarily equivalent to the direct integral of the operators $[\Lambda]P_m$, we have

$$\|[\Lambda]\Pi_m\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} = \|[\Lambda]P_m\|_{L_2(\Omega)\to L_2(\Omega)} \le |\Omega|^{-1/2} \|\Lambda\|_{L_2(\Omega)} \le C_\Lambda^{(1)}. \tag{7.8}$$

We have used (5.49). Next, using (4.4), (5.34), and (7.3), we obtain

$$
\begin{aligned}
&\varepsilon^{2p-1}\big\|b(\mathbf{D})(A^0+\varepsilon^{2p}I)^{-1}\Pi\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\\
&\quad=\varepsilon^{2p-1}\sup_{\boldsymbol{\xi}\in\widetilde{\Omega}}\big|b(\boldsymbol{\xi})(b(\boldsymbol{\xi})^*g^0b(\boldsymbol{\xi})+\varepsilon^{2p}\mathbf{1}_n)^{-1}\big|\\
&\quad\le\varepsilon^{2p-1}\alpha_1^{1/2}\sup_{\boldsymbol{\xi}\in\widetilde{\Omega}}|\boldsymbol{\xi}|^p\big(c_*|\boldsymbol{\xi}|^{2p}+\varepsilon^{2p}\big)^{-1}\\
&\quad\le\alpha_1^{1/2}c_*^{-1/2p}\sup_{\boldsymbol{\xi}\in\widetilde{\Omega}}|\boldsymbol{\xi}|^{p-1}\le\alpha_1^{1/2}c_*^{-1/2p}r_1^{p-1},\quad\varepsilon>0.
\end{aligned}\tag{7.9}
$$

As a result, relations (7.2), (7.8), and (7.9) imply estimate (7.5) with the constant $C_A^{(1)}=\widetilde{C}_A+C_\Lambda^{(1)}\alpha_1^{1/2}c_*^{-1/2p}r_1^{p-1}$.

Now we prove (7.6) with the help of (7.4). By (4.4), (5.34), and (7.3), we have

$$
\begin{aligned}
&\big\|A^{1/2}(A^0+\varepsilon^{2p}I)^{-1}(I-\Pi)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\\
&\quad=\big\|hb(\mathbf{D})(A^0+\varepsilon^{2p}I)^{-1}(I-\Pi)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\\
&\quad\le\|g\|_{L_\infty}^{1/2}\sup_{|\boldsymbol{\xi}|\ge r_0}\big|b(\boldsymbol{\xi})(b(\boldsymbol{\xi})^*g^0b(\boldsymbol{\xi})+\varepsilon^{2p}\mathbf{1}_n)^{-1}\big|\\
&\quad\le\alpha_1^{1/2}\|g\|_{L_\infty}^{1/2}\sup_{|\boldsymbol{\xi}|\ge r_0}|\boldsymbol{\xi}|^p(c_*|\boldsymbol{\xi}|^{2p}+\varepsilon^{2p})^{-1}.
\end{aligned}\tag{7.10}
$$

Hence,

$$
\varepsilon^{p-1}v\big\|A^{1/2}(A^0+\varepsilon^{2p}I)^{-1}(I-\Pi)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\le\alpha_1^{1/2}\|g\|_{L_\infty}^{1/2}c_*^{-1/2-1/2p}r_0^{-1}.
$$

Combined with (7.4), this implies (7.6) with the constant

$$
C_A^{(2)}=C_A^\circ+\alpha_1^{1/2}\|g\|_{L_\infty}^{1/2}c_*^{-1/2-1/2p}r_0^{-1}.
$$

We proceed to the proof of (7.7). Denote

$$
\mathcal{G}(\varepsilon):=gb(\mathbf{D})\big((A+\varepsilon^{2p}I)^{-1}-(I+\Lambda b(\mathbf{D}))(A^0+\varepsilon^{2p}I)^{-1}\Pi\big).\tag{7.11}
$$

From (7.4) it follows that

$$
\varepsilon^{p-1}\|\mathcal{G}(\varepsilon)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\le C_A^\circ\|g\|_{L_\infty}^{1/2},\quad\varepsilon>0.\tag{7.12}
$$

By (4.3), for any sufficiently smooth function $\mathbf{u}(\mathbf{x})$ in $\mathbb{R}^d$ we have

$$
b(\mathbf{D})(\Lambda\mathbf{u})=(b(\mathbf{D})\Lambda)\mathbf{u}+\sum_{|\alpha|=p}b_\alpha\sum_{\beta\le\alpha\,:\,|\beta|\ge1}C_\alpha^\beta(\mathbf{D}^{\alpha-\beta}\Lambda)\mathbf{D}^\beta\mathbf{u}.
$$

Then, recalling the notation (5.30), we represent (7.11) as

$$
\mathcal{G}(\varepsilon)=gb(\mathbf{D})(A+\varepsilon^{2p}I)^{-1}-\widetilde{g}b(\mathbf{D})(A^0+\varepsilon^{2p}I)^{-1}\Pi-\widetilde{\mathcal{G}}(\varepsilon),\tag{7.13}
$$

where

$$
\widetilde{\mathcal{G}}(\varepsilon):=g\sum_{|\alpha|=p}b_\alpha\sum_{\beta\le\alpha\,:\,|\beta|\ge1}C_\alpha^\beta(\mathbf{D}^{\alpha-\beta}\Lambda)\Pi_m\mathbf{D}^\beta b(\mathbf{D})(A^0+\varepsilon^{2p}I)^{-1}\Pi.
$$

As in (7.8),

$$
\big\|[\mathbf{D}^{\alpha-\beta}\Lambda]\Pi_m\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\le|\Omega|^{-1/2}\|\mathbf{D}^{\alpha-\beta}\Lambda\|_{L_2(\Omega)}\le C_\Lambda.\tag{7.14}
$$

In the last inequality, (5.51) was used. By (4.5) and (7.14),

$$
\|\widetilde{\mathcal{G}}(\varepsilon)\|_{L_2\to L_2}\le\|g\|_{L_\infty}\alpha_1^{1/2}C_\Lambda\sum_{|\alpha|=p}\sum_{\beta\le\alpha\,:\,|\beta|\ge1}C_\alpha^\beta\|\mathbf{D}^\beta b(\mathbf{D})(A^0+\varepsilon^{2p}I)^{-1}\Pi\|_{L_2\to L_2}.\tag{7.15}
$$

Like in (7.9), from (4.4), (5.34), and (7.3) it follows that

$$\big\|\mathbf{D}^\beta b(\mathbf{D})(A^0+\varepsilon^{2p}I)^{-1}\Pi\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \alpha_1^{1/2} \sup_{\boldsymbol{\xi}\in\widetilde{\Omega}} |\boldsymbol{\xi}|^{p+|\beta|}(c_*|\boldsymbol{\xi}|^{2p}+\varepsilon^{2p})^{-1},$$

whence

$$\varepsilon^{p-1}\big\|\mathbf{D}^\beta b(\mathbf{D})(A^0+\varepsilon^{2p}I)^{-1}\Pi\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \alpha_1^{1/2} c_*^{-1/2-1/2p} r_1^{|\beta|-1}, \quad |\beta|\ge 1. \tag{7.16}$$

Inequalities (7.15) and (7.16) imply

$$\varepsilon^{p-1}\|\widetilde{\mathcal{G}}(\varepsilon)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le C_{10}, \quad \varepsilon>0, \tag{7.17}$$

where

$$C_{10} := \|g\|_{L_\infty}\alpha_1 C_\Lambda c_*^{-1/2-1/2p}\Big(\sum_{|\alpha|=p}\ \sum_{\beta\le\alpha\,:\,|\beta|\ge 1} C_\alpha^\beta r_1^{|\beta|-1}\Big).$$

As a result, relations (7.12), (7.13), and (7.17) yield the required inequality (7.7) with the constant $C_A^{(3)} = C_A^\circ \|g\|_{L_\infty}^{1/2} + C_{10}$. □

Now we distinguish special cases. If $g^0=\overline{g}$, then conditions (5.41) are satisfied, so that the solution Λ of problem (5.26) is equal to zero. Therefore, (7.6) simplifies, and we arrive at the following statement.

Proposition 7.3. *If $g^0=\overline{g}$ (i.e., conditions (5.41) are satisfied), then*

$$\big\|A^{1/2}\big((A+\varepsilon^{2p}I)^{-1}-(A^0+\varepsilon^{2p}I)^{-1}\big)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le C_A^{(2)}\varepsilon^{1-p}, \quad \varepsilon>0. \tag{7.18}$$

Now we consider the case where $g^0=\underline{g}$.

Proposition 7.4. *If $g^0=\underline{g}$ (i.e., the representations (5.43) are valid), then*

$$\big\|g b(\mathbf{D})(A+\varepsilon^{2p}I)^{-1} - g^0 b(\mathbf{D})(A^0+\varepsilon^{2p}I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \widetilde{C}_A^{(3)}\varepsilon^{1-p}, \quad \varepsilon>0. \tag{7.19}$$

The constant $\widetilde{C}_A^{(3)}$ depends only on m, d, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.

Proof. By Remark 5.6, in the case under consideration we have $\widetilde{g}(\mathbf{x}) = g^0 = \underline{g}$. Then inequality (7.7) takes the form

$$\big\|g b(\mathbf{D})(A+\varepsilon^{2p}I)^{-1} - g^0 b(\mathbf{D})(A^0+\varepsilon^{2p}I)^{-1}\Pi\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le C_A^{(3)}\varepsilon^{1-p}, \quad \varepsilon>0. \tag{7.20}$$

As in (7.10), by (5.40) we have

$$\big\|g^0 b(\mathbf{D})(A^0+\varepsilon^{2p}I)^{-1}(I-\Pi)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \|g\|_{L_\infty}\alpha_1^{1/2}\sup_{|\boldsymbol{\xi}|\ge r_0} |\boldsymbol{\xi}|^p\left(c_*|\boldsymbol{\xi}|^{2p}+\varepsilon^{2p}\right)^{-1}.$$

Hence,

$$\varepsilon^{p-1}\big\|g^0 b(\mathbf{D})(A^0+\varepsilon^{2p}I)^{-1}(I-\Pi)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \alpha_1^{1/2}\|g\|_{L_\infty} c_*^{-1/2-1/2p} r_0^{-1}. \tag{7.21}$$

Relations (7.20) and (7.21) imply (7.19) with the constant

$$\widetilde{C}_A^{(3)} = C_A^{(3)} + \alpha_1^{1/2}\|g\|_{L_\infty} c_*^{-1/2-1/2p} r_0^{-1}.$$

□

7.3. Approximation of the resolvent $\big(A-\zeta\varepsilon^{2p}I\big)^{-1}$ for $\zeta\in\mathbb{C}\setminus\mathbb{R}_+$. Now we carry the results of Theorems 7.1 and 7.2 over to the case of the resolvent at an arbitrary regular point in $\mathbb{C}\setminus\mathbb{R}_+$. For this, we apply appropriate identities for resolvents (cf. [Su]).

Let $\zeta\in\mathbb{C}\setminus\mathbb{R}_+$. We put $\zeta=|\zeta|e^{i\varphi}$, $0<\varphi<2\pi$, and denote

$$c(\varphi)=\begin{cases}|\sin\varphi|^{-1}, & \varphi\in(0,\pi/2)\cup(3\pi/2,2\pi),\\ 1, & \varphi\in[\pi/2,3\pi/2].\end{cases}\tag{7.22}$$

Theorem 7.5. *Under the assumptions of Theorem* 7.1, *let $\zeta=|\zeta|e^{i\varphi}\in\mathbb{C}\setminus\mathbb{R}_+$, and let $c(\varphi)$ be defined by* (7.22). *Then for $\varepsilon>0$ we have*

$$\big\|(A-\zeta\varepsilon^{2p}I)^{-1}-(A^0-\zeta\varepsilon^{2p}I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\le\mathcal{C}_1c(\varphi)^2\varepsilon^{1-2p}|\zeta|^{1/2p-1}.\tag{7.23}$$

The constant $\mathcal{C}_1=4\widetilde{C}_A$ depends only on d, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.

Proof. Let $\widehat{\zeta}=e^{i\varphi}$, $0<\varphi<2\pi$. We employ the identity

$$\begin{aligned}(A-\widehat{\zeta}\varepsilon^{2p}I)^{-1}-(A^0-\widehat{\zeta}\varepsilon^{2p}I)^{-1}&=(A+\varepsilon^{2p}I)(A-\widehat{\zeta}\varepsilon^{2p}I)^{-1}\\ \times\big((A+\varepsilon^{2p}I)^{-1}-(A^0+\varepsilon^{2p}I)^{-1}\big)&\,(A^0+\varepsilon^{2p}I)(A^0-\widehat{\zeta}\varepsilon^{2p}I)^{-1}.\end{aligned}\tag{7.24}$$

Since the spectrum of A is contained in $\mathbb{R}_+$, we have

$$\begin{aligned}\big\|(A+\varepsilon^{2p}I)(A-\widehat{\zeta}\varepsilon^{2p}I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}&\le\sup_{x\ge0}(x+\varepsilon^{2p})|x-\widehat{\zeta}\varepsilon^{2p}|^{-1}\\ &=\sup_{y\ge0}(y+1)|y-\widehat{\zeta}|^{-1}\le2c(\varphi).\end{aligned}\tag{7.25}$$

Similarly,

$$\big\|(A^0+\varepsilon^{2p}I)(A^0-\widehat{\zeta}\varepsilon^{2p}I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\le2c(\varphi).\tag{7.26}$$

From (7.2), (7.24)–(7.26) it follows that

$$\big\|(A-\widehat{\zeta}\varepsilon^{2p}I)^{-1}-(A^0-\widehat{\zeta}\varepsilon^{2p}I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\le4c(\varphi)^2\widetilde{C}_A\varepsilon^{1-2p},\quad\varepsilon>0.\tag{7.27}$$

Replacing ε by $\varepsilon|\zeta|^{1/2p}$ in (7.27), we arrive at the required inequality (7.23). □

Theorem 7.6. *Under the assumptions of Theorem* 7.2, *let $\zeta=|\zeta|e^{i\varphi}\in\mathbb{C}\setminus\mathbb{R}_+$, and let $c(\varphi)$ be given by* (7.22). *Then for $\varepsilon>0$ we have*

$$\begin{aligned}\big\|(A-\zeta\varepsilon^{2p}I)^{-1}-(I+\Lambda b(\mathbf{D})\Pi)(A^0-\zeta\varepsilon^{2p}I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}&\\ \le\mathcal{C}_2c(\varphi)^2\varepsilon^{1-2p}|\zeta|^{1/2p-1},&\end{aligned}\tag{7.28}$$

$$\begin{aligned}\big\|A^{1/2}\big((A-\zeta\varepsilon^{2p}I)^{-1}-(I+\Lambda b(\mathbf{D})\Pi)(A^0-\zeta\varepsilon^{2p}I)^{-1}\big)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}&\\ \le\mathcal{C}_3c(\varphi)^2\varepsilon^{1-p}|\zeta|^{1/2p-1/2},&\end{aligned}\tag{7.29}$$

$$\begin{aligned}\big\|gb(\mathbf{D})(A-\zeta\varepsilon^{2p}I)^{-1}-\widetilde{g}b(\mathbf{D})(A^0-\zeta\varepsilon^{2p}I)^{-1}\Pi\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}&\\ \le\mathcal{C}_4c(\varphi)^2\varepsilon^{1-p}|\zeta|^{1/2p-1/2}.&\end{aligned}\tag{7.30}$$

The constants $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$ depend only on m, d, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.

Proof. Let $\widehat{\zeta}=e^{i\varphi}$, $0<\varphi<2\pi$. We use the identity

$$\begin{aligned}(A-\widehat{\zeta}\varepsilon^{2p}I)^{-1}-(I+\Lambda b(\mathbf{D})\Pi)(A^0-\widehat{\zeta}\varepsilon^{2p}I)^{-1}&=(A+\varepsilon^{2p}I)(A-\widehat{\zeta}\varepsilon^{2p}I)^{-1}\\ \times\big((A+\varepsilon^{2p}I)^{-1}-(I+\Lambda b(\mathbf{D})\Pi)(A^0+\varepsilon^{2p}I)^{-1}\big)&(A^0+\varepsilon^{2p}I)(A^0-\widehat{\zeta}\varepsilon^{2p}I)^{-1}\\ +\varepsilon^{2p}(\widehat{\zeta}+1)(A-\widehat{\zeta}\varepsilon^{2p}I)^{-1}&\Lambda b(\mathbf{D})\Pi(A^0-\widehat{\zeta}\varepsilon^{2p}I)^{-1}.\end{aligned}\tag{7.31}$$

Denote the consecutive terms on the right-hand side of (7.31) by $\mathcal{J}_1(\widehat{\zeta},\varepsilon)$ and $\mathcal{J}_2(\widehat{\zeta},\varepsilon)$.

From (7.5), (7.25), and (7.26) it follows that

$$\varepsilon^{2p-1}\|\mathcal{J}_1(\widehat{\zeta},\varepsilon)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le 4c(\varphi)^2 C_A^{(1)}. \tag{7.32}$$

The second term satisfies

$$\begin{aligned}\|\mathcal{J}_2(\widehat{\zeta},\varepsilon)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} &\le 2\varepsilon^{2p}\big\|(A-\widehat{\zeta}\varepsilon^{2p}I)^{-1}\big\|_{L_2\to L_2}\\ &\times \big\|\Lambda\Pi_m b(\mathbf{D})(A^0+\varepsilon^{2p}I)^{-1}\Pi\big\|_{L_2\to L_2}\big\|(A^0+\varepsilon^{2p}I)(A^0-\widehat{\zeta}\varepsilon^{2p}I)^{-1}\big\|_{L_2\to L_2}.\end{aligned} \tag{7.33}$$

Note that the norm of the resolvent $(A-\widehat{\zeta}\varepsilon^{2p}I)^{-1}$ does not exceed the inverse distance from the point $\widehat{\zeta}\varepsilon^{2p}$ to $\mathbb{R}_+$:

$$\big\|(A-\widehat{\zeta}\varepsilon^{2p}I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \varepsilon^{-2p}c(\varphi). \tag{7.34}$$

Combining (7.8), (7.9), (7.26), (7.33), and (7.34), we obtain

$$\varepsilon^{2p-1}\|\mathcal{J}_2(\widehat{\zeta},\varepsilon)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le 4c(\varphi)^2 C_\Lambda^{(1)}\alpha_1^{1/2}c_*^{-1/2p}r_1^{p-1}. \tag{7.35}$$

Relations (7.31), (7.32), and (7.35) imply that

$$\begin{aligned}\big\|(A-\widehat{\zeta}\varepsilon^{2p}I)^{-1} - (I+\Lambda b(\mathbf{D})\Pi)(A^0-\widehat{\zeta}\varepsilon^{2p}I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} &\le \mathcal{C}_2 c(\varphi)^2\varepsilon^{1-2p},\\ &\varepsilon>0,\end{aligned} \tag{7.36}$$

with the constant $\mathcal{C}_2 = 4C_A^{(1)} + 4C_\Lambda^{(1)}\alpha_1^{1/2}c_*^{-1/2p}r_1^{p-1}$. Replacing ε by $\varepsilon|\zeta|^{1/2p}$ in (7.36), we arrive at (7.28).

To check (7.29), we apply the operator $A^{1/2}$ to both sides of (7.31). We have

$$A^{1/2}\big((A-\widehat{\zeta}\varepsilon^{2p}I)^{-1} - (I+\Lambda b(\mathbf{D})\Pi)(A^0-\widehat{\zeta}\varepsilon^{2p}I)^{-1}\big) = \mathcal{T}_1(\widehat{\zeta},\varepsilon)+\mathcal{T}_2(\widehat{\zeta},\varepsilon), \tag{7.37}$$

where

$$\begin{aligned}\mathcal{T}_1(\widehat{\zeta},\varepsilon) &= (A+\varepsilon^{2p}I)(A-\widehat{\zeta}\varepsilon^{2p}I)^{-1}A^{1/2}\,\big((A+\varepsilon^{2p}I)^{-1} - (I+\Lambda b(\mathbf{D})\Pi)(A^0+\varepsilon^{2p}I)^{-1}\big)\\ &\qquad\times(A^0+\varepsilon^{2p}I)(A^0-\widehat{\zeta}\varepsilon^{2p}I)^{-1},\\ \mathcal{T}_2(\widehat{\zeta},\varepsilon) &= \varepsilon^{2p}(\widehat{\zeta}+1)A^{1/2}(A-\widehat{\zeta}\varepsilon^{2p}I)^{-1}\Lambda b(\mathbf{D})\Pi(A^0-\widehat{\zeta}\varepsilon^{2p}I)^{-1}.\end{aligned}$$

The first term is estimated with the help of (7.6), (7.25), and (7.26):

$$\varepsilon^{p-1}\|\mathcal{T}_1(\widehat{\zeta},\varepsilon)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le 4C_A^{(2)}c(\varphi)^2, \quad \varepsilon>0. \tag{7.38}$$

The second term satisfies

$$\begin{aligned}\|\mathcal{T}_2(\widehat{\zeta},\varepsilon)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} &\le 2\varepsilon^{2p}\big\|A^{1/2}(A-\widehat{\zeta}\varepsilon^{2p}I)^{-1}\big\|_{L_2\to L_2}\\ &\times \big\|\Lambda\Pi_m b(\mathbf{D})(A^0+\varepsilon^{2p}I)^{-1}\Pi\big\|_{L_2\to L_2}\big\|(A^0+\varepsilon^{2p}I)(A^0-\widehat{\zeta}\varepsilon^{2p}I)^{-1}\big\|_{L_2\to L_2}.\end{aligned} \tag{7.39}$$

From (4.4), (5.34), (7.3), and (7.8) it follows that

$$\big\|\Lambda\Pi_m b(\mathbf{D})(A^0+\varepsilon^{2p}I)^{-1}\Pi\big\|_{L_2\to L_2} \le C_\Lambda^{(1)}\alpha_1^{1/2}\sup_{\boldsymbol{\xi}\in\widetilde{\Omega}}|\boldsymbol{\xi}|^p(c_*|\boldsymbol{\xi}|^{2p}+\varepsilon^{2p})^{-1}. \tag{7.40}$$

By (7.25), we have

$$\big\|A^{1/2}(A-\widehat{\zeta}\varepsilon^{2p}I)^{-1}\big\|_{L_2\to L_2} \le 2c(\varphi)\big\|A^{1/2}(A+\varepsilon^{2p}I)^{-1}\big\|_{L_2\to L_2} \le 2c(\varphi)\varepsilon^{-p}. \tag{7.41}$$

Now, relations (7.26) and (7.39)–(7.41) imply

$$\begin{aligned}\varepsilon^{p-1}\|\mathcal{T}_2(\widehat{\zeta},\varepsilon)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} &\le 8c(\varphi)^2 C_\Lambda^{(1)}\alpha_1^{1/2}\varepsilon^{2p-1}\sup_{\boldsymbol{\xi}\in\widetilde{\Omega}}|\boldsymbol{\xi}|^p(c_*|\boldsymbol{\xi}|^{2p}+\varepsilon^{2p})^{-1}\\ &\le 8c(\varphi)^2 C_\Lambda^{(1)}\alpha_1^{1/2}c_*^{-1/2p}r_1^{p-1}, \quad \varepsilon>0.\end{aligned} \tag{7.42}$$

As a result, relations (7.37), (7.38), and (7.42) yield

$$\begin{aligned}(7.43)\qquad &\big\| A^{1/2}\big((A-\widehat{\zeta}\varepsilon^{2p}I)^{-1}-(I+\Lambda b(\mathbf{D})\Pi)(A^0-\widehat{\zeta}\varepsilon^{2p}I)^{-1}\big)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\\ &\qquad\le \mathcal{C}_3 c(\varphi)^2\varepsilon^{1-p},\quad \varepsilon>0,\end{aligned}$$

with the constant $\mathcal{C}_3 = 4C_A^{(2)} + 8C_\Lambda^{(1)}\alpha_1^{1/2}c_*^{-1/2p}r_1^{p-1}$. Replacing ε by $\varepsilon|\zeta|^{1/2p}$ in (7.43), we arrive at the required inequality (7.29).

It remains to check (7.30). We use the identity

$$\begin{aligned}(7.44)\qquad & gb(\mathbf{D})(A-\widehat{\zeta}\varepsilon^{2p}I)^{-1}-\widetilde{g}b(\mathbf{D})(A^0-\widehat{\zeta}\varepsilon^{2p}I)^{-1}\Pi\\ &=\big(gb(\mathbf{D})(A+\varepsilon^{2p}I)^{-1}-\widetilde{g}b(\mathbf{D})(A^0+\varepsilon^{2p}I)^{-1}\Pi\big)(A^0+\varepsilon^{2p}I)(A^0-\widehat{\zeta}\varepsilon^{2p}I)^{-1}\\ &\quad+(\widehat{\zeta}+1)\varepsilon^{2p}gb(\mathbf{D})(A+\varepsilon^{2p}I)^{-1}\big((A-\widehat{\zeta}\varepsilon^{2p}I)^{-1}-(A^0-\widehat{\zeta}\varepsilon^{2p}I)^{-1}\big).\end{aligned}$$

We denote the consecutive summands on the right-hand side of (7.44) by $\mathcal{L}_1(\widehat{\zeta},\varepsilon)$ and $\mathcal{L}_2(\widehat{\zeta},\varepsilon)$. From (7.7) and (7.26) it follows that

$$(7.45)\qquad \|\mathcal{L}_1(\widehat{\zeta},\varepsilon)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\le 2C_A^{(3)}c(\varphi)\varepsilon^{1-p}.$$

The second term is estimated with the help of (7.27):

$$\begin{aligned}(7.46)\qquad \|\mathcal{L}_2(\widehat{\zeta},\varepsilon)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} &\le 8\varepsilon c(\varphi)^2\widetilde{C}_A\|g\|_{L_\infty}^{1/2}\|A^{1/2}(A+\varepsilon^{2p}I)^{-1}\|_{L_2\to L_2}\\ &\le 8\varepsilon^{1-p}c(\varphi)^2\widetilde{C}_A\|g\|_{L_\infty}^{1/2}.\end{aligned}$$

Now relations (7.44)–(7.46) imply that

$$(7.47)\qquad \big\|gb(\mathbf{D})(A-\widehat{\zeta}\varepsilon^{2p}I)^{-1}-\widetilde{g}b(\mathbf{D})(A^0-\widehat{\zeta}\varepsilon^{2p}I)^{-1}\Pi\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\le \mathcal{C}_4c(\varphi)^2\varepsilon^{1-p}$$

for $\varepsilon>0$, where $\mathcal{C}_4 = 2C_A^{(3)} + 8\widetilde{C}_A\|g\|_{L_\infty}^{1/2}$. Replacing ε by $\varepsilon|\zeta|^{1/2p}$ in (7.47), we arrive at (7.30). □

7.4. Special cases. Now we prove analogs of Propositions 7.3 and 7.4 for the resolvent $(A-\zeta\varepsilon^{2p}I)^{-1}$. The following statement is deduced from (7.29) and from the fact that $\Lambda = 0$ if $g^0=\overline{g}$.

Proposition 7.7. *Under the assumptions of Theorem* 7.5, *if* $g^0=\overline{g}$ (*i.e., conditions* (5.41) *are satisfied*), *then*

$$\big\|A^{1/2}\big((A-\zeta\varepsilon^{2p}I)^{-1}-(A^0-\zeta\varepsilon^{2p}I)^{-1}\big)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\le \mathcal{C}_3c(\varphi)^2\varepsilon^{1-p}|\zeta|^{1/2p-1/2},\quad \varepsilon>0.$$

Now we consider the case where $g^0=\underline{g}$.

Proposition 7.8. *Under the assumptions of Theorem* 7.5, *if* $g^0=\underline{g}$ (*i.e., representations* (5.43) *are valid*), *then*

$$\begin{aligned}(7.48)\qquad &\big\|gb(\mathbf{D})(A-\zeta\varepsilon^{2p}I)^{-1}-g^0b(\mathbf{D})(A^0-\zeta\varepsilon^{2p}I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\\ &\qquad\le \mathcal{C}_4^\circ c(\varphi)^2\varepsilon^{1-p}|\zeta|^{1/2p-1/2},\quad \varepsilon>0.\end{aligned}$$

The constant $\mathcal{C}_4^\circ$ *depends only on* m, d, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *and the parameters of the lattice* Γ.

Proof. By Remark 5.6, in the case under consideration we have $\widetilde{g}(\mathbf{x})=g^0=\underline{g}$. First, we consider the resolvent at the point $\widehat{\zeta}\varepsilon^{2p}$, where $\widehat{\zeta}=e^{i\varphi}$. The following analog of identity (7.44) is true:

$$gb(\mathbf{D})(A-\widehat{\zeta}\varepsilon^{2p}I)^{-1}-g^0b(\mathbf{D})(A^0-\widehat{\zeta}\varepsilon^{2p}I)^{-1}=\mathcal{L}_1^\circ(\widehat{\zeta},\varepsilon)+\mathcal{L}_2(\widehat{\zeta},\varepsilon),$$

where

$$\mathcal{L}_1^\circ(\widehat{\zeta},\varepsilon) = \big(gb(\mathbf{D})(A+\varepsilon^{2p}I)^{-1} - g^0 b(\mathbf{D})(A^0+\varepsilon^{2p}I)^{-1}\big)(A^0+\varepsilon^{2p}I)(A^0-\widehat{\zeta}\varepsilon^{2p}I)^{-1},$$

and the second term $\mathcal{L}_2(\widehat{\zeta},\varepsilon)$ is the same as in (7.44).

From (7.19) and (7.26) it follows that

$$\|\mathcal{L}_1^\circ(\widehat{\zeta},\varepsilon)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le 2c(\varphi)\widetilde{C}_A^{(3)}\varepsilon^{1-p}, \quad \varepsilon>0.$$

Combining this with (7.46), we obtain

$$\big\|gb(\mathbf{D})(A-\widehat{\zeta}\varepsilon^{2p}I)^{-1} - g^0 b(\mathbf{D})(A^0-\widehat{\zeta}\varepsilon^{2p}I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \mathcal{C}_4^\circ c(\varphi)^2\varepsilon^{1-p} \tag{7.49}$$

for $\varepsilon>0$, where $\mathcal{C}_4^\circ = 2\widetilde{C}_A^{(3)} + 8\widetilde{C}_A\|g\|_{L_\infty}^{1/2}$. Replacing ε by $\varepsilon|\zeta|^{1/2p}$ in (7.49), we arrive at (7.48). □

7.5. Removal of the smoothing operator. It turns out that, under some additional assumptions on the matrix-valued function $\Lambda(\mathbf{x})$, it is possible to remove the smoothing operator Π from the approximations (7.28)–(7.30). However, the order of estimates for the terms containing $I-\Pi$ differs from the that of estimates (7.28)–(7.30); see Proposition 7.12 below.

Condition 7.9. *Suppose that the Γ-periodic solution $\Lambda\in\widetilde{H}^p(\Omega)$ of problem* (5.26) *is bounded and is a multiplier from $H^p(\mathbb{R}^d;\mathbb{C}^m)$ to $H^p(\mathbb{R}^d;\mathbb{C}^n)$:*

$$\Lambda\in L_\infty(\mathbb{R}^d)\cap M(H^p(\mathbb{R}^d;\mathbb{C}^m)\to H^p(\mathbb{R}^d;\mathbb{C}^n)).$$

Since the matrix-valued function Λ is periodic, Condition 7.9 is equivalent to the fact that $\Lambda\in L_\infty(\Omega)\cap M(H^p(\Omega;\mathbb{C}^m)\to H^p(\Omega;\mathbb{C}^n))$. The norm of the operator $[\Lambda]$ of multiplication by $\Lambda(\mathbf{x})$ is denoted by

$$M_\Lambda := \|[\Lambda]\|_{H^p(\mathbb{R}^d)\to H^p(\mathbb{R}^d)}. \tag{7.50}$$

The spaces of multipliers in the Sobolev classes were described in the book [MSh]. The following statement gives some sufficient conditions ensuring that Condition 7.9 is satisfied.

Proposition 7.10. *Suppose that at least one of the following two assumptions is fulfilled:*

1°. *$2p>d$;*

2°. *$g^0=\underline{g}$, i.e., the representations* (5.43) *are valid.*

Then Condition 7.9 *is satisfied. Moreover, $\|\Lambda\|_{L_\infty}$ and the multiplier norm* (7.50) *are controlled in terms of m, n, d, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.*

Proof. Since $\Lambda\in\widetilde{H}^p(\Omega)$, in the case where $2p>d$ the Sobolev embedding theorem and the theorem in [MSh, Subsection 1.3.3] show that Condition 7.9 is satisfied. Herewith, $\|\Lambda\|_{L_\infty}$ and M_Λ are estimated by $C\|\Lambda\|_{H^p(\Omega)}$, where C depends on m, n, d, p, and the domain Ω. Taking estimate (5.51) into account, we prove the first statement.

Now we prove the second statement. We assume that $2p\le d$ (otherwise, the first statement can be applied). Suppose $g^0=\underline{g}$. By Remark 5.6, $\widetilde{g}=g(b(\mathbf{D})\Lambda+\mathbf{1}_m)=g^0$. Then $\Lambda\in\widetilde{H}^p(\Omega)$ is the Γ-periodic solution of the problem

$$b(\mathbf{D})^*b(\mathbf{D})\Lambda(\mathbf{x}) = b(\mathbf{D})^*g(\mathbf{x})^{-1}g^0, \quad \int_\Omega \Lambda(\mathbf{x})\,d\mathbf{x}=0. \tag{7.51}$$

The operator $b(\mathbf{D})^*b(\mathbf{D})$ is a matrix elliptic operator with constant coefficients. Therefore, the solution of the problem (7.51) can be described in terms of Fourier coefficients:

$$\widehat{\Lambda}_{\mathbf{0}}=0;\quad \widehat{\Lambda}_{\mathbf{s}} = (b(\mathbf{s})^*b(\mathbf{s}))^{-1}\,b(\mathbf{s})^*\widehat{(g^{-1})}_{\mathbf{s}}g^0, \quad \mathbf{0}\ne\mathbf{s}\in\widetilde{\Gamma}.$$

Since $g^{-1}g^0 \in L_\infty \subset L_q(\Omega)$ for any $q < \infty$, the well-known Marcinkiewicz theorem on multipliers for Fourier series (see [Ma]) shows that $\Lambda \in \widetilde{W}^p_q(\Omega)$ for any $q < \infty$. We fix q such that $pq > d$ (e.g., $q = p^{-1}(d+1)$). By the Marcinkiewicz theorem, the norm $\|\Lambda\|_{W^p_q(\Omega)}$ is controlled in terms of m, n, d, p, α_0, α_1, $\|g\|_{L_\infty}$, and $\|g^{-1}\|_{L_\infty}$. Next, by the Sobolev embedding theorem, Proposition 3 and Corollary 1 in [MSh, Subsection 1.3.4], relation $\Lambda \in \widetilde{W}^p_q(\Omega)$ ensures that Condition 7.9 is satisfied. Herewith, $\|\Lambda\|_{L_\infty}$ and M_Λ are estimated by $C\|\Lambda\|_{W^p_q(\Omega)}$, where C depends on m, n, d, p, and the domain Ω. This completes the proof of the second statement. □

Now we estimate the operator $b(\mathbf{D})(I-\Pi)(A^0 - \zeta\varepsilon^{2p}I)^{-1}$ in the $(L_2 \to H^p)$-norm.

Lemma 7.11. *For $\varepsilon > 0$ and $\zeta \in \mathbb{C}\setminus\mathbb{R}_+$ we have*

$$\left\|b(\mathbf{D})(I-\Pi)(A^0 - \zeta\varepsilon^{2p}I)^{-1}\right\|_{L_2(\mathbb{R}^d)\to H^p(\mathbb{R}^d)} \leq C_{11}c(\varphi), \tag{7.52}$$

where $C_{11} = 2\alpha_1^{1/2}c_*^{-1}\left(1 + r_0^{-2}\right)^{p/2}$.

Proof. Using (4.4), (5.34), and (7.3), we obtain

$$\begin{aligned}
&\left\|b(\mathbf{D})(I-\Pi)(A^0 + |\zeta|\varepsilon^{2p}I)^{-1}\right\|_{L_2(\mathbb{R}^d)\to H^p(\mathbb{R}^d)} \\
&\quad = \sup_{\boldsymbol{\xi}\in\mathbb{R}^d}(1 - \chi_{\widetilde{\Omega}}(\boldsymbol{\xi}))(1+|\boldsymbol{\xi}|^2)^{p/2}\left|b(\boldsymbol{\xi})\big(b(\boldsymbol{\xi})^* g^0 b(\boldsymbol{\xi}) + |\zeta|\varepsilon^{2p}\mathbf{1}_n\big)^{-1}\right| \\
&\quad \leq \alpha_1^{1/2}\sup_{|\boldsymbol{\xi}|\geq r_0}(1+|\boldsymbol{\xi}|^2)^{p/2}|\boldsymbol{\xi}|^p\big(c_*|\boldsymbol{\xi}|^{2p} + |\zeta|\varepsilon^{2p}\big)^{-1} \leq \alpha_1^{1/2}c_*^{-1}(1+r_0^{-2})^{p/2}.
\end{aligned} \tag{7.53}$$

Obviously,

$$\begin{aligned}
\left\|(A^0 + |\zeta|\varepsilon^{2p}I)(A^0 - \zeta\varepsilon^{2p}I)^{-1}\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} &\leq \sup_{x\geq 0}(x + |\zeta|\varepsilon^{2p})|x - \zeta\varepsilon^{2p}|^{-1} \\
&= \sup_{y\geq 0}(y+1)|y - \widehat{\zeta}|^{-1} \leq 2c(\varphi).
\end{aligned} \tag{7.54}$$

Relations (7.53) and (7.54) imply (7.52). □

Proposition 7.12. *Under the assumptions of Theorem* 7.6, *let Condition* 7.9 *be satisfied. Then for $\varepsilon > 0$ we have*

$$\left\|\Lambda b(\mathbf{D})(I-\Pi)(A^0 - \zeta\varepsilon^{2p}I)^{-1}\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \mathcal{C}_5 c(\varphi), \tag{7.55}$$

$$\left\|A^{1/2}\big(\Lambda b(\mathbf{D})(I-\Pi)(A^0 - \zeta\varepsilon^{2p}I)^{-1}\big)\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \mathcal{C}_6 c(\varphi), \tag{7.56}$$

$$\left\|\widetilde{g} b(\mathbf{D})(I-\Pi)(A^0 - \zeta\varepsilon^{2p}I)^{-1}\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \mathcal{C}_7 c(\varphi). \tag{7.57}$$

The constants $\mathcal{C}_5$, $\mathcal{C}_6$, and $\mathcal{C}_7$ depend only on m, d, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, and also on M_Λ and $\|\Lambda\|_{L_\infty}$.

Proof. Estimate (7.55) with $\mathcal{C}_5 = \|\Lambda\|_{L_\infty}C_{11}$ follows from Condition 7.9 and estimate (7.52).

To prove (7.56), observe that

$$\begin{aligned}
&\left\|A^{1/2}\big(\Lambda b(\mathbf{D})(I-\Pi)(A^0 - \zeta\varepsilon^{2p}I)^{-1}\big)\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\
&\quad \leq \|g\|_{L_\infty}^{1/2}\alpha_1^{1/2}\left\|\Lambda b(\mathbf{D})(I-\Pi)(A^0 - \zeta\varepsilon^{2p}I)^{-1}\right\|_{L_2(\mathbb{R}^d)\to H^p(\mathbb{R}^d)}.
\end{aligned} \tag{7.58}$$

Combining Condition 7.9 and inequalities (7.52), (7.58), we obtain (7.56) with the constant $\mathcal{C}_6 = \alpha_1^{1/2}\|g\|_{L_\infty}^{1/2}M_\Lambda C_{11}$.

To prove (7.57), note that, by Lemma 1 in [MSh, Subsection 1.3.2], Condition 7.9 implies that $b(\mathbf{D})\Lambda$ is a multiplier from $H^p(\mathbb{R}^d;\mathbb{C}^m)$ to $L_2(\mathbb{R}^d;\mathbb{C}^m)$, and its multiplier norm is controlled in terms of α_1, $\|\Lambda\|_{L_\infty}$, and M_Λ:

$$\|[b(\mathbf{D})\Lambda]\|_{H^p(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \mathfrak{C}_\Lambda = \mathfrak{C}_\Lambda(\alpha_1, \|\Lambda\|_{L_\infty}, M_\Lambda).$$

Then the matrix-valued function $\widetilde{g} = g(b(\mathbf{D})\Lambda + \mathbf{1}_m)$ is a multiplier from $H^p(\mathbb{R}^d;\mathbb{C}^m)$ to $L_2(\mathbb{R}^d;\mathbb{C}^m)$, and $\|[\widetilde{g}]\|_{H^p(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \|g\|_{L_\infty}(\mathfrak{C}_\Lambda + 1)$. Together with (7.52) this implies (7.57) with $\mathcal{C}_7 = \|g\|_{L_\infty}(\mathfrak{C}_\Lambda + 1)C_{11}$. □

Now, Theorem 7.6 and Proposition 7.12 yield the following result.

Theorem 7.13. *Under the assumptions of Theorem* 7.6*, let Condition* 7.9 *be satisfied. Then for* $\varepsilon > 0$ *we have*

$$\begin{aligned}
&\big\|(A - \zeta\varepsilon^{2p}I)^{-1} - (I + \Lambda b(\mathbf{D}))(A^0 - \zeta\varepsilon^{2p}I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\
&\qquad\qquad \le \mathcal{C}_2 c(\varphi)^2 \varepsilon^{1-2p}|\zeta|^{1/2p-1} + \mathcal{C}_5 c(\varphi), \\
&\big\|A^{1/2}\left((A - \zeta\varepsilon^{2p}I)^{-1} - (I + \Lambda b(\mathbf{D}))(A^0 - \zeta\varepsilon^{2p}I)^{-1}\right)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\
&\qquad\qquad \le \mathcal{C}_3 c(\varphi)^2 \varepsilon^{1-p}|\zeta|^{1/2p-1/2} + \mathcal{C}_6 c(\varphi), \\
&\big\|g b(\mathbf{D})(A - \zeta\varepsilon^{2p}I)^{-1} - \widetilde{g} b(\mathbf{D})(A^0 - \zeta\varepsilon^{2p}I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\
&\qquad\qquad \le \mathcal{C}_4 c(\varphi)^2 \varepsilon^{1-p}|\zeta|^{1/2p-1/2} + \mathcal{C}_7 c(\varphi).
\end{aligned}$$

The constants $\mathcal{C}_2$*,* $\mathcal{C}_3$*, and* $\mathcal{C}_4$ *depend only on* m*,* d*,* p*,* α_0*,* α_1*,* $\|g\|_{L_\infty}$*,* $\|g^{-1}\|_{L_\infty}$*, and the parameters of the lattice* Γ*. The constants* $\mathcal{C}_5$*,* $\mathcal{C}_6$*, and* $\mathcal{C}_7$ *depend on the same parameters and also on* $\|\Lambda\|_{L_\infty}$ *and* M_Λ*.*

§8. Homogenization of the operator A_ε

8.1. Approximation of the resolvent of the operator A_ε in the operator norm in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. For any Γ-periodic function $\varphi(\mathbf{x})$ in $\mathbb{R}^d$ we denote

$$\varphi^\varepsilon(\mathbf{x}) := \varphi(\varepsilon^{-1}\mathbf{x}), \quad \varepsilon > 0.$$

In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, we consider the operator A_ε, $\varepsilon > 0$, given formally by the differential expression

$$A_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}), \quad \varepsilon > 0. \tag{8.1}$$

As usual, the precise definition of A_ε is given in terms of the corresponding closed quadratic form

$$a_\varepsilon[\mathbf{u},\mathbf{u}] = \int_{\mathbb{R}^d} \langle g^\varepsilon(\mathbf{x}) b(\mathbf{D})\mathbf{u}, b(\mathbf{D})\mathbf{u}\rangle\, d\mathbf{x}, \quad \mathbf{u} \in H^p(\mathbb{R}^d;\mathbb{C}^n).$$

The form a_ε is subject to the following estimates similar to (4.8):

$$\alpha_0 \|g^{-1}\|_{L_\infty}^{-1} \int_{\mathbb{R}^d} |\boldsymbol{\xi}|^{2p} |\widehat{\mathbf{u}}(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi} \le a_\varepsilon[\mathbf{u},\mathbf{u}] \le \alpha_1 \|g\|_{L_\infty} \int_{\mathbb{R}^d} |\boldsymbol{\xi}|^{2p} |\widehat{\mathbf{u}}(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi}. \tag{8.2}$$

For small ε the coefficients of the operator (8.1) oscillate rapidly. As applied to the operator (8.1), a typical homogenization problem is to approximate its resolvent for small ε. Using the results of §7 and the scaling transformation, we deduce theorems about approximation of the resolvent $(A_\varepsilon - \zeta I)^{-1}$ for $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$.

Let T_ε be the unitary scaling transformation in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ given by

$$(T_\varepsilon \mathbf{u})(\mathbf{x}) := \varepsilon^{d/2}\mathbf{u}(\varepsilon\mathbf{x}).$$

It is easily checked that

$$A_\varepsilon = \varepsilon^{-2p} T_\varepsilon^* A T_\varepsilon,$$

where A is the operator (4.1). Then

$$(A_\varepsilon - \zeta I)^{-1} = \varepsilon^{2p} T_\varepsilon^* (A - \zeta\varepsilon^{2p} I)^{-1} T_\varepsilon. \tag{8.3}$$

A similar identity is true for the operator A^0:

$$(A^0 - \zeta I)^{-1} = \varepsilon^{2p} T_\varepsilon^* (A^0 - \zeta\varepsilon^{2p} I)^{-1} T_\varepsilon. \tag{8.4}$$

Subtracting (8.4) from (8.3) and using the fact that the operator T_ε is unitary, we obtain

$$\begin{aligned} &\big\|(A_\varepsilon - \zeta I)^{-1} - (A^0 - \zeta I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\ &\qquad = \varepsilon^{2p}\big\|(A - \zeta\varepsilon^{2p} I)^{-1} - (A^0 - \zeta\varepsilon^{2p} I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}. \end{aligned} \tag{8.5}$$

Theorem 7.5 together with (8.5) imply the following result.

Theorem 8.1. *Let A_ε be the operator* (8.1), *and let A^0 be the effective operator* (5.33). *Let $\zeta = |\zeta| e^{i\varphi} \in \mathbb{C}\setminus\mathbb{R}_+$, and let $c(\varphi)$ be defined by* (7.22). *For $\varepsilon > 0$ we have*

$$\big\|(A_\varepsilon - \zeta I)^{-1} - (A^0 - \zeta I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \mathcal{C}_1 c(\varphi)^2 \varepsilon |\zeta|^{1/2p-1}. \tag{8.6}$$

The constant $\mathcal{C}_1$ depends only on d, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.

8.2. Approximation of the resolvent of the operator A_ε in the energy norm. Now we use Theorem 7.6 to approximate the resolvent $(A_\varepsilon - \zeta I)^{-1}$ in the norm of operators acting from $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to the Sobolev space $H^p(\mathbb{R}^d;\mathbb{C}^n)$, and also approximate the operator $g^\varepsilon b(\mathbf{D})(A_\varepsilon - \zeta I)^{-1}$ (corresponding to the "flux") in the norm of operators acting from $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to $L_2(\mathbb{R}^d;\mathbb{C}^m)$.

Let Π_ε be the pseudodifferential operator in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ with the symbol $\chi_{\widetilde\Omega/\varepsilon}(\boldsymbol{\xi})$, i.e.,

$$(\Pi_\varepsilon \mathbf{u})(\mathbf{x}) = (2\pi)^{-d/2} \int_{\widetilde\Omega/\varepsilon} e^{i\langle \mathbf{x},\boldsymbol{\xi}\rangle} \widehat{\mathbf{u}}(\boldsymbol{\xi})\, d\boldsymbol{\xi}. \tag{8.7}$$

The operators (7.3) and (8.7) satisfy the identity

$$\Pi_\varepsilon = T_\varepsilon^* \Pi T_\varepsilon. \tag{8.8}$$

We put

$$K(\zeta;\varepsilon) := \Lambda^\varepsilon b(\mathbf{D})(A^0 - \zeta I)^{-1}\Pi_\varepsilon, \tag{8.9}$$

which is called a *corrector*; it is a continuous mapping of $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to $H^p(\mathbb{R}^d;\mathbb{C}^n)$.

Theorem 8.2. *Under the assumptions of Theorem* 8.1, *let Π_ε be the operator* (8.8), *let $K(\zeta;\varepsilon)$ be given by* (8.9), *and let $\widetilde g(\mathbf{x})$ be the matrix-valued function* (5.30). *Then for $\varepsilon > 0$ we have*

$$\begin{aligned} &\big\|(A_\varepsilon - \zeta I)^{-1} - (A^0 - \zeta I)^{-1} - \varepsilon^p K(\zeta;\varepsilon)\big\|_{L_2(\mathbb{R}^d)\to H^p(\mathbb{R}^d)} \\ &\qquad\qquad \le \varepsilon c(\varphi)^2 |\zeta|^{1/2p}(\mathcal{C}'|\zeta|^{-1} + \mathcal{C}''|\zeta|^{-1/2}), \end{aligned} \tag{8.10}$$

$$\begin{aligned} &\big\|g^\varepsilon b(\mathbf{D})(A_\varepsilon - \zeta I)^{-1} - \widetilde g^\varepsilon b(\mathbf{D})(A^0 - \zeta I)^{-1}\Pi_\varepsilon\big\|_{L_2(\mathbb{R}^d)\to H^p(\mathbb{R}^d)} \\ &\qquad\qquad \le \varepsilon c(\varphi)^2 \mathcal{C}_4 |\zeta|^{1/2p-1/2}. \end{aligned} \tag{8.11}$$

The constants $\mathcal{C}'$, $\mathcal{C}''$, and $\mathcal{C}_4$ depend only on m, d, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.

Proof. As in (8.3), by (8.8) we have

$$K(\zeta;\varepsilon) = \varepsilon^p T_\varepsilon^* \Lambda b(\mathbf{D})(A^0 - \zeta\varepsilon^{2p} I)^{-1}\Pi T_\varepsilon. \tag{8.12}$$

From (8.3), (8.4), and (8.12) it follows that

$$\begin{aligned}(8.13)\quad &\big\|(A_\varepsilon-\zeta I)^{-1}-(A^0-\zeta I)^{-1}-\varepsilon^p K(\zeta;\varepsilon)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\\ &\quad=\varepsilon^{2p}\big\|(A-\zeta\varepsilon^{2p}I)^{-1}-(I+\Lambda b(\mathbf{D})\Pi)(A^0-\zeta\varepsilon^{2p}I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}.\end{aligned}$$

Using (7.28) and (8.13), for $\varepsilon>0$ we get

$$(8.14)\quad \big\|(A_\varepsilon-\zeta I)^{-1}-(A^0-\zeta I)^{-1}-\varepsilon^p K(\zeta;\varepsilon)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\le \mathcal{C}_2 c(\varphi)^2\varepsilon|\zeta|^{1/2p-1}.$$

Similarly,

$$\begin{aligned}&\big\|A_\varepsilon^{1/2}\big((A_\varepsilon-\zeta I)^{-1}-(A^0-\zeta I)^{-1}-\varepsilon^p K(\zeta;\varepsilon)\big)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\\ &\quad=\varepsilon^{p}\big\|A^{1/2}\big((A-\zeta\varepsilon^{2p}I)^{-1}-(I+\Lambda b(\mathbf{D})\Pi)(A^0-\zeta\varepsilon^{2p}I)^{-1}\big)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}.\end{aligned}$$

Together with (7.29) this yields

$$(8.15)\quad \big\|A_\varepsilon^{1/2}\big((A_\varepsilon-\zeta I)^{-1}-(A^0-\zeta I)^{-1}-\varepsilon^p K(\zeta;\varepsilon)\big)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\le \mathcal{C}_3 c(\varphi)^2\varepsilon|\zeta|^{1/2p-1/2}.$$

Since $(1+|\boldsymbol{\xi}|^2)^p\le 2^{p-1}(1+|\boldsymbol{\xi}|^{2p})$, taking the lower estimate (8.2) into account, for any $\mathbf{u}\in H^p(\mathbb{R}^d;\mathbb{C}^n)$ we have

$$\begin{aligned}\|\mathbf{u}\|^2_{H^p(\mathbb{R}^d)}&=\int_{\mathbb{R}^d}(1+|\boldsymbol{\xi}|^2)^p|\widehat{\mathbf{u}}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi}\le 2^{p-1}\int_{\mathbb{R}^d}(1+|\boldsymbol{\xi}|^{2p})|\widehat{\mathbf{u}}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi}\\ &\le 2^{p-1}\big(\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}+\alpha_0^{-1}\|g^{-1}\|_{L_\infty}\|A_\varepsilon^{1/2}\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}\big).\end{aligned}$$

Combining this with (8.14) and (8.15), we deduce (8.10) with the constants

$$\mathcal{C}'=2^{(p-1)/2}\mathcal{C}_2,\quad \mathcal{C}''=2^{(p-1)/2}\mathcal{C}_3\alpha_0^{-1/2}\|g^{-1}\|^{1/2}_{L_\infty}.$$

Inequality (8.11) follows from (7.30) with the help of the scaling transformation. $\square$

Remark 8.3. 1) For a fixed $\zeta\in\mathbb{C}\setminus\mathbb{R}_+$, estimates of Theorems 8.1 and 8.2 are of the sharp order $O(\varepsilon)$. For large $|\zeta|$ the order improves, due to the presence of the factors $|\zeta|^{-s}$ (with $s>0$) on the right-hand sides. 2) Estimates (8.6), (8.10), and (8.11) are uniform with respect to φ in any sector of the form $\{\zeta=|\zeta|e^{i\varphi}\in\mathbb{C}\,:\,\varphi_0\le\varphi\le 2\pi-\varphi_0\}$ with arbitrarily small φ_0.

8.3. Special cases. If $g^0=\overline{g}$, then $\Lambda=0$ and the corrector (8.9) is equal to zero. In this case (8.10) simplifies.

Proposition 8.4. *Under the assumptions of Theorem* 8.1, *suppose that* $g^0=\overline{g}$ (*i.e., conditions* (5.41) *are satisfied*). *Then for* $\varepsilon>0$ *we have*

$$\big\|(A_\varepsilon-\zeta I)^{-1}-(A^0-\zeta I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to H^p(\mathbb{R}^d)}\le \varepsilon c(\varphi)^2|\zeta|^{1/2p}\big(\mathcal{C}'|\zeta|^{-1}+\mathcal{C}''|\zeta|^{-1/2}\big).$$

The following statement is deduced from Proposition 7.8 by the scaling transformation.

Proposition 8.5. *Under the assumptions of Theorem* 8.1, *if* $g^0=\underline{g}$ (*i.e., the representations* (5.43) *are valid*), *then for* $\varepsilon>0$ *we have*

$$\big\|g^\varepsilon b(\mathbf{D})(A-\zeta I)^{-1}-g^0 b(\mathbf{D})(A^0-\zeta I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\le \mathcal{C}_4^\circ c(\varphi)^2\varepsilon|\zeta|^{1/2p-1/2}.$$

8.4. Removal of the smoothing operator. Now, we suppose that Condition 7.9 is satisfied. Then instead of the corrector (8.9) one can use the operator

$$K^0(\zeta;\varepsilon) := \Lambda^\varepsilon b(\mathbf{D})(A^0 - \zeta I)^{-1}, \tag{8.16}$$

which in this case is a continuous mapping of $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to $H^p(\mathbb{R}^d;\mathbb{C}^n)$. Note that (8.16) is the traditional corrector used in the homogenization theory.

The following result is deduced from Theorem 7.13 by the scaling transformation (cf. the proof of Theorem 8.2).

Theorem 8.6. *Under the assumptions of Theorem* 8.1*, let Condition* 7.9 *be satisfied. Let* $K^0(\zeta;\varepsilon)$ *be given by* (8.16)*, and let* $\widetilde{g}(\mathbf{x})$ *be the matrix-valued function* (5.30)*. Then for* $0 < \varepsilon \le 1$ *we have*

$$\begin{aligned}&\big\|(A_\varepsilon - \zeta I)^{-1} - (A^0 - \zeta I)^{-1} - \varepsilon^p K^0(\zeta;\varepsilon)\big\|_{L_2(\mathbb{R}^d)\to H^p(\mathbb{R}^d)}\\ &\qquad\le \varepsilon c(\varphi)^2|\zeta|^{1/2p}\big(\mathcal{C}'|\zeta|^{-1} + \mathcal{C}''|\zeta|^{-1/2}\big) + \mathcal{C}_8\varepsilon^p c(\varphi),\end{aligned} \tag{8.17}$$

$$\begin{aligned}&\big\|g^\varepsilon b(\mathbf{D})(A - \zeta I)^{-1} - \widetilde{g}^\varepsilon b(\mathbf{D})(A^0 - \zeta I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\\ &\qquad\le \varepsilon\mathcal{C}_4 c(\varphi)^2|\zeta|^{1/2p-1/2} + \mathcal{C}_7\varepsilon^p c(\varphi).\end{aligned} \tag{8.18}$$

The constants $\mathcal{C}'$, $\mathcal{C}''$, *and* $\mathcal{C}_4$ *depend only on* m, d, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *and the parameters of the lattice* Γ. *The constants* $\mathcal{C}_7$ *and* $\mathcal{C}_8$ *depend on the same parameters and also on* $\|\Lambda\|_{L_\infty}$ *and* M_Λ.

Note that in Theorem 8.6 we assume that $0 < \varepsilon \le 1$, because in the proof of (8.17) the inequality $\varepsilon^{2p} \le \varepsilon^p$ was used. Moreover, estimates (8.17) and (8.18) are of interest for small ε. The constant $\mathcal{C}_8$ is given by $\mathcal{C}_8 = 2^{(p-1)/2}\big(\mathcal{C}_5 + \alpha_0^{-1/2}\|g^{-1}\|_{L_\infty}^{1/2}\mathcal{C}_6\big)$.

Comparing Proposition 7.10 and Theorem 8.6, we arrive at the following statement.

Corollary 8.7. *Under the assumptions of Theorem* 8.1*, let* $K^0(\zeta;\varepsilon)$ *be the operator* (8.16)*, and let* $\widetilde{g}(\mathbf{x})$ *be the matrix-valued function* (5.30)*. Moreover, suppose that at least one of the following two assumptions is fulfilled*:

1°. $2p > d$;

2°. $g^0 = \underline{g}$, *i.e., the representations* (5.43) *are valid.*

Then estimates (8.17) *and* (8.18) *are true for* $0 < \varepsilon \le 1$. *All the constants in these estimates depend only on* m, n, d, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *and the parameters of the lattice* Γ.

Remark 8.8. 1) For fixed $\zeta \in \mathbb{C}\backslash\mathbb{R}_+$, estimates of Theorem 8.6 are of the sharp order $O(\varepsilon)$.

2) Estimates (8.17) and (8.18) are uniform with respect to φ in any sector of the form $\{\zeta = |\zeta|e^{i\varphi} \in \mathbb{C} : \varphi_0 \le \varphi \le 2\pi - \varphi_0\}$ with arbitrarily small φ_0.

3) The assumptions of Corollary 8.7 are satisfied in the following cases, which are interesting for applications: a) if $p = 2$ and $d = 2$ or $d = 3$, then $2p > d$; b) if $m = n$, then $g^0 = \underline{g}$. For instance, this is the case for the operator $A_\varepsilon = \Delta g^\varepsilon(\mathbf{x})\Delta$.

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