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DECOMPOSITION OF TRANSVECTIONS: AN ALGEBRO-GEOMETRIC APPROACH

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Abstract. A simple and uniform algebro-geometric proof is given for the decomposition of transvections for Chevalley groups in minuscule representations.

§1. Introduction

The decomposition of transvections technique proposed by Stepanov and Vavilov in [3] is successfully used to prove the normality of the elementary subgroups in Chevalley groups and the structure theorem on subgroups normalized by an elementary subgroup, to estimate the width of Chevalley groups in various generators, to describe automorphisms, and so on. Various versions of this method were proposed in [8, 9, 10, 11, 6].

While being simple for classical groups, in the case of exceptional groups this method relies on the surprising (at the first sight) fact that some cleverly chosen combination of elementary transvections changes a column only up to multiples of some equations on the highest weight orbit. This combination arises *ad hoc*, and the fact itself is checked by tedious computations that appeal to the signs of the action of elementary transvections and the explicit equations on the orbit. We cite [9] on this account:

"A major technical point is the necessity to make the signs in these products consistent with the signs of action constants and especially with the signs of coefficients in the equations describing the highest weight orbit. Another serious complication is that now the number of free parameters in the unipotents is much less than that of the affected coordinates of the vector. This shows that there is no way of stabilizing an arbitrary column; the only reason why the extra summands may possibly cancel lies in the equations satisfied by a column of a matrix $g \in G(E_l, R)$. The fact that such a proof does work is a sort of a miracle rather than a natural phenomenon."

In the present paper we propose a new proof of the decomposition of transvections in the minuscule case, which relies only on a simple algebro-geometric consideration ("Chernousov–Gille–Merkurjev filtration") and employs neither the signs of the action of elementary transvections nor the explicit equations on the orbit. Moreover, from the proof one can extract equations on the highest weight orbit. In a forthcoming paper the author will apply similar considerations to prove the decomposition of transvections in the adjoint case (which was announced for E_6 in [9], but, as far as the author knows, has never been published in detail).

§2. Definitions and examples

Throughout the paper we assume that a root system Φ is simply laced. The general facts on Chevalley groups and Weyl representations can be found, e.g., in [5] or [7].

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Definition 1. We say that the Chevalley group $G(\Phi, R)$ in a Weyl representation V *admits decomposition of transvections* if there exists a set of roots $A \subset \Phi^-$ such that the one-parameter subgroups $t_\alpha(*)$ mutually commute, and a set of linear forms $\phi_\alpha \colon V \to R$, $\alpha \in A$, such that the closed subvariety in V given by the equation

$$\prod_{\alpha \in A} t_\alpha(\phi_\alpha(v))v = v \tag{2.1}$$

contains the affine cones of the orbits $G(\Phi, R)\langle v^\lambda \rangle$ for an admissible basis $\{v^\lambda\}$ of V, and, moreover, there exist $\beta \in A$ and $v_0 \in V$ such that

$$\phi_\alpha(v_0) = \delta_{\alpha\beta} = \begin{cases} 1, & \alpha = \beta, \\ 0, & \alpha \neq \beta. \end{cases} \tag{2.2}$$

Usually, decomposition of transvections is employed as follows. First, observe that on the affine cones as in Definition 1 a more general identity is valid:

$$\prod_{\alpha \in A} t_\alpha(\phi_\alpha(v\xi))v = v \tag{2.3}$$

for any $\xi \in R$. Indeed, it suffices to replace v by vX in (2.1) for an indeterminate X, cancel X, and put $X = \xi$.

Now, let g be an element in $G(\Phi, R)$, $\xi \in R$. Denote by x_λ the coefficients of gv_0 with respect to the basis $\{v^\lambda\}$. Then

$$\sum_\lambda \phi_\alpha(g^{-1}v^\lambda)x_\lambda = \phi_\alpha\Big(g^{-1}\Big(\sum_\lambda v^\lambda x_\lambda\Big)\Big) = \delta_{\alpha\beta},$$

whence

$$gt_\beta(\xi)g^{-1} = \prod_\lambda g\Big(\prod_{\alpha \in A} t_\alpha(\phi_\alpha(g^{-1}v^\lambda)x_\lambda\xi)\Big)g^{-1}. \tag{2.4}$$

By (2.3), every factor stabilizes the basis vector v^λ.

Thus, every transvection can be decomposed into a product of factors lying in some fixed proper subgroups (maximal parabolic subgroups in the minuscule case). This immediately implies (in the case of a simple group) the normality of the elementary subgroup; with little more effort one can obtain a description of subgroups normalized by the elementary subgroup (see [12]), and so on.

We give a simple example.

Lemma 1. *The group* $\mathrm{SL}_n(R)$ *in the natural representation admits decomposition of transvections whenever* $n \geq 3$.

Proof. In this case $\Phi = A_{n-1}$; we enumerate the admissible basis as $e_1, \dots, e_n$, where e_1 has the weight ϖ_1, e_n has the weight $-\varpi_{n-1}$, and the weight of e_i equals $\varpi_i - \varpi_{i-1}$ for $2 \leq i \leq n-1$. We denote the basis dual to $\{e_i\}$ by $\{\varepsilon_i\}$.

Take $A = \{-\alpha_2, -\alpha_1 - \alpha_2\}$, $\phi_{-\alpha_2} = \varepsilon_1$, $\phi_{-\alpha_1-\alpha_2} = -\varepsilon_2$, $\beta = -\alpha_2$, and $v_0 = e_1$. Obviously, condition (2.2) is satisfied. We check that (2.1) holds true for any v:

$$t_{32}(\varepsilon_1(v))t_{31}(-\varepsilon_2(v))v = t_{32}(\varepsilon_1(v))(v - \varepsilon_1(v)\varepsilon_2(v)e_3) = v. \qquad \square$$

The case of the even orthogonal group is only slightly more complicated.

Lemma 2. *The group* $\mathrm{O}^+_{2n}(R)$ *in the natural representation admits decomposition of transvections whenever* $n \geq 3$.

Proof. In this case $\Phi = D_n$; we enumerate the admissible basis as $e_1, \dots, e_n, e_{-n}, \dots, e_{-1}$, where e_1 has the weight ϖ_1, e_{n-2} has the weight $\varpi_{n-1} + \varpi_n - \varpi_{n-2}$, the weight of e_i equals $\varpi_i - \varpi_{i-1}$ when $2 \le i \le n$, $i \ne n-2$, and the sum of the weights of e_i and e_{-i} equals zero. We denote the basis dual to $\{e_i\}$ by $\{\varepsilon_i\}$.

Take $A = \{-\alpha_2, -\alpha_1 - \alpha_2, -\widetilde{\alpha}\}$, where

$$\widetilde{\alpha} = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$$

stands for the maximal root. Set $\phi_{-\alpha_2} = \varepsilon_1$, $\phi_{-\alpha_1-\alpha_2} = -\varepsilon_2$, $\phi_{-\widetilde{\alpha}} = \varepsilon_{-3}$, $\beta = -\alpha_2$, $v_0 = e_1$. Obviously, condition (2.2) is satisfied. We check that (2.1) holds true for any v:

$$\begin{aligned} &T_{32}(\varepsilon_1(v))T_{31}(-\varepsilon_2(v))T_{-21}(\varepsilon_{-3}(v))v \\ &\quad = T_{32}(\varepsilon_1(v))T_{31}(-\varepsilon_2(v))(v + \varepsilon_{-3}(v)\varepsilon_1(v)e_{-2} - \varepsilon_{-3}(v)\varepsilon_2(v)e_{-1}) \\ &\quad = T_{32}(\varepsilon_1(v))(v - \varepsilon_2(v)\varepsilon_1(v)e_3 + \varepsilon_{-3}(v)\varepsilon_1(v)e_{-2}) = v. \qquad \square \end{aligned}$$

In the examples above equation (2.1) was satisfied by any vector v. In the case of groups of type E_l the condition becomes nontrivial, and it is necessary to check that the corresponding subvariety contains the orbits of the weight vectors.

The following useful fact reduces the case of semisimple Chevalley groups to its factors.

Lemma 3. *Let G and H be Chevalley groups, and let V and W be their representations. Assume that G in the representation V admits decomposition of transvections. Then $G \times H$ in the representation $V \otimes W$ also admits decomposition of transvections.*

Proof. Let G be of type Φ and H of type Ψ, then $G \times H$ is of type $\Phi + \Psi$. Take $A \subset \Phi^-$, ϕ_α, and v_0 as in Definition 1. We fix an arbitrary $w_0 \in W$ and a linear form $\theta\colon W \to R$ such that $\theta(w_0) = 1$. Set $\widetilde{A}$ to be equal to A viewed as a subset in $(\Phi + \Psi)^-$, and $\widetilde{\phi}_\alpha = \phi_\alpha \otimes \theta$, $\widetilde{v}_0 = v_0 \otimes w_0$.

An element in the cone of the orbit $(G \times H)(v^\lambda \otimes w^\mu)$ has the form $v \otimes w$, where v lies in the cone of the orbit Gv^λ. By (2.3) we have

$$\prod_{\alpha \in \widetilde{A}} t_\alpha(\widetilde{\phi}_\alpha(v \otimes w))(v \otimes w) = \prod_{\alpha \in \widetilde{A}} t_\alpha(\phi_\alpha(v)\theta(w))v \otimes w = v \otimes w,$$

so that the conditions of Definition 1 are satisfied by $G \times H$. $\square$

§3. Chernousov–Gille–Merkurjev filtration

As above, consider a Chevalley group G in a representation V with the highest weight vector e. Then the stabilizer of the line $\langle e \rangle$ is a parabolic subgroup P in G. There exists a cocharacter $\chi\colon \mathbb{G}_m \to G$ such that

$$P = P(\chi) = \{g \in G \mid \lim_{t \to \infty} \chi(t) g \chi(t)^{-1} \text{ exists}\}$$

(cf. [1, § 6]), and the opposite cocharacter χ^{-1} determines a filtration on $\mathbb{P}(V)$ and the induced filtration on $G/P = G\langle e \rangle$, see [1, Theorem 3.2]. This filtration was introduced in [2]; we call it the *Chernousov–Gille–Merkurjev filtration*.

Let us describe this filtration in more detail. The cocharacter χ^{-1} determines a decomposition of V into weight submodules that are invariant under the action of the Levi subgroup L in P. The highest weight submodule coincides with $\langle e \rangle$; we denote the next submodule by U and the direct complement to these two submodules by W:

$$V = \langle e \rangle \oplus U \oplus W. \tag{3.1}$$

As a representation of L, U can be identified with $U^-/[U^-, U^-]$, where U^- stands for the unipotent radical of the opposite parabolic subgroup P^-. In particular, in the cominuscule case U can be identified with U^-.

Denote the homogeneous coordinates on $\mathbb{P}(V)$ corresponding to the decomposition (3.1) by $[a : u : w]$. Then the Chernousov–Gille–Merkurjev filtration starts with

$$G/P \supset G/P \cap \{a = 0\} \supset G/P \cap \{a = 0, u = 0\},$$

where the difference of the first two terms is the big cell of G/P isomorphic to U^- (hence, to an affine space). The difference of the next two terms is taken to $\mathbb{P}(U)$ by the projection

$$[0 : u : w] \mapsto u,$$

and this map is an affine bundle over a projective homogeneous variety L/Q embedded into $\mathbb{P}(U)$, where Q is a parabolic subgroup in L.

Note that in the big cell (when $a \neq 0$) the entry w is uniquely determined by the entries a and u in the cominuscule case, because U parametrizes U^- in this case.

Lemma 4. *Under the projection map from V to U, the affine cone over G/P goes to the affine cone over L/Q.*

Proof. The claim follows from the description above and the definition of the affine cone as the preimage under the map $V \setminus \{0\} \to \mathbb{P}(V)$ together with the origin. □

§4. Decomposition of transvections in the minuscule case

Now we state and prove the main result.

Theorem 1. *Let G be a Chevalley group corresponding to a simply laced root system Φ such that not all the irreducible components of Φ are of type A_1, and let V be a minuscule Weyl representation of G. Then G admits decomposition of transvections in V.*

Proof. The proof goes by induction on the dimension of G. By Lemma 3, we may assume that Φ is irreducible and is not of type A_1. However, we shall use the inductive hypothesis also for reducible root systems.

Consider the decomposition (3.1). Since Φ is simply laced, V is also cominuscule, hence U can be identified with the unipotent radical U^-. Furthermore, U is a minuscule representation of the derived subgroup $[L, L]$. Denote the root system of $[L, L]$ by Ψ. In all cases except where $G = A_2$, $V = V(\varpi_1)$, and $G = D_3$, $V = V(\varpi_1)$ (which were settled in Lemma 1 and Lemma 2, respectively) not all components of Ψ are of type A_1, so we may apply the inductive hypothesis to $[L, L]$ in the representation U. This means that there is a collection $B \subset \Psi^-$, a collection of linear forms $\psi_\alpha \colon U \to R$, an element $\beta \in B$, and $v_0 \in U$ such that the conditions Definition 1 are fulfilled. Set ϕ_α to be equal to ψ_α composed with the projection from V onto U.

We take a vector v from the affine cone of an orbit $G\langle v^\lambda\rangle$. Since the representation is minuscule, this vector lies in the affine cone of the orbit $G\langle e\rangle = G/P$. Denote the coefficients of v with respect to the decomposition (3.1) by a, u, and w, that is

$$v = ae + u + w, \quad u \in U, \;\; w \in W.$$

Pass to the ring $R/(a)$; then, by Lemma 4, in this ring u lies in the affine cone of the orbit L/Q. Since the derived subgroup $[L, L]$ acts trivially on $\langle e\rangle$, over R we have

$$\prod_{\alpha \in B} t_\alpha(\phi_\alpha(v))v = ae + u + au' + w' \tag{4.1}$$

for some $u' \in U$ and $w' \in W$. Moreover, the coefficients on the left-hand side depend quadratically on v, so that u' depends linearly on v. Taking into account the grading given by the action of the torus via the cocharacter χ, we see that u' depends only on w. Identifying U and U^-, we may write

$$e - u' - w'' = \prod_{\gamma \in C} t_\gamma(\phi_\gamma(v))e \tag{4.2}$$

for some $C \subset \Phi^-$, linear forms ϕ_γ vanishing on U, and $w'' \in W$.

Since U^- is Abelian, the t_γ's with $\gamma \in C$ mutually commute; we show that they also commute with the t_α's, $\alpha \in B$. By the formula expressing the action of a root element in a minuscule representation (see [4] or [7]), every t_α adds some coefficients of weight vectors v^λ, $\lambda \in \Lambda$, to the coefficients of weight vectors v^μ, $\mu \in M$, for some sets of weights Λ and M; since the t_α commute, Λ and M are disjoint. Hence, u' has nonzero coefficients only at v^μ with $\mu \in M$, while the t_γ add the coefficient of e to the coefficients of v^μ with $\mu \in M$ and so commute with the t_α.

Now we set $A = B \cup C$ (here Ψ is viewed as a subset of Φ), and let β and v_0 be as above (here U is viewed as a submodule in V). The linear forms ϕ_α with $\alpha \in A$ are already defined. Since the ϕ_γ's with $\gamma \in C$ vanish on U, condition (2.2) is fulfilled, and it remains to check condition (2.1). It suffices to check it over the ring of regular functions on the affine cone over G/P over $\mathbb{Z}$, so we may assume that the base ring R is a domain.

From (4.1) and (4.2) it follows that

$$\prod_{\alpha \in A} t_\alpha(\phi_\alpha(v))v = ae + u + w'''$$

for some $w''' \in W$. In the localization $R[a^{-1}]$ the vector v belongs to the big cell; therefore, as was noted in §3, we must have $w''' = w$. Since R is a domain, the same identity holds true in R. □

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