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ON THE COORDINATE FUNCTIONS OF PEANO CURVES

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Abstract. A construction of "nonsymmetric" plane Peano curves is described whose coordinate functions satisfy the Lipschitz conditions of orders α and $1-\alpha$ for some α. It is proved that these curves are metric isomorphisms between the interval $[0,1]$ and the square $[0,1]^2$. This fact is used to show that the graphs of their coordinate functions have the maximum possible Hausdorff dimension for a given smoothness.

Introduction

This note is devoted to the properties of the coordinate functions of Peano curves and their graphs. By a Peano curve we mean a continuous map from an interval onto a multidimensional cube (see [1, 2, 3]). We shall show that the Hausdorff dimension of the graph of one of the coordinate functions of a "nonsymmetric" plane Peano curve may be arbitrarily close to two (or even equal to two), while the dimension of the graph of the other coordinate function is arbitrarily close (or equal) to one.

For the first time, examples of continuous functions whose graphs have Hausdorff dimension greater than one were constructed in [4]. In that paper, it was shown that the Hausdorff dimension of the graph of a function satisfying some Lipschitz condition can be arbitrarily close to two, and exact bounds were found for the interval where this dimension can lie. These examples show, in particular, that the Hausdorff measures μ_p for $p>1$ do not satisfy Cavalieri's principle, which says that the areas of two regions in a plane coincide provided that every vertical line intersects them in congruent intervals. Indeed, every vertical cross-section of the graph of a function, as well as that of an interval, reduces to a single point. However, as can be seen from the examples in [4], for $p\in(1,2)$ the measure μ_p of the graph can be positive, while it is easily seen that this measure vanishes for the interval.

In [5] it was established, in particular, that the Hausdorff dimension of graphs can take any value in the interval $(1,2)$ for functions arising in a different way compared to [4], namely, for the coordinate functions of some Peano curves. Like in [4], where examples of appropriate functions were constructed with the help of series with highly oscillating terms (see also [6]), and in [5], the derivation of the desired result required considerable effort.

The proof we present below does not involve any subtle analytic tools and is based on the study of the "occupation time" spent by the graph of a coordinate function in a given range and on the fact that the Peano curves under consideration are metric isomorphisms between the interval $[0,1]$ and the unit square regarded as spaces endowed with Lebesgue measures. Unexpectedly, it turned out that it is not easy to provide a reference to the latter result, which undoubtedly belongs to "mathematical folklore". We could not find any sources apart from Steinhaus's paper [7] published in 1936, where this fact (for the curve constructed by Peano) was mentioned without any indication of

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a proof, and the paper [8], where it was established (for the same curve) in quite an indirect way; therefore, we present a proof below (see Theorem 2).

Recall that a function f defined on an interval Δ satisfies a Lipschitz condition of order α, with $0 < \alpha \le 1$, if

$$|f(s) - f(t)| \le C|s-t|^\alpha \quad \text{for all} \quad s, t \in \Delta$$

for some $C > 0$ (called the Lipschitz constant). Clearly, one can regard α as a measure of smoothness for f. For uniformity, we shall say that a continuous function satisfies a Lipschitz condition of order 0.

§1. On the possible smoothness of the coordinate functions of Peano curves

In what follows, by a path we mean, as usual, a continuous map from an interval into a Euclidean space, and the support of a path is the image of that interval. In particular, a d-dimensional Peano curve is a path whose support is a cube in $\mathbb{R}^d$. Clearly, the coordinate functions of a Peano curve must be sufficiently "bad." The following theorem is an immediate consequence of a result obtained in [9].

Theorem 1. *Let $\gamma\colon [0,1] \to [0,1]^d$ be a path whose coordinate functions satisfy Lipschitz conditions of orders $\alpha_1, \dots, \alpha_d \in [0,1]$. If at least one of the following conditions is satisfied:*

1) $\alpha_1 + \dots + \alpha_d > 1$,

2) *one of the coordinate functions has finite variation,*

then the support of γ has zero volume (d*-dimensional Lebesgue measure*) *and, consequently, γ is not a Peano curve.*

In particular, from the first claim it follows that for every Peano curve, the sum $\alpha_1 + \dots + \alpha_d$ is at most one.

A another result of this type is as follows.

Proposition. *Let $\gamma\colon [0,1] \to [0,1]^d$ be a path whose coordinate functions satisfy Lipschitz conditions of orders $\alpha_1, \dots, \alpha_d \in [0,1]$. If $\sum_{j \ne k} \alpha_j = 1$ for some k, then the support of γ has zero volume and, consequently, γ is not a Peano curve.*

In particular, it follows that the classical plane Peano curve, the two coordinate functions of which satisfy Lipschitz conditions of order $\frac{1}{2}$, is "too good" to be the projection of some three-dimensional Peano curve (while, at the same time, like any plane path, it is the projection of a Jordan curve in $\mathbb{R}^3$).

Proof. Without loss of generality, we assume that $\alpha_1 + \dots + \alpha_{d-1} = 1$. We prove that the support of γ can be embedded into a polyhedron of arbitrarily small volume. To see this, divide the interval $[0,1]$ into N congruent parts $\delta_1, \dots, \delta_N$. Due to the Lipschitz condition, for $j < d$ the variation of the jth coordinate function at every interval δ_i (of length $\frac{1}{N}$) is $O(N^{-\alpha_j})$. At the same time, the variation of the last coordinate function is at most $\omega\big(\frac{1}{N}\big)$, where ω is the modulus of continuity of this function. Thus, the set $\gamma(\delta_i)$ is contained in a rectangular parallelepiped whose volume does not exceed

$$\text{const}\, N^{-\alpha_1} \cdots N^{-\alpha_{d-1}} \omega\Big(\frac{1}{N}\Big) = \frac{\text{const}}{N} \omega\Big(\frac{1}{N}\Big).$$

The union of all such parallelepipeds (there are N of them) contains the support of the path, and its volume is at most $\text{const}\, \omega\big(\frac{1}{N}\big) \xrightarrow[N \to +\infty]{} 0$. $\square$

§2. Examples of Peano curves and the smoothness of their coordinate functions

We describe a construction of a Peano curve that goes back to [1] (see also [2] and [3]). Fix two positive integers $m > 1$ and $n > 1$. To simplify the construction, in what follows we always assume that these integers are odd.

Consider the unit square $[0,1]^2$ and divide its horizontal and vertical sides into m and n equal parts, respectively. Thus, the entire square is divided into $N = mn$ congruent rectangles, which will be called rectangles[1] of rank 1. Simultaneously, we divide the unit interval into $N = mn$ intervals of equal length, which will be called segments of rank 1 and will be numbered from left to right by the integers from 0 to $N-1$ (i.e., the "digits" of the base N number system): $\delta_0, \dots, \delta_{N-1}$. Thus, the number of an interval coincides with the first digit of the base N representation of every interior point of this interval. We establish a correspondence between the segments and the rectangles of rank 1 in such a way that neighboring intervals correspond to rectangles that share a side. This can be done, for example, as shown in Figure 1, where (for $m = 5$, $n = 3$) the order of visiting the rectangles of rank 1 is indicated by arrows. We label rectangles of rank 1 by the same numbers as the corresponding segments.[2]

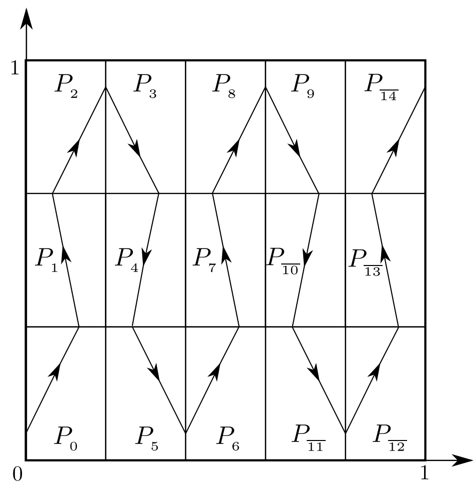

Figure 1

Then we repeat this procedure: as at the first step, we divide the horizontal and vertical side of each rectangle of rank 1 into m and n parts of equal length, respectively, obtaining N rectangles of rank 2 in each such rectangle. Simultaneously, we divide each segment δ_a $(0 \le a < N)$ of rank 1 into N congruent segments, which will be called segments of rank 2. Again, we number these segments from left to right by the integers $0, 1, \dots, N-1$. Thus, segments of rank 2 are labeled by two numbers, δ_{ab}, the first of which coincides with the number of the ambient segment of rank 1. These numbers are the first two digits of the N base representation of interior points of the segment under consideration.

With segments of rank 2 contained in a segment δ_a of rank 1, we associate rectangles of rank 2 contained in the rectangle P_a of rank 1 corresponding to δ_a. In doing so, we

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[1] We consider only closed rectangles and intervals.

[2] Double-digit numbers of rectangles are overlined to emphasize that they form a single index — a "digit" of the N base representation.

take into account the direction of the arrows shown in Figure 1 in rectangles of rank 1. Namely, if P_a contains the arrow $\nearrow$, then its parts are numbered in the same way as at the first step; if P_a contains the arrow $\nwarrow$, then its numbering corresponds to the symmetric reflection of Figure 1 about the vertical axis; finally, if the arrow in P_a is directed downward, then one should use the reflection of one of the first two diagrams about the horizontal axis. Figure 2 shows the numbering of the rectangles of rank 2 contained in P_0 (Figure 2, a) and in P_1 (Figure 2, b).

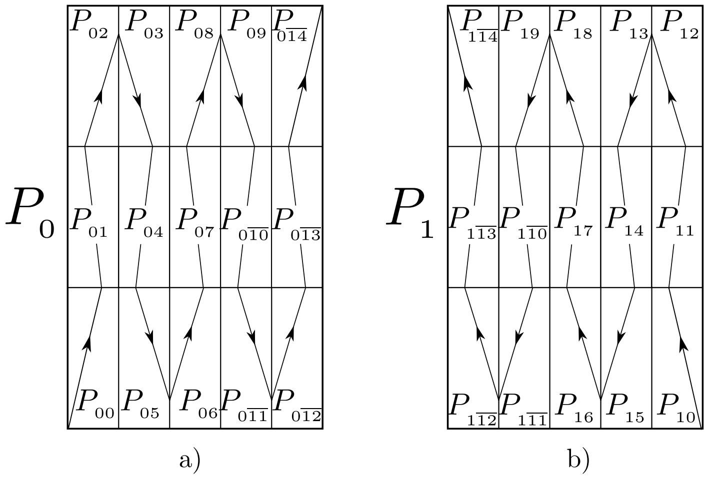

FIGURE 2

Continuing this process, we obtain segments and the corresponding rectangles of rank 3, 4, etc. The segments and rectangles of rank k are labeled by the collections of k base N "digits" $0, 1, \ldots, N-1$; such a collection is the collection of the first k digits of the base N representation of interior points of the interval under consideration. Note that the established correspondence between the segments and rectangles of a given rank is one-to-one, so that one may say that they correspond to each other.

Now we turn to constructing a desired curve. Every point t in $[0,1]$ is the intersection point of a system of nested intervals $\delta^{(1)} \supset \delta^{(2)} \supset \ldots$ of ranks 1, 2, etc. This system of intervals corresponds to a system of nested rectangles $P^{(1)} \supset P^{(2)} \supset \ldots$, whose diameters, obviously, tend to zero. Hence, the intersection of these rectangles consists of a single point, which we denote by $\gamma_{mn}(t)$, or merely $\gamma(t)$. Thus, we obtain a map γ from $[0,1]$ to the unit square. Note that γ is well defined. Indeed, a point t can be the intersection point of two different systems of nested intervals of the form under consideration only if it eventually coincides with a common endpoint of these intervals. Then, for sufficiently large k, the rectangles of rank k corresponding to these intervals share a side (for small k, these rectangles, as well as the corresponding intervals, may coincide). Hence the intersection points of both sequences of rectangles must coincide, which shows that γ is well defined.

Since every point z of the unit square belongs to some rectangle $P_z^{(1)}$ of rank 1, and also belongs to some rectangle $P_z^{(2)}$ of rank 2 nested in $P_z^{(1)}$, etc., from the construction of γ it follows that z coincides with $\gamma(t)$, where t is the intersection point of the (nested) segments corresponding to the rectangles $P_z^{(k)}$. Thus, γ maps $[0,1]$ onto the entire unit square. Furthermore, for every k, it obviously maps every interval of rank k onto the corresponding rectangle, which implies that γ is continuous. Moreover, each of the coordinate functions x and y of the map γ satisfies some Lipschitz condition. Indeed, since the image of an interval δ of rank k coincides with a rectangle of the same rank and with sides of length m^{-k} (the horizontal side) and n^{-k} (the vertical side), for $s, t \in \delta$ we have

$$|x(s) - x(t)| \le m^{-k}, \quad |y(s) - y(t)| \le n^{-k}. \tag{1}$$

Therefore, if

$$N^{-(k+1)} < |s - t| \le N^{-k} \tag{2}$$

and points s, t belong to the same interval of rank k, from (1) it follows that

$$\begin{aligned} |x(s) - x(t)| &\le m^{-k} = N^{-\alpha k} = N^{\alpha} N^{-\alpha(k+1)} \le N^{\alpha} |s - t|^{\alpha}, \\ |y(s) - y(t)| &\le n^{-k} = N^{-\beta k} = N^{\beta} N^{-\beta(k+1)} \le N^{\beta} |s - t|^{\beta}, \end{aligned} \tag{1'}$$

where $\alpha = \log_N m$, $\beta = \log_N n$. If points $s, t \in [0, 1]$ satisfy condition (2), but do not lie in the same interval of rank k, then they belong to neighboring intervals, and hence inequalities (1′) remain valid with the right-hand sides multiplied by 2. Thus, the coordinate functions of the resulting Peano curve satisfy Lipschitz conditions of orders α and β. Note that $\alpha + \beta = \log_N m + \log_N n = 1$. If $m = n$, then $\alpha = \beta = \frac{1}{2}$, but if one of the numbers m, n is sufficiently large in comparison with the other, then one of the numbers α, β may be arbitrarily small while the other may be arbitrarily close to one.

Since the γ-image of an interval of rank k coincides with a rectangle of the same rank, the variation of each coordinate function on such an interval is large in comparison with the length of the interval, which is equal to N^{-k} (the variation of the first coordinate is m^{-k}, and the variation of the second coordinate is n^{-k}). It follows that these functions are not differentiable at any point of $[0, 1]$. We leave a detailed check of this claim to the reader as an easy exercise.

Replacing the unit square by the d-dimensional cube $[0, 1]^d$ and dividing its edges at each step into an odd number of parts $n_1, \dots, n_d$, one can construct, in a similar way, a d-dimensional Peano curve, i. e., a path whose support coincides with this cube. Its coordinate functions $x_1, \dots, x_d$ will satisfy Lipschitz conditions of orders $\log_N n_1, \dots, \log_N n_d$, respectively, where $N = n_1 \cdots n_d$.

Remark. If we construct a plane Peano curve in the same way, but increase the number m at each step (keeping n fixed), then we obtain a Peano curve whose first coordinate function satisfies a Lipschitz condition of any order smaller than one. A similar generalization can also be obtained in the multidimensional case.

§3. Proof of the metric isomorphism theorem

In this and the subsequent sections, λ_1 and λ_2 stand for the Lebesgue measures on the line and plane, respectively.

Lemma. *Let γ be the plane Peano curve constructed in* §2. *For a rectangle P of any rank, the following bound holds*:

$$\lambda_1\bigl(\gamma^{-1}(P)\bigr) \le 9\, \lambda_2(P). \tag{3}$$

A similar bound is valid also for the d-dimensional Peano curve whose construction is described in the preceding section. In this case, the coefficient 9 on the right-hand side should be replaced by 3^d.

Proof. There are at most eight rectangles of the same rank neighboring to P (having common points with P). Let $\delta^{(0)}, \delta^{(1)}, \dots, \delta^{(L)}$ ($L \le 8$) be the intervals corresponding to P and these neighboring rectangles. We check that

$$\gamma^{-1}(P) \subset \bigcup_{\ell=0}^{L} \delta^{(\ell)}.$$

Indeed, if δ is an interval of the same rank different from $\delta^{(0)}, \delta^{(1)}, \ldots, \delta^{(L)}$, then its image is not neighboring to P and hence has no points in common with P. This means that $\delta \cap \gamma^{-1}(P) = \varnothing$. Since δ is arbitrary, no point lying outside $\bigcup_{\ell=0}^{L} \delta^{(\ell)}$ can belong to $\gamma^{-1}(P)$, as required.

Inequality (3) obviously follows from the inclusion obtained. □

Observe a useful corollary to the above lemma.

Corollary. *If $\lambda_2(e) = 0$, then $\lambda_1\bigl(\gamma^{-1}(e)\bigr) = 0$.*

Proof. We may assume without loss of generality that $e \subset [0,1]^2$. Let G be an arbitrary open set in $[0,1]^2$ that contains e. Clearly, G can be exhausted by a sequence of pairwise nonoverlapping rectangles $P^{(k)}$ of different ranks: $G = \bigcup_{k=1}^{\infty} P^{(k)}$. Hence,

$$\lambda_1\bigl(\gamma^{-1}(e)\bigr) \le \lambda_1\bigl(\gamma^{-1}(G)\bigr) \le \sum_{k=1}^{\infty} \lambda_1\bigl(\gamma^{-1}(P^{(k)})\bigr) \le 9 \sum_{k=1}^{\infty} \lambda_2\bigl(P^{(k)}\bigr) = 9\lambda_2(G).$$

Since the measure of G can be arbitrarily small, the corollary follows. □

One can easily check that the corollary holds true also for the curves described in the remark in §2.

Of course, the map γ constructed above is not a bijection, but it is, however, "almost" one-to-one. Let us discuss this issue in more detail. Distinct points s, t from $[0,1]$ belong, for sufficiently large k, to distinct intervals of rank k. The relation $\gamma(s) = \gamma(t)$ is possible only if the rectangles corresponding to these intervals share a side, and the point $z_0 = \gamma(s)$ $(= \gamma(t))$ belongs exactly to this side, i.e., the points s and t belong to the γ-preimage of the boundary of a rectangle. Therefore, at least one of the coordinates of the point z_0 must be rational (more precisely, either the first coordinate of this point is a rational number whose denominator is a power of m, i.e., can be written in the form $\frac{p}{m^q}$ where $p, q \in \mathbb{Z}_+$, or the second coordinate is a rational number whose denominator is a power of n).

Let e_0 be the collection of all points of the unit square for which at least one coordinate is rational. Clearly, for a point z in $[0,1]^2 \backslash e_0$, a chain of rectangles containing z is uniquely determined, and hence, the preimage of the point z is also uniquely determined. In other words, the set $[0,1] \setminus \gamma^{-1}(e_0)$ is mapped one-to-one onto $[0,1]^2 \setminus e_0$. Since the set e_0 has zero Lebesgue measure, the corollary implies that the same is true for its preimage. Thus, the map γ is one-to-one on a set of full measure.

For any m and n, the constructed Peano curve has the following property, which is important for our purposes and (as far as we know) which was first observed for the case where $m = n = 3$ in [7] (see also [8]).

Theorem 2. *The Peano curve γ constructed in §2 is measure-preserving, i.e., for any measurable set E contained in the square $[0,1]^2$,*

$$\lambda_1\bigl(\gamma^{-1}(E)\bigr) = \lambda_2(E). \tag{4}$$

Since the restriction of the constructed Peano curve to a set of full measure is one-to-one, the assertion of the lemma means that it is a metric isomorphism modulo 0 between the unit interval and the square regarded as measure spaces (with the measures λ_1 and λ_2, respectively).

The above theorem is valid also for the Peano curve described in the remark (including the multidimensional case).

Proof. First, we check that (4) is true if E is a rectangle of some rank. Let $E = \gamma(\delta)$, where δ is the interval corresponding to E. Then

$$\delta \subset \gamma^{-1}(E) \subset \delta \cup \gamma^{-1}(e_0),$$

where e_0 is the set of points in $[0,1]^2$ for which at least one coordinate is rational. Since $\lambda_2(e_0) = 0$, from the above corollary it follows that $\lambda_1\big(\gamma^{-1}(e_0)\big) = 0$. Hence, $\lambda_2(E) = \lambda_1(\delta) = \lambda_1\big(\gamma^{-1}(E)\big)$, which proves (4) in the case under consideration. Therefore, this identity is valid for all open sets contained in $[0,1]^2$, because, obviously, such a set can be exhausted by a sequence of pairwise nonoverlapping rectangles of different ranks. Now the proof can be completed by standard arguments, which we leave to the reader. □

Note that the coordinate functions x, y of Peano curves for which (4) is true have an interesting property, which was first observed in [7] (see also [10]): they are statistically independent. By definition, this means that for any two Borel subsets A and B of the interval $[0,1]$, we have

$$\lambda_1\big(\{t \in [0,1] \,:\, x(t) \in A, y(t) \in B\}\big) = \lambda_1(A)\,\lambda_1(B).$$

Clearly, this identity is a special case of (4) for $E = A \times B$. It is easily seen (and this was observed in [7] and [10]) that a path whose coordinate functions are statistically independent is a Peano curve.

§4. The Hausdorff measures

In order to state our main result, we need the notion of the p-dimensional Hausdorff measure (with $p > 0$), which we denote by μ_p.

The Hausdorff measure $\mu_p(A)$ of a subset A of an Euclidean space is defined via ε-covers. For an arbitrary positive number ε, a sequence of sets $\{e_i\}_{i=1}^{\infty}$ is called an ε-cover of A if

$$A \subset \bigcup_{i=1}^{\infty} e_i, \quad \operatorname{diam}(e_i) \le \varepsilon \ \text{ for all } \ i = 1, 2, \dots$$

(here $\operatorname{diam}(e)$ denotes the diameter of a set e, which is equal, by definition, to the quantity $\sup_{x,y\in e} \|x - y\|$). The value $\mu_p(A)$ is defined as

$$\mu_p(A) = \lim_{\varepsilon\to 0}\Big\{\inf \sum_{i=1}^{\infty} \operatorname{diam}^p(e_i)\Big\},$$

where the infimum is taken over all ε-covers of A. For Lebesgue-measurable subsets of the space $\mathbb{R}^d$, the measure μ_p for $p = d$ is proportional to the Lebesgue measure. This definition implies that $\mu_p(A) > 0$ if the sums $\sum_i \operatorname{diam}^p(e_i)$ are bounded away from zero for all covers $\{e_i\}_{i\ge 1}$ of A. One can easily deduce that if $\mu_p(A) < \infty$, then $\mu_q(A) = 0$ for $q > p$, and if $\mu_p(A) > 0$, then $\mu_q(A) = \infty$ for $q < p$. The value $\inf\{q \,:\, \mu_q(A) = 0\}$ (which coincides with $\sup\{q \,:\, \mu_q(A) = \infty\}$) is called the Hausdorff dimension of the set A and is denoted by $\dim_H(A)$. For a more detailed introduction to the notions of Hausdorff measure and Hausdorff dimension, see, e.g., the books [6, 11]. Observe another simple and well-known result (see, e.g., [6, Theorem 8.1]), which will be useful in §5.

Theorem 3. *If Γ is the graph of a function defined on a finite interval and satisfying a Lipschitz condition of order α (with $0 \le \alpha \le 1$), then $\mu_{2-\alpha}(\Gamma) < \infty$, and hence $\dim_H(\Gamma) \le 2 - \alpha$.*

Also, we have the following generalization of Theorem 3.

Theorem 3′. *Let L be the support of a d-dimensional path whose coordinate functions satisfy Lipschitz conditions of orders $\alpha_1, \dots, \alpha_d$. Suppose that $\sigma = \alpha_1 + \dots + \alpha_d \geq 1$, and let $\theta = \max_j \alpha_j$, $p = d - \frac{\sigma - 1}{\theta}$. Then $\mu_p(L) < \infty$.*

This can be shown by embedding L into the collection of parallelepipeds constructed in the proof of the proposition and dividing each of them into $O(N^{d\theta-\sigma})$ cubes with side $N^{-\theta}$. The total number of such cubes does not exceed const $\cdot N^{1+d\theta-\sigma}$, which implies the required bound for the measure μ_p.

§5. The main theorem

Consider an arbitrary plane path $\gamma\colon [0,1] \to [0,1]^2$ with coordinate functions x and y. Generalizing the property established in Theorem 2, we assume that γ satisfies the following condition.

There exists a number $C > 0$ such that for every rectangle P (*with sides parallel to coordinate axes*), *the following inequality is fulfilled*:

$$\lambda_1\bigl(\gamma^{-1}(P)\bigr) \leq C\,\lambda_2(P). \tag{5}$$

Note that if $0 < \lambda_1\bigl(\gamma^{-1}(P)\bigr)$ for all (nondegenerate) rectangles P contained in $[0,1]^2$, then, obviously, γ is a Peano curve.

For condition (5) to be satisfied, it is necessary that the area of the support of γ be positive. Indeed, otherwise the support of γ can be covered by a sequence of rectangles whose total area is arbitrarily small, which is incompatible with (5), because the preimages of these rectangles cover $[0,1]$. Hence, if (5) is true, the conditions of Theorem 1 or the proposition in §1 cannot be satisfied. In particular, the sum of the Lipschitz exponents of the coordinate functions of γ does not exceed one, and none of these functions can have finite variation.

Condition (5) has a natural probabilistic interpretation. Let δ and δ' be arbitrary intervals, and let u, v be their lengths, $\delta \subset [0,1]$. Consider the projection to the abscissa axis of the intersection of the graph Γ_y of the function y with the rectangle $\delta \times \delta'$, i.e., the set T of points from δ at which the values of the function y lie in δ':

$$T = \mathrm{Pr}\bigl(\Gamma_y \cap (\delta \times \delta')\bigr) = \{t \in \delta \,:\, y(t) \in \delta'\}.$$

Interpreting the parameter t as time, we may say that the measure $\lambda_1(T)$ is the "occupation time" spent by the point $(t, y(t))$ in the rectangle $\delta \times \delta'$.

The ratio $\frac{\lambda_1(T)}{\lambda_1(\delta)}$ can be viewed as the probability that the point $y(t)$ falls into δ' under the condition that $t \in \delta$. We apply inequality (5) to estimate this conditional probability, assuming that the first coordinate function x satisfies a Lipschitz condition of order α. Let $\Delta = x(\delta)$, $\Delta' = y(\delta)$. Clearly, $\gamma(\delta) \subset \Delta \times \Delta'$, and the length of the interval Δ does not exceed $C_x u^\alpha$ (where C_x is a Lipschitz constant for the function x).

If $P = \Delta \times \delta'$, then

$$T = \{t \in \delta \,:\, y(t) \in \delta',\ x(t) \in \Delta\} = \{t \in \delta \,:\, \gamma(t) \in P\} \subset \gamma^{-1}(P).$$

Hence,

$$\lambda_1(T) \leq \lambda_1\bigl(\gamma^{-1}(P)\bigr) \leq C\lambda_2(P) \leq C\,C_x u^\alpha\, v. \tag{6}$$

Dividing by $\lambda_1(\delta) = u$, we rewrite (6) in an equivalent form:

$$\frac{\lambda_1(T)}{\lambda_1(\delta)} \leq C\,C_x u^{\alpha-1} v.$$

This inequality shows a kind of equidistribution of the values of the function y in the interval Δ': the conditional probability that $y(t)$ falls into an interval δ' has a bound that does not depend on the position of this interval and is proportional to its length. If γ is a metric isomorphism, then the values of the function y are equidistributed.

Theorem 4. *Let γ be a path with coordinate functions x and y satisfying Lipschitz conditions of orders α and β, respectively, and let Γ_x and Γ_y be their graphs. If $\alpha+\beta \le 1$ and condition* (5) *is fulfilled, then*

$$\mu_{1+\beta}(\Gamma_x) > 0, \quad \mu_{1+\alpha}(\Gamma_y) > 0,$$

whence

$$\dim_H(\Gamma_x) \ge 1+\beta, \quad \dim_H(\Gamma_y) \ge 1+\alpha.$$

By Theorem 3,

$$\dim_H(\Gamma_x) \le 2-\alpha, \quad \dim_H(\Gamma_y) \le 2-\beta.$$

Hence, if the conditions of Theorem 4 are satisfied, then we have the two-sided bounds

$$1+\beta \le \dim_H(\Gamma_x) \le 2-\alpha, \quad 1+\alpha \le \dim_H(\Gamma_y) \le 2-\beta$$

(recall that $\alpha+\beta \le 1$ if γ is a Peano curve, as follows from Theorem 1). If $\alpha+\beta = 1$, as in the examples from §2, then the left- and right-hand sides of these inequalities coincide and, moreover, we see that

$$0 < \mu_{2-\alpha}(\Gamma_x) < \infty, \quad 0 < \mu_{2-\beta}(\Gamma_y) < \infty.$$

Proof of the theorem. Since the roles of the coordinates are interchangeable, it suffices to prove that the measure $\mu_{1+\alpha}(\Gamma_y)$ is positive. For this, we must check that for all covers $\mathcal{E} = \{e_1, e_2, \dots\}$ of the graph Γ_y, the corresponding sums

$$\Sigma(\mathcal{E}) = \sum_i \operatorname{diam}^p(e_i),$$

where $p = 1+\alpha$, are bounded away from zero. As can easily be seen, here we may assume without loss of generality that all sets e_i are squares with sides parallel to the coordinate axes.

Let $\mathcal{E} = \{e_1, e_2, \dots\}$ be such a cover. To estimate the sum $\Sigma(\mathcal{E})$, consider one of its elements, a square $e = e_i = \delta \times \delta'$. Put

$$T = \Pr(\Gamma_y \cap e) = \{t \in \delta \,:\, y(t) \in \delta'\}, \quad u = \lambda_1(\delta) = \lambda_1(\delta').$$

By (6) with $v = u$, we have

$$\lambda_1(T) \le C\, C_x u^{1+\alpha} = C\, C_x u^p. \tag{7}$$

Using this inequality, we can easily complete the proof of the theorem. Indeed, returning to the cover $\mathcal{E} = \{e_1, e_2, \dots\}$ that we have fixed and assuming that the sides of the squares e_i are equal to u_i, in view of (7) we have

$$\begin{aligned}\Sigma(\mathcal{E}) &= \sum_i (\operatorname{diam}(e_i))^p = \sum_i (\sqrt{2}u_i)^p \\ &\ge 2^{\frac{p}{2}}(C\, C_x)^{-1} \sum_i \lambda_1\big(\Pr(\Gamma_y \cap e_i)\big) \ge 2^{\frac{p}{2}}(C\, C_x)^{-1}\end{aligned}$$

(the last inequality follows from the countable semiadditivity of the measure λ_1, because the projections of the sets $\Gamma_y \cap e_i$ cover $[0,1]$). Since $\mathcal{E}$ is an arbitrary cover, the obtained bound implies that

$$\mu_p(\Gamma_y) \ge 2^{\frac{p}{2}}(C\, C_x)^{-1} > 0. \qquad \square$$

In the proof of the inequality $\mu_{1+\alpha}(\Gamma_y) > 0$, we used the Lipschitz property of the first coordinate function without imposing any conditions on the second coordinate function. Therefore, for the Peano curve described in the remark in §2, whose first coordinate function satisfies a Lipschitz condition of any order $\alpha < 1$, we have $\dim_H(\Gamma_y) \geq 1 + \alpha$, and hence $\dim_H(\Gamma_y) \geq 2$. Since the reverse inequality is trivial, $\dim_H(\Gamma_y) = 2$ (though, of course, $\mu_2(\Gamma_y) = \lambda_2(\Gamma_y) = 0$).

§6. Generalizations

Let $\gamma\colon [0,1] \to [0,1]^d$ be a path with coordinate functions $x_1, \dots, x_r, y_1, \dots, y_s$ ($r+s = d$); assume that the functions x_k, y_j satisfy Lipschitz conditions of orders α_k ($k = 1, \dots, r$) and β_j ($j = 1, \dots, s$), respectively. Put

$$\sigma_x = \alpha_1 + \cdots + \alpha_r, \quad \sigma_y = \beta_1 + \cdots + \beta_s, \quad \sigma = \sigma_x + \sigma_y.$$

Consider the path $y(t) = \bigl(y_1(t), \dots, y_s(t)\bigr)$ ($t \in [0,1]$), and let L_y be its "graph," i.e., the Jordan arc

$$L_y = \{(t, y(t)) \,:\, t \in [0,1]\} \subset \mathbb{R}^{s+1}.$$

The following generalization of Theorem 4 holds, in which we use the notation introduced above.

Theorem 4′. *Let $\gamma\colon [0,1] \to [0,1]^d$ be a path whose coordinate functions satisfy Lipschitz conditions. Suppose that $\sigma \leq 1$ and γ satisfies the following analog of condition* (5): *for some $C > 0$,*

$$\lambda_1\bigl(\gamma^{-1}(P)\bigr) \leq C\,\lambda_d(P) \tag{5′}$$

for any d-dimensional parallelepiped P (*with edges parallel to coordinate axes*). *Then*

$$\mu_{s+\sigma_x}\bigl(L_y\bigr) > 0,$$

and hence $s + \sigma_x \leq \dim_H\bigl(L_y\bigr) \leq s + 1 - \sigma_y$.

The last inequality follows from Theorem 3′ applied to L_y.

If $\sigma = 1$, then $\dim_H\bigl(L_y\bigr) = s + 1 - \sigma_y$ and, moreover, $0 < \mu_{s+1-\sigma_y}\bigl(L_y\bigr) < \infty$. As in the case of (5), for condition (5′) to be satisfied it is necessary that the support of the path γ have a positive measure. Together with Theorem 1, this shows that $\sigma \leq 1$ in this case. Moreover, for $\sigma = 1$ the relation $\sigma_y = 0$ is impossible, as follows from the proposition in §1. For $\sigma_x < 1$, the relation $\sigma_y = 0$ is possible, which can be seen from a multidimensional analog of the example mentioned in the remark in §2.

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