Алгебра и анализ
Том 28 (2016), № 3

St. Petersburg Math. J.
Vol. 28 (2017), No. 3, Pages 377–410
http://dx.doi.org/10.1090/spmj/1455
Article electronically published on March 29, 2017

ALMOST STANDING WAVES IN A PERIODIC WAVEGUIDE WITH RESONATOR, AND NEAR-THRESHOLD EIGENVALUES

S. A. NAZAROV

Abstract. The definition and an existence criterion are given for the standing waves at the threshold of the continuous spectrum for a periodic quantum waveguide with a resonator (the Dirichlet problem for the Laplace operator). Such waves and their linear combinations do not transfer energy to infinity, and they only differ from the standing waves with the zero Floquet parameter by an exponentially decaying term. It is shown that the almost standing and trapped waves at the threshold generate eigenvalues in the discrete spectrum of a waveguide with a regular sloping local perturbation of the wall.

§1. Introduction

1.1. Setting of the problem. Let $\Xi = \Pi_- \cup \Theta \cup \Pi_+$ be a waveguide (see Figure 1) with two semi-infinite periodic trunks

$$\Pi_\pm = \{x = (y, z) \in \Pi : \pm z = \pm x_d > l > 0\} \subset \mathbb{R}^d, \quad d \geq 2. \tag{1.1}$$

Here Π is a quasicylinder, i.e., a 1-periodic domain

$$\Pi = \{(y, z) \in \mathbb{R}^{d-1} \times \mathbb{R} : (y, z \pm 1) \in \Pi\} \tag{1.2}$$

with a $(d-1)$-dimensional boundary $\partial\Pi$ (of class C^4 – to simplify justification of asymptotics; cf. items 4 and 5 in §6 and also item 1 in §2), and Θ is a resonator (shaded in Figure 1), i.e., a set lying in the layer $\{(y, z) : y = (x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1}, |z| \leq l\}$ that links the trunks (1.1) and turns Ξ into a domain with piecewise smooth boundary $\partial\Xi$.

Viewing Ξ as a quantum waveguide, consider the spectral Dirichlet problem

$$-\Delta_x u(x) = \lambda u(x), \quad x \in \Xi, \quad u(x) = 0, \quad x \in \partial\Xi. \tag{1.3}$$

The variational statement of problem (1.3) invokes the symmetric closed bilinear form $(\nabla_x u, \nabla_x v)_\Xi$ in the Sobolev space $H^1_0(\Xi)$ of functions that vanish on $\partial\Xi$; here $\nabla_x = \mathrm{grad}$ and $(\, , \,)_\Xi$ is the scalar product in $L^2(\Xi)$. Thus, with problem (1.3) we associate (see [1, Chapter 10]) an unbounded positive self-adjoint operator A in $L^2(\Xi)$ with a domain $\mathcal{D}(A) \subset H^1_0(\Xi)$. Its essential spectrum σ_{es} has a positive lower bound $\lambda^\dagger$ (see Lemma 2.1), and the discrete spectrum may be located in the interval $(0, \lambda^\dagger)$.

The classical result of [2] shows (cf. Subsection 6.1) that the full multiplicity $\#\sigma_d$ of the discrete spectrum grows unboundedly if, e.g., the middle part Π_0 of the waveguide is blown out in all directions. The discrete spectrum points of the Dirichlet problem move downward when the domain grows, so that the increment of the multiplicity $\#\sigma_d$ happens exclusively when eigenvalues drop down from the edge $\lambda^\dagger$ of the essential spectrum. Precisely this phenomenon is studied in the present paper.

2010 *Mathematics Subject Classification.* Primary 81Q10.

Key words and phrases. Periodic waveguide, resonator, discrete spectrum, almost standing waves, threshold scattering matrix, asymptotics.

Supported by RFBR (project no. 15-01-02175).

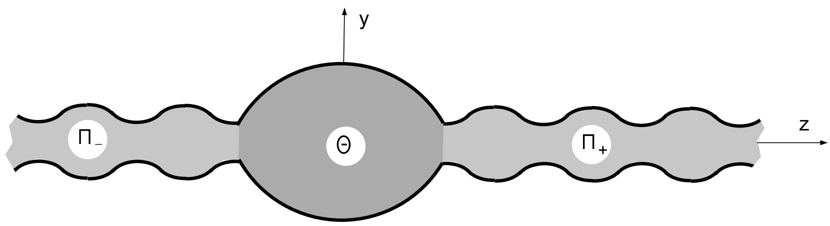

FIGURE 1. A waveguide with periodic sleeves and a resonator.

1.2. The content of the paper. In §2 we collect the general, mostly known, results on the structure of the continuous spectrum near its lower edge, and at the beginning of §3 we state the diffraction problem at the threshold $\lambda = \lambda^\dagger$ and introduce the threshold scattering matrix, which is unitary and symmetric. After that, in Subsections 3.3 and 3.4, we establish the fact that, when an eigenvalue drops down, this possess is necessarily accompanied by the arising of an "almost-standing" wave at the threshold, while in Subsection 3.5 we present a criterion for the arising of such waves, expressed in terms of the threshold scattering matrix. In §4, the traditional asymptotic analysis of the spectrum of cylindrical waveguides is adapted to periodic outlets to infinity, and we check that the truly chosen small regular perturbation of the wall results in the arising of near-threshold discrete spectrum eigenvalues the number of which is equal to the dimension of the subspace of standing and trapped waves. Finally, in the last section we analyze examples and pose open questions.

1.3. On terminology. Some objects that arise at the threshold values of the spectral parameter will be used throughout the paper and need clarification. First of all, the thresholds themselves are the points of the continuous spectrum at which the linear space of oscillatory Floquet waves (the spectral solutions of the problem in a periodic quasicylinder Π that decay neither as $z \to +\infty$, nor as $z \to -\infty$) changes dimension. A characteristic peculiarity of the thresholds consists in the arising, besides bounded Floguet waves, of polynomial ones, in particular, those linearly depending on z. Then the role of the leading term is played by *standing waves*, and if no quadratic waves occur, then the linear waves are said to be *resonance*. In the case where quadratic (in z) Floquet waves do occur, no stable terminology exists up to now.

In the paper, we only deal with one threshold, the lower bound $\lambda^\dagger$ of the essential spectrum. As is checked in §2, this threshold gives rise to one standing and one resonance Floquet waves, and in the vicinity of the threshold the essential spectrum turns out to be continuous.

Solutions of problem (1.3) in a waveguide Ξ with resonator Θ are also called *waves*, and those among them that decay in each of the two sleeves, and, therefore, are eigenfunctions of the operator A, are called *trapped waves*. Each point of the discrete spectrum in the interval $(0, \lambda^\dagger)$ gives rise to a trapped wave belonging to the Sobolev space $H_0^1(\Xi)$ and therefore decaying exponentially at infinity. For $\lambda = \lambda^\dagger$, the space of trapped waves, necessarily finite-dimensional, is denoted by $\mathcal{L}^{\rm tr}$. The *almost standing* waves are the solutions of the same problem at the threshold that differ from standing waves in each of the trunks $\Pi_\pm$ by only exponentially decaying terms as $z \to \pm\infty$; they form the space $\mathcal{L}^{\rm st}$, and only the zero standing wave falls into the space $H_0^1(\Xi)$. Denoting by $\mathcal{L}^{bo}$ the space of all bounded solutions of the problem with $\lambda = \lambda^\dagger$, we see that the subspace $\mathcal{L}^{\rm st}$ is a direct complement of $\mathcal{L}^{\rm tr}$ in $\mathcal{L}^{bo}$, i.e.,

$$\mathcal{L}^{\rm st} = \mathcal{L}^{bo} \ominus \mathcal{L}^{\rm tr}. \tag{1.4}$$

Since the almost standing waves are defined up to summands belonging to $\mathcal{L}^{\mathrm{tr}}$, they should be viewed more consistently as equivalence classes in the factor-space $\mathcal{L}^{bo}/\mathcal{L}^{\mathrm{tr}}$, but we shall not go into such pettifoggery.

§2. Floquet waves

2.1. Jordan chain. In accordance with the Eloquet–Bloch–Gelfand theory (see, e.g., [3, 4] and [5, 6]), the following model problem on the periodicity cell $\varpi = \{x \in \Pi : 0 < z < 1\}$ is associated with the Dirichlet problem in the quasicylinder (1.2) (without a resonator):

$$-\Delta_y U(\eta; x) - (\partial_z + i\eta)^2 U(\eta; x) = \Lambda(\eta) U(\eta, x), \quad x \in \varpi, \tag{2.1}$$

$$U(\eta; x) = 0, \quad x \in \gamma = \partial\varpi \setminus (\overline{\tau} \times \{0, 1\}), \tag{2.2}$$

$$U(\eta; y, 1) = U(\eta; y, 0), \quad \partial_z U(\eta; y, 1) = \partial_z U(\eta; y, 0), \quad y \in \tau. \tag{2.3}$$

Here $\eta \in [-\pi, \pi]$ is the Floquet parameter (the dual variable of the Gelfand transformation), and $\Lambda(\eta)$ is the new notation for the spectral parameter. The Dirichlet condition (2.2) is kept on the lateral surface of the cell, but on its ends $\tau(0) = \tau \times \{0\}$ and $\tau(1) = \tau \times \{1\}$ the periodicity conditions are prescribed. The variational statement of (2.1)–(2.3) invokes the integral identify

$$(\nabla_y U, \nabla_y V)_\varpi + \big((\partial_z + i\eta)U, (\partial_z + i\overline{\eta})V\big)_\varpi = \Lambda (U, V)_\varpi, \quad V \in H^1_{0\ \mathrm{per}}(\varpi), \tag{2.4}$$

where $H^1_{0\ \mathrm{per}}(\varpi)$ is the Sobolev space of functions that satisfy the first periodicity conditions (2.3) and the Dirichlet condition (2.2).

The spectral boundary-value problem (2.1)–(2.3) or the variational problem (2.4) can be seen from two viewpoints. First, for a fixed Floque parameter $\eta \in [-\pi, \pi]$ (assumed to be real: the complex conjugation sign on the left-hand side in (2.4) is not needed as yet), since the embedding $H^1(\varpi) \subset L^2(\varpi)$ is compact, the problem has a purely discrete spectrum formed by the unbounded positive sequence of eigenvalues

$$0 < \Lambda_1(\eta) \le \Lambda_2(\eta) \le \Lambda_3(\eta) \le \cdots \le \Lambda_n(\eta) \le \cdots \to +\infty.$$

The functions $\eta \mapsto \Lambda_n(\eta)$ are continuous and 2π-periodic, and the union of the spectral segments (connected, bounded, and closed sets)

$$\left\{\Lambda_n(\eta) \,\middle|\, \eta \in [-\pi, \pi]\right\}, \quad n \in \mathbb{N} = \{1, 2, 3, \dots\},$$

coincides with the essential spectrum σ_{es} of the Dirichlet problems in Π and in Ξ.

As a result,

$$\lambda^\dagger = \min\left\{\Lambda_1(\eta) \,\middle|\, \eta \in [-\pi, \pi]\right\}. \tag{2.5}$$

The author knows no the origin of the next statement. Regarding it as a "mathematical folklore", we present a brief proof for the reader's convenience.

Lemma 2.1. *For* $|\eta| \in (0, \pi]$ *we have* $\Lambda_1(0) < \Lambda_1(\eta)$, *i.e.,* $\lambda^\dagger = \Lambda_1(0)$.

Proof. For $\eta = 0$, the operator on the left in (2.1) becomes the Laplace operator (with the minus sign), so that $\Lambda_1^0 = \Lambda_1(0)$ is a simple eigenvalue, in particular, we have $\Lambda_1^0 < \Lambda_2(0)$. The eigenfunction $U_1^0 = U_1(0; \cdot) \in C^3_{\mathrm{per}}(\overline{\varpi})$ can be taken positive in $\overline{\varpi} \setminus \overline{\gamma}$, and $\partial_n U_1^0(x) < 0$ on $\overline{\gamma}$, where ∂_n is the exterior normal derivative on the smooth (by assumption) surface $\overline{\gamma}$. We put $U_1^\eta(x) = U_1(\eta; x)$, observing that $\nabla_x U_1^\eta \in C^2_{\mathrm{per}}(\overline{\varpi})$ and, as before, $U_1^\eta(x) = O(\mathrm{dist}(x, \gamma))$. Thus, the ratio $R^\eta = U_1^\eta / U_1^0$ is continuously differentiable in $\overline{\varphi}$ and certainly belongs to the Sobolev space $H^1_{\mathrm{per}}(\varpi)$ (this function fails to vanish on γ). After easy transformations (differentiation and integration by parts), identify (2.4) with $\Lambda = \Lambda_1(\eta)$ and $U = V = U_1^\eta$ becomes

$$\begin{aligned}0 &= \|\nabla_y(R^\eta U_1^0); L^2(\varpi)\|^2 + \|(\partial_z + i\eta)(R^\eta U_1^0); L^2(\varpi)\|^2 - \Lambda_1(\eta)\|R^\eta U_1^0; L^2(\varpi)\|^2\\ &= \|U_1^0 \nabla_y R^\eta; L^2(\varpi)\|^2 + \|U_1^0(\partial_z + i\eta)R^\eta; L^2(\varpi)\|^2 + (\Lambda_1(0) - \Lambda_1(\eta))\|U_1^\eta; L^2(\varpi)\|^2,\end{aligned}$$

which finishes the proof of the lemma, because the identify $R^\eta(x) = e^{iz\eta}$ is impossible for $|\eta| \in (0, \pi]$, in view of the periodicity of the functions U_1^η and U_1^0. □

Now, we fix $\lambda \in \mathbb{R}_+$, allow the Floquet parameter η to be complex, and interpreter problem (2.4) as a quadratic pencil (see, e.g., [7, Chapter 1])

$$\mathbb{C} \ni \eta \mapsto \left(\mathfrak{A}_\lambda(\eta) \,:\, H^1_{0\ \mathrm{per}}(\varpi) \to H^1_{0\ \mathrm{per}}(\varpi)^*\right). \tag{2.6}$$

Here the adjoint space $H^1_{0\ \mathrm{per}}(\varpi)^*$ occurs; since the Dirichlet problem for the Laplace operator on the cell ϖ with identified ends is uniquely solvable, this space is formed by the functionals $F(V) = (\nabla_x U^F, \nabla_x V)_\varpi$ with $U^F \in H^1_{0\ \mathrm{per}}(\varpi)^*$.

If $\lambda = \Lambda_n(\eta_n)$ for some $n \in \mathbb{N}$ and $\eta_n \in [-\pi, \pi]$, then η_n is an eigenvalue of the pencil (2.6). Note that, for pencils, eigenspaces can be more involved, compared to the case of usual self-adjoint operators: besides the eigenvectors, pencils may have associated vectors, which form Jordan chains (the details can be found, e.g., in [5, Chapters 1 and 5]). In what follows, we shall need full information about the Jordan chain of the pencil $\mathfrak{A}_{\lambda^\dagger}$ that corresponds to its eigenvalue $\eta = 0$.

Lemma 2.2. *The chain $\{U^0, U^1\}$ mentioned above consists of the eigenfunction $U^0 = U_1(0;\,\cdot\,)$ and the solution $U^1 \in H^1_{0\ \mathrm{per}}(\varpi) \cap H^2(\varpi)$ of the variational problem*

$$(\nabla_x U^1, \nabla_x V)_\varpi - \lambda^\dagger (U^1, V)_\varpi = F^1(V) := i(\partial_z U^0, V)_\varpi - i(U^0, \partial_z V),\ V \in H^1_{0\ \mathrm{per}}(\varpi). \tag{2.7}$$

This chain is unextendible and is unique for the eigenvalue $\eta = 0$ of the pencil $\mathfrak{A}_{\lambda^\dagger}$.

Proof. By definition, a first order associated vector is a solution of the abstract equation

$$\mathfrak{A}_{\lambda^\dagger}(0)U^1 = -\partial_\eta \mathfrak{A}_{\lambda^\dagger}(0)U^0,$$

which takes the form (2.4) in view of the integral identify (2.7). Note that the fact that $\eta = 0$ allows us to use the full gradient $\nabla_x = (\nabla_y, \partial_z)$ in (2.7).

Since $\lambda^\dagger = \Lambda_1(0)$ is a simple eigenvalue, the Fredholm alternative provides a single compatibility condition $F^1(U^0) = 0$ for problem (2.7). Since the eigenfunction $U^0 = U_1(0;\,\cdot\,)$ is real, the above identity is obvious, and the orthogonality condition

$$(U^1, U^0)_\varpi = 0 \tag{2.8}$$

makes the solution U^1 unique and, moreover, pure imaginary.

We cheek that no second order associated vector may exist. By definition, such a vector U^2 satisfies the abstract equation

$$\mathfrak{A}_{\lambda^\dagger}(0)U^2 = -\partial_\eta \mathfrak{A}_{\lambda^\dagger}(0)U^1 - \frac{1}{2}\partial_\eta^2 \mathfrak{A}_{\lambda^\dagger}(0)U^0,$$

which turns into the integral identify

$$(\nabla_x U^2, \nabla_x V)_\varpi - \lambda^\dagger (U^2, V)_\varpi = F^2(V) := i(\partial_z U^1, V)_\varpi - i(U^1, \partial_z V)_\varpi - (U^0, V)_\varpi,\quad V \in H^1_{0\ \mathrm{per}}(\varpi).$$

We must verify that $F^2(U^0) \neq 0$. Using (2.7) with $V = U^1$, we get

$$\begin{aligned}J &= -i(\partial_z U^1, U^0)_\varpi + i(U^1, \partial_z U^0)_\varpi + (U^0, U^0)_\varpi\\ &= \left(\|\nabla_x U^1; L^2(\varpi)\|^2 - \lambda^\dagger \|U^1; L^2(\varpi)\|^2\right) + \|U^0; L^2(\varpi)\|^2 > 0.\end{aligned} \tag{2.9}$$

Here we have used the Poincaré inequality

$$\|U^1; L^2(\varpi)\| \le (\lambda^\dagger)^{-1/2} \|\nabla_x U^1; L^2(\varpi)\|,$$

which follows from (2.8) and shows that the difference in brackets in (2.9) is nonnegative. □

2.2. Floquet waves. Extending the elements U^0 and U^1 that form the Jordan chain in Lemma 2.2 by periodicity to the quasicylinder Π, we get two Floque waves

$$w_0(y,z) = U^0(y,z) \quad \text{and} \quad w_1(y,z) = izU^0(y,z) + U^1(y,z), \tag{2.10}$$

which satisfy the Dirichlet problem for the Helmholtz operator $\Delta_x + \lambda^\dagger$ in Π (this is checked by direct calculations) and are called, respectively, the standing wave and the resonance wave (cf. Subsection 1.3), because the former corresponds to the zero Floquet parameter, and the latter has a linearly growing amplitude. The problem in question admits no other solutions that are bounded or have polynomial growth.

In the sequel we shall need the Floquet waves that correspond to the spectral parameter

$$\lambda^\delta = \lambda^\dagger + \delta \tag{2.11}$$

located above ($\delta > 0$) or below ($\delta < 0$) the threshold (2.5). The theorem on the stability of the total multiplicity of the spectrum (see [7, Theorem 1.3.1]) shows that the pencil $\mathfrak{A}_{\lambda^\delta}$ has precisely two eigenvalues $\eta^\delta_\pm$ in a neighborhood of the point $\eta = 0 \in \mathbb{C}$.

In accordance with the general procedure[1] [8, Chapter 9]), the asymptotics of these eigenvalues, as well as of the corresponding eigenfunctions, can be sought in the form

$$\begin{aligned} \eta^\delta_\pm &= 0 \pm |\delta|^{1/2}\xi + |\delta|\xi_\pm + \widetilde{\eta}^\delta_\pm, \\ U^\delta_\pm(x) &= U^0(x) \pm |\delta|^{1/2}\xi U^1(x) + |\delta| Z_1(x) + \widetilde{U}^\delta_\pm(x), \end{aligned} \tag{2.12}$$

where U^0 and U^1 are the elements of the Jordan chain occurring in Lemma 2.2, and the quantities ξ, $\xi_\pm$ and the function Z_1 are to be determined. We substitute the expressions (2.12) in problem (2.4) with $\Lambda = \lambda^\delta$ and collect the coefficients of $1 = |\delta|^0$, $|\delta|^{1/2}$, and $|\delta|$. As a result, we arrive at three problems of the form (2.4) with the same parameter $\Lambda = \Lambda_1(0) = \lambda^\dagger$, the first two of which are satisfied automatically due to the definition of an eigenvector and an adjoined vector. For the third problem, the usual compatibility condition (the Fredholm alternative)

$$\begin{aligned} &(\nabla_x Z_1, \nabla_x V)_\varpi - \lambda^\dagger (Z_1, V)_\varpi \\ &\quad = \operatorname{sgn}(\delta)(U^0, V)_\varpi + \xi_\pm \left(i(\partial_z U^0, V)_\varpi - i(U^0, \partial_z V)_\varpi \right) \\ &\qquad + \xi^2 \left(i(\partial_z U^1, V)_\varpi - i(U^1, \partial_z V)_\varpi - (U^0, V)_\varpi \right), \quad V \in H^1_{0\,\mathrm{per}}(\varpi), \end{aligned}$$

turns into the quadratic equation

$$J\xi^2 = \operatorname{sgn}(\delta) \|U^0; L^2(\varpi)\|^2, \tag{2.13}$$

(we have used formulas from the proof of Lemma 2.2). This equation has a nonzero root

$$\xi = \begin{cases} |\xi| & \text{for } \delta > 0, \\ i|\xi| & \text{for } \delta < 0, \end{cases} \qquad |\xi| = J^{-1/2} \|U^0; L^2(\varpi)\|, \tag{2.14}$$

where $J > 0$ is the quantity (2.9).

The correction terms in the expansions (2.12) have been found. We present the estimates of remainders ensured by the general results of [8, Chapter 9].

---

[1] In this book, the linear but nonselfadjoint pencils are studied, and reduction of a quadratic pencil to a linear nonselfadjoint one is easy.

Lemma 2.3. *There exist positive numbers δ_ϖ and ρ, c such that for any $\delta \in (-\delta_\varpi, 0) \cup (0, \delta_\varpi)$, in the rectangle*

$$\mathbb{Q}_\rho = \{\eta \in \mathbb{C} : |\operatorname{Re}\eta| \le \pi, |\operatorname{Im}\eta| < \rho\}, \tag{2.15}$$

the pencil $\mathfrak{A}_{\lambda^\dagger+\delta}$ has only two eigenvalues, which are geometrically and algebraically simple and, together with the corresponding eigenfunctions, obey the asymptotic formulas

$$\left|\eta_\pm^\delta \mp |\delta|^{1/2}\xi\right| \le c|\delta|, \quad \left\|U_\pm^\delta - U^0 \mp |\delta|^{1/2}\xi U^1; H^2(\varpi)\right\| \le c|\delta|, \tag{2.16}$$

where $\{U^0, U^1\}$ is the Jordan chain from Lemma 2.2, and ξ is the root (2.14) *of equation* (2.13).

It should be noted that, in Lemma 2.3, the eigenfunctions $U_\pm^\varepsilon$ are in no way normalized. Precisely for that reason, the second estimate in (2.16) may involve an arbitrary associated vector U^1, not necessarily the one satisfying (2.8).

The Floquet waves

$$w_\pm^\delta(y, z) = e^{i\eta_\pm^\delta z} U_\pm^\delta(y, z) \tag{2.17}$$

constructed by the eigenpairs $\{\eta_\pm^\delta, U_\pm^\delta\}$ behave differently above and below the threshold. In case $\delta > 0$, both waves (2.17) are oscillating and bounded, but if $\delta < 0$, the wave $w_\pm^\delta(y, z)$ grows exponentially as $z \to \mp\infty$, but decays as $z \to \pm\infty$.

§3. Scattering problem at the threshold

3.1. Operator setting of the problem. We describe the well-known settings of the diffraction problem (1.1) in a waveguide Ξ with periodic trunks. For this, we introduce the weighted Lebesgue space $L^2_\beta(\Xi)$ and Sobolev space $W^1_\beta(\Xi)$ as the completion of the linear set $C_c^\infty(\Xi)$ (infinitely differentiable functions with compact support) respectively in the norms

$$\|v; L^2_\beta(\Xi)\| = \|e^{\beta|z|}v; L^2(\Xi)\| \quad \text{and} \quad \|v; W^1_\beta(\Xi)\| = \|e^{\beta|z|}v; H^1(\Xi)\|. \tag{3.1}$$

The number $\beta \in \mathbb{R}$ is called the weight index. The subspace $W^1_{\beta,0}(\Xi)$ is formed by the functions $v \in W^1_\beta(\Xi)$ that vanish on the boundary $\partial\Xi$. It should be noted that the spaces introduced above are characterized by identical behavior "decay/growh" in the two trunks $\Pi_\pm$ and are applied mostly in the description of the properties of problem (1.3) in the waveguide Ξ with a resonator Θ. The use of the Gelfand transformation when passing from the problem in a periodic quasicylinder Π to the model problem (2.1)–(2.3) in the cell ϖ requires introduction of other weighted spaces with norms like (3.1) in which the weight factor $e^{\beta|z|}$ is replaced with $e^{\beta z}$ (see [3] and also [5, §3.4] and [9]).

The generalized setting of the nonhomogeneous (with source f) problem (1.1) in a weighted class consists in the search of a function $u \in W^1_{-\beta,0}(\Xi)$ that satisfies the integral identify

$$(\nabla_x u, \nabla_x v)_\Xi - \lambda(u, v)_\Xi = f(v), \quad v \in W^1_{\beta,0}(\Xi), \tag{3.2}$$

where $f \in W^1_{\beta,0}(\Xi)^*$ is an (anti)linear functional on $W^1_{\beta,0}(\Xi)$, for example, $f(v) = (f, v)_\Xi$ with $f \in L^2_{-\beta}(\Xi)$, and the operation $(\,,)_\Xi$ on the left in (3.2) is understood as the extension of the scalar product in $L^2(\Xi)$ up to duality between the spaces $L^2_{-\beta}(\Xi)$ and $L^2_\beta(\Xi)$. Observe that each element f of the space $W^1_{\beta,0}(\Xi)^*$ can be described, e.g., by the formula $f(v) = (\nabla_x u^f, \nabla_x v)_\Xi + k_\beta^2(u^f, v)_\Xi$, where $u^f \in W^1_{-\beta,0}(\Xi)$, and the number $k_\beta > 0$ is chosen so that the Dirichlet problem for the Helmholtz operator $-\Delta_x + k_\beta^2$ in the unbounded domain Ξ is uniquely solvable in the space $W^1_{-\beta,0}(\Xi)$.

Problem (3.2) gives rise to the mapping

$$B_{-\beta}(\lambda)\colon W^1_{-\beta,0}(\Xi) \to W^1_{\beta,0}(\Xi)^*,$$

which is continuous for all $\beta \in \mathbb{R}$, but acquires useful properties only under certain restrictions on the weight index. We state a pair of claims borrowed from [3] (see also the book [5, §3.3, §5.1] and the survey [9]).

Proposition 3.1. 1) *The mutually adjoint operators $B_{-\beta}(\lambda)$ and $B_\beta(\lambda)$ are Fredholm if and only if the segment $\{\eta \in \mathbb{C} : |\operatorname{Re}\eta| \le \pi, \operatorname{Im}\eta = \beta\}$ is free from the spectrum of the pencil $\mathfrak{A}_\lambda$. If this condition is fulfilled, then the solution $u \in W^1_{-\beta,0}(\Xi)$ of problem* (3.2) *satisfies the estimate*

$$\|u; W^1_{-\beta,0}(\Xi)\| \le c_K(\lambda)(\|f; W^1_{\beta,0}(\Xi)^*\| + \|u; L^2(K)\|), \tag{3.3}$$

where K is an arbitrary compact set in $\bar{\Xi}$ containing a ball of positive radius, and the factor $c_K(\lambda)$ depends on the choice of K and, of course, on the spectral parameter λ.

2) *Let $\lambda = \lambda^\dagger$, and let $\beta \in (0, \rho)$, where $\rho > 0$ is a fixed number such that only one eigenvalue $\eta = 0$ of the pencil $\mathfrak{A}_{\lambda^\dagger}$ lies in the rectangle* (2.15). *If $u \in W^1_{-\beta,0}(\Xi)$ is a solution of problem* (3.2) *with $\lambda = \lambda^\dagger$ and $f \in W^1_{-\beta,0}(\Xi)^* \subset W^1_{\beta,0}(\Xi)^*$, then we have*

$$u = \sum_\pm \chi_\pm \sum_{j=0,1} a_j^\pm w_j + \widetilde{u}, \tag{3.4}$$

where $\chi_\pm \in C^\infty(\mathbb{R})$ are cut-off functions,

$$\chi_\pm(z) = 0 \quad \text{for} \quad \pm z < L, \qquad \chi_\pm(z) = 1 \quad \text{for} \quad \pm z > L + 1, \tag{3.5}$$

with $L \in \mathbb{N}$ and $L \in (l, l + l]$. Also, w_0 and w_1 are the Floquet waves (2.10), *and the coefficients $a_j^\pm$ and the remainder term $\widetilde{u} \in W^1_{\beta,0}(\Xi)$ obey the inequality*

$$\sum_\pm \sum_{j=0,1} |a_j^\pm| + \|\widetilde{u}; W^1_\beta(\Xi)\| \le c_\beta \left(\|f; W^1_{-\beta,0}(\Xi)^*\| + \|u; W^1_{-\beta,0}(\Xi)\|\right). \tag{3.6}$$

Note that the relation $f \in W^1_{-\beta,0}(\Xi)^*$, valid, e.g., for $f \in L^2_\beta(\Xi)$, means in some sense that the functional f decays exponentially.

3.2. Radiation conditions and scattering matrix at the threshold. By Proposition 3.1, the operator $B_0(\lambda^\dagger)$ at the threshold is not Fredholm. At the same time, for $\beta \in (0, \rho)$, the kernel of $B_\beta(\lambda^\dagger)$ consists of the solutions of (1.3) that decay exponentially at infinity, i.e., the trapped waves in the waveguide Ξ which form the subspace $\mathcal{L}^{\mathrm{tr}} = \ker B_\beta(\lambda^\dagger)$ (cf. Subsection 1.3). The asymptotic formula (3.4), involving the four coefficients $a_j^\pm$, shows that the dimension of the kernel of the adjoint operator $B_{-\beta}(\lambda^\dagger) = B_\beta(\lambda^\dagger)^*$ is equal to $2 + \dim \mathcal{L}^{\mathrm{tr}}$. This means that the realization of problem (3.2) as a Fredholm operator with zero index requires posing two radiation conditions involving the coefficients in the expansion (3.4). Since neither the Sommerfeld radiation principle, nor the limit absorption principle work at the threshold (see [10, Chapter 1] and [11]), we invoke Mandelshtam's energy radiation conditions (see [12] and, e.g., [10, Sec. 1], [5, Sec. 5], [13, 11]), which employs the symplectic, i.e., sesquilenear and anti-Hermitian, form

$$q(u, v) = \int_{v(R)} \left(\overline{v(y, R)}\, \partial_z u(y, R) - u(y, R)\, \overline{\partial_z v(y, R)}\right) dy, \tag{3.7}$$

where $v(R) = \{x \in \Pi : z = R\}$ is a cross-section of the quasicylinder (1.2). The expression (3.7) arises as a surface integral in the Green formula for the Helmholtz operator, and hence, does not depend on the parameter R for the Floquet waves (2.10), (2.17), or others. In particular, after integration of (3.7) over $R \in (0,1)$, we find

$$q(u,v) = \int_{\varpi} \big(\overline{v(x)}\,\partial_z u(x) - u(x)\,\overline{\partial_z v(x)}\big)\,dx. \tag{3.8}$$

As was checked in [5, §5.6] (see also [11, §3]), the quantity $\operatorname{Im} q(w,w)$ is proportional to the z projection of the Umov–Pointing vector (see [14, 15]) calculated for the wave w, and, therefore, is called the form of energy transfer. Since w_0 and iw_1 are real functions, we have $q(w_p,w_p) = 0$, so that the standing and resonance waves (2.10) themselves do not transfer energy.

Following [16, 17] (see also [5, Sec. 5]), we introduce the following combined waves (wave packet) in the quasicylinder Π:

$$w_\pm = 2^{-1/2} i (w_1 \pm J^{-1} w_0). \tag{3.9}$$

It should be noted that the limits $w^0_\pm$ of the functions $w^\delta_\pm$ (see (2.17)) differ from the wave packets (3.9); this observation will play an important role in the construction of asymptotics in §4.

Lemma 3.4. *We have*

$$q(w_\pm, w_\pm) = \pm i, \quad q(w_\pm, w_\mp) = 0. \tag{3.10}$$

Proof. The definitions (2.10) and (3.8) show that

$$\begin{aligned} q(w_1,w_0) &= i\int_{\varpi} z\big(\overline{U^0}\partial_z U^0 - U^0\overline{\partial_z U^0}\big)\,dx + \int_{\varpi}\big(\overline{U^0}\partial_z U^1 - U^1\,\overline{\partial_z U^0} + i|U^0|^2\big)\,dx \\ &= 0 + iJ, \end{aligned} \tag{3.11}$$

because the eigenfunction U^0 is real; we have used formula (2.9). Now the verification of (3.10) is obvious. □

The energy radiation principle identifies the propagation direction "from $\mp\infty$ to $\pm\infty$" for the wave $w_\pm$ by the sign of the quantity $\operatorname{Im} q(w_\pm, w_\pm)$, i.e., the rows

$$w^{\mathrm{out}} = (\chi_+ w_+, \chi_- w_-), \quad w^{\mathrm{in}} = (\chi_+ w_-, \chi_- w_+) \tag{3.12}$$

yield the outgoing (*out*) and ingoing (*in*) waves. The cut-off functions $\chi_\pm$ are introduced for localizing the waves in the sleeves $\Pi_\pm$; despite their presence, we call the products in (3.12) waves as before. Observe the important relation

$$w^{\mathrm{in}}(y,z) = \overline{w^{\mathrm{out}}(y,z)}, \tag{3.13}$$

ensured by the definition (3.9) because the functions w_0 and iw_1 are real.

Consider the preimage $\mathcal{W}_\beta(\Xi;\lambda^\dagger) = B_{-\beta}(\lambda^\dagger)^{-1} W^1_{-\beta,0}(\Xi)^*$; by Proposition 3.1 (2), this space consists of functions representable as in (3.4), or equivalently,

$$u = w^{\mathrm{in}} a^{\mathrm{in}} + w^{\mathrm{out}} a^{\mathrm{out}} + \widetilde{u} \tag{3.14}$$

with an exponentially decaying remainder $\widetilde{u}$ and the columns of coefficients a^{in}, $a^{\mathrm{out}} \in \mathbb{C}^2$. Estimate (3.6) turns $\mathcal{W}_\beta(\Xi;\lambda^\dagger)$ into a Hilbert space with the norm

$$\big(|a^{\mathrm{in}}|^2 + |a^{\mathrm{out}}|^2 + \|\widetilde{u}; W^1_\beta(\Xi)\|^2\big)^{1/2}.$$

However, we shall not need this Hilbert structure in this paper. Note that the factor-space $\mathcal{W}_\beta(\Xi;\lambda^\dagger)/W^1_\beta(\Xi)$ has dimension 4 and can be identified with the linear space of Floquet waves in the trunks Π_+ and Π_-.

Problem (3.2) with energy radiation conditions is set on the subspace

$$\mathcal{W}^{\mathrm{out}}_\beta(\Xi;\lambda^\dagger) = \{u \in \mathcal{W}_\beta(\Xi;\lambda^\dagger) : a^{\mathrm{in}} = 0 \in \mathbb{C}^2 \text{ in the expansion (3.14)}\}.$$

The above constructions erase the specifics of periodic trunks and the threshold situation, reducing the proofs to the algebraic operations described in detail, e.g., in [5, Chapter 5] for the case of general formally selfadjoint elliptic problems or in [18] for the case of Dirichlet problems for the Laplace operator.

Proposition 3.2. 1) *For any* $f \in W^1_{-\beta,0}(\Xi)^*$ *satisfying*

$$f(v) = 0, \quad v \in \mathcal{L}^{\mathrm{tr}} = \ker B_\beta(\lambda^\dagger), \tag{3.15}$$

problem (3.2) *admits a solution* $u \in \mathcal{W}^{out}_\beta(\Xi; \lambda^\dagger)$ *that is determined up to an element of the subspace* $\mathcal{L}^{\mathrm{tr}}$*, i.e., up to a trapped wave. The orthogonality conditions*

$$(u, v)_\Xi = 0, \quad v \in \mathcal{L}^{\mathrm{tr}},$$

make the solution unique and ensure the estimate

$$\|u; \mathcal{W}_\beta(\Xi; \lambda^\dagger)\| \le c_\beta \|f; W^1_{\beta,0}(\Xi)^*\|.$$

2) *The subspace* $\ker B_{-\beta}(\lambda^\dagger) \ominus \mathcal{L}^{\mathrm{tr}}$ *has dimension two and is spanned by the two solutions* $\zeta_\pm \in W^1_{-\beta,0}(\Xi)$ *of the homogeneous* ($f = 0$) *problem* (3.2)*, the row of which can be written as*

$$\zeta = (\zeta_+, \zeta_-) = w^{\mathrm{in}} + w^{\mathrm{out}} S + \widetilde{\zeta},$$

where $\widetilde{\zeta} \in W^1_{\beta,0}(\Xi)^2$*, and* S *is the threshold scattering matrix of size* 2×2*, unitary and symmetric*[2]*, i.e.,* $S^* = S^{-1}$ *and* $S = S^\top$*, where* $S^\top$ *and* $S^* = (\bar{S})^\top$ *are, respectively, the transposed matrix and the adjoint matrix.*

3.3. Almost standing waves as limits of trapped ones. Consider a waveguide Ξ^ε with regularly perturbed boundary, namely, on a smooth (as before, of class C^4 for simplicity) open part Γ of the surface $\partial\Theta \cap \partial\Pi$ we define a profile function $h \in C^2_c(\Gamma)$ and put

$$\Gamma^\varepsilon = \{x \in \mathcal{V} \,:\, s \in \Gamma, n = \varepsilon h(s)\}, \tag{3.16}$$

where $\mathcal{V}$ is a d-dimensional neighborhood of Γ, $\mathcal{V} \cap \partial\Theta = \Gamma$, n is the oriented distance to $\partial\Theta$, $n < 0$ inside Θ, and s is an atlas of charts on $\overline{\Gamma}$. In the waveguide with the boundary

$$\partial\Xi^\varepsilon = (\partial\Xi \setminus \Gamma) \cup \Gamma^\varepsilon, \tag{3.17}$$

we consider the following spectral Dirichlet problem similar to (1.3):

$$-\Delta_x u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x), \;\; x \in \Xi^\varepsilon, \quad u^\varepsilon(x) = 0, \;\; x \in \partial\Xi^\varepsilon. \tag{3.18}$$

$$(\nabla_x u^\varepsilon, \nabla_x v^\varepsilon) = \lambda^\varepsilon (u^\varepsilon, v^\varepsilon), \quad v^\varepsilon \in H^1_0(\Xi^\varepsilon), \tag{3.19}$$

The variational setting of this problem gives rise (see [1, Chapter 10]) to a self-adjoint positive definite operator A^ε. Since the boundary is perturbed locally (see [19, §7.6.5]), the essential spectra σ^ε_{es} and σ_{es} of the operators A^ε and $A = A^0$ coincide, but the discrete spectra may differ, first, because of small variations of isolated eigenvalues in the discrete spectrum σ^ε_{di}, and second, because of the possible dropping of eigenvalues from the edge of the essential spectrum σ_{di}. The latter phenomenon is the subject of study in the present paper.

Suppose that, for some positive infinitely small sequence $\{\varepsilon_k\}$, $k \in \mathbb{N}$, the variational problem (3.19) (or (3.18) in the differential setting) has eigenvalues $\lambda^{\varepsilon_k} \in (0, \lambda^\dagger)$, and that

$$\lambda^{\varepsilon_k} \to \lambda^\dagger - 0 \;\; \text{as} \;\; \varepsilon_k \to +0. \tag{3.20}$$

The corresponding eigenfunctions will be normalized by the condition

$$\|u^{\varepsilon_k}; L^2(\Pi^L_\#)\| = 1, \tag{3.21}$$

[2]These two properties are implied by formulas (3.10) and (3.13); see, e.g., [18].

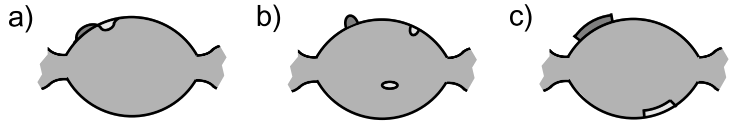

FIGURE 2. The resonator's perturbation: regular (a), singular (b), and step (c). The volume increment is shown by dark color, and volume reduction by light color.

where $\Pi_{\#}^{L} = \{x \in \Xi : L < |z| < L+3\}$ is the union of the bounded parts of the trunks $\Pi_{\pm}$ not intersecting the resonator Θ, because $L > l$ (see comments on the definition of the cut-off functions (3.5)). Each part comprises three periodicity cells.

Theorem 3.1. *Under condition* (3.20), *problem* (3.2) *at the threshold* $\lambda = \lambda^{\dagger}$ *admits a nontrivial solution* $u^0 \in \mathcal{W}_\beta(\Xi; \lambda^{\dagger})$ *with the following coefficients* (*in the expansion* (3.4)):

$$a_1^+ = a_1^- = 0. \tag{3.22}$$

The solution u^{st} described in Theorem 3.1 will be called an almost standing wave, because in each of the trunks $\Pi_{\pm}$ it only differs from the true standing waves $a_0^{\pm} w_0(y,z)$ by summands that decay exponentially as $z \to \pm\infty$: identities (3.22) mean precisely that the expansion (3.4) of this solution has no resonance components. At the threshold we also have a resonance wave $w_1(y,z)$ which, in contrast to $w_0(y,z)$, grows linearly as $z \to \infty$. Therefore, the question on the existence of almost standing waves, the subspace of which will be denoted by $\mathcal{L}^{\mathrm{st}}$ (cf. Subsection 1.3), requires separate study, see Theorem 3.2 below.

3.4. Proof of Theorem 3.1. We start with constructing a coordinate change $x \mapsto x^\varepsilon$ that transforms Ξ^ε into Ξ, differs little from the identical mapping Id and coincides with it for $|z| > L$. Namely, we perform the change of the local coordinates

$$(n,s) \mapsto (n^\varepsilon, s^\varepsilon) = (n - \varepsilon h(s), s),$$

which rectifies the perturbed part Γ^ε of the boundary (3.17) and yields the mapping $x \mapsto x^\varepsilon$ in a neighborhood of ε, and then glue this mapping with the identical one via a suitable partition of unity. The resulting coordinate change is nonsingular for small ε, and its Jacobi matrix satisfies

$$\begin{aligned} &|\mathfrak{J}^\varepsilon(x) - \mathbb{I}_2| \le c\varepsilon,\ |\nabla_x \mathfrak{J}^\varepsilon(x)| \le c\varepsilon, \quad x \in \Theta(L), \\ &\mathfrak{J}^\varepsilon(x) = \mathbb{I}_2 = \operatorname{diag}\{1,1\}, \quad x \in \Xi \setminus \overline{\Theta(L)}. \end{aligned} \tag{3.23}$$

In other words, $x^\varepsilon = x$ for $|z| > L$, i.e., outside of the extended resonator $\Theta(L) = \{x \in \Xi : |z| < L\}$ (recall that $L > l$; cf. the definition (1.1)).

Now we omit the index k at the parameter ε_k. The function $\mathbf{u}^\varepsilon$ given by $\mathbf{u}^\varepsilon(x^\varepsilon) = u^\varepsilon(x)$ remains in the Sobolev space $H_0^1(\Xi)$ and satisfies the integral identify

$$\mathbf{a}^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon) = \lambda^\varepsilon \mathbf{b}^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon), \quad \mathbf{v}^\varepsilon \in C_c^\infty(\Xi), \tag{3.24}$$

where, by (3.23), the bilinear forms $\mathbf{a}^\varepsilon$ and $\mathbf{b}^\varepsilon$, obtained by the change $x \mapsto x^\varepsilon$ from the forms $(\nabla_x u^\varepsilon, \nabla_x v^\varepsilon)_{\Xi^\varepsilon}$ and $(u^\varepsilon, v^\varepsilon)_{\Xi^\varepsilon}$ respectively, obey the estimate

$$\begin{aligned} &|\mathbf{a}^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon) - (\nabla_x \mathbf{u}^\varepsilon, \nabla_x \mathbf{v}^\varepsilon)_\Xi| + |\mathbf{b}^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon) - (\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon)_\Xi| \\ &\qquad \le c\varepsilon \|\mathbf{u}^\varepsilon; H^1(\Theta(L))\| \, \|\mathbf{v}^\varepsilon; H^1(\Theta(L))\|. \end{aligned} \tag{3.25}$$

Note that the direct limit passage as $\varepsilon \to +0$ in (3.24) is illegal, because the Sobolev norms $\|u^\varepsilon; H^1(\Xi^\varepsilon)\|$ grow unboundedly whenever the expansions of the eigenfunctions

u^ε involve slowly decaying exponential Floquet waves (2.17). This forces us to make a workaround.

First, we apply the following local estimate for solutions of the Dirichlet problem for the Helmholtz operator (see, e.g., [20]):

$$\begin{aligned} \|u^\varepsilon; H^2(\varpi_\pm^{L+1})\| &\le c\big(\big\|\Delta u^\varepsilon + \lambda^\varepsilon u^\varepsilon; L^2(\widehat{\varpi}_\pm^{L+1})\big\|^2 + \big\|u^\varepsilon; L^2(\widehat{\varpi}_\pm^{L+1})\big\|^2\big) \\ &\le c(0+1) = c. \end{aligned} \tag{3.26}$$

Here the $\varpi_\pm^{L+1} = \{x \in \Pi : \pm z \in (L+1, L+2)\}$ are shifted periodicity cells, and the $\widehat{\varpi}_\pm^{L+1} = \{x \in \Pi : \pm z \in (L-1/2, L+5/2)\} \subset \Pi_\#^L$ are expanded (at double length) periodicity cells. At the end of formula (3.26) we have used the normalization (3.21). Now we multiply u^ε by the cut-off functions $\chi_\pm$ and observe that the products $\chi_\pm u^\varepsilon \in H_0^1(\Pi_\pm) \subset W_{-\beta,0}^1(\Pi_\pm)$ satisfy problems similar to (3.2) in the clipped trunks $\Pi_\pm$ with the parameter $\lambda = \lambda^\varepsilon$ and compactly supported right-hand sides $f_\pm^\varepsilon$,

$$\|f_\pm^\varepsilon; W_{\beta,0}^1(\Pi_\pm)^*\| \le \|u^\varepsilon; H^1(\varpi_\pm^{L+1})\| \le C.$$

On the truncating surfaces $\{x \in \Pi : z = \pm l\}$, the Dirichlet conditions can be prescribed.

As a result, rewriting estimate (3.3) for the problem stated in $\Pi_\pm$, we get

$$\begin{aligned} &\|\chi_\pm u^\varepsilon; W_{-\beta}^1(\Pi_\pm)\| \\ &\quad\le c\big(\|f_\pm^\varepsilon; W_{\beta,0}^1(\Pi_\pm)^*\| + |\lambda^\varepsilon - \lambda^\dagger|\,\|\chi_\pm u^\varepsilon; L_{-\beta}^2(\Pi_\pm)\| + \|\chi_\pm u^\varepsilon; L^2(\varpi_\pm^{L+1})\|\big) \\ &\quad\le C + c|\lambda^\varepsilon - \lambda^\dagger|\,\|\chi_\pm u^\varepsilon; L_{-\beta}^2(\Pi_\pm)\|. \end{aligned}$$

Since the factor $|\lambda^\varepsilon - \lambda^\dagger|$ is small, these inequalities establish the uniform boundedness of the norms $\|\chi_\pm u^\varepsilon; W_{-\beta}^1(\Pi_\pm)\|$ for small ε. Finally, we plug the test function $v^\varepsilon = X^2 u^\varepsilon$, where $X = 1 - \chi_+ - \chi_- \in C_c^\infty(\bar{\Xi})$, in the integral identify (3.18). Recalling (3.26), after simple calculations we get

$$\|\nabla_x(Xu^\varepsilon); L^2(\Xi^\varepsilon)\|^2 \le \lambda^\varepsilon \|Xu^\varepsilon; L^2(\Xi^\varepsilon)\|^2 + c\|u^\varepsilon; H^1(\varpi_+^{L+1} \cup \varpi_-^{L+1})\|^2 \le \lambda^\dagger + c^2.$$

Thus,

$$\|\mathbf{u}^\varepsilon; W_{-\beta}^1(\Xi)\| \le c\|u^\varepsilon; W_{-\beta}^1(\Xi^\varepsilon)\| \le C, \quad \|\mathbf{u}^\varepsilon; H^2(\varpi_+^{L+1})\| \le c,$$

and we can choose a subsequence $\{\varepsilon_k\}$ (we keep the previous notation) along which

$$\mathbf{u}^\varepsilon \to u^0 \ \text{ weekly in } \ W_{-\beta,0}^1(\Xi) \ \text{ and strongly in } \ L^2(\Pi_\#^L), \tag{3.27}$$

because, clearly, the limit passage preserves the zero trace on the boundary $\partial\Xi$, and the domains $\Pi_\#^L$ and $\varpi_\pm^{L+1}$ are bounded.

The week convergence in (3.27), together with formulas (3.20) and (3.25) ensures that (3.24) turns into the integral identify

$$(\nabla_x u^0, \nabla_x v)_\Xi = \lambda^\dagger (u^0, v)_\Xi,$$

and, by closure, the test functions can be taken from the weighted Sobolev space $W_{\beta,0}^1(\Xi)$, rather than from the linear set $C_c^\infty(\Xi)$. The strong convergence in $L^2(\Pi_\#^L)$ and formulas (3.21), (3.23) show that

$$1 = \|u^\varepsilon; L^2(\Pi_\#^{\varepsilon L})\| \ \to \ \|u^0; L^2(\Pi_\#^L)\| = 1$$

(recall that, by definition, $\Pi_\#^L = \Pi_\#^{\varepsilon L}$). Thus, $u^0 \in W_{-\beta,0}^1(\Xi)$ is a nontrivial solution of problem (3.2) with $\lambda = \lambda^\dagger$ and $f = 0$. It admits expansion as in (3.4), i.e., it remains to check identifies (3.22) themselves.

For $\lambda^\varepsilon < \lambda^\dagger$, the Dirichlet problem in Π acquires an exponentially decaying (as $z \to +\infty$) Floquet wave

$$\mathbf{w}_+^\delta(y, z) = e^{i\eta_+^\delta z}\mathbf{U}_+^\delta(y, z), \tag{3.28}$$

where $\delta := \lambda^\dagger - \lambda^\varepsilon > 0$, and

$$\eta^\delta_\pm = \pm i\delta^{1/2}\zeta + O(\delta), \ \ \zeta > 0, \quad \mathbf{U}^\delta_\pm(y,z) = U^0(y,z) \pm \delta^{1/2}\zeta U^1(y,z) + O(\delta) \tag{3.29}$$

(see Lemma 2.3 and the second estimate in (2.16), which interprets the last relation in (3.29)). We substitute the wave (3.28) together with the exponentially decaying eigenfunction u^ε in the Green formula on the semicylinder $\Pi_+(R) = \{x \in \Pi : z > R\}$ with the end $\tau(R)$, and then integrate additionally over $R \in (L+1, L+2)$:

$$\begin{aligned} 0 &= \int_{L+1}^{L+2}\int_{\tau(R)} \big(\overline{\mathbf{w}^\delta_+(y,R)}\,\partial_z u^\varepsilon(y,R) - u^\varepsilon(y,R)\,\overline{\partial_z \mathbf{w}^\delta_+(y,R)}\big)\,dy\,dR \\ &= \int_{\varpi^{L+1}_+} \big(\overline{\mathbf{w}^\delta_+}\,\partial_z u^\varepsilon - u^\varepsilon\,\partial_z\overline{\mathbf{w}^\delta_+}\big)\,dx \to \int_{\varpi^{L+1}_+}\big(\overline{w_0}\,\partial_z u^0 - u^0\,\overline{\partial_z w_0}\big)\,dx \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{as } \varepsilon \to +0. \end{aligned} \tag{3.30}$$

The limit passage is justified by the asymptotic representations (3.29) and the strong convergence in the space $H^1(\varpi^{L+1}_+)$; the latter is ensured by the following local estimate of the solution of the Dirichlet problem for the Poisson equation with the right-hand side $\lambda^\varepsilon u^\varepsilon$:

$$\|u^\varepsilon; H^1(\omega^{L+1}_\pm)\| \le c\big(\|\lambda^\varepsilon u^\varepsilon; L^2(\Pi^L_\#)\| + \|u^\varepsilon; L^2(\Pi^L_\#)\|\big) = c(1+\lambda^\varepsilon) \le C.$$

Recalling that, by the definition (3.8) and calculation (3.11), the last integral is equal to iJa^+_1, we see that $a^+_1 = 0$. The identify $a^-_1 = 0$ is checked similarly, but with the use of the Floquet wave $\mathbf{w}^\delta_-$ instead of (3.28). Theorem 3.1 is proved. □

The limit passage $u^{\varepsilon_k} \to u^0$ shows that, at the threshold $\lambda = \lambda^\dagger$, problem (1.3) admits solutions bounded at infinity, the subspace of which will be denoted by $\mathcal{L}^{bo}$ (see Subsection 1.3). The trapped waves $u^0 \in \mathcal{L}^{\rm tr} \subset W^1_{\beta,0}(\Xi)$ are certainly in this subspace. A nontrivial element of the subspace (1.4) can be written as

$$u^{\rm st} = \sum_\pm \chi_\pm K_\pm w_0 + \widetilde{u}^{\rm st} \tag{3.31}$$

with a nontrivial column of coefficients $K = (K_+, K_-)^\top \in \mathbb{C}^2$, i.e., (1.4) is indeed the subspace of almost standing waves.

The result established in Theorem 3.1 does not allow us to distinguish between the elements of $\mathcal{L}^{\rm st}$ and $\mathcal{L}^{\rm tr}$. We present two statements that follow from the calculations made above and give different characterizations for the almost standing and trapped waves in the waveguide Ξ.

Corollary 3.1. *Suppose that, in the situation* (3.20), (3.21), *the limit of at least one of the integrals over the ends of the trunks* $\Pi_\pm$

$$\lim_{\varepsilon\to+0}\int_{\tau(\pm l)} \big(\overline{U^1}\,\partial_z u^{\varepsilon_k} + \big(i\overline{U^0} - \partial_z\overline{U^1}\big)u^{\varepsilon_k}\big)\,dy \tag{3.32}$$

is nonzero. Then the limit u^0 *of the eigenfunctions* u^{ε_k} *in the space* $W^1_{-\beta,0}(\Xi)$ *falls into the set* $\mathcal{L}^{\rm st} \setminus \{0\}$.

Corollary 3.2. *Suppose that there is a positive infinitely small sequence* $\{\varepsilon_k\}$ *of values of the parameter* ε *for which problem* (3.19) (*or* (3.18) *in the differential setting*) *has eigenvalues* $\lambda^{\varepsilon_k} > \lambda^\dagger$ *above the threshold, and*

$$\lambda^{\varepsilon_k} \to \lambda^\dagger + 0 \quad as \quad \varepsilon_k \to +0. \tag{3.33}$$

Then the eigenfunctions u^{ε_k} *normalized as in* (3.21) *converge weekly in* $W^1_{-\beta,0}(\Xi)$ *to an eigenfunction* $u^0 \in \mathcal{L}^{\rm tr} \setminus \{0\}$ *of problem* (1.3) *at the threshold* $\lambda = \lambda^\dagger$.

Comments on these two corollaries are in order. The situation (3.33), (3.21) differs only little from that treated in Theorem 3.1. However, at the end of the proof, the exponential decay of the functions u^{ε_k} in the Green formula on $\Pi_\pm(R)$ allows us to use in (3.30) two bounded Floquet waves $\mathbf{w}^\delta_\pm$ defined, in accordance with (2.17), by the positive parameter $\delta = \lambda^\varepsilon - \lambda^\dagger$ and the real root (2.14) of equation (2.13); more precisely, there waves are the linear combinations $\mathbf{w}^\delta_+ + \mathbf{w}^\delta_-$ and $\delta^{-1/2}(\mathbf{w}^\delta_+ - \mathbf{w}^\delta_-)$, which, by Lemma 2.3, have limits $2w_0$ and $2\xi w_1$. As a result, all four coefficients $a^\pm_j$ in the representation (3.4) of the function u^0 vanish.

Finally, for $z = \pm l$ we have

$$w_1 \mp ilw_0 = U^1, \quad \partial_z(w_1 \mp ilw_0) = iU^0 + \partial_z U^1,$$

whence, since we know that $a^\pm_1 = 0$, the limit (3.32) is equal to

$$\int_{\tau(\pm l)} \big(\big(\overline{w_1 \pm ilw_0}\big)\partial_z u^0 + u^0\big(\overline{\partial_z w_1 \pm il\partial_z w_0}\big)\big)\, dy = q(u^0, w_1 \pm ilw_0) = iJa^\pm_0.$$

by the assumptions in Corollary 3.1, at least one of the coefficients $a^\pm_0$ of the standing wave w_0 in the expansion(3.4) of the limit function u^0 is nonzero.

3.5. Criterion for the arising of almost standing waves. A criterion for the existence of trapped waves, in particular, at thresholds of the continuous spectrum of cylindrical waveguides, was developed in [21, 22]. This criterion describes the subspace $\mathcal{L}^{\rm tr} = \ker B_\beta(\lambda^\dagger)$ of exponentially decaying solutions of problem (1.3) in the case of cylindrical outlets $\Pi_\pm = \{(y, z) : y \in \omega, \pm z > l\}$ to infinity in terms of a root subspace of the operator $\mathfrak{S} + \mathsf{Id}$, where $\mathfrak{S}$ is a unitary operator, called the fictitious scattering operator, which is defined with the help of solutions of a special boundary-value problem in the bounded resonator Θ of the waveguide Ξ. No similar results are known in the case of periodic exist to infinity; the author has not been able so far to find conditions ensuring the arising of trapped waves at the threshold (cf. Subsection 6.7). Note that the subspace $\mathcal{L}^{\rm st}$ of almost standing waves is closely related to the threshold scattering matrix occurring in Proposition 3.2.

Theorem 3.2. *We have* $\dim \mathcal{L}^{\rm st} = \dim \ker(S + \mathbb{I}_2)$*, where the right-hand side is the multiplicity of the eigenvalue* -1 *of the threshold scattering matrix* S*. Moreover,*

$$\mathcal{L}^{\rm st} = \left\{u^{\rm st} = \zeta b \,:\, Sb + b = 0,\ b \in \mathbb{C}^2\right\}. \tag{3.34}$$

Proof. Any function $u^{\rm st} \in \mathcal{L}^{\rm st} = \ker B_{-\beta}(\lambda^\dagger) \ominus \ker B_\beta(\lambda^\dagger)$ can be written as the linear combination ζb with a coefficient column $b \in \mathbb{C}^2 \setminus \{(0, 0)\}$ and the row of solutions (3.18). Hence, by the definitions (3.9) and (3.12), we have

$$\begin{aligned} u^{\rm st} &= w^{\rm in} b + w^{\rm out} Sb + \widetilde{u}^{\rm st} \\ &= 2^{-1/2} i(\chi_+ w_1, \chi_- w_1)(b + Sb) - J^{-1} i(\chi_+ w_0, -\chi_- w_0)(b - Sb) + \widetilde{u}^{\rm st}. \end{aligned} \tag{3.35}$$

In (3.35), the resonance wave w_1 is absent if and only if $b + Sb = 0$. In this case, $b - Sb = 2b \neq 0$, i.e., $u^{\rm st}$ cannot become a trapped wave. Finally, obviously, we have $\zeta b \in \mathcal{L}^{\rm st}$ whenever $b \in \ker(S + \mathbb{I}_2)$. □

§4. Asymptotics of near-threshold eigenvalues

4.1. Perturbation of trapped waves. To start with, suppose that $\det(S+\mathbb{I}_2) \neq 0$, but at the threshold $\lambda = \lambda^\dagger$ problem (1.3) in the waveguide Ξ has (real) linearly independent eigenfunctions $u^{\rm tr}_1, \dots, u^{\rm tr}_N$, where $N = \dim \mathcal{L}^{\rm tr}$. We shall show that, by choosing an appropriate profile h of a regular perturbation (3.16) of a part Γ of the waveguide's wall, we can form the following N eigenvalues in the discrete spectrum $\sigma^{\varepsilon}_{di}$:

$$\lambda^{\varepsilon}_j = \lambda^{\dagger} - \varepsilon\lambda'_j + \widetilde{\lambda}^{\varepsilon}_j, \quad \lambda'_j > 0, \quad j = 1, \dots, N. \tag{4.1}$$

The corresponding eigenfunctions will be sought in the form

$$u^{\varepsilon}_j = u^0_j + \varepsilon u'_j + \widetilde{u}^{\varepsilon}_j, \tag{4.2}$$

$$u^0_j = u^{\mathrm{tr}} b^j, \ \ b^j \in \mathbb{R}^N, \quad (b^k)^{\top} b^j = \delta_{j,k}, \ \ j, k = 1, \dots, N, \tag{4.3}$$

where $\delta_{j,k}$ is the Kronecker symbol, $u^{\mathrm{tr}} = (u^{\mathrm{tr}}_1, \dots, u^{\mathrm{tr}}_N)$ is a row of trapped waves subject to the orthogonality and normalization conditions

$$(u^{\mathrm{tr}}_j, u^{\mathrm{tr}}_k)_{\Xi} = \delta_{j,k}, \quad j, k = 1, \dots, N. \tag{4.4}$$

The quality of the small remainders $\widetilde{\lambda}^{\varepsilon}_j$ and $\widetilde{u}^{\varepsilon}_j$ in (4.1) and (4.2) will be described in §5. The functions u^0_j and u'_j on the right-hand side in (4.2), defined initially on Ξ, are extended with preservation of smoothness to a neighborhood $\mathcal{V}$ of the set $\bar{\Gamma} \subset \partial\Xi$; in particular, they are defined everywhere on the perturbed waveguide Ξ^{ε}.

We plug the expressions (4.1), (4.2) in (3.18) and collect the coefficients of ε. As a result, we see that the correction terms u'_j and λ'_j, the factors of ε in (4.1) and (4.2) independent of the small parameter, must satisfy the differential equation

$$-\Delta_x u'_j(x) - \lambda^{\dagger} u'_j(x) = -\lambda'_j u^0_j(x), \quad x \in \Xi, \tag{4.5}$$

and the homogeneous Dirichlet condition on $\partial\Xi \setminus \Gamma$. To deduce the boundary condition on the smooth part Γ of the boundary, we employ the Taylor formula, obtaining

$$\begin{aligned} u^{\varepsilon}_j(\varepsilon h(s), s) &= u^0_j(\varepsilon h(s), s) + \varepsilon u'_j(\varepsilon h(s), s) + \dots \\ &= u^0_j(0, s) + \varepsilon \left(u'_j(0, s) + h(s)\partial_n u^0_y(0, s) \right) + \dots, \end{aligned} \tag{4.6}$$

where $v(n, s)$ is the function v written in the local coordinates (n, s) (see Subsection 3.3), and the dots hide the lower terms unessential for our formal asymptotic analysis. To kill the factor of ε on the right in (4.6), we prescribe the boundary condition

$$u'_j(x) = g'_j(x), \quad x \in \partial\Xi, \tag{4.7}$$

where

$$g'_j(x) = \begin{cases} 0, & \text{if } \ x \in \partial\Xi \setminus \bar{\Gamma}, \\ -h(s)\partial_n u^0_j(x), & \text{if } \ x \in \Gamma. \end{cases} \tag{4.8}$$

As usual, the smooth nonhomogeneity (4.8) in the Dirichlet condition (4.7) will be removed with the help of an extension $G'_j \in H^1(\Xi)$ with support in the set $\overline{\Theta(L)}$. As a result, for the difference $u'_j - G'_j$ we obtain problem (3.2) with the ingredients $\lambda = \lambda^{\dagger}$ and

$$f(v) = \lambda(u^0_j, v)_{\Xi} - (\nabla_x G'_j, \nabla_x v)_{\Xi} + \lambda^{\dagger}(G'_j, v)_{\Xi}. \tag{4.9}$$

Since u^0_j decays exponentially at infinity and G'_j vanishes for $|z| > L$, the functional (4.9) belongs to $W^1_{-\beta,0}(\Xi)^*$ and Proposition 3.2(1) provides N compatibility conditions (3.15) in the resulting problem in the space $\mathcal{W}^{\mathrm{out}}_{\beta}(\Xi; \lambda^{\dagger})$. Thus, after integration by parts, we deduce the following compatibility conditions for problem (4.5), (4.7):

$$\lambda'_j \int_{\Xi} u^0_j u^{\mathrm{tr}}_p \, dx = -\int_{\partial\Xi} u'_j \partial_n u^{\mathrm{tr}}_p \, ds_x, \quad j = 1, \dots, N.$$

Using (4.3), (4.4), and (4.7), we turn these conditions into the system of algebraic equations

$$\lambda_j' b^j = P(h) b^j \tag{4.10}$$

with a symmetric $(N \times N)$-matrix $P(h) = (P_{pq}(h))$, where

$$P_{pq}(h) = \int_\Gamma h(s) \partial_n u_p^{\mathrm{tr}}(0, s) \partial_n u_q^{\mathrm{tr}}(0, s) \, ds_x. \tag{4.11}$$

The theorem on a unique extension[3] implies that neither normal derivatives of the functions u_p^{tr}, nor their nontrivial linear combinations can vanish identically on Γ. Therefore, if $h \geq 0$, $h \not\equiv 0$, then the matrix $P(h)$ is positive definite. Thus, system (4.10) has positive eigenvalues $\lambda_1', \dots, \lambda_N'$, and the corresponding eigenvectors $b^1, \dots, b^N \in \mathbb{R}^N$ obey the orthogonality and normalization conditions as in (4.3). There is no need to require that the volume increment be positive, which is ensured by the nonnegativity of the nontrivial profile h in (3.16) (cf. Remark 4.2). Therefore, in what follows we only assume that the function h is fixed in such a way that the matrix P is positive definite.

The above calculations specify the asymptotic formulas (4.1) and (4.2), the justification of which will be postponed till §5. We only note that the solution of problem (4.5), (4.7) is taken in the form

$$u' = u^{\mathrm{out}} + \zeta c = \sum_\pm \chi_\pm K_\pm' w_0 + \widetilde{u}'. \tag{4.12}$$

The particular solution u^{out}, satisfying the energy radiation conditions, and the elements $\zeta_\pm$ are subject to the orthogonality conditions, and the expansion (3.18) free of the resonance waves is achieved by choosing the coefficient column $c = (c_+, c_-)^\top \in \mathbb{C}^2$. By the calculation (3.35), the last is possible because $\dim \mathcal{L}^{\mathrm{st}} = 0$ by assumption, and the matrix $S + \mathbb{I}_2$ is nonsingular by Theorem 3.2.

4.2. Perturbation of almost standing waves. Now, let $\dim \mathcal{L}^{\mathrm{tr}} = 0$, but $\dim \mathcal{L}^{\mathrm{st}} > 0$, i.e., there are no trapped waves, but almost standing ones do exist.

First, we consider the case of one bounded solution u_1^{st}, which, of course, can be fixed real and admitting formula (3.31) with a coefficient column $K_{(1)} = (K_{1+}, K_{1-})^\top \in \mathbb{R}^2$, $|K_{(1)}| = 1$. The other, unbounded, basis element in the two-dimensional (we have $\dim \ker B_\beta(\lambda^\dagger) = 0$ by assumption) subspace $\ker B_{-\beta}(\lambda^\dagger)$ can be chosen so that

$$u_1^{\mathrm{res}} = \sum_\pm \chi_\pm K_{1\mp} (w_1 \mp K_0 w_0) + \widetilde{u}_1^{res}. \tag{4.13}$$

The specific form of the coefficients of the Floquet waves (2.10) is explained as follows. The coefficients $a_1^\pm = K_{1\mp}$ of w_1 in formula (3.4) for u_1^{res} can be found with the help of normalization and the Green formula, together with relation (3.11):

$$\begin{aligned} 0 &= \lim_{R + \infty} \sum_\pm \int_{v(\pm R)} \big(u_1^{\mathrm{st}}(y, \pm R) \, \partial_z u_1^{\mathrm{res}}(y, \pm R) - u_1^{\mathrm{res}}(y, \pm R) \, \partial_z u_1^{\mathrm{st}}(y, \pm R) \big) \, dy \\ &= \sum_\pm \pm q(a_1^\pm w_1 + a_0^\pm w_0, K_{1\pm} w_0) = K_{1+} a_1^+ - K_{1-} a_1^-, \end{aligned} \tag{4.14}$$

and the coefficients $a_0^\pm = \mp K_0 K_{1\mp}$ are obtained via the change $u_1^{\mathrm{res}} \mapsto u_1^{\mathrm{res}} - c_0 u_1^{\mathrm{st}}$, while the quantities c_0 and K_0 are determined by an algebraic system with the orthogonal

[3] The homogeneous Cauchy problem for the Helmholtz equation has only zero solution (see, e.g., [23, Part 1, Chapter 3]).

matrix $\mathbb{K}_+$:

$$\mathbb{K}_+ \begin{pmatrix} c_0 \\ K_0 \end{pmatrix} = \begin{pmatrix} a_0^+ \\ a_0^- \end{pmatrix} \Leftrightarrow \begin{pmatrix} c_0 \\ K_0 \end{pmatrix} = \mathbb{K}_- \begin{pmatrix} a_0^+ \\ a_0^- \end{pmatrix},$$

$$\mathbb{K}_\pm = \begin{pmatrix} K_{1+} & \mp K_{1-} \\ \pm K_{1-} & K_{1+} \end{pmatrix}.$$

We shall use the method of matched asymptotic expansions (see [24, 25], etc.) in the version of [26, 18]. Namely, for the eigenvalue we assume the asymptotic *Ansatz*

$$\lambda_m^\varepsilon = \lambda^\dagger - \varepsilon^2 \lambda_m' + \widetilde{\lambda}_m^\varepsilon, \quad \lambda_m' > 0, \tag{4.15}$$

and the *Ansatz*

$$u_m^\varepsilon = u_m^0 + \varepsilon u_m' + \dots, \tag{4.16}$$

similar to (4.2), for the eigenfunction will be viewed as the inner expansion valid is a finite but long part of the waveguide Ξ. As in (4.2), the functions u_m^0 and u_m' are extended with preservation of smoothness from Ξ to the perturbed waveguide Ξ^ε.

Formulas (4.15) and (4.16) will also be needed in the sequel; now we assume that $m = 1$ and $u_1^0 = u_1^{\mathrm{st}}$. Two exterior expansions available for large $\pm z$, are of the form

(4.17)

$$\begin{aligned} u_m^\varepsilon(x) &= K_{m\pm}^\varepsilon w_\pm^{\delta_m}(y,z) + \dots = K_{m\pm}^\varepsilon e^{i\eta_\pm^{\delta_m} z} U_\pm^\varepsilon(y,z) + \dots \\ &= K_{m\pm}^0 U_0^0(y,z) + \varepsilon\big(\mp K_{m\pm}^0 i\sqrt{\lambda_m'}|\xi| \big(izU^0(y,z) + U^1(y,z)\big) + K_{m\pm}' U_0(y,z)\big) + \dots \\ &= K_{m\pm}^0 w_0(y,z) + \varepsilon\big(\mp K_{m\pm}^0 i\sqrt{\lambda_m'}\,|\xi| w_1(y,z) + K_{m\pm}' w_0(y,z)\big) + \dots \end{aligned}$$

Here, $K_{m\pm}^\varepsilon = K_{m\pm}^0 + \varepsilon K_{m\pm}' + \dots$ is an unknown coefficient, and the $w_\pm^{\delta_m}(y,z)$ are the Floquet waves as in (2.17), exponentially decaying as $z \to \pm\infty$, with $\delta_m = -\varepsilon^2 \lambda_m' < 0$ (see (2.11) and (4.15)), $|\delta_m|^{1/2} = \varepsilon\sqrt{\lambda_m'}$, and $\xi \in i\mathbb{R}_+$ is the root (2.14) of equation (2.13).

In accordance with the general matching procedure, the exterior expansions (4.17) prescribe the behavior as $z \to \pm\infty$ of the terms of the inner expansions (4.16). Therefore, recalling (3.31), we put at once

$$K_{m\pm}^0 = K_{m\pm}.$$

The problem

$$\begin{aligned} &-\Delta_x u_m'(x) - \lambda^\dagger u_m'(x) = 0, \quad x \in \Xi, \\ &u_m'(x) = 0, \quad x \in \partial\Xi \setminus \Gamma, \quad u_m'(x) = -h(s)\partial_n u_m^0(x), \quad x \in \Gamma, \end{aligned} \tag{4.18}$$

for the correction term u_m' is deduced like problem (4.5), (4.7), with the exception that, since the variation of the spectral parameter (see (4.15)) is of order of ε^2, the term $\lambda_m' u_m^0(x)$ disappeared from the differential equation. Due to Proposition 3.2 and the requirement $\dim \mathcal{L}^{\mathrm{tr}} = 0$, problem (4.18) with the energy radiation conditions has a unique particular solution u_1^{out}, and the general solution in the space $W_{-\beta}^1(\Xi)$ looks like this:

$$u_1' = u_1^{\mathrm{out}} + c_1^{\mathrm{st}} u_1^{\mathrm{st}} + c_1^{\mathrm{res}} u_1^{\mathrm{res}}. \tag{4.19}$$

The coefficients in formula (3.4) for u_1^{out} will be denoted by $a_j^\pm$, as before. Repeating the calculation (4.14), we get

$$\begin{aligned} iJ(K_{1+}a_1^+ - K_{1-}a_1^-) &= \lim_{R\to+\infty} \pm \int_{v(\pm R)} (u_1^{\mathrm{st}} \partial_z u_1^{\mathrm{out}} - u_1^{\mathrm{out}} \partial_z u_1^{\mathrm{st}})\, dx \\ &= -\int_\Gamma (u_1^{\mathrm{st}} \partial_n u_1^{\mathrm{out}} - u_1^{\mathrm{out}} \partial_n u_1^{\mathrm{st}})\, ds = \int_\Gamma h(s)|\partial_n u_1^{\mathrm{st}}(0,s)|^2\, ds_x. \end{aligned} \tag{4.20}$$

Comparison of (4.17) with (3.4), (3.31), and (4.13) shows that the matching procedure requires that at least the following identities be fulfilled:

$$-K_{1+}i\sqrt{\lambda_1'}\,|\xi| = a_1^+ + K_{1-}c_1^{\mathrm{res}},\qquad K_{1-}i\sqrt{\lambda_1'}\,|\xi| = a_1^- + K_{1+}c_1^{\mathrm{res}}. \tag{4.21}$$

Solving this system, we find

$$\sqrt{\lambda_1'}|\xi| = i(K_{1+}a_1^+ - K_{1-}a_1^-) = J^{-1}I_{11}(h).$$

Here $I_{11}(h)$ is the last integral in (4.20). Since, like in Subsection 4.1, the normal derivative $\partial_n u_1^{\mathrm{st}}$ cannot vanish everywhere on Γ, the proper choice of a profile function h allows us to give this integral any prescribed value, in particular, a positive value (e.g., for $h \geq 0$, $h \not\equiv 0$). Thus, using (2.14), we see that

$$I_{11}(h) > 0 \quad \Rightarrow \quad \lambda_1' = |\xi|^{-2}J^{-2}I_{11}(h)^2 = J^{-1}I_{11}(h)^2\|U^0; L^2(\varpi)\|^2, \tag{4.22}$$

completing thereby the calculation of the terms of the *Ansätze* (4.15) and (4.16).

It remains to consider the case of two linearly independent almost standing waves, because if no such waves exist, then, by assumption $\dim \mathcal{L}^{\mathrm{st}} = 0$, the near-threshold eigenvalues are absent by Theorem 3.1. The almost standing waves u_1^{st} and u_2^{st} can be fixed so that the coefficient columns in (3.31) take the form

$$K_{(1)} = (1,0)^\top, \quad K_{(2)} = (0,1)^\top. \tag{4.23}$$

Now we describe the necessary modifications to be made in the asymptotic procedure above.

The *Ansätze* (4.15)–(4.17) with $m = 1, 2$ preserve with the exception that

$$u_m^0 = b_1^m u_1^{\mathrm{st}} + b_2^m u_2^{\mathrm{st}}, \quad m = 1, 2, \tag{4.24}$$

where the $b^m = (b_1^m, b_2^m)^\top \in \mathbb{C}^2$ are orthonormal columns to be determined. The correction terms u_1' and u_2' can be found from problem (4.18) as before, but formula (4.19) for the general solution is replaced with the relation

$$u_m' = u_m^{\mathrm{out}} + c_{m1}^{\mathrm{st}}u_1^{\mathrm{st}} + c_{m2}^{\mathrm{st}}u_2^{\mathrm{st}}, \quad m = 1, 2. \tag{4.25}$$

Since the last two summands are bounded, the linear growth at infinity of the functions (4.25) is only caused by the terms u_m^{out}; the coefficients of the expansion (3.4) for these terms will be denoted by $a_{mj}^{\pm}$. By (4.23). the calculation (4.20) yields the identifies

$$a_{m1}^+ = -iJ^{-1}\int_\Gamma h(s)\,\partial_n u_m^0(0,s)\,\partial_n u_1^{\mathrm{st}}(0,s)\,ds,\qquad a_{m1}^- = iJ^{-1}\int_\Gamma h(s)\,\partial_n u_m^0(0,s)\,\partial_n u_2^{\mathrm{st}}(0,s)\,ds. \tag{4.26}$$

The matching procedure for the exterior expansions (4.17) and the inner expansion (4.16) refined by formulas (3.31), (4.23) for u_m^0 and formulas (4.25), (3.4), and (4.26) for u_m' result in the relations

$$K_{m+}^0 = b_1^m, \quad K_{m-}^0 = b_0^m \tag{4.27}$$

and

$$-K_{m+}^0 i\sqrt{\lambda_m'}\,|\xi| = -iJ^{-1}\left(I_{11}(h)b_1^m + I_{12}(h)b_2^m\right),\qquad K_{m-}^0 i\sqrt{\lambda_m'}\,|\xi| = iJ^{-1}\left(I_{21}(h)b_1^m + I_{22}(h)b_2^m\right)$$

or, in the vector form,

$$\sqrt{\lambda_m'}\,b^m = Q(h)b^m.$$

Here $Q(h)$ and $I(h)$ are symmetric (2×2)-matrices with the entries

$$Q_{jk}(h) = |\xi|^{-1}J^{-1}I_{jk}(h) = \|U^0; L^2(\varpi)\|J^{-1/2}I_{jk}(h), \tag{4.28}$$

$$I_{jk}(h) = \int_\Gamma h(s)\,\partial_n u_j^{\mathrm{st}}(0,s)\,\partial_n u_k^{\mathrm{st}}(0,s)\,ds_x. \tag{4.29}$$

Both matrices can be made positive definite by choosing an appropriate profile function h. This follows from formula (4.29) itself and the next simple claim.

Lemma 4.5. *The traces of the functions* $(\partial_n u_1^{\mathrm{st}})^2$, $\partial_n u_1^{\mathrm{st}}\,\partial_n u_2^{\mathrm{st}}$, *and* $(\partial_n u_2^{\mathrm{st}})^2$ *on* Γ *are linearly independent.*

Proof. By a unique extension theorem, the functions $\partial_n u_1^{\mathrm{st}}$, $\partial_n u_2^{\mathrm{st}}$ and their nontrivial linear combinations cannot vanish on a subset $\gamma\subset\Gamma$ of positive $(d-1)$-measure. Consequently, the products like $\partial_n u_1^{\mathrm{st}}\,\partial_n u_2^{\mathrm{st}}$ also possess this property. Therefore, the identify

$$0 = \alpha_{11}(\partial_n u_1^{\mathrm{st}})^2 + 2\alpha_{12}\partial_n u_1^{\mathrm{st}}\partial_n u_2^{\mathrm{st}} + \alpha_{22}(\partial_n u_2^{\mathrm{st}})^2 \text{ on } \gamma \tag{4.30}$$

implies that $\alpha_{11}^2+\alpha_{22}^2\neq 0$. Suppose $\alpha_{11}=1$ (we can multiply (4.30) by α_{11}^{-1}); then

$$0 = (\partial_n u_1^{\mathrm{st}} + \alpha_{12}\partial_n u_2^{\mathrm{st}})^2 + (\alpha_{22}-\alpha_{12}^2)(\partial_n u_2^{\mathrm{st}})^2.$$

Thus, $\alpha_0^2 = \alpha_{12}^2-\alpha_{22} > 0$ and

$$0 = (\partial_n u_1^{\mathrm{st}} + (\alpha_{12}-\alpha_0)\partial_n u_2^{\mathrm{st}})(\partial_n u_1^{\mathrm{st}} + (\alpha_{12}+\alpha_0)\partial_n u_2^{\mathrm{st}}) \text{ on } \gamma,$$

but above it was explained that this is impossible. □

So, the *Ansätze* (4.10) and (4.16) with $m=1,2$ will be fixed as follows: the positive corrections λ'_m are the squares of (necessarily positive) eigenvalues of the matrix $Q(h)$, and the coefficients of the linear combinations (4.24) form the corresponding orthonormal eigenvector-rows.

Remark 4.1. As in Subsection 4.1, the matrix $Q(h)$ will certainly be positive definite whenever the profile h of wall's distortion (see (3.16)) is nonnegative and nontrivial, which leads to the enlargement of the volume of the waveguide. Lemma 4.5 shows that the needed property of the matrix $Q(h)$ can be kept even if the profile function h changes its sign on Γ, i.e., in principle, the total volume's increment can be made negative.

4.3. The general case. Suppose that at the threshold $\lambda=\lambda^\dagger$ we have $N=\dim\mathcal{L}^{\mathrm{tr}}$ trapped waves and $M=\dim\mathcal{L}^{\mathrm{st}}$ almost standing waves. The perturbation (3.16) of the waveguide's wall, ensuring the arising of $M+N$ near-threshold eigenvalues (4.1) and (4.15), is constructed as earlier, but requires new arguments.

First of all, by Theorem 3.2, the presence of almost standing waves ($M=1$ or $M=2$) makes it impossible to achieve the boundedness of the correction terms (4.12) for formula (4.2), which is necessary for constructing the global asymptotic approximations to the eigenvalues of problem (3.18) (see Subsection 5.3). The integral representations (4.20), (4.14), and (4.26) show that the coefficients $a_1^\pm$ of the linearly growing terms will vanish under the following conditions:

$$\int_\Gamma h(s)\,\partial_n u_j^{\mathrm{tr}}(0,s)\,\partial_n u_k^{\mathrm{st}}(0,s)\,ds_x = 0, \quad j=1,\dots,N,\ \ k=1,\dots,M. \tag{4.31}$$

By Proposition 3.2.1, the same orthogonality conditions establish the existence of a component u_m^{out} that is involved in the correction terms (4.19) or (4.25) of the inner expansion (4.16) and solves problem (4.18) with the energy radiation conditions.

Of course, we can find a profile function h satisfying (4.31), but it is difficult to arrange that $h\geq 0$, $h\not\equiv 0$, which ensures the positive definiteness of the matrix $P(h)$ with the entries (4.11). The conditions needed for the construction of the eigenvalues (4.1) and (4.15) can easily be satisfied in the case where the products

$$\begin{aligned}&\partial_n u_j^{\mathrm{tr}} \partial_n u_p^{\mathrm{tr}},\ \ \partial_n u_j^{\mathrm{tr}} \partial_n u_k^{\mathrm{st}},\ \ \partial_n u_k^{\mathrm{st}} \partial_n u_q^{\mathrm{st}},\\ &j,p=1,\dots,N,\ \ p\le j,\quad k,q=1,\dots,M,\ \ q\le k,\end{aligned} \tag{4.32}$$

(in total, $\frac{1}{2}(N+M+1)(N+M)$ functions) are linearly independent in $L^2(\Gamma)$. Assuming this (cf. the discussion in Subsection 6.6), we choose a profile h in such a way that the two matrices $P(h)$ and $Q(h)$ are positive definite and relations (4.31) are fulfilled. As a result, we find all ingredients of the *Ansätze* adopted, completing the formal asymptotic analysis.

Remark 4.2. If the requirements imposed on the matrices $P(h)$ and $Q(h)$ cannot be respected completely, then the asymptotic *Ansätze* can be kept as before, but they provide fewer good approximations to eigenvalues in the discrete spectrum eigenvalues.

§5. Justification of asymptotics

5.1. Abstract setting. Because of variation of the waveguide's boundary, the Lebesgue norm estimates of the discrepancies left by the approximate eigenvalues in problem f^ε do not give any acceptable accuracy in handling the asymptotic remainders (cf. Remark 5.3). It is convenient to relax the norm of the space where the discrepancy in question lie. In the space $\mathcal{H}^\varepsilon = H_0^1(\Xi^\varepsilon)$ we introduce the scalar product

$$\langle u^\varepsilon, v^\varepsilon\rangle = (\nabla_x u^\varepsilon, \nabla_x v^\varepsilon)_{\Xi^\varepsilon} \tag{5.1}$$

and the continuous positive self-adjoint operator $\mathcal{K}^\varepsilon$ given by the identify

$$\langle \mathcal{K}^\varepsilon u^\varepsilon, v^\varepsilon\rangle = (u^\varepsilon, v^\varepsilon)_{\Xi^\varepsilon},\quad u^\varepsilon, v^\varepsilon\in\mathcal{H}^\varepsilon. \tag{5.2}$$

Comparing formulas (5.1), (5.2), and (3.19), we see that the variational problem (3.18) is equivalent to the abstract equation

$$\mathcal{K}^\varepsilon u^\varepsilon = \kappa^\varepsilon u^\varepsilon \text{ in } \mathcal{H}^\varepsilon \tag{5.3}$$

with the new spectral parameter

$$\kappa^\varepsilon = (\lambda^\varepsilon)^{-1}. \tag{5.4}$$

The link (5.4) shows that the essential spectrum $\varsigma_{es}^\varepsilon$ of the operator $\mathcal{K}^\varepsilon$ lies on the (closed) segment $[0,\kappa^\dagger]$, while on the ray $(\kappa^\dagger,+\infty)$ we may find only points of the discrete spectrum $\varsigma_{di}^\varepsilon$, which will be the object of study in what follows.

The one-dimensional Hardy inequality yields the relation

$$\|\varrho_\varepsilon^{-1} u^\varepsilon; L^2(\Xi^\varepsilon)\| \le C_\varrho \|\nabla_x u^\varepsilon; L^2(\Xi^\varepsilon)\| = C_\varrho\|u^\varepsilon;\mathcal{H}^\varepsilon\|, \tag{5.5}$$

where $\varrho_\varepsilon = \operatorname{dist}(x,\partial\Xi^\varepsilon)$ and C_ϱ is a constant independent of ε and u^ε. As a result, the discrepancy f^ε should be estimated in the adjoint space $H^{-1}(\Xi^\varepsilon) = (\mathcal{H}^\varepsilon)^*$ or in the weighted Lebesgue space with the norm $\|\varrho_\varepsilon f^\varepsilon; L^2(\Xi^\varepsilon)\|$ (cf. the left hand side in (5.5)). The weight factor ϱ_ε becomes small in a narrow neighborhood of the perturbed part Γ^ε of the waveguide's wall, and the errors of asymptotic analysis caused by the boundary's variation are concentrated precisely there. Note that, in a neighborhood of the mentioned smooth part (3.16) of $\partial\Xi^\varepsilon$, the above inequality takes the form

$$\int_{-d}^{\varepsilon h(s)} (n-\varepsilon h(s))^{-2} |u^\varepsilon(n,s)|^2\,dn \le 4\int_{-d}^{\varepsilon h(s)} \left|\frac{\partial u^\varepsilon}{\partial n}(n,s)\right|^2 dn$$

and is true in the case where $u^\varepsilon(\varepsilon h(s),s)=0$ for $s\in\Gamma$. In fact, inequality (5.5) is valid for Lipschitz boundaries.

5.2. Spectral measure. In order to verify the existence of a single near-threshold eigenvalue, it suffices to possess the asymptotic constructions of approximate "eigenpairs" of problem (3.18) or equation (5.3), as presented in the preceding section. In particular, these constructions provide a lower estimate for the norm of the resolvent of the operator A^ε or $\mathcal{K}^\varepsilon$, and, thus, an upper estimate for the distance from the spectrum (see comments in Remark 5.3). Since the eigenvalues in question can be multiple, we are forced to deal also with eigenfunctions, for which we need elementary information from the theory of spectral measure (see, e.g., [1, Chapter 6]).

The positive self-adjoint operator $\mathcal{K}^\varepsilon$ gives rise to a spectral measure E^ε, which in its turn generates a nonnegative scalar measure

$$\mu^\varepsilon_{\mathcal{U}} = \langle E^\varepsilon \mathcal{U}^\varepsilon, \mathcal{U}^\varepsilon \rangle$$

for each $\mathcal{U}^\varepsilon \in \mathcal{H}^\varepsilon$. We have

$$\int_{\mathbb{R}} d\mu^\varepsilon_{\mathcal{U}}(t) = \|\mathcal{U}^\varepsilon; \mathcal{H}^\varepsilon\|^2, \tag{5.6}$$
$$\int_{\mathbb{R}} (t - k^\varepsilon)^2 d\mu^\varepsilon_{\mathcal{U}}(t) = \|\mathcal{K}^\varepsilon \mathcal{U}^\varepsilon - k^\varepsilon \mathcal{U}^\varepsilon; \mathcal{H}^\varepsilon\|^2.$$

(cf. the proof of Theorem 6.1.3 in [1]). Also, assuming that the segment

$$\theta^\varepsilon = [k^\varepsilon - \gamma_\varepsilon, k^\varepsilon + \gamma_\varepsilon] \tag{5.7}$$

is free of the spectrum ς^ε, we get

$$\int_{\mathbb{R}} (t - k^\varepsilon)^2 d\mu^\varepsilon_{\mathcal{U}}(t) = \int_{\mathbb{R}\setminus\theta^\varepsilon} (t - k^\varepsilon)^2 d\mu^\varepsilon_{\mathcal{U}}(t) \geq \gamma_\varepsilon^2 \int_{\mathbb{R}\setminus\theta^\varepsilon} d\mu^\varepsilon_{\mathcal{U}}(t) = \gamma_\varepsilon^2 \int_{\mathbb{R}} d\mu^\varepsilon_{\mathcal{U}}(t). \tag{5.8}$$

In what follows, on the basis of the asymptotic procedures of §4, we shall find pairs $\{k^\varepsilon_j, \mathcal{U}^\varepsilon_j\}$, $j = 1, \dots, J$, such that

$$\|\mathcal{K}^\varepsilon \mathcal{U}^\varepsilon_j - k^\varepsilon_j \mathcal{U}^\varepsilon_j; \mathcal{H}^\varepsilon\| < \gamma_\varepsilon \|\mathcal{U}^\varepsilon_j; \mathcal{H}^\varepsilon\| \tag{5.9}$$

with some small γ_ε. Relation (5.9) contradicts formulas (5.6), (5.8); hence, the segment (5.7) includes points of the spectrum ς^ε of the operator $\mathcal{K}^\varepsilon$, and if we know additionally that $k^\varepsilon - \gamma^\varepsilon > \kappa^\dagger$, then this segment necessarily includes at least one point of the discrete spectrum σ^ε_{di}.

Remark 5.3. Inequality (5.9) means that the norm of the resolvent $(\mathcal{K}^\varepsilon - k^\varepsilon\,\mathsf{Id})^{-1}$ exceeds γ_ε^{-1}. Consequently, the well-known relation

$$\operatorname{dist}(k^\varepsilon, \varsigma^\varepsilon) = \|(\mathcal{K}^\varepsilon - k^\varepsilon\,\mathsf{Id})^{-1}\|^{-1},$$

implied by the spectral expansion of the resolvent (see, e.g., [1, Chapter 6]) shows that $\theta^\varepsilon \cap \varsigma^\varepsilon \neq \varnothing$. If, moreover, $\theta^\varepsilon \cap \varsigma^\varepsilon_{es} = \varnothing$, then on the segment (5.7) we have at least one eigenvalue. However, in this way we cannot check that the spectrum $\varsigma^\varepsilon_{di}$ has many point.

Suppose that the factor γ_ε on the right in (5.9), playing the role of approximation error, is so small that the segment (5.8) with center k^ε_j contains, among $k^\varepsilon_1, \dots, k^\varepsilon_J$, only the quantities

$$k^\varepsilon_j = \dots = k^\varepsilon_{j+\varkappa-1}$$

with some $j \geq 1$ and $\varkappa \geq 1$. We check that, under some additional requirement (see (5.12) below) imposed on the pairs $\{k^\varepsilon_p, \mathcal{U}^\varepsilon_p\}$, $p = j, \dots, j + \varkappa - 1$, the wider segment

$$\Theta^\varepsilon_j = [k^\varepsilon_j - \gamma'_\varepsilon, k^\varepsilon_j + \gamma'_\varepsilon], \quad \gamma'_\varepsilon = T\gamma_\varepsilon \geq \gamma_\varepsilon, \tag{5.10}$$

includes at least $\varkappa$ eigenvalues of $\mathcal{K}^\varepsilon$. This result will allow us to identify the required number of points in the discrete spectrum.

We normalize the approximate eigenfunctions,

$$\mathcal{W}_p^\varepsilon = \|\mathcal{U}_p^\varepsilon; \mathcal{H}^\varepsilon\|^{-1} \mathcal{U}_p^\varepsilon, \quad p = j, \dots, j + \varkappa - 1, \tag{5.11}$$

and suppose that

$$|\langle \mathcal{W}_p^\varepsilon, \mathcal{W}_q^\varepsilon \rangle - \delta_{p,q}| \le \tau_\varepsilon, \quad p, q = j, \dots, j + \varkappa - 1, \tag{5.12}$$

(in fact, defining the number τ_ε). Introducing the orthogonal projection

$$\mathcal{P}_j^\varepsilon = \int_{\Theta_j^\varepsilon} dE^\varepsilon(t), \tag{5.13}$$

we verify that, for small τ_ε in (5.12) and large T in (5.10), the projections $\mathcal{P}_j^\varepsilon \mathcal{W}_j^\varepsilon, \dots, \mathcal{P}_j^\varepsilon \mathcal{W}_{j+\varkappa-1}^\varepsilon$ are linearly independent, i.e., on the segment $\Theta_j^\varepsilon \subset (\kappa^\dagger, +\infty)$ we have at least $\varkappa$ eigenvalues of the operator $\mathcal{K}^\varepsilon$.

By formulas (5.9), (5.11) and (5.13), (5.10), we have

$$\begin{aligned} \|\mathcal{U}_p^\varepsilon - \mathcal{P}_j^\varepsilon \mathcal{U}_p^\varepsilon; \mathcal{H}^\varepsilon\|^2 &= \|\mathcal{U}_p^\varepsilon; \mathcal{H}^\varepsilon\|^2 - \|\mathcal{P}_j^\varepsilon \mathcal{U}_p^\varepsilon; \mathcal{H}^\varepsilon\|^2 = \int_{\mathbb{R}} d\mu_{\mathcal{U}_p}^\varepsilon(t) - \int_{\Theta_j^\varepsilon} d\mu_{\mathcal{U}_p}^\varepsilon(t) \\ &= \int_{\mathbb{R}\setminus\Theta_j^\varepsilon} d\mu_{\mathcal{U}_p}^\varepsilon(t) \le \frac{1}{(\gamma_\varepsilon')^2} \int_{\mathbb{R}\setminus\Theta_j^\varepsilon} (t - k_j^\varepsilon)^2 \, d\mu_{\mathcal{U}_p}^\varepsilon(t) = \frac{1}{(\gamma_\varepsilon')^2} \|\mathcal{K}^\varepsilon \mathcal{U}_p^\varepsilon - k_j^\varepsilon \mathcal{U}_p^\varepsilon; \mathcal{H}^\varepsilon\|^2 \\ &= \Big(\frac{\gamma_\varepsilon}{\gamma_\varepsilon'}\Big)^2 \|\mathcal{U}_p^\varepsilon; \mathcal{H}^\varepsilon\|^2 \quad \Rightarrow \quad \|\mathcal{W}_p^\varepsilon - \mathcal{P}_j^\varepsilon \mathcal{W}_p^\varepsilon; \mathcal{H}^\varepsilon\| \le \frac{1}{T}. \end{aligned}$$

Now we deduce the inequalities

$$\begin{aligned} |\langle \mathcal{P}_j^\varepsilon \mathcal{W}_p^\varepsilon, \mathcal{P}_j^\varepsilon \mathcal{W}_q^\varepsilon \rangle - \delta_{p,q}| &\le |\langle \mathcal{W}_p^\varepsilon, \mathcal{W}_q^\varepsilon \rangle - \delta_{p,q}| + |\langle \mathcal{P}_j^\varepsilon \mathcal{W}_p^\varepsilon, \mathcal{P}_j^\varepsilon \mathcal{W}_q^\varepsilon \rangle - \langle \mathcal{W}_p^\varepsilon, \mathcal{W}_q^\varepsilon \rangle| \\ &\le \tau_\varepsilon + |\langle \mathcal{P}_j^\varepsilon \mathcal{W}_p^\varepsilon - \mathcal{W}_p^\varepsilon, \mathcal{P}_j^\varepsilon \mathcal{W}_q^\varepsilon \rangle| + |\langle \mathcal{W}_p^\varepsilon, \mathcal{P}_j^\varepsilon \mathcal{W}_q^\varepsilon - \mathcal{W}_q^\varepsilon \rangle| \\ &\le \tau_\varepsilon + 2T^{-1}. \end{aligned}$$

Thus, indeed, the projections $\mathcal{P}_j^\varepsilon \mathcal{W}_j^\varepsilon, \dots, \mathcal{P}_j^\varepsilon \mathcal{W}_{j+\varkappa-1}^\varepsilon$ are "almost orthonormal" for small τ_ε and $1/T$, i.e., the dimension of the subspace $\mathcal{P}_j^\varepsilon \mathcal{H}^\varepsilon$ is at least $\varkappa$, as claimed.

We shall not formulate the general form of the resulting simple statement, known as the lemma on "near eigenvalues and eigenvectors" (precisely this property is given to the pairs $\{k_p^\varepsilon, \mathcal{W}_p^\varepsilon\}$ by inequalities (5.9) and (5.12)) for small γ_ε and τ_ε; instead, we shall deal with the specific situations as in §4.

5.3. No almost standing waves. Consider the case treated in Subsection 4.1. In accordance with (5.4) and (4.1), the role of the approximate eigenvalues of the operator $\mathcal{K}^\varepsilon$ (see (5.2)) is given to the quantities

$$k_j^\varepsilon = (\lambda^\dagger - \varepsilon \lambda_j')^{-1} \in (\kappa^\dagger, +\infty), \quad j = 1, \dots, N, \tag{5.14}$$

where $\lambda_1', \dots, \lambda_N'$ are positive (by construction) eigenvalues of the $(N \times N)$-matrix $P(h)$ with the entries (4.11). Starting with the asymptotic formula (4.2), we build global approximations $\mathcal{U}_1^\varepsilon, \dots, \mathcal{U}_N^\varepsilon$ to eigenfunctions of problem (3.18). First, we ensure the Dirichlet boundary condition on the perturbed wall (3.16).

Both terms u_j^0 and u_j' of formula (4.2) are smooth in the neighborhood $\mathcal{V}$ of the set $\bar{\Gamma}$ where the local coordinates (n, s) are introduced (recall the extension outside Ξ). Therefore, we have the Taylor formulas

$$\begin{aligned} &u_j^0(x) = 0 + n\partial_n u_j^0(0, s) + \widehat{u}_j^0(x), \quad u_j'(x) = u_j'(0, s) + \widehat{u'}_j(x), \\ &|\widehat{u}_j^0(x)| \le cn^2, \ \ |\nabla_x \widehat{u}_j^0(x)| \le cn, \quad |\widehat{u'}_j(x)| \le cn, \quad |\nabla_x \widehat{u'}_j(x)| \le c, \ \ x \in \mathcal{V}. \end{aligned} \tag{5.15}$$

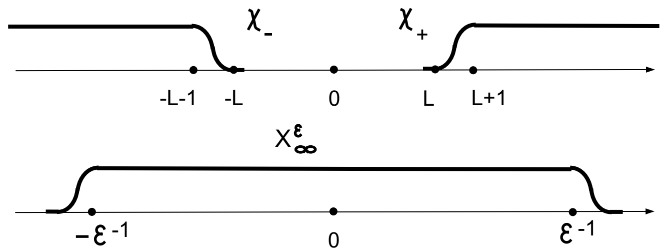

FIGURE 3. The graphs of the cut-off functions $\chi_\pm$ and X^ε_∞.

Let $\chi_\Gamma \in C^\infty_c(\Gamma)$ and $\chi^\varepsilon_\Gamma \in C^\infty_c(\mathbb{R})$ be cut-off functions such that $\chi_\Gamma = 1$ on the support of the profile h, and

$$\begin{aligned} &\chi^\varepsilon_\Gamma(n) = 1 \quad \text{for} \quad |n| \le 2\varepsilon h_0, \quad \chi^\varepsilon_\Gamma(n) = 0 \quad \text{for} \quad |n| \ge 3\varepsilon h_0, \\ &h_0 = \max\{|h(s)| \,:\, s \in \Gamma\}, \quad |\nabla^q_x \chi^\varepsilon_\Gamma(n)| \le c_q \varepsilon^{-q}, \;\; q = 0,1,2,\dots. \end{aligned} \tag{5.16}$$

Suppose also that $\mathcal{X}^\varepsilon_\Gamma = 1$ outside of $\mathcal{V}$ and $\mathcal{X}^\varepsilon_\Gamma(x) = 1 - \chi_\Gamma(s)\chi^\varepsilon_\Gamma(n)$ inside $\mathcal{V}$. By the boundary conditions in problem (1.3) and (4.5), (4.7), the function

$$\mathcal{U}^\varepsilon_{\Gamma j}(x) = \mathcal{X}^\varepsilon_\Gamma(x)\big(u^0_j(x) + \varepsilon u'_j(x)\big) + \chi^\varepsilon_\Gamma(n)\chi_\Gamma(s)\big(n\partial_n u^0_j(0,s) + \varepsilon u'_j(0,s)\big) \tag{5.17}$$

vanishes everywhere on the boundary $\partial\Xi^\varepsilon$.

Since the correction term (4.12) may fail to decay, so that it does not belong to $H^1(\Xi^\varepsilon)$, the function (5.17) need modification near infinity. Namely, we match it with the Floquet waves $w^{\delta_j}_\pm(y,z)$ that decay exponentially as $z \to \pm\infty$ and are defined by (2.17) with $\delta_j = (k^\varepsilon_j)^{-1} - \lambda^\dagger = -\varepsilon\lambda'_j < 0$ (cf. (2.11) and (5.14), (5.4)). We introduce yet another cut-off function $X^\varepsilon_\infty \in C^\infty(\mathbb{R})$,

$$X^\varepsilon_\infty(z) = 1 \quad \text{for} \quad |z| \le \varepsilon^{-1}, \quad X^\varepsilon_\infty(z) = 0 \quad \text{for} \quad |z| \ge 1 + \varepsilon^{-1}, \quad |\partial^q_z X^\varepsilon_\infty(z)| \le c_q, \tag{5.18}$$

the support of which overlaps those of the cut-off functions (3.5) used before (see the graphs in Figure 3).

We apply the asymptotic constructions that combine the methods of composite and matched expansions (see, e.g., [27, Chapter 2] and [28]). Set

$$\mathcal{U}^\varepsilon_j = X^\varepsilon_\infty \mathcal{U}^\varepsilon_{\Gamma j} + \varepsilon \sum_\pm \chi_\pm K'_{j\pm}\big(w^{\delta j}_\pm - X^\varepsilon_\infty w_0\big), \tag{5.19}$$

where the coefficients $K'_{j\pm}$ are taken from the expansion (4.12) of the function u'_j.

We calculate the scalar products $\langle \mathcal{U}^\varepsilon_p, \mathcal{U}^\varepsilon_q \rangle = (\nabla_x \mathcal{U}^\varepsilon_p, \nabla_x \mathcal{U}^\varepsilon_q)_{\Xi^\varepsilon}$. On the central part $\Theta^\varepsilon(l) = \{x \in \Xi^\varepsilon \,:\, |z| < l\}$ of the waveguide we have

$$\begin{aligned} \nabla_x \mathcal{U}^\varepsilon_j &= \nabla_x\big(u^0_j + \varepsilon u'_j - \chi^\varepsilon_\Gamma \chi_\Gamma(\widehat{u}^0_j + \varepsilon \widehat{u'}_j)\big) \\ &= \nabla_x u^0_j + \varepsilon \nabla_x u'_j - \chi^\varepsilon_\Gamma \chi_\Gamma \nabla_x\big(\widehat{u}^0_j + \varepsilon\widehat{u'}_j\big) - \big(\widehat{u}^0_j + \varepsilon\widehat{u'}_j\big)\nabla_x(\chi^\varepsilon_\Gamma\chi_\Gamma). \end{aligned} \tag{5.20}$$

As always, the functions u^0_j and u'_j are assumed to be smoothly extended from Ξ to $\Xi \cup \mathcal{V}$. Estimating the remainders in the Taylor formulas (5.15) and using the definition (5.16) of the cut-off functions, we see that the last two summands in (5.20) are concentrated on a set of volume $O(\varepsilon)$, and that their modules do not exceed $c\varepsilon$. Consequently, since the small factor ε is present in the second term on the right in (5.20), we have

$$\|\nabla_x \mathcal{U}^\varepsilon_j - \nabla_x u^0_j; L^2(\Theta^\varepsilon(l))\| \le c\varepsilon^{3/2}. \tag{5.21}$$

Since $\mathcal{X}^\varepsilon_\Gamma = 1$ and $\chi^\varepsilon_\Gamma \chi_\Gamma = 0$ on the trunks $\Pi_\pm$ of the waveguide Ξ, recalling the expansion (4.12) of the solution u'_j of problem (4.5), (4.7) and formula (5.19), we get the identity

$$\nabla_x \mathcal{U}_j^\varepsilon = X_\infty^\varepsilon \nabla_x\big(u_j^0 + \varepsilon \widetilde{u'}_j\big) + \big(u_j^0 + \varepsilon \widetilde{u'}_j\big)\nabla_x X_\infty^\varepsilon + \varepsilon \sum_\pm K'_{j\pm}\nabla_\pm\big(\chi_\pm w_\pm^{\delta_j}\big). \tag{5.22}$$

Due to the exponential decay of the trapped wave u_j^0 and the remainder $\widetilde{u'}_j = u'_j - \sum_\pm \chi_\pm K'_{j\pm} w_0$ in the expansion (4.12), we have

$$\|X_\infty^\varepsilon \nabla_x(u_j^0 + \varepsilon \widetilde{u'}_j) - \nabla_x u_j^0; L^2(\Pi_\pm)\| \le c\varepsilon,$$

and the $L^2(\Pi_\pm)$-norm of last but one term in (5.22) is exponentially small as $\varepsilon \to +0$, because the gradient $\nabla_x X_\infty^\varepsilon$ can be nonzero only for $|z| \in [\varepsilon^{-1}, \varepsilon^{-1}+1]$. Finally, since the Floquet waves $w_\pm^{\delta_j}(y,z)$ decay slowly as $z \to \pm\infty$, we have

$$\varepsilon\big\|\nabla_x\big(\chi_\pm w_\pm^{\delta_j}\big); L^2(\Pi_\pm)\big\| \le c\varepsilon\bigg(\int_l^{+\infty} e^{-2\xi_j\sqrt{\varepsilon} z}\, dz\bigg)^{1/2} \le c\varepsilon^{3/4}.$$

Thus, we have obtained the formulas

$$|(\nabla_x \mathcal{U}_p^\varepsilon, \nabla_x \mathcal{U}_q^\varepsilon)_{\Xi^\varepsilon} - (\nabla_x u_p^0, \nabla_x u_q^0)_\Xi| \le c\varepsilon^{3/4}, \quad p,q = 1,\dots,N. \tag{5.23}$$

The subtrahend under the module sign is equal to $\lambda^\dagger(u_p^0, u_q^0)_\Xi = \lambda^\dagger \delta_{p,q}$, because the functions $u_p^0, u_q^0 \in \mathcal{L}^{\rm tr}$ solve problem (1.3) at the threshold $\lambda = \lambda^\dagger$ and the orthogonality and normalization conditions (4.4) and (4.3) are fulfilled. As a result, the normalized asymptotic approximations (5.11) satisfy the requirement (5.12) with the infinitely small quantity $\tau_\varepsilon = C\varepsilon^{3/4}$ on the right-hand side.

Now we estimate the factor γ_ε occurring in (5.9). Using formulas (5.1), (5.3) and one of the definitions of the Hilbert norm, we get

$$\begin{aligned}\|\mathcal{K}^\varepsilon \mathcal{U}_j^\varepsilon - k_j^\varepsilon \mathcal{U}_j^\varepsilon; \mathcal{H}^\varepsilon\| &= \sup|\langle \mathcal{K}^\varepsilon \mathcal{U}_j^\varepsilon - k_j^\varepsilon \mathcal{U}_j^\varepsilon, \mathcal{Z}^\varepsilon\rangle| \\ &= (\lambda^\dagger - \varepsilon\lambda'_j)^{-1}\sup\big|(\lambda^\dagger - \varepsilon\lambda'_j)(\mathcal{U}_j^\varepsilon, \mathcal{Z}^\varepsilon)_{\Xi^\varepsilon} - (\nabla_x \mathcal{U}_j^\varepsilon, \nabla_z \mathcal{Z}^\varepsilon)_{\Xi^\varepsilon}\big| \\ &= (\lambda^\dagger - \varepsilon\lambda'_j)^{-1}\sup\big|((\Delta_x + \lambda^\dagger - \varepsilon\lambda'_j)\mathcal{U}_j^\varepsilon, \mathcal{Z}^\varepsilon)_{\Xi^\varepsilon}\big|,\end{aligned} \tag{5.24}$$

where the supremum is computed over all $\mathcal{Z}^\varepsilon \in \mathcal{H}^\varepsilon$ such that $\|\mathcal{Z}^\varepsilon; \mathcal{H}^\varepsilon\| = 1$, and hence, by (5.5), we have

$$\|\varrho_\varepsilon^{-1}\mathcal{Z}^\varepsilon; L^2(\Xi^\varepsilon)\| \le C_\rho.$$

To process the last scalar product in (5.24), first we note that on the sleeves $\Pi_\pm$ its first factor takes the form

$$\begin{aligned}&X_\infty^\varepsilon(\Delta_x + \lambda^\dagger - \varepsilon\lambda'_j)(u_j^0 + \varepsilon u'_j) + [\Delta_x, X_\infty^\varepsilon](u_j^0 + \varepsilon\widetilde{u'}_j) \\ &\quad + \varepsilon\sum_\pm \chi_\pm K'_{j\pm}(\Delta_x + \lambda^\dagger - \varepsilon\lambda'_j)w_\pm^{\delta_j} + \varepsilon\sum_\pm K'_{j\pm}[\Delta_x, \chi_\pm]\big(w_\pm^{\delta_j} - w_0\big).\end{aligned} \tag{5.25}$$

Here, the first term is equal to $-\varepsilon^2\lambda'_j X_\infty^\varepsilon u'_j$, and admits the estimate

$$\varepsilon^2\lambda'_j\|X_\infty^\varepsilon u'_j; L^2(\Pi_\pm)\| \le c\varepsilon^2\bigg(\int_l^{1+\varepsilon^{-1}} dz\bigg)^{1/2} \le c\varepsilon^{3/2}.$$

because the function (4.12) is bounded.

The supports of the coefficients of the commutators $[\Delta_x, X_\infty^\varepsilon] = 2\nabla_x X_\infty^\varepsilon \cdot \nabla_x + (\Delta_x X_\infty^\varepsilon)$ lie in the set $\{x \in \overline{\Pi} : |z| \in [\varepsilon^{-1}, 1+\varepsilon^{-1}]\}$ where the functions u_j^0 and $\widetilde{u}'_j$ are exponentially small (see (5.18)). Each term of the last but one sum in (5.25) vanishes by the definition of the Floquet waves (2.17). The supports of the coefficients of the commutator $[\Delta_z, \chi_\pm]$ lie in the set $\overline{\varpi_\pm^L} = \{x \in \overline{\Pi} : \pm z \in [L, L+1]\}$ (see (3.5)). Hence, by estimate (2.16) in

Lemma 2.3, we have $\|w_{\pm}^{\delta_j} - w_0; H^1(\varpi_{\pm}^L)\| \le c\delta_j \le c_j\sqrt{\varepsilon}$, so that the $L^2(\Pi_{\pm})$-norms of the last two summands do not exceed $c\varepsilon^{3/2}$. As a result, we get

$$\big|((\Delta_x + \lambda^\dagger - \varepsilon\lambda_j')\,\mathcal{U}_j^\varepsilon, \mathcal{Z}^\varepsilon)_{\Pi_\pm}\big| \le c\varepsilon^{3/2}.$$

On the middle part $\Theta^\varepsilon(l)$ of the perturbed waveguide, the expression $(\Delta_x+\lambda^\dagger-\varepsilon\lambda_j')\mathcal{U}_j^\varepsilon$ can be written as a sum:

$$\begin{aligned}\mathcal{X}_\Gamma^\varepsilon\left(\Delta_x + \lambda^\dagger - \varepsilon\lambda_j'\right)(u_j^0 + \varepsilon u_j') + \chi_\Gamma^\varepsilon\chi_\Gamma\left(\Delta_x + \lambda^\dagger - \varepsilon\lambda_j'\right)\left(n\partial_n u_j^0\big|_{n=0} + \varepsilon u_j'\big|_{n=0}\right)\\ -[\Delta_x, \chi_\Gamma^\varepsilon\chi_\Gamma]\big(\widehat{u}_j^0 + \varepsilon\widehat{u'}_j\big) =:\ \mathcal{Y}_1^\varepsilon + \mathcal{Y}_2^\varepsilon + \mathcal{Y}_3^\varepsilon.\end{aligned} \tag{5.26}$$

The differential equation (4.5) shows that the first term $\mathcal{Y}_1^\varepsilon$ takes the form $-\varepsilon^2\lambda_j'\mathcal{X}_\Gamma^\varepsilon u_j'$ on Ξ and becomes $O(\varepsilon)$ on the narrow set $\Theta^\varepsilon(l)\setminus\Xi$, because of the smooth extension of the functions u_j^0 and u_j'. Hence, using the Hardy inequality (5.5), we obtain

$$\begin{aligned}|(\mathcal{Y}_1^\varepsilon, \mathcal{Z}^\varepsilon)_{\Xi^\varepsilon}| &\le c(\varepsilon^2\|\mathcal{Z}^\varepsilon; L^2(\Xi)\| + \varepsilon\|\mathcal{Z}^\varepsilon; L^2(\Theta^\varepsilon(l)\setminus\Xi)\|)\\ &\le c\varepsilon^2\|\varrho_\varepsilon^{-1}\mathcal{Z}^\varepsilon; L^2(\Xi^\varepsilon)\| \le c\varepsilon^2.\end{aligned} \tag{5.27}$$

The second term $\mathcal{Y}_2^\varepsilon$ is bounded uniformly in ε and is supported in the set $\mathcal{T}_\varepsilon = \{x \in \mathcal{V} : s \in \bar{\Gamma}, |n| \le 2\varepsilon h_0\}$ (see (5.16)); the volume $\operatorname{meas}_d \mathcal{T}_\varepsilon$ is $O(\varepsilon)$, and for $x \in \mathcal{T}_\varepsilon$ the weight factor ϱ_ε is at most $2\varepsilon h_0$. Thus, we have

$$\begin{aligned}|(\mathcal{Y}_2^\varepsilon, \mathcal{Z}^\varepsilon)_{\Theta^\varepsilon(l)}| &\le \|\mathcal{Y}_2^\varepsilon; L^2(\mathcal{T}_\varepsilon)\|\,\|\mathcal{Z}^\varepsilon; L^2(\mathcal{T}_\varepsilon)\|\\ &\le c\varepsilon^{1/2}\varepsilon\|\varrho_\varepsilon^{-1}\mathcal{Z}^\varepsilon; L^2(\Theta^\varepsilon(l))\| \le c\varepsilon^{3/2}.\end{aligned} \tag{5.28}$$

Finally, we apply the first order differential operator $[\Delta_x, \chi_\Gamma^\varepsilon\chi_\Gamma]$ (its coefficients are supported in $\mathcal{T}_\varepsilon$) to the remainder terms $\widehat{u}_j^0$ and $\widehat{u'}_j$ of the Taylor formulas (5.15), obtaining the estimate

$$\begin{aligned}&\|(\mathcal{Y}_3^\varepsilon, \mathcal{Z}^\varepsilon)_{\Theta^\varepsilon(l)}\|\\ &\le c\big(\varepsilon^{-2}\|\widehat{u}_j^0 + \varepsilon\widehat{u'}_j; L^2(\mathcal{T}^\varepsilon)\| + \varepsilon^{-1}\|\nabla_x\widehat{u}_j^0 + \varepsilon\nabla_x\widehat{u'}_j; L^2(\mathcal{T}_\varepsilon)\|\big)\|\mathcal{Z}^\varepsilon; L^2(\mathcal{T}_\varepsilon)\|\\ &\le c\varepsilon^{1/2}\big(\varepsilon^{-2}\varepsilon^2 + \varepsilon^{-1}\varepsilon\big)\,\varepsilon\,\|\varrho_\varepsilon^{-1}\mathcal{Z}^\varepsilon; L^2(\Theta^\varepsilon(l))\| \le c\varepsilon^{3/2}.\end{aligned} \tag{5.29}$$

The factors ε^{-2} and ε^{-1} have arisen from estimating the second and first derivative of the cut-off function χ_Γ^ε (see (5.16)), and the factor $\varepsilon^{1/2}$, as usual, is related to the smallness order of the quantity $(\operatorname{meas}_d \mathcal{T}_\varepsilon)^{1/2}$.

Remark 5.4. Should we use the standard (not weighted) Lebesgue norm of the test function $\mathcal{Z}^\varepsilon$, none of estimates (5.27)–(5.29) would acquire the desired majorant $c\varepsilon^{3/2}$.

Thus, we have verified the two requirements (5.9) and (5.12) of the abstract method described in Subsection 5.2, and the norm (5.24) does not exceed $c\varepsilon^{3/2}$, so that $\gamma_\varepsilon \le c\varepsilon^{3/2}$ because $\|\mathcal{U}_j^\varepsilon; \mathcal{H}^\varepsilon\| \ge c > 0$ by (5.23). It remains to state the theorem on the asymptotics of eigenvalues.

Theorem 5.3. *In the situation of Subsection* 4.1, *there exist positive numbers* ε_0 *and* c_0 *such that, for* $\varepsilon \in (0, \varepsilon_0)$, *the interval* $(0, \lambda^\dagger)$ *contains* N *points* $\lambda_1^\varepsilon, \dots, \lambda_N^\varepsilon$ *of the spectrum* σ_{di}^ε. *These points obey the asymptotic formulas* (4.1), *where* $\lambda_1', \dots, \lambda_N'$ *are positive eigenvalues of the symmetric positive definite* $(N \times N)$*-matrix* $P(h)$ *with the entries* (4.11), *and the remainder terms satisfy*

$$\big|\widetilde{\lambda}_j^\varepsilon\big| \le c_0\varepsilon^{3/2}, \quad j = 1, \dots, N. \tag{5.30}$$

It should be noted that the membership relation

$$(\lambda_j^\varepsilon)^{-1} = \kappa_j^\varepsilon \in \big[k_j^\varepsilon - C_j\varepsilon^{3/2}, k_j^\varepsilon + C_j\varepsilon^{3/2}\big], \tag{5.31}$$

ensured by the inequality $\gamma_\varepsilon \le C\varepsilon^{3/2}$ established above, together with the arguments of Subsection 5.2, can easily be reshaped into estimate (5.30).

5.4. No trapped waves. We restrict our arguments to the case where $M = \dim \mathcal{L}^{\rm st} = 2$, because, by Remark 5.3, the case of $M = 1$ (only one standing wave) is considerably simpler.

So, suppose that problem (1.3) at the threshold $\lambda = \lambda^\dagger$ admits no trapped waves, but has two almost standing waves (3.31) with coefficient columns as in (4.23). We follow the same pattern as in Subsection 5.3 and refer to formulas already written. First of all, in accordance with (4.15) and (5.1), the role of approximate eigenvalues of the operator $\mathcal{K}^\varepsilon$ will be played by the quantities

$$k_m^\varepsilon = (\lambda^\dagger - \varepsilon^2 \lambda_m')^{-1} \in (\kappa^\dagger, +\infty), \quad m = 1, 2. \tag{5.32}$$

However, for several reasons, the asymptotic constructions (5.17) and (5.19) need modification. First, to the inner expansion (4.2), a correction term $\varepsilon^2 u_m''(x)$ of the next order should be added. It could be found from a boundary-value problem similar to (4.5), (4.7), but, aiming at justification of the asymptotics constructed, for the role of u_m'' we may take any smooth function supported in $\Theta(l) \cup \Gamma$ and satisfying the boundary condition

$$u_m''(x) = -\frac{1}{2} h(s)^2 \partial_n^2 u_m^0(0, s) - h(s)\, \partial_n u_m'(0, s), \quad x \in \Gamma,$$

(cf. the deduction (4.6) of relation (4.7)). Since the surface Γ is smooth, the Taylor formulas (5.15) admit the following refinement:

$$\begin{aligned} &u_m^0(x) = n\partial_n u_m^0(0, s) + \frac{1}{2} n^2 \partial_n^2 u_m^0(0, s) + \widehat{u}_m^0(x), \\ &u_m'(x) = u_m'(0, s) + n\partial_n u_m'(0, s) + \widehat{u'}_m(x), \quad u_m''(x) = u_m''(0, s) + \widehat{u''}_m(x), \\ &\left|\nabla_x^p \widehat{u}_m^0(x)\right| \le c n^{3-p}, \ \ p = 0, 1, 2, 3, \quad \left|\nabla_x^p \widehat{u'}_m(x)\right| \le c n^{2-p}, \ \ p = 0, 1, 2, \\ &\left|\nabla_x^p \widehat{u''}_m(x)\right| \le c n^{1-p}, \ \ p = 0, 1. \end{aligned} \tag{5.33}$$

Using the same cut-off functions χ_Γ and χ_Γ^ε as before, we put

$$\begin{aligned} \mathcal{U}_{\Gamma m}^\varepsilon(x) = \mathcal{X}_\Gamma^\varepsilon(x) \left(u_m^0(x) + \varepsilon u_m'(x) + \varepsilon^2 u_m''(x)\right) + \chi_\Gamma^\varepsilon(n)\chi_\Gamma(s)\Big(n\partial_n u_m^0(0, s) \\ + \frac{1}{2} n^2 \partial_n^2 u_j^0(0, s) + \varepsilon(u_m'(0, s) + n\partial_n u_m'(0, s)) + \varepsilon^2 u_m''(0, s)\Big). \end{aligned} \tag{5.34}$$

by analogy with (5.17).

Using this new notation, we get a sum similar to (5.26):

$$\begin{aligned} (\Delta_x + \lambda^\dagger - \varepsilon\lambda_m')\,\mathcal{U}_m^\varepsilon &= \mathcal{X}_\Gamma^\varepsilon(\Delta_x + \lambda^\dagger - \varepsilon^2\lambda_m')(u_m^0 + \varepsilon u_m' + \varepsilon^2 u_m'') \\ &+ \chi_\Gamma^\varepsilon \chi_\Gamma(\Delta_x + \lambda^\dagger - \varepsilon\lambda_m')\Big(n\partial_n u_m^0 + \frac{1}{2} n^2 \partial_n^2 u_m^0 + \varepsilon\,(u_m' + n\partial_n u_m') + \varepsilon^2 u_m''\Big) \\ &- [\Delta_x, \chi_\Gamma^\varepsilon \chi_\Gamma]\big(\widehat{u}_m^0 + \varepsilon \widehat{u'}_m + \varepsilon^2 \widehat{u''}_m\big) =: \mathcal{Y}_1^\varepsilon + \mathcal{Y}_2^\varepsilon + \mathcal{Y}_3^\varepsilon. \end{aligned} \tag{5.35}$$

On the initial waveguide Ξ, the first term $\mathcal{Y}_1^\varepsilon$ on the right takes the form

$$\varepsilon^2 \lambda_m'(u_m^0 + \varepsilon u_m' + \varepsilon^2 u_m'') + \varepsilon^2(\Delta_x + \lambda^\dagger) u_m'',$$

and on the thin set $\Theta^\varepsilon(l) \setminus \Xi$ we have

$$|\mathcal{Y}_1^\varepsilon(x)| \le c(|n| + \varepsilon) \le c(1 + h_0)\varepsilon.$$

Recall that the functions u_m^0, u_m', and u_m'' are smoothly extended outside of Ξ and, in particular, $\left|(\Delta_x + \lambda^\dagger) u_m^0(x)\right| \le c|n|$. Thus, estimate (5.27) is preserved completely.

The third and second terms in (5.35) are concentrated on the set $\mathcal{T}_\varepsilon$ of thickness $2\varepsilon h_0$. Since the quadratic terms are detached in the first Taylor formula (5.33), we have

$$\mathcal{Y}_2^\varepsilon = \chi_\Gamma^\varepsilon \chi_\Gamma (\Delta_x + \lambda^\dagger) u_m^0(x) + \varepsilon \widehat{\mathcal{Y}}_2^\varepsilon,$$
$$|\widehat{\mathcal{Y}}_2^\varepsilon(x)| \le c \;\Rightarrow\; |\mathcal{Y}_2^\varepsilon(x)| \le c(|n| + \varepsilon) \le c(1+h_0)\varepsilon.$$

As a result, relation (5.28) is replaced with

$$|(\mathcal{Y}_2^\varepsilon, \mathcal{Z}^\varepsilon)_{\Theta^\varepsilon(l)}| \le c\varepsilon\varepsilon^{1/2} \|\mathcal{Z}^\varepsilon; L^2(\mathcal{T}_\varepsilon)\| \le c\varepsilon^{3/2}\,\varepsilon \|\rho_\varepsilon^{-1}\mathcal{Z}^\varepsilon; L^2(\Theta^\varepsilon(l))\| \le c\varepsilon^{5/2}. \tag{5.36}$$

The refinement (5.33) of the Taylor formula (5.15) reduces the remainders and improves estimate (5.29) as follows:

$$\begin{aligned}\big|(\mathcal{Y}_3^\varepsilon, \mathcal{Z}^\varepsilon)_{\Theta^\varepsilon(l)}\big| &\le c\big(\varepsilon^{-2}\|\widehat{u}_m^0 + \varepsilon \widehat{u'}_m + \varepsilon \widehat{u'}_m + \varepsilon^2 \widehat{u''}_m; L^2(\mathcal{T}_\varepsilon)\| \\ &\quad + \varepsilon^{-1}\|\nabla_x \widehat{u}_m^0 + \varepsilon \nabla_x \widehat{u'}_m + \varepsilon^2 \nabla_x \widehat{u''}_m; L^2(\mathcal{T}_\varepsilon)\|\big)\, \|\mathcal{Z}^\varepsilon; L^2(\mathcal{T}_\varepsilon)\| \\ &\le c(\varepsilon^{-2}\varepsilon^3 + \varepsilon^{-1}\varepsilon^2)\,\varepsilon^{1/2}\,\varepsilon\|\rho_\varepsilon^{-1}\mathcal{Z}^\varepsilon; L^2(\Theta^\varepsilon(l))\| \le c\varepsilon^{5/2}.\end{aligned} \tag{5.37}$$

The modification (see Subsection 4.2) of the asymptotic construction for the eigenfunctions generated by almost standing waves forces us to complicate the structure (5.19) of the global asymptotic approximation. Namely, using the "overlapping" cut-off functions (5.18) and (3.5) (see Figure 3 and also [27, Chapter 2], [28], as well as [18, 29] as applied to cylindrical waveguides), we put

$$\mathcal{U}_m^\varepsilon = X_\infty^\varepsilon \mathcal{U}_{\Gamma_m}^\varepsilon + \sum_\pm \chi_\pm K_{m\pm}^0 w_\pm^{\delta_m} - X_\infty^\varepsilon \sum_\pm \chi_\pm K_{m\pm}^0 \big(w_0 \mp \varepsilon i \sqrt{\lambda'_m} |\xi| w_1\big). \tag{5.38}$$

Here $\mathcal{U}_{\Gamma m}^\varepsilon$ is the function (5.34), the $K_{m\pm}^0$ are the coefficients (4.27), and the $w_\pm^{\delta_\pm}$ are the Floquet waves (2.17) the expansion (4.17) of which involves the terms collected in the last sum in (5.38). There terms were subject to matching in Subsection 4.2, and they are present simultaneously in the first two summands on the right in (5.26), i.e., the subtrahend removes doubling.

Since $\chi_\pm = 0$ on $\Theta^\varepsilon(l)$, formulas (5.27), (5.36), and (5.37) provide estimates for a fragment of the last scalar product in an identity similar to (5.24):

$$\|\mathcal{K}^\varepsilon \mathcal{U}_m^\varepsilon - \kappa_m^\varepsilon \mathcal{U}_m^\varepsilon; \mathcal{H}^\varepsilon\| = (\lambda^\dagger - \varepsilon^2\lambda'_m)^{-1} \sup \big|((\Delta_x + \lambda^\dagger - \varepsilon^2 \lambda'_m)\mathcal{U}_m^\varepsilon, \mathcal{Z}^\varepsilon)_{\Xi^\varepsilon}\big|.$$

It remains to handle the scalar product restricted to the trunks $\Pi_\pm$, where we have

$$\begin{aligned}(\Delta_x + \lambda^\dagger - \lambda'_m)\mathcal{U}_m^\varepsilon &= X_\infty^\varepsilon (\Delta_x + \lambda^\dagger - \varepsilon^2\lambda'_m)(u_m^0 + \varepsilon u'_m) \\ &+ [\Delta_x, X_\infty^\varepsilon]\Big(u_m^0 + \varepsilon u'_m - \sum_\pm \chi_\pm K_{m\pm}^0 \big(w_0 \mp \varepsilon i \sqrt{\lambda'_m}\,|\xi| w_1\big)\Big) \\ &- X_\infty^\varepsilon \sum_\pm \chi_\pm K_{m\pm}^0 (\Delta_x + \lambda^\dagger - \varepsilon^2\lambda'_m)\big(w_0 \mp \varepsilon i\sqrt{\lambda'_m}\,|\xi| w_1\big) \\ &+ \sum_\pm \chi_\pm K_{m\pm}^0 (\Delta_x + \lambda' - \varepsilon^2 \lambda'_m) w_\pm^{\delta_m} \\ &- \sum_\pm K_{m\pm}^0 [\Delta_x, \chi_\pm]\big(w_\pm^{\delta_m} - w_0 \pm \varepsilon i \sqrt{\lambda'_m}\,|\xi| w_1\big) \\ &=: \mathcal{F}_1^\varepsilon + \mathcal{F}_2^\varepsilon + \mathcal{F}_3^\varepsilon + \mathcal{F}_4^\varepsilon + \mathcal{F}_5^\varepsilon.\end{aligned}$$

Observe at once that $\mathcal{F}_4^\varepsilon = 0$ by the definition (2.17) of the Floquet waves $w_\pm^{\delta_m}$ with $\delta_m = (k_m^\varepsilon)^{-1} - \lambda^\dagger = -\varepsilon^2\lambda'_m < 0$ (cf. (2.11) and (5.32), (5.4)). The functions u_m^0, u'_m and

w_0, w_1 satisfy the Helmholtz equation in $\Pi_\pm$, i.e.,

$$\begin{aligned}\mathcal{F}_1^\varepsilon + \mathcal{F}_3^\varepsilon &= -X_\infty^\varepsilon \varepsilon^2 \lambda'_m \Big(u_m^0 + \varepsilon u'_m - \sum_\pm \chi_\pm K_{m\pm}^0 \big(w_0 \mp \varepsilon i \sqrt{\lambda'_m}\, |\xi| w_1 \big) \Big) \\ &= -X_\infty^\varepsilon \varepsilon^2 \lambda'_m (\widetilde{u}_m^0 + \varepsilon \widetilde{u'}_m).\end{aligned} \tag{5.39}$$

We have used the terms of the expansions (3.4) of the solutions u_m^0 and u'_m, obtained by the matching procedure in Subsection 4.2; at the end of formula (5.39) we have exponentially decaying remainders $\widetilde{u}_m^0$ and $\widetilde{u'}_m$. Thus,

$$|(\mathcal{F}_1^\varepsilon + \mathcal{F}_3^\varepsilon, \mathcal{Z}^\varepsilon)_{\Pi_\pm}| \le c\varepsilon^2.$$

Also, the expression $\mathcal{F}_2^\varepsilon = [\Delta_x, X_\infty^\varepsilon](\widetilde{u}_m^0 + \varepsilon \widetilde{u'}_m)$ may be nonzero only if $\pm z \in [\varepsilon^{-1}, \varepsilon^{-1}+1]$, and the functions $\widetilde{u}_m^0$ and $\widetilde{u'}_m$ decay exponentially as $z \to \pm\infty$. Therefore,

$$|(\mathcal{F}_2^\varepsilon, \mathcal{Z}^\varepsilon)_{\Pi_\pm}| \le c e^{-\beta/\varepsilon}, \quad \beta > 0.$$

Finally, the coefficients of the commutators $[\Delta_x, \chi_\pm]$ are supported in the sets $\{x \in \overline{\Pi} : \pm z \in [L, L+1]\}$; therefore, using formula (2.17) for $w_\pm^{\delta_m}$ and inequality (2.16) from Lemma 2.3 with the parameter $|\delta_m| = O(\varepsilon^2)$, we see that

$$|(\mathcal{F}_5^\varepsilon, \mathcal{Z}^\varepsilon)_{\Pi_\pm}| \le c|\delta_m| \le C\varepsilon^2.$$

Collecting the estimates, we conclude that

$$\|\mathcal{K}^\varepsilon \mathcal{U}_m^\varepsilon - k_m^\varepsilon \mathcal{U}_m^\varepsilon; \mathcal{H}^\varepsilon\| \le C\varepsilon^2.$$

In the sequel, we shall verify the formulas

$$|\langle \mathcal{U}_m^\varepsilon, \mathcal{U}_p^\varepsilon \rangle - \varepsilon^{-1} \mu_m \delta_{m,p}| \le c, \quad m, p = 1, 2, \tag{5.40}$$

with a positive quantity μ_m, see (5.42) below. Consequently, $\|\mathcal{U}_m^\varepsilon; \mathcal{H}^\varepsilon\| \ge c\varepsilon^{-1/2}$, $c > 0$. Thus we have $\gamma_\varepsilon \le c\varepsilon^{5/2}$ and $\tau_\varepsilon \le c\varepsilon^{1/2}$ under conditions (5.9) and (5.12), respectively. This conclusion completes the verification of the theorem on the asymptotics of eigenvalues. We state this theorem for both cases, $M = 2$ and $M = 1$ – as has been already mentioned, the latter case is much simpler than the former.

Theorem 5.4. *Suppose that* $\dim \mathcal{L}^{\mathrm{tr}} = 0$ *and* $M = \dim \mathcal{L}^{\mathrm{tr}} > 0$ (*see Subsection* 4.2). *Then there exist positive numbers* ε_0 *and* c_0 *such that for any* $\varepsilon \in (0, \varepsilon_0)$ *there exist* M *points of the spectrum* σ_{di}^ε *of problem* (3.18), *and these points admit the asymptotic representations* (4.15) *with*

$$\big|\widetilde{\lambda}_m^\varepsilon\big| \le c_0 \varepsilon^{5/2}, \tag{5.41}$$

and $\lambda'_1 > 0$ *is the quantity* (4.22) *in the case where* $M = 1$, *and* $\sqrt{\lambda'_1}$, $\sqrt{\lambda'_2}$ *are the* (*positive*) *eigenvalues of the symmetric positive definite* (2×2)*-matrix* $Q(h)$ *with the entries* (4.28) *in the case where* $M = 2$.

We refer to the commentary to Theorem 5.3 and establish the simple result announced above.

Lemma 5.6. *Inequality* (5.40) *is valid with*

$$\mu_m = (4\lambda'_m)^{-1/2} J^{1/2} \lambda^\dagger \|U^0; L^2(\varpi)\| > 0, \tag{5.42}$$

where J *is as in* (2.9).

Proof. Like in (5.39), we write the restrictions $\mathcal{U}_m^{\varepsilon\pm}$ of the global approximations (5.38) to the trunks $\Pi_\pm$ as follows:

$$\mathcal{U}_m^{\varepsilon\pm} = \sum_\pm K_{m\pm}^0 w_\pm^{\delta_m} + X_\infty^\varepsilon \big(\widetilde{u}_m^{\,0} + \varepsilon \widetilde{u'}_m\big).$$

The last summand decays exponentially at infinity and has finite Sobolev $H^1(\Pi_\pm)$-norm. Moreover, the calculations in Subsection 5.3 that resulted in formula (5.21), which pertained to the function (5.17) but remain valid for the function (5.21) because the added terms are small, show that the norm $\|\mathcal{U}_m^\varepsilon; H^1(\Xi^\varepsilon(l))\|$ is also bounded uniformly in ε. Simultaneously, the norms $w_\pm^{\delta_m}(y,z)$ grow unboundedly as $z \to \pm\infty$, because the Floquet waves $\|\nabla_x w_\pm^{\delta_m}; L^2(\Pi_\pm)\|$ decay slowly as $\varepsilon \to +0$. We calculate the asymptotics of there waves, recalling that

$$\begin{aligned} &\delta_m = -\varepsilon^2 \lambda'_m \quad \Rightarrow \quad |\delta_m|^{1/2} = \varepsilon (\lambda'_m)^{1/2}, \\ &\beta_m := (\lambda'_m)^{1/2} |\xi| = (\lambda'_m)^{1/2} J^{-1/2} \|U^0; L^2(\varpi)\|. \end{aligned}$$

In accordance with (2.12) and (2.17), the last quantity indicates the leading term of the asymptotics of the exponent $i\eta_\pm^{\delta_m} = \mp\varepsilon\beta_m + O(\varepsilon)$ in the Floquet waves $w_\pm^{\delta_m}$ (cf. (2.17)). We have

$$\begin{aligned} \|w_+^{\delta_m}; L^2(\Pi_+)\|^2 &= \sum_{n=l}^{+\infty} \int_{\varpi_+^n} e^{2\varepsilon(-\beta_m + O(\varepsilon))z} \big(|\nabla_x U^0(y,z)|^2 + O(\varepsilon)\big)\, dy\, dz \\ &= \big(\|\nabla_x U^0; L^2(\varpi)\|^2 + O(\varepsilon)\big) \sum_{n=l}^{+\infty} e^{2\varepsilon(-\beta_m + O(\varepsilon))n} (1 + O(\varepsilon)) \\ &= \big(\lambda^\dagger \|U^0; L^2(\varpi)\|^2 + O(\varepsilon)\big) \int_l^{+\infty} e^{2\varepsilon(-\beta_m + O(\varepsilon))z}\, dz\, (1 + O(\varepsilon)) \\ &= \frac{1}{\varepsilon} \frac{1}{2\beta_m} \lambda^\dagger \|U^0; L^2(\varpi)\|^2 (1 + O(\varepsilon)). \end{aligned} \tag{5.43}$$

In (5.43), we can replace the formal $O(\varepsilon)$ with infinitely small quantities $c\,\varepsilon$ or $-c\,\varepsilon$ with appropriate c, and the equality sign "=" with the signs "≤" or "≥" respectively, obtaining a two-sided estimate of the form

$$-c \le \|w_\pm^{0m}; L^2(\Pi_\pm)\|^2 - \varepsilon^{-1}\mu_m \le c, \quad c > 0, \tag{5.44}$$

(the norm $\|w_-^{0m}; L^2(\Pi_-)\|$ is listed in (5.44) due to the symmetry of the waveguide).

It remains to observe that, by (4.27) and (4.24), the expansion (3.31) of the almost standing waves $u_1^{\rm st}$ and $u_2^{\rm st}$ involves the orthonormal columns $(K_{1+}^0, K_{1-}^0)^\top$ and $(K_{2+}^0, K_{2-}^0)^\top$. □

5.5. Asymptotics of eigenvalues in the general case. Suppose that the products (4.32) are linearly independent in $L^2(\Gamma)$ and that the profile function h is chosen so that, first, identities (4.31) are true, and second, the matrices $P(h)$ and $Q(h)$ (the latter is a scalar if $M = 1$) with the entries (4.11) and (4.28) are positive definite. The formal asymptotic *Ansätze* (4.1) and (4.15) differ substantially: the approximate eigenvalues (5.14) caused by the trapped waves lie at a distance of $O(\varepsilon)$ from the threshold $\kappa^\dagger$, and the approximate eigenvalues caused by the almost standing waves lie at a distance of $O(\varepsilon^2)$ from the same threshold (cf. Subsection 6.7). This observation allows us, when implementing the abstract approach of Subsection 5.2, to treat the groups of approximations mentioned above separately. Therefore the calculations of Subsections 5.3 and 5.4 lead to the following statement.

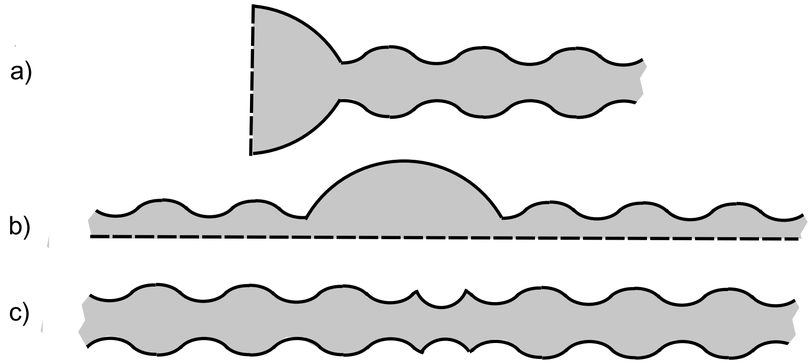

FIGURE 4. The halves (a and b) of symmetric waveguides, and a periodic waveguide with notches (c).

Theorem 5.5. *Under the restrictions indicated above* (*see Subsection* 4.3), *there exist positive numbers* ε_0 *and* c_0 *such that for any* $\varepsilon \in (0, \varepsilon_0)$ *the interval* $(0, \lambda^\dagger)$ *carries* $N + M = \dim \mathcal{L}^{\rm tr} + \dim \mathcal{L}^{\rm st}$ *eigenvalues of problem* (3.18) *with the asymptotic representations* (4.1), (5.30) *and* (4.15), (5.41).

If the choice of the profile h (see (3.16)) of wall's perturbation ensures a smaller number of positive (of course, nonzero) eigenvalues of the matrices $P(h)$ and $Q(h)$, then, as was said in Remark 4.2, the number of the points of the discrete spectrum $\sigma_{di}^{\varepsilon}$ detected in $(0, \lambda^\dagger)$ also becomes smaller. However, the modifications needed in Theorems 5.5, 5.3, and 5.4 are clear, and we shall not state the corresponding new claims.

§6. EXAMPLES, REMARKS, AND OPEN QUESTIONS

6.1. Existence of almost standing and trapped waves at the threshold. Suppose that the domain Ξ is symmetric relative to the plane $\{x : z = 0\}$. On its half $\Xi^+ = \{x \in \Xi : z > 0\}$ (see Figure 4,a) we pose the boundary-value problem

$$-\Delta_x u(x) = \lambda u(x),\ \ x \in \Xi^+, \quad u(x) = 0,\ \ x \in \partial\Xi^+ \setminus \bar{\Upsilon}, \tag{6.1}$$

with the fictitious Dirichlet or Neumann conditions

$$u^D(x) = 0,\ \ x \in \Upsilon, \quad \text{or} \quad -\partial_z u^N(x) = 0,\ \ x \in \Upsilon, \tag{6.2}$$

on the cross-section $\Upsilon = \{x \in \Xi : z = 0\}$ of the waveguide. Denoting by $\Pi_\pm(l)$ the trunks ((1.1)), we consider the family ($L \geq l$) of the waveguides $\Pi_\pm^L = \Pi_-(L) \cup \Theta(L) \cup \Pi_+(L)$ with the resonators $\Theta(L) = \{x : L^{-1} l x \in \Theta\}$ obtained from Θ by uniform Ll^{-1} times stretching. Problem (6.1), (6.2) is also considered in $\Xi^+(L)$. If the domain Θ is star-shaped relative to the origin, then $\Xi^+(L_2) \supset \Xi^+(L_1)$ whenever $L_2 > L_1$, so that the classical result [2] based on the maximin principle (see, e.g., [1, Theorem 10.2.2]) shows that the total multiplicity $\#\sigma_{di}(L)$ of the discrete spectrum on the interval $(0, \lambda^\dagger)$ grows unboundedly as $L \to +\infty$ for each problem (6.1), (6.2). Since the eigenvalues $\lambda_k(L)$ are monotone decreasing functions of the variable L, the increment of the discrete spectrum may only happen as a result of dropping eigenvalues down from the lower edge $\lambda^\dagger$ of the essential spectrum. Since Theorem 3.1 can easily be adapted to the two problems (6.1), (6.2), we see that there are two (D and N) unbounded sequences

$$l < L_1^{D/N} \leq L_2^{D/N} \leq \cdots \leq L_j^{D/N} \leq \cdots \to +\infty \tag{6.3}$$

of critical values of the parameter L for which the Diriclet/Neumann problem (6.1), (6.2) in $\Xi^+(L_j^{D/N})$ at the threshold $\lambda = \lambda^\dagger$ admits an almost standing wave or a trapped wave.

Unfortunately, the author was not able to find techniques for distinguishing these two types of waves. Corollary 3.1 can only help for realizing computational or asymptotic algorithms. Yet another open question is related to the simultaneous arising of two standing waves at the threshold. Such waves, odd and even in the variable z, come into being, e.g., whenever some terms L_j^D and L_k^N of the two sequences (6.3) coincide, but the possibility of such coincidence is an open question.

It is not difficult to give an example of an almost standing wave: such is the Floquet wave w_0 itself in the quasicylinder Π (formally: $\Theta = \{x \in \Pi : |z| \leq l\}$). An example of a trapped wave at the threshold frequency is built with the help of an elegant trick [30]: fictitions Dirichlet conditions are posed on the longitudinal cross-section $\Sigma = \{x \in \Xi : y_1 = 0\}$ of a symmetric waveguide

$$\Xi = \{x : (-y_1, y_2, \dots, y_{d-1}, z) \in \Xi\}.$$

The Dirichlet conditions on Σ admit smooth odd extension of solutions from the half $\Xi^\wedge = \{x \in \Xi : y_1 > 0\}$ (see Figure 4, b) to the entire waveguide Ξ. In particular, the lower bound $\lambda^\wedge$ of the essential spectrum of the Dirichlet problem in $\Xi^\wedge$ lies strictly above the previous threshold $\lambda^\dagger$; indeed, by Lemma 2.1, the new threshold $\lambda^\wedge = \Lambda_1^\wedge(0)$ is the first eigenvalue of problem (2.1)–(2.3) on the half-cell $\varpi^\wedge$, which coincides with some eigenvalue $\Lambda_q(0)$ of the same problem on the entire cell ϖ, but then, necessarily, we have $q > 1$, because the eigenfunction $U_1(0; x)$ vanishes for $y_1 = 0$. Then, like in Subsection 6.1, we dilate the resonator $\Theta^\wedge(L)$, watching the eigenvalue that drops down from threshold $\lambda^\wedge$. This eigenvalue is infinitely small as $L \to +\infty$ (by the maximin principle and a similar property of the eigenvalues of the Dirichlet problem in $\Theta^\wedge(L) = \{x : lL^{-1}x \in \Theta, y_1 > 0\}$). Since the dependence on $L \in (l, +\infty)$ is continuous and monotone (this is yet another consequence of the maximin principle), the eigenvalue under discussion turns into $\lambda^\dagger \in (0, \lambda^\wedge)$ at a certain point $L = L^\dagger$. The odd y_1-extension from the half $\Xi^\wedge(L^\dagger)$ to the entire waveguide $\Xi(L^\dagger)$ yields a trapped wave.

6.2. Absence of almost standing and trapped waves at the threshold. If the resonator is contained inside the part $\{x \in \Pi : |z| \leq l\}$ of the quasicylinder (1.2) (Figure 4,c), then the waves mentioned in the heading are absent. Indeed, acting as in §4 we can construct a perturbed waveguide Ξ^ε contained in Π as before, but yielding an eigenvalue $\lambda^\varepsilon \in (0, \lambda^\dagger)$ of the Dirichlet problem. We plug the corresponding eigenfunction $u^\varepsilon \in H_0^1(\Xi^\varepsilon)$, extended by zero to the entire quasicylinder Π, into the minimal principle (see, e.g., [1, Theorem 10.2.1]) for the operator A_Π of the Dirichlet problem in Π. This gives us the inequality

$$\inf_{u \in H_0^1(\Pi)} \frac{(\nabla_x u, \nabla_x u)}{(u, u)_\Pi} \leq \frac{(\nabla_x u^\varepsilon, \nabla_x u^\varepsilon)_{\Xi^\varepsilon}}{(u^\varepsilon, u^\varepsilon)_{\Xi^\varepsilon}} = \lambda^\varepsilon < \lambda^\dagger,$$

which means that the discrete spectrum of A_Π is nonempty, but this is well known to be impossible (see, e.g., [3], or [5, §3.4]).

6.3. Instability of almost standing waves. Since a trapped wave at the threshold $\lambda = \lambda^\dagger$ corresponds to an eigenvalue impregnated in the continuous spectrum, a small perturbation of the operator A of problem (1.3), e.g., a regular perturbation of the boundary (see [19, §7.6.5] and Subsection 3.3), can lead the eigenvalue out of the spectrum, turning it into a point of complex resonance (see the papers [30, 31], the survey [32], and other publications). However, certain asymptotic procedures including a “fine tuning” of one or several parameters $t = (t_1, \dots, t_N)$ in the profile function $h(s; t)$ defined as in (3.16), ensure the enforced stability (see [31, 29]) of the eigenvalue in question, i.e., it keeps staying in the spectrum.

The intrinsic instability is also inherent to almost standing waves at least for the reason that the criterion given by Theorem 3.2, in particular, formula (3.34), relates the arising of them to the eigenspace $\ker(S + \mathbb{I}_2)$ of the threshold scattering matrix, depending analytically on the parameters t occurring in the profile function $h(s;t)$. Thus, the quantity $\dim\ker(S(t) + \mathbb{I}_2)$ rarely remains stable when the (vector) parameter t varies – stability requires fine turning, as in [18, 31, 29].

In Subsection 4.2, everything was ready for the deduction of a simple condition ensuring the disappearance of a single almost standing wave under a regular perturbation of the waveguide's wall. Assuming that $\lambda_1^\varepsilon = \lambda^\dagger$, in particular, $\lambda_1' = 0$, we seek a solution of problem (3.18) in the form (4.16), arriving at problem (4.18) for the correction term u_1'. If the last integral $I_{11}(h)$ in (4.20) is nonzero, then the solution (4.19) cannot be bounded, because system (4.21) with zero ($\lambda_1' = 0$) left hand sides is unsolvable. Thus, in the case where $I_{11}(h) \neq 0$, problem (3.18) with sufficiently small $\varepsilon > 0$ admits no almost standing wave at the threshold.

6.4. Irregular variation of the waveguide's form. The dropping of an eigenvalue down from the threshold $\lambda = \lambda^\dagger$ may be ensured by not only regular, but also other perturbations of the wall $\partial\Xi$, in particular, by forming juts with small diameter or shallow steps (Figure 2,b and 2,c, respectively). The asymptotic methods that work in the case of an irregular variation of the boundary were fully elaborated in [27, Chapters 2, 4, 5, 9].

Specifics of the Dirichlet boundary conditions (narrowing the domain causes the growth of eigenvalues) show that the required effect cannot be achieved by forming cavities or inner holes, but volume's reduction in one part of the waveguide can be compensated for by growth in another part (cf. Figure 2). Similarly, matrices with entries (4.11) or (4.28) can become positive definite not only when the profile function in formula (3.16) is nonnegative.

If the surfaces $\partial\Pi$ and Γ, or the profile function h are not as smooth as required in Subsections 1.1 and 3.3, but are only piecewise smooth, then the methods of the book [27] allow us to obtain the same results as in §§4, 5, but the required asymptotic analysis complicates considerably.

6.5. Variational method. We reproduce an observation made in [33] in a similar situation: the smallness requirement imposed on the parameter $\varepsilon > 0$ in Theorem 5.3–5.5 in the part concerning the existence of eigenvalues is not essential provided $h \geq 0$. Indeed, if ε grows, then so does the domain Ξ^ε, and the maximum principle finds at least N eigenvalues on the interval $(0, \lambda^\dagger)$ for any ε such that the surface (3.16) lies inside the neighborhood $\mathcal{V}$ on which we have the local coordinates (n, s). As has already been mentioned, the construction of asymptotics allows us to identity the eigenvalues also in the case of profile functions of variable sign.

A more detailed comparison of the variational and asymptotic methods can be found in [26]. We mention, e.g. the publications [33, 34, 35, 36, 18, 37], where the search of eigenvalues is done via asymptotic constructions, and, e.g., [2, 30, 38, 39, 40], where the fact that the discrete spectrum is nonempty was proved via calculating the Rayleigh ratio and finding an appropriate test function that places this ratio below the cut-off point of the continuous spectrum.

It should be noted that, usually, smoothness requirements needed in the variational approach are weaker than those in the asymptotic methods.

6.6. On the choice of the profile of perturbation. In constructing the asymptotics of the set of eigenvalues lying an the interval $(0, \lambda^\dagger)$, one of the principal points is the linear independence of the products (4.32) of the traces on Γ of normal derivatives of trapped and almost standing waves. Lemma 4.5 verifies independence only if $N+M=2$, but in the general case this question remains open.

It should be noted that oscillating waves may fail to possess this property. For example, for $\lambda \in (\pi^2, 4\pi^2)$, the Dirichlet problem on the unit strip $\Pi = (0,1) \times \mathbb{R}$ admits two pairs of propagating waves

$$v^1_\pm(y,z) = e^{\pm iz\sqrt{\lambda-\pi^2}} \sin(\pi y), \quad v^2_\pm(y,z) = e^{\pm iz\sqrt{\lambda-4\pi^2}} \sin(2\pi y), \tag{6.4}$$

and if $\lambda = \frac{25}{8}\pi^2$, we have

$$\overline{\partial_y v^2_-(y,z)}\partial_y v^2_+(y,z) = 2\overline{\partial_y v^2_+(y,z)}\partial_y v^1_+(y,z) \quad \text{for} \quad y=0 \tag{6.5}$$

(we may also pass to real functions). Thus, the products in question are linearly dependent on the lower base $\Gamma_0\{0\} \times \mathbb{R}$ of the strip Π. However, for $y=1$ (on the upper base Γ_1) identity (6.5) fails. An example of linear dependence on the entire boundary $\partial\Pi = \Gamma_0 \cup \Gamma_1$ is obtained for $\lambda \in (9\pi^2, 16\pi^2)$, when yet another wave

$$v^3_\pm(y,z) = e^{\pm iz\sqrt{\lambda-9\pi^2}} \sin(3\pi y)$$

is added to the waves (6.4), and for $\lambda = 10\pi^2$ we have

$$\overline{\partial_y v^3_-(y,z)}\partial_y v^3_+(y,z) = 3\overline{\partial_y v^3_+(y,z)}\partial_y v^1_+(y,z) \text{ for } y=0,1.$$

It is highly plansible that the absence of oscillation in trapped and almost standing waves still results in the linear independence of the traces (4.32).

6.7. One more characteristic of trapped waves. The rates $O(\varepsilon)$ and $O(\varepsilon^2)$ at which the threshold is approached by the eigenvalues (4.1) and (4.15) generated by the trapped and the almost standing waves (respectively), differ in order. This observation supports the following conjecture, which unfortunately has remained unproved in the present paper: if there is a positive infinitely small sequence $\{\varepsilon_k\}$, $k \in \mathbb{N}$, of values of the small parameter ε for which problem (3.18) in the perturbed waveguide Ξ^ε has eigenvalues $\lambda^{\varepsilon_k} \in (0, \lambda^\dagger)$ such that

$$c\varepsilon \le \lambda^\dagger - \lambda^{\varepsilon_k} \le C\varepsilon, \quad c > 0, \tag{6.6}$$

then $\dim \mathcal{L}^{\mathrm{tr}} > 0$, and, at the threshold $\lambda = \lambda^\dagger$, problem (1.3) in the initial waveguide Ξ admits a trapped wave.

It should be noted that the change $\varepsilon \mapsto \varepsilon^2$ in inequalities (6.6) does not yield a sufficient condition for the existence of a standing wave at the threshold, because in no way does the relation $\lambda'_j = 0$ mean that the eigenvalue (4.1) cannot move downward from the cut-off point $\lambda^\dagger$, e.g., at the expense of the lower asymptotic term $-\varepsilon^2\lambda''_j$, not touched on in Subsection 4.1.

6.8. Discrete spectrum inside spectral gaps. In some situations, the asymptotics analysis presented above fits also for the study of near-threshold eigenvalues inside spectral gaps, not only below the essential spectrum (cf. [41]). However, the spectrum's nature near the ends of a spectral gap may differ substantially from that described in Lemma 2.1, which requires complication of the asymptotic procedure and leads to new conditions for dropping an eigenvalue down from the threshold.

References

[1] M. Š. Birman and M. Z. Solomjak, *Spectral theory of self-adjoint operators in Hilbert space*, Leningrad. Univ., Leningrad, 1980. MR0609148

[2] D. S. Jones, *The eigenvalues of $\nabla^2 u + \lambda u = 0$ when the boundary conditions are given on semi-infinite domains*, Proc. Cambridge Philos. Soc., v. 49, Cambridge Univ. Press., Cambridge, 1953, pp. 668–684. MR0058086

[3] S. A. Nazarov, *Elliptic boundary value problems with periodic coefficients in a cylinder*, Izv. Akad. Nauk SSSR Ser. Mat. **45** (1981), no. 1, 101–112; English transl., Math. USSR-Izv. **18** (1982), no. 1, 89–98. MR607578

[4] P. A. Kuchment, *Floquet theory for partial differential equations*, Uspekhi Mat. Nauk **37** (1982), no. 4, 3–52; English transl., Russian Math. Surveys **37** (1982), no. 4, 1–60. MR0667973

[5] S. A. Nazarov and B. A. Plamenevsky, *Elliptic problems in domains with piecewise smooth boundaries*, de Gruyter Expos. Math., vol. 13, Walter de Gruyter, Berlin, 1994. MR1283387

[6] P. A. Kuchment, *Floquet theory for partial differential equations*, Operator theory: Adv. Appl., vol. 60, Birchäuser, Basel, 1993. MR1232660

[7] I. C. Gohberg and M. G. Kreĭn, *Introduction to the theory of linear non-selfadjoint operators in Hilbert space*, Nauka, Moscow, 1965; English transl., Transl. Math. Monogr., vol. 18, Amer. Math. Soc., Providence, RI, 1969. MR0246142

[8] M. M. Vainberg and V. A. Trenogin, *The theory of branching solutions of nonlinear equations*, Nauka, Moscow, 1969. (Russian) MR0261416

[9] S. A. Nazarov, *Properties of spectra of boundary value problems in cylindrical and quasicylindrical domains*, Sobolev Spaces in Math. II, Int. Math. Ser. (N.Y.), vol. 9, Springer-Verlag, New York, 2008, pp. 261–309. MR2484629

[10] I. I. Vorovič and V. A. Babeško, *Dynamic mixed problems of the theory of elasticity for nonclassical domains*, Nauka, Moscow, 1979. (Russian) MR0566583

[11] S. A. Nazarov, *Umov–Mandel'shtam radiation conditions in elastic periodic waveguides*, Mat. Sb. **205** (2014), no. 7, 43–72; English transl., Sb. Math. **205** (2014), no. 7–8, 953–982. MR3242645

[12] L. I. Mandel′shtam, *Lectures on optics, relativity and quantum mechanics*, Izdat. AN SSSR, Moscow, 1947. (Russian)

[13] S. A. Nazarov, *The Mandel'stam energy radiation conditions and the Umov–Poynting vector in elastic waveguides*, Probl. Mat. Anal., vyp. 72, T. Rozkovskaya, Novisibirsk, 2013, pp. 101–146; English transl., J. Math. Sci. (N. J.) **195** (2013), no. 5, 676–729. MR3155335

[14] N. A. Umov, *The equations of motion of the energy in bodies*, Odessa, 1874. (Russian)

[15] J. H. Poynting, *On the transfer of energy in the electromagnetic field*, Philos. Trans. Roy. Soc. London, v. 175, London, 1884. pp. 343–361.

[16] S. A. Nazarov and B. A. Plamenevskiĭ, *Radiation conditions for selfadjoint elliptic problems*, Dokl. Akad. Nauk SSSR **311** (1990), no. 3, 532–536; English transl., Soviet Math. Dokl. **41** (1990), no. 2, 274–277. MR1075678

[17] ———, *Radiation principles for selfadjoint elliptic problems*, Probl. Mat. Fiz., vyp. 13, Leningrad. Univ., Leningrad, 1991, 192–244. (Russian) MR1341639

[18] S. A. Nazarov, *Asymptotic expansions of eigenvalues in the confinuous spectrum of a regularly perturbed quantum waveguide*, Teor. Mat. Fiz. **167** (2011), no. 2, 239–263; English transl., Theoret. and Math. Phys. **167** (2011), no. 2, 606–627. MR3166368

[19] T. Kato, *Perturbation theory for linear operators*, 2nd ed., Grundlehren Math. Wiss., Bd. 132, Springer-Verlag, Berlin, 1976. MR0407617

[20] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for soluitons of elliptic partial differential equations satisfying general boundary conditions*. I, Comm. Pure Appl. Math. **12** (1959), no. 1, 623–727. MR0125307

[21] S. A. Nazarov, *A criterion for the existence of decaying solutions in the problem of a resonator with a cylindrical waveguide*, Functsional. Anal. i Prilozhen. **40** (2006), no. 2. 20–32; English transl., Funct. Anal. Appl. **40** (2006), no. 2, 97–107. MR2256860

[22] ———, *Artifical boundary conditons for finding surface waves in the problem of diffraction by a periodic boundary*, Zh. Vychisl. Mat. Mat. Fiz. **46** (2006), no. 12, 2265–2276; English transl., Comput. Math. Phys. **46** (2006), no. 12, 2164–2175. MR2344971

[23] L. Bers, J. Fitz, and M. Schechter, *Partial differential equations*, Lectures in Appl. Math., vol. 3, Intersci. Publ. J.Wiley& Sons, Inc., New York, 1964. MR0163043

[24] M. van Dyke, *Perturbation methods in fluid mechanics*, Appl. Math. and Mech., vol. 8, Acad. Press, New York, 1964. MR0176702

[25] A. M. Il′in, *Matching of asymptotic expansions of solutions of boundary value problems*, Nauka, Moscow, 1989. (Russian) MR1007834

[26] S. A. Nazarov, *Variational and asymptotic methods for finding eigenvalues below the continuous spectrum threshold*, Sibirsk. Mat. Zh. **51** (2010), no. 5, 1086–1101; English transl., Sib. Math. J. **51** (2010), no. 5, 866–878. MR2768506

[27] W. G. Mazja, S. A. Nasarow, and B. A. Plamenewski, *Asymptotische Theorie elliptischer Randwertaufgaben in singulär gestörten Gebieten.* I, II, Akademie-Verlag, Berlin, 1991. MR1101139; English transl. V. Maz′ya, S. Nazarov, and B. Plamenevskij, *Asymptotic theory of elliptic boundary value problems in singularly perturbed domains.* Vol. 1, 2, Birkhäuser Verlag, Basel, 2000. MR1779978

[28] S. A. Nazarov, *Asymptotic conditions at a point, selfadjoint extensions of operators, and the method of matched asymptotic expansions*, Tr. St.-Peterburg. Math. Obshch. **5** (1998), 112–183; English transl., Amer. Math. Soc. Transl. Ser. 2, vol. 193, Amer. Math. Soc., Providence, RI, 1999, pp. 77–125. MR1736907

[29] ———, *Forced stability of a simple eigenvalue in the continuous spectrum of a waveguide*, Functsional. Anal. i Prilozhen. **47** (2013), no. 3, 37–53; English transl., Funct. Anal. Appl. **47** (2013), no. 3, 195–209. MR3154838

[30] D. V. Evans, M. Levitin, and D. Vasil′ev, *Existence theorems for trapped modes*, J. Fluid Mech. **261** (1994), 21–31. MR1265871

[31] S. A. Nazarov, *Enforced stability of an eigenvalue in the continuous spectrum of a waveguide with an obstacle*, Zh. Vychisl. Mat. Mat. Fiz. **52** (2012), no. 3, 521–538; English transl., Comput. Math. Math. Phys. **52** (2012), no. 3, 448–464. MR3244976

[32] C. M. Linton and P. McIver, *Embedded trapped modes in water waves and acoustics*, Wave Motion. **45** (2007), 16–29. MR2441664

[33] W. Bulla, F. Gesztesy, W. Renger, and B. Simon, *Weakly coupled bound states in quantum waveguides*, Proc. Amer. Math. Soc. **125** (1997), no. 8, 1487–1495. MR1371117

[34] V. V. Grushin, *On the eigenvalues of a finitely perturbed Laplace operator in infinite cylindrical domains*, Mat. Zametki **75** (2004), no. 3, 360–370; English transl., Math. Notes **75** (2004), no. 3–4, 331–340. MR2068799

[35] R. R. Gadyl′shin, *On local perturbations of quantum waveguides*, Teoret. Mat. Fiz. **145** (2005), no. 3, 358–371; English transl., Theoret. and Math. Phys. **145** (2005), no. 3, 1678–1690. MR2243437

[36] D. Borisov, P. Exner, and R. Gadyl′shin, *Geometric coupling thresholds in a two-dimensional strip*, J. Math. Phys. **43** (2005), no. 12, 6265–6278. MR1939643

[37] S. A. Nazarov, *Asymptotics of an eigenvalue on the continuous spectrum of two quantum waveguides coupled through narrow windows*, Mat. Zametki **93** (2013), no. 2, 227–245; English transl., Math. Notes **93** (2013), no. 2, 266–281. MR3205969

[38] I. V. Kamotskii and S. A. Nazarov, *Wood's anomalies and surface waves in the problem of scattering by a periodic boundary* I, II, Math. Sb. **190** (1999), no. 1, 109–138; no. 2, 227–245; English transl., Sb. Math. **190** (1999), no. 1–2, 111–141, 205–231. MR1700697 MR1701000

[39] S. A. Nazarov, *Trapped modes for a cylindrical elastic waveguide with a damping gasket*, Zh. Vychisl. Mat. Mat. Fiz. **48** (2008), no. 5, 863–881; English transl., Comput. Math. Math. Phys. **48** (2008), no. 5, 816–833. MR2433645

[40] ———, *Sufficient conditions for the existence of trapped modes in problems of the linear theory of surface waves*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **369** (2009), 202–223; English transl., J. Math. Sci. (N. J.) **167** (2010), no. 5, 713–725. MR2749207

[41] ———, *Gaps and eigenfrequencies in the spectrum of a periodic acoustic waveguide*, Akustich. Zh. **59** (2013), no. 3, 312–321. (Russian)

Department of Mathematics and Mechanics, St. Petersburg State University, Universitetskiĭ pr. 28, Petrodvoretz, 198504 St. Petersburg, Russia;

Peter the Great St. Petersburg Polytechnical University, Polytechnicheskaya ul. 29, 195251 St. Petersburg, Russia;

Institute of Mechanical Engineering Problems, Bol′shoĭ pr. V. O. 61, 199178 St. Petersburg, Russia

E-mail address: srgnazarov@yahoo.co.uk

E-mail address: s.nazarov@spbu.ru

Received 20/NOV/2015

Translated by A. PLOTKIN